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TAYLOR SERIES EXPANSIONS FOR POISSON-DRIVEN (max, +)-LINEAR SYSTEMS[1]

BY FRANÇOIS BACCELLI[2] AND VOLKER SCHMIDT

INRIA–Sophia Valbonne and University of Ulm

We give a Taylor series expansion for the mean value of the canonical stationary state variables of open (max, +)-linear stochastic systems with Poisson input process. Probabilistic expressions are derived for coefficients of all orders, under certain integrability conditions. The coefficients in the series expansion are the expectations of polynomials, known in explicit form, of certain random variables defined from the data of the (max, +)-linear system.

These polynomials are of independent combinatorial interest: their monomials belong to a subset of those obtained in the multinomial expansion; they are also invariant under cyclic permutation and under translations along the main diagonal.

The method for proving these results is based on two ingredients: (1) the (max, +)-linear representation of the stationary state variables as functionals of the input point process; (2) the series expansion representation of functionals of marked point processes and, in particular, of Poisson point processes.

Several applications of these results are proposed in queueing theory and within the framework of stochastic Petri nets. It is well known that (max, +)-linear systems allow one to represent stochastic Petri nets belonging to the class of event graphs. This class contains various instances of queueing networks like acyclic or cyclic fork-join queueing networks, finite or infinite capacity tandem queueing networks with various types of blocking (manufacturing and communication), synchronized queueing networks and so on. It also contains some basic manufacturing models such as Kanban networks, Job-Shop systems and so forth. The applicability of this expansion method is discussed for several systems of this type. In the *M/D* case (i.e., all service times are deterministic), the approach is quite practical, as all coefficients of the expansion are obtained in closed form. In the *M/GI* case, the computation of the coefficient of order k can be seen as that of joint distributions in a stochastic PERT graph of an order which is linear in k.

1. Introduction. Under the notion of an open (max, +)-linear stochastic system, one understands a sequence $\{X_n\}$ of random vectors satisfying the recursion $X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1}$ where the addition $\oplus$ means taking the maximum and multiplication $\otimes$ means $+$. Here $\{T_n\}$ is an increasing sequence of real-valued random variables, and $\{A_n\}$ and $\{B_n\}$ are station-

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ary sequences of random matrices. Such systems allow one to represent the dynamics of stochastic Petri nets belonging to the class of event graphs (see [2] and [6]). In particular, this class contains various instances of queueing networks like acyclic or cyclic fork-join queueing networks, finite or infinite capacity tandem queueing networks with various types of blocking (manufacturing and communication), synchronized queueing networks and so on. It also contains some basic manufacturing models such as Kanban networks, Job-Shop systems and so forth. In all these models, T_n is the arrival epoch of the nth customer in the network and the coordinates X_n^i of the state vector $X_n = (X_n^1, \dots, X_n^\alpha)$ represent absolute times (like the beginning of the nth service in the ith queue) which grow to ∞ when n increases unboundedly. For this reason, one is actually more interested in the differences $W_n^i = X_n^i - T_n$ (like the waiting time of the nth customer until the beginning of his service in queue i), which are expected to admit a stationary state $W^i = \lim_{n\to\infty} W_n^i$ (in distribution) under certain rate conditions. Unfortunately, in most cases, and particularly for systems of dimension α larger than 2, it is impossible to determine characteristics of the random vector $W = (W^1, \dots, W^\alpha)$ in closed form (e.g., by complex-variable techniques, which are essentially limited to two unbounded coordinates). Even in the case when all system data are exponential, analytical formulas for the expectation vector $\mathbb{E}W = (\mathbb{E}W^1, \dots, \mathbb{E}W^\alpha)$ are only known for rather specific models; see, for example, [16] and [33]. The only case in which the stationary (or more precisely the periodic) regime(s) of such multidimensional (max, +)-linear systems is known in explicit form seems to be the purely deterministic case ([6], [18] and [19]). This motivated our research to derive a method which makes it possible to determine an expansion for $\mathbb{E}W$ holding for stochastic systems of any dimension.

Assuming that $\{T_n\}$ is a homogeneous Poisson process with intensity λ and that the sequences $\{A_n\}$ and $\{B_n\}$ have certain independence properties, we derive a series expansion for $\mathbb{E}W$ with respect to the arrival intensity λ. For this, we use a general method which consists of expanding the expectation of a vector-valued functional of a marked point process using its factorial moment measures. The roots of this method can be traced back to the following papers: [3], [10], [11], [13], [14], [21], [30], [34] and [37]. For univariate (nonmarked) point processes, this concept has been developed in [10] starting from a corresponding first-order expansion obtained in [3]. Related higher-order expansions for functionals of independently marked Poisson processes have been considered in [21], [30] and [34], and for more general marked point processes in [11] and [13]; see also the survey given in [14].

Under certain monotonicity and integrability conditions on $\{A_n, B_n\}$, we derive a probabilistic expression for the coefficients c_k^i of all orders k of the expansion

$$\mathbb{E}W^i = \sum_{k=0}^{m} c_k^i \lambda^k + \mathscr{O}(\lambda^{m+1}).$$

Namely, we show that $c_k^i = \mathbb{E}p_{k+1}(D_0^i, D_1^i, \dots, D_k^i)$, where D_n^i is the ith component of the random vector $D_n = A_{-1} \otimes \cdots \otimes A_{-n} \otimes B_{-n}$. The mappings

p_k: $\mathbb{R}^k \to \mathbb{R}$ are polynomials which are determined explicitly. They are of independent combinatorial interest. Their monomials belong to a subset of those obtained in the multinomial expansion; they are also invariant under cyclic permutation and under translation along the main diagonal.

The applicability of the derived algorithm to determine an approximation of $\mathbb{E}W$ is discussed for several examples of queueing networks. In the M/D case (i.e., all service times are deterministic), our approach is quite practical as all coefficients of the expansion are obtained in closed form. In the M/GI case, the computation of the coefficient of order k can be seen as that of joint distributions in a stochastic PERT graph of an order which is linear in k, a problem for which no polynomial algorithms are apparently known. We nevertheless show that expansions of limited order can be obtained in explicit form along these lines.

One can find several other earlier attempts to approximate characteristics of queueing systems by expanding them into a series. The derivation of expansions for characteristics of continuous-time Markov chains associated with single-server queues and stochastic networks has been studied by several authors (see, e.g., [8], [9], [24] and [35]). There is also an extensive literature on expansions for non-Markov queues in isolation (see [7], [23], [25]–[27], [32] and [38]).

In relation to this, we would like to stress that our new approach applies to:

1. networks with general (and possibly correlated) (see subsection 4.2.2) service times, whereas earlier approaches for networks apply essentially to the all exponential case;
2. a class of systems [(max, +)-linear systems] which is defined via its structural properties, and not via properties of its Markovian generator.

The paper is organized as follows. In Section 2 some preliminaries are given including the basic recurrence equations and the $(\max, +)$-representation of stationary state variables. Section 3 contains the main expansion formula and the conditions under which this expansion holds. General properties of the polynomials p_k appearing in the coefficients of the expansion are stated and the polynomials $p_1, p_2, \dots, p_5$ up to order 5 are calculated explicitly. In Section 4 we discuss several examples of discrete event systems, the state variables of which satisfy the basic recurrence equations, in particular stochastic event graphs such that all places and transitions are FIFO. For practical examples of queueing, communication and manufacturing systems that fall in the class of stochastic event graphs, it is demonstrated how expansions of $\mathbb{E}W$ can be found. The general method of factorial-moment expansion is stated in Section 5, together with the proof that it is allowed to use this method for expanding $\mathbb{E}W$. Section 6 is devoted to the calculation of the mappings p_k which appear in the expansion coefficients. First we show that the p_k's satisfy a recursive integral equation and next we derive an explicit polynomial solution of this equation. In Section 7 we show that, under certain tail conditions on the random variables D_n^i, the functions $f^i(\lambda) = \mathbb{E}W^i$ are infinitely differentiable in λ in a right neighborhood of 0, and that, for all $n \geq 1$,

$$\lim_{\lambda \downarrow 0} (f^i(\lambda))^{(n)} = \mathbb{E} p_{n+1}(D_0^i, D_1^i, \ldots, D_n^i).$$

In that sense, the expansion stated above for $\mathbb{E}W^i$ is of Taylor type.

2. Preliminaries.

2.1. *Basic equations.* The basic reference algebra throughout this paper is the so-called (max, +)-algebra on the real line $\mathbb{R}$, namely the semi-field with the two operations $(\oplus, \otimes)$, where $\oplus$ is max and $\otimes$ is $+$. As in the conventional algebra, $\otimes$ has priority over $\oplus$ in all arithmetic expressions.

Let $\alpha \in \mathbb{N} = \{1, 2, \ldots\}$ be any given natural number. For convenience, we denote the entries of a matrix A by $A_{i,j}$, and the components of an (α-dimensional) column vector X by X^i; that is, considering X as an $\alpha \times 1$ matrix, we have $X_{i,1} = X^i$. As in the conventional algebra, we use the same symbols $\oplus, \otimes$ to represent the sum and the product of two reals and two matrices, respectively. The $\otimes$-product of two matrices, say A of size $p \times q$ and B of size $q \times r$, is the $p \times r$ matrix $A \otimes B$ with entries

$$(A \otimes B)_{i,j} = \bigoplus_{k=1}^{q} A_{i,k} \otimes B_{k,j} \tag{1}$$

and the $\oplus$-sum of two matrices, say A and B both of size $p \times q$, is the $p \times q$ matrix $A \oplus B$ with entries

$$(A \oplus B)_{i,j} = A_{i,j} \oplus B_{i,j}. \tag{2}$$

The main topic of this paper is the set of α-dimensional vectorial recurrence equations

$$X_{n+1} = A_n \otimes X_n \oplus B_{n+1} \otimes T_{n+1}, \tag{3}$$

with initial condition X_0, where:

1. $\{T_n\}$ is a nondecreasing sequence of real-valued random variables (the epochs of the *arrival point process*—a Poisson point process in most examples below);
2. $\{A_n\}$ is a sequence of $\alpha \times \alpha$ random matrices with real-valued random entries;
3. $\{B_n\}$ is a sequence of $\alpha \times 1$ random matrices with real-valued random entries;
4. $\{X_n\}$ is the sequence of α-dimensional state vectors.

Various examples of discrete event systems with state variables satisfying an equation of type (3) are provided in Section 4.

In all the applications presented below, the coordinates of the state vector X_n represent absolute times (like the beginning of the nth service in a queue) which grow to ∞ when n increases unboundedly, and one is actually more interested in the differences

$$W_n^i = X_n^i - T_n \tag{4}$$

(like the waiting time of the nth customer until the beginning of his service in queue i; see subsection 4.2), which are expected to admit a stationary regime under certain rate conditions. Let $\tau_n = T_{n+1} - T_n$, $n \geq 0$. By subtracting T_{n+1} on both sides of (3), it is easily checked that the (new) state vector W_n, given by (4) and being of dimension α as well, satisfies the linear evolution equation

$$W_{n+1} = A_n \otimes C(\tau_n) \otimes W_n \oplus B_{n+1}, \tag{5}$$

where, for all $x \in \mathbb{R}$, $C(x)$ is the $\alpha \times \alpha$ matrix with all diagonal entries equal to $-x$ and all nondiagonal entries equal to $\varepsilon = -\infty$.

2.2. *Ergodic theorem.* Although in this paper we will focus on the case when the random counting measure $\sum_n \delta_{T_n}$ (where δ_x denotes the Dirac measure at point x) is a stationary Poisson process, for the moment let us consider the more general model where $\sum_n \delta_{(T_n, A_n, B_n)}$ is an arbitrary stationary and ergodic marked point process N on the real line, defined on a probability space $(\Omega, \mathscr{F}, P)$, and such that (A_n, B_n) is a *mark* of point T_n. We further assume that $(\Omega, \mathscr{F}, P)$ is equipped with a group $\{\theta_t\}_{t\in\mathbb{R}}$ of measurable shift operators θ_t: $\Omega \to \Omega$ such that P is invariant with respect to $\{\theta_t\}$ and that N is consistent with $\{\theta_t\}$, that is, $P \circ \theta_t = P$ and $N \circ \theta_t = N(\cdot + t)$.

Moreover, we assume that N is simple. That is, with probability 1 there are no multiple points, and that N has a positive and finite intensity λ. Point processes on the real line and, in particular, queueing systems with arrival epochs forming a stationary ergodic marked point process were studied in [4], [20] and [29]. An important special case of that is the case of a renewal arrival process and an independent (i.i.d. or Markovian) sequence $\{A_n, B_n\}$, which is sometimes referred to as the renewal-Markov case.

Let P^0 denote the Palm probability of N, and θ the discrete (pointwise) shift associated with the continuous-time shifts θ_t. By T_0 we denote the smallest nonnegative point of N. That is, $P^0(T_0 = 0) = 1$ and T_n, $n < 0$, is the nth point of N on the negative half-line $\mathbb{R}^- = (-\infty, 0)$. Let $A = A_0$ and $B = B_0$; so, for all integers n,

$$A_n = A \circ \theta^n, \qquad B_n = B \circ \theta^n, \qquad P^0\text{-a.s.} \tag{6}$$

Similarly, let $C = C(\tau_0)$. That is, $C(\tau_n) = C \circ \theta^n$ under P^0. Note that the sequence $\{A_n, B_n\}$ is stationary under both P and P^0 provided that $\{T_n\}$ and $\{A_n, B_n\}$ are independent.

The following result is proved in Chapter 7 of [6].

THEOREM 1. *Assume that the matrices A_n, B_n are P^0-integrable. If $\rho < 1$, where $\rho = \lambda a$ and a is the maximal* (max, +)*-Liapounov exponent of the sequence $\{A_n\}$, then the sequence W_n couples P^0-a.s. with a unique stationary* *sequence* $\{W \circ \theta^n\}$ *on* $(\Omega, \mathscr{F}, P^0)$, *where W is the unique finite random-variable solution of the functional equation*

$$W \circ \theta = A \otimes C \otimes W \oplus B \circ \theta, \tag{7}$$

which is given by the following matrix-series:

$$W = B \oplus \bigoplus_{n \geq 1} A \circ \theta^{-1} \otimes C \circ \theta^{-1} \otimes \cdots \otimes A \circ \theta^{-n} \otimes C \circ \theta^{-n} \otimes B \circ \theta^{-n}. \tag{8}$$

REMARK. Since $C(x)$ commutes with any matrix, W admits the following equivalent representation:

$$W = B_0 \oplus \bigoplus_{n \geq 1} C(-T_{-n}) \otimes D_n, \tag{9}$$

where

$$D_n = A_{-1} \otimes \cdots \otimes A_{-n} \otimes B_{-n}. \tag{10}$$

The main object of this paper are characteristics of the law F^0 of W under P^0. In a single-server queue, F^0 boils down to the distribution of the stationary actual waiting time. In some cases, one is also interested in the law F of W as defined in (9) under P. This is the law of the X_0 vector that a tagged customer arriving at time 0 would experience, superimposed to the time-stationary arrival pattern. In a single-server queue, F boils down to the distribution of the stationary virtual waiting time. If N is Poisson, then F and F^0 coincide provided the marks are independent of the epochs of N.

3. Main results. We show that, under some assumptions stated below, the expectation $\mathbb{E}W$ of the stationary state variable W given in (9) is finite and that the components of $\mathbb{E}W$ can be expanded into a finite power series with respect to the arrival intensity λ. Moreover, we derive an explicit polynomial expression for the coefficients of this expansion.

As we will see in the next section (see particularly Lemmas 1 to 3), the assumptions stated below are satisfied whenever the recurrence equations (3) originate from a so-called *open stochastic event graph.*

3.1. *Support and monotonicity assumptions.* We assume that each entry of A_n is either a.s. nonnegative or a.s. equal to ε; that is,

$$(A_n)_{i,j} \geq 0 \quad \text{or} \quad (A_n)_{i,j} = \varepsilon, \quad P^0\text{-a.s.}, \tag{11}$$

and that all entries on the diagonal of A_n are nonnegative; that is, $(A_n)_{i,i} \geq 0$.

We also assume that there exists an integer $0 < \alpha' \leq \alpha$ such that the first α' coordinates of B_n are nonnegative. That is, $B_n^i \geq 0$ for all $i \leq \alpha'$. Moreover, the α-dimensional vectors $D_0, D_1, \ldots$ with $D_0 = B_0$ and

$$D_k = \left(\bigotimes_{n=1}^{k} A_{-n} \right) \otimes B_{-k} \quad \text{for } k \geq 1, \tag{12}$$

are assumed to be such that

$$0 \le D_0^i \le D_1^i \le \cdots \tag{13}$$

for all $i = 1, \ldots, \alpha'$.

3.2. *Stochastic assumptions.* Throughout the rest of this paper we assume that $\{T_n\}$ is a stationary Poisson process with intensity λ and $\{A_n, B_n\}$ is a stationary sequence of random matrices which is independent of $\{T_n\}$. Under these assumptions, the law of $\{A_n, B_n\}$ is the same under P and P^0. We assume that

$$\lim_{n\to\infty} \bigoplus_i \{(A_{-1} \otimes A_{-2} \otimes \cdots \otimes A_{-n} \otimes (B_{-n} \oplus O))^i\} = \infty, \quad P^0\text{-a.s.}, \tag{14}$$

where O is the α-dimensional column vector with all its components equal to 0. Note that (14) is practically always fulfilled because it is sufficient for (14) that $\mathbb{E}A_{i,i} > 0$ for some i, where $\mathbb{E}$ denotes expectation with respect to P^0 (or equivalently with respect to P for this specific expression). Besides this we will assume that, for $r \in \mathbb{N}$ large enough,

$$\lambda < r\left[\mathbb{E}\bigoplus_i \{(A_{-1} \otimes A_{-2} \otimes \cdots \otimes A_{-r} \otimes (B_{-r} \oplus O))^i\}\right]^{-1}. \tag{15}$$

We also assume that, for the same r as above, $\{H_n\}$ with

$$H_n = \bigoplus_i \{(A_{-(nr+1)} \otimes A_{-(nr+2)} \otimes \cdots \otimes A_{-(n+1)r} \otimes (B_{-(n+1)r} \oplus O))^i\} \tag{16}$$

is a sequence of 1-dependent random variables. Finally, we assume that

$$\mathbb{E}[(H_n)^{m+3}] < \infty \tag{17}$$

for some $m \in \mathbb{N}$.

3.3. *Main theorem.*

THEOREM 2. *Under the above assumptions on $\{A_n, B_n\}$, for all $1 \le i \le \alpha'$,*

$$\mathbb{E}W^i = \sum_{k=0}^{m} \lambda^k \mathbb{E} p_{k+1}(D_0^i, D_1^i, \ldots, D_k^i) + \mathscr{O}(\lambda^{m+1}). \tag{18}$$

The functions p_k are the following polynomials:

$$p_k(x_0, x_1, \ldots, x_{k-1}) = \sum_{(i_0, i_1, \ldots, i_{k-1}) \in S_k} (-1)^{q_k(i_0, i_1, \ldots, i_{k-1})} \frac{x_0^{i_0}}{i_0!} \frac{x_1^{i_1}}{i_1!} \cdots \frac{x_{k-1}^{i_{k-1}}}{i_{k-1}!}, \tag{19}$$

where

$$S_k = \{(i_0, i_1, \ldots, i_{k-1}) \in \{0, 1, \ldots\}^k \colon i_0 + i_1 + \cdots + i_{k-1} = k \text{ and if } i_s = l > 1, \text{ then } i_{s-1} = i_{s-2} = \cdots = i_{s-l+1} = 0\}$$

(*the* $s - j$ *are modulo* k) *and*

$$q_k(i_0, i_1, \dots, i_{k-1}) = 1 + \sum_{s=0}^{k-1} \mathbf{1}(i_s > 0).$$

In particular, *we get*

$$p_1(x_0) = x_0, \qquad p_2(x_0, x_1) = \tfrac{1}{2}\big[x_0^2 + x_1^2 - 2x_0x_1\big], \tag{20}$$

$$p_3(x_0, x_1, x_2) = \tfrac{1}{6}\big[x_0^3 + x_1^3 + x_2^3 - 3(x_0^2x_1 + x_1^2x_2 + x_2^2x_0) + 6x_0x_1x_2\big], \tag{21}$$

$$\begin{aligned} p_4(x_0, x_1, x_2, x_3) = \tfrac{1}{24}\big[&x_0^4 + x_1^4 + x_2^4 + x_3^4 \\ &- 4(x_0^3x_1 + x_1^3x_2 + x_2^3x_3 + x_3^3x_0) \\ &- 6(x_0^2x_2^2 + x_1^2x_3^2) \\ &+ 12(x_0^2x_1x_2 + x_1^2x_2x_3 + x_2^2x_3x_0 + x_3^2x_0x_1) \\ &- 24x_0x_1x_2x_3\big], \end{aligned} \tag{22}$$

$$\begin{aligned} &p_5(x_0, x_1, x_2, x_3, x_4) \\ &\quad = \tfrac{1}{120}\big[x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 \\ &\qquad - 5(x_0^4x_1 + x_1^4x_2 + x_2^4x_3 + x_3^4x_4 + x_4^4x_0) \\ &\qquad - 10(x_0^3x_2^2 + x_1^3x_3^2 + x_2^3x_4^2 + x_3^3x_0^2 + x_4^3x_1^2) \\ &\qquad + 20(x_0^3x_1x_2 + x_1^3x_2x_3 + x_2^3x_3x_4 + x_3^3x_4x_0 + x_4^3x_0x_1) \\ &\qquad + 30(x_0^2x_2^2x_3 + x_1^2x_3^2x_4 + x_2^2x_4^2x_0 + x_3^2x_0^2x_1 + x_4^2x_1^2x_2) \\ &\qquad - 60(x_0^2x_1x_2x_3 + x_1^2x_2x_3x_4 + x_2^2x_3x_4x_0 \\ &\qquad\qquad + x_3^2x_4x_0x_1 + x_4^2x_0x_1x_2) + 120x_0x_1x_2x_3x_4\big]. \end{aligned} \tag{23}$$

The proof of Theorem 2 is given in Sections 5 and 6. First, in Section 5, a general expansion technique for functionals of marked point processes is used in order to show that an expansion of $\mathbb{E}W^i$ of the form (18) exists. The polynomial representation (19) of the coefficients $\mathbb{E}p_{k+1}(D_0^i, D_1^i, \dots, D_k^i)$ is derived in Section 6. In particular, it is shown that (18) is equivalent to

$$\mathbb{E}W^i = \mathbb{E}p_1(D_0^i) + \sum_{k=1}^{m} \lambda^k \mathbb{E}p_{k+1}(0, D_1^i - D_0^i, \dots, D_k^i - D_0^i) + \mathscr{O}(\lambda^{m+1}). \tag{24}$$

However, before stating the proof of Theorem 2 in detail, we give some examples of application in Section 4.

REMARK. We found no earlier use of this class of polynomials in the literature. Below we summarize some of their key properties (see also Section 6):

1. The polynomials p_k, $k \geq 1$, are invariant with respect to circular permutation. That is,
$$p_k(x_0, x_1, \ldots, x_{k-1}) = p_k(x_1, x_2, \ldots, x_{k-1}, x_0)$$
for all $x_0, \ldots, x_{k-1} \in \mathbb{R}$.
2. The polynomials p_k, $k \geq 2$, are 1-invariant. That is, for all $t \in \mathbb{R}$,
$$p_k(x_0, x_1, \ldots, x_{k-1}) = p_k(x_0 + t, x_1 + t, \ldots, x_{k-1} + t).$$
3. The polynomials p_k, $k \geq 1$, satisfy the integral recurrence relation:
$$\begin{aligned} p_{k+1}&(x_0, x_1, \ldots, x_k) \\ &= \sum_{n=0}^{k-1} \int_{x_n - x_0}^{x_{n+1}-x_0} \Big[p_k(\underbrace{x_0, \ldots, x_0}_{n}, x_{n+1} - u, \ldots, x_k - u) \\ &\qquad\qquad - p_k(\underbrace{x_0, \ldots, x_0}_{n+1}, x_{n+1} - u, \ldots, x_{k-1} - u) \Big] \, du. \end{aligned} \tag{25}$$
4. For all $k \geq 2$,
$$p_k(0, \ldots, 0) = 0,$$
$$p_k(1, \ldots, k) = \tfrac{1}{2}.$$
5. For all $t \in \mathbb{R}$,
$$p_k(tx_0, tx_1, \ldots, tx_{k-1}) = t^k p_k(x_0, x_1, \ldots, x_{k-1}).$$

4. Examples. In this section we provide several examples of discrete event systems with the state variables satisfying a vectorial recurrence equation of type (3).

4.1. *Stochastic event graphs.* A *Petri net* is defined as a tuple $PN = (\mathscr{P}, \mathscr{T}, \mathscr{F}, \mathscr{M}_0)$, where
$$\mathscr{P} = \{p_1, p_2, \ldots, p_P\} \text{ is the set of places,}$$
$$\mathscr{T} = \{u_1, u_2, \ldots, u_\beta\} \text{ is the set of transitions,}$$
$$\mathscr{F} \subseteq (\mathscr{P} \times \mathscr{T}) \cup (\mathscr{T} \times \mathscr{P}) \text{ is the set of arcs,}$$
$$\mathscr{M}_0\colon \mathscr{P} \to \{0, 1, 2, 3, \ldots, M\} \text{ is the initial marking.}$$
A Petri net is an *event graph* if each place has not more than one input and one output arc. A *timed Petri net* is a net with firing times associated with the transitions. The firing time of a transition is the time that elapses between the starting and the completion of the firing of the transition. If firing times are random variables, we speak of a *stochastic Petri net*. A typical situation is when the sequence $\{\sigma_n^1, \ldots, \sigma_n^\beta\}_n$, where σ_n^i is the nth firing time of transition i, is i.i.d. In what follows, these independence conditions are assumed to be satisfied and will be referred to as a *GI-stochastic event graph* later on. This

admits as a particular case the situation when the successive firing times of transition i are i.i.d. for all i, and, in addition, the firing times of the various transitions are all mutually independent.

Besides this we always assume that the expected firing times are finite. We remark, however, that all our arguments used below (in particular, those used in subsection 5.3) remain true in the case when the nth random firing times of several transitions are realized at once and, therefore, are not independent. In this way, the tandem queues with identical successive service times considered, for example, in [15] and [31] can also be investigated by our expansion results. Moreover, our arguments easily extend to the case that, within the sequence of nth, $(n+1)$th, ... firing-time vectors, there is a correlation structure of finite range.

With any stochastic event graph, we associate a set of random matrices $A_0(n), A_1(n), \dots, A_M(n)$, all of dimension $\beta\times\beta$, defined as follows: $M < \infty$ is the maximal initial marking. The entry i, j of matrix $A_k(n)$ is the firing time σ^j_{n-k} of the $(n-k)$th firing of transition u_j, whenever there is a place p with k tokens in the initial marking and a path $u_j \to p \to u_i$ (namely an arc from u_j to p and one from p to u_i). If there is no such place, this entry is given the value ε $(= -\infty)$.

To the above event graph, we may add an input structure, namely an input transition u with no input arcs, input places that connect u to the internal transitions of the net, and a real-valued increasing sequence input function T_n (with the interpretation that T_n is the epoch of the nth external arrival to the input transition u). For the sake of simplicity, we will assume that all input places have a 0 initial marking, so that M will again denote the maximal initial marking in all places (including input places). Associated with such a structure is the sequence of matrices $B_0(n)$, of dimension $\beta \times 1$, defined as follows: the entry i, 1 of matrix $B_0(n)$ is 0 whenever there is an input place p and a path $u \to p \to u_i$, and ε otherwise.

The following results are taken from [2]. Theorem 3 concerns FIFO event graphs, that is, event graphs such that all its places and transitions are FIFO. A sufficient condition for the event graph to be FIFO is that the matrices $A_1(n)$ have no ε on their principal diagonal. We will also assume that this condition holds. A deterministic event graph [i.e., all internal transitions have a constant (deterministic) firing time sequence, the value of which may depend on the transition] is always FIFO, even whenever the last condition is not satisfied.

THEOREM 3. *For any FIFO event graph, let x^i_n denote the epoch when transition u_i starts firing for the nth time, and let x_n be the column vector given by $x'_n = (x^1_n, x^2_n, \dots, x^\beta_n)$, where x'_n denotes transposition of x_n. Then the x_n satisfy the following* (max, +)*-recurrence equation, for all $n \geq M$*:

$$x_n = \bigoplus_{k=0}^{M} A_k(n) \otimes x_{n-k} \oplus B_0(n) \otimes T_n, \tag{26}$$

with the initial conditions $x_0, x_1, \dots, x_{M-1}$.

Theorem 4 concerns FIFO *deadlock-free* event graphs, that is, event graphs such that, for each marking $\nu\colon \mathscr{P} \to \{0, 1, \dots\}$ reachable from the initial marking $\mathscr{M}_0$ and for each transition $\tau \in \mathscr{T}$, there exists a marking $\mu\colon \mathscr{P} \to \{0, 1, \dots\}$ which is reachable from ν such that τ is enabled from μ. A necessary and sufficient condition for the event graph to be deadlock-free is that the matrices $A_0(n)$ are strictly lower triangular for an appropriate numbering of the transitions; that is, all entries i, j of the matrices $A_0(n)$ with $i \le j$ are equal to ε. In that case, we define the following matrices:

1. $A_0^*(n)$ is the $\beta \times \beta$ matrix

$$A_0^*(n) = \bigoplus_{k \ge 0} A_0^k(n), \tag{27}$$

   where, for all A, $A^0 = E$ is the (max, +)-identity matrix (i.e., all diagonal elements are 0, and all nondiagonal elements are ε), and $A^{k+1} = A^k \otimes A$. The series defined in (27) converges whenever $A_0(n)$ is strictly lower triangular.
2. $\overline{A}_k(n)$ is the $\beta \times \beta$ matrix

$$\overline{A}_k(n) = A_0^*(n) \otimes A_k(n). \tag{28}$$

3. $\overline{B}_0(n)$ is the $\beta \times 1$ matrix

$$\overline{B}_0(n) = A_0^*(n) \otimes B_0(n). \tag{29}$$

4. A_n is the $M\beta \times M\beta$ matrix

$$A_n = \begin{pmatrix} \overline{A}_1(n+1) & \overline{A}_2(n+1) & \cdots & \cdots & \overline{A}_M(n+1) \\ E & \mathscr{E} & \cdots & \mathscr{E} & \mathscr{E} \\ \mathscr{E} & E & \ddots & \vdots & \vdots \\ \vdots & \ddots & E & \mathscr{E} & \mathscr{E} \\ \mathscr{E} & \dots & \mathscr{E} & E & \mathscr{E} \end{pmatrix}, \tag{30}$$

   where E is the $\beta \times \beta$-(max, +)-identity matrix and $\mathscr{E}$ denotes the (max, +)-zero matrix (i.e., the $\beta \times \beta$ matrix with all entries equal to ε).
5. B_n is the $M\beta \times 1$ matrix

$$B_n = \begin{pmatrix} \overline{B}_0(n) \\ \varepsilon \\ \vdots \\ \varepsilon \end{pmatrix}. \tag{31}$$

6. X_n is the $M\beta$-dimensional vector

$$X_n = \begin{pmatrix} x_n \\ x_{n-1} \\ \vdots \\ x_{n-M+1} \end{pmatrix}. \tag{32}$$

THEOREM 4. *For any FIFO deadlock-free event graph and for $n \geq M-1$, the X_n satisfy* (3), *with $\alpha = M\beta$, and with A_n and B_n defined in* (30) *and* (31), *respectively*.

REMARK. There is a converse theorem, proved in [6], which states that, for all (max, +)-linear equations of type (3), one can construct a FIFO stochastic event graph with these evolution equations.

Moreover, note that, since the event graph is FIFO, we always have $x_n^i \geq x_{n-1}^i$. So, in the definition (30) of A_n, we can replace each $\mathscr{E}$ matrix of the main diagonal by E, without altering the solution of the recurrence equations. Thus, there is an equivalent representation of the system where the A_n matrix is

$$A_n = \begin{pmatrix} \overline{A}_1(n+1) & \overline{A}_2(n+1) & \cdots & \cdots & \overline{A}_M(n+1) \\ E & E & \cdots & \mathscr{E} & \mathscr{E} \\ \mathscr{E} & E & \ddots & \vdots & \vdots \\ \vdots & \ddots & E & E & \mathscr{E} \\ \mathscr{E} & \cdots & \mathscr{E} & E & E \end{pmatrix} \tag{33}$$

and has all its diagonal entries nonnegative.

In what follows, we will assume that the network is "input connected," namely that $\overline{B}_0(n) \geq 0$ for all n. We conclude this section by a few lemmas showing that, under the stability condition $\rho < 1$ of Theorem 1, stochastic Petri nets of this class satisfy the assumptions stated in subsections 3.1 and 3.2 (with $\alpha = M\beta$ and $\alpha' = \beta$).

LEMMA 1. *Let a denote the maximal* (max, +)*-Liapounov exponent of* $\{A_n\}$. *If $\lambda a < 1$, then, for r large enough*,

$$\lambda < r\Big[\mathbb{E}\max_i\big\{(A_{-1} \otimes A_{-2} \otimes \cdots \otimes A_{-r} \otimes (B_{-r} \oplus O))^i\big\}\Big]^{-1}, \tag{34}$$

where O denotes the $M\beta$-dimensional vector with all its coordinates equal to 0.

PROOF. One of the characterizations of the Liapounov exponent a is

$$a = \lim_{r\to\infty} \frac{\mathbb{E}\max_{i,j}\{(A_{-1} \otimes \cdots \otimes A_{-r})_{i,j}\}}{r} \tag{35}$$

(see [6]). So, under the condition $\lambda a < 1$, there exists an integer R such that, for all $r \geq R$,

$$\lambda \frac{\mathbb{E}\max_{i,j}\{(A_{-1} \otimes \cdots \otimes A_{-r})_{i,j}\}}{r} < 1. \tag{36}$$

But

$$\begin{aligned} &\mathbb{E}\max_i\{(A_{-1} \otimes \cdots \otimes A_{-r} \otimes (B_{-r} \oplus O))^i\} \\ &\quad \leq \mathbb{E}\Big[\max_{i,j}\{(A_{-1} \otimes \cdots \otimes A_{-r})_{i,j}\}\Big] + \mathbb{E}\Big[\max_i\{(B_{-r})^i\}\Big]. \end{aligned} \tag{37}$$

Since the B_n's are identically distributed and all non-ε entries of B_0 are supposed to be integrable, we have

$$\lim_{r \to \infty} \frac{\mathbb{E}\max_i\{B^i_{-r}\}}{r} = 0. \tag{38}$$

Thus, (36) and (37) imply that there exists an integer R' such that (34) holds for all $r \geq R'$. $\square$

LEMMA 2. *Consider an input-connected stochastic event graph such that the sequence* $\{B_0(n)\}$ *is constant* (*we assumed this to be the case above*). *Then*, *for all integers* s, *the sequence*

$$A_{-s} \otimes A_{-s-1} \otimes \cdots \otimes A_{-s-n} \otimes B_{-s-n}$$

is nondecreasing in n; *in particular*, *for all* $i = 1, \ldots, \beta$,

$$0 \leq D_0^i \leq \cdots \leq D_n^i \leq D_{n+1}^i \cdots.$$

PROOF. Since $A_0^*(n) \geq E$, we obtain from (29) that

$$\overline{B}_0(n) \geq J, \tag{39}$$

where J denotes the constant vector equal to $B_0(n)$ for all n. In view of the definition of B_n in (31), we obtain

$$B_n \geq \begin{pmatrix} J \\ \varepsilon \\ \vdots \\ \varepsilon \end{pmatrix}.$$

Similarly, since all transitions are recycled, the matrices $A_1(n)$ are such that $A_1(n) \geq E$. So $\overline{A}_1(n)$ defined in (28) is such that

$$\overline{A}_1(n) \geq A_0^*(n). \tag{40}$$

From (39) and (40), we obtain

$$\overline{A}_1(-n+1) \otimes \overline{B}_0(-n) \geq A_0^*(-n+1) \otimes J = \overline{B}_0(-n+1). \tag{41}$$

When making use of this inequality in the definitions of A_n and D_n, we obtain

$$A_{-n} \otimes B_{-n} \geq B_{-n+1}, \tag{42}$$

which completes the proof of the first monotonicity property.

So, in particular, $D_n \geq D_{n-1}$. The only additional property to prove is that, for the first β coordinates of D_0, $D_0^i = B_0^i \geq 0$. But this follows from the assumption that the network is input-connected. □

LEMMA 3. *For all GI-stochastic event graphs* (*i.e., the firing-time sequences* $\{\sigma_n^1, \dots, \sigma_n^\beta\}_n$ *are i.i.d. in* n) *with maximal initial marking equal to* M, *the sequence* $\{A_n, B_n\}$ *is* M-*dependent.*

PROOF. In view of (30) and (31), the random matrices A_n and B_n are functions of the random variables

$$\{\sigma_{n+1}^j, \sigma_n^j, \dots, \sigma_{n+1-M}^j, \ j \in \mathscr{T}\}$$

only. This means that, under the above independence assumptions, for all n, the random matrices $\{(A_{n-l}, B_{n-l})\}, l \geq 0\}$ are independent of the random matrices $\{(A_{n+M+k}, B_{n+M+k})\}, \ k \geq 1\}$. □

4.2. *Queueing networks.* The aim of this subsection is to give a few practical examples of queueing, communication and manufacturing systems that fall in the class of stochastic event graphs and to apply our main theorem to these systems. For all of the examples given here, the maximal initial marking is $M = 1$, with the exception of the Kanban system where we take $M = 2$. We start with a toy example, the interest of which is purely pedagogical.

4.2.1. *Single-server queue.* Consider a single-server FIFO queue with infinite capacity which is initially empty. This is the system of Figure 1. Here,

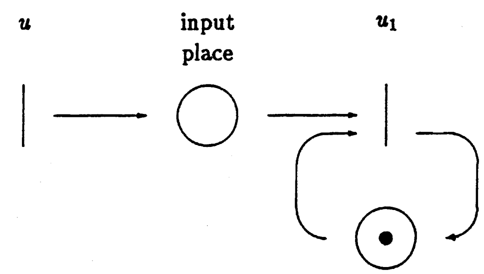

FIG. 1. *Single-server queue.*

$\beta = 1$, $A_n = \sigma_n$ represents the service time of the nth customer, and $B_n = 0$, so that (5) reads

$$W_{n+1} = (\sigma_n \otimes (-\tau_n) \otimes W_n) \oplus 0 \quad \text{with } W_0 = 0, \tag{43}$$

which is Lindley's equation for the actual waiting time W_{n+1} of the $(n+1)$th customer.

In this simple case, $D_0 = 0$ and $D_k = \sum_{n=1}^{k} \sigma_{-n}$, and direct computations give

$$\begin{aligned}
p_2(D_0, D_1) &= \tfrac{1}{2}\sigma_{-1}^2,\\
p_3(D_0, D_1, D_2) &= \tfrac{1}{6}(\sigma_{-2}^3 - \sigma_{-1}^3 + 3\sigma_{-1}\sigma_{-2}^2),\\
p_4(D_0, D_1, D_2, D_3) &= \tfrac{1}{24}\big(\sigma_{-1}^4 - 2\sigma_{-2}^4 + \sigma_{-3}^4 + 4(\sigma_{-1}\sigma_{-3}^3 - 2\sigma_{-1}\sigma_{-2}^3 + \sigma_{-2}\sigma_{-3}^3)\\
&\qquad - 6(\sigma_{-1}^2\sigma_{-2}^2 - \sigma_{-2}^2\sigma_{-3}^2) + 12(\sigma_{-1}\sigma_{-2}\sigma_{-3}^2)\big).
\end{aligned}$$

Assume that the stability condition $\lambda\mathbb{E}[\sigma_n] < 1$ is satisfied and that $E[\sigma_n^6] < \infty$, so that we can apply Theorem 2 for the expansion of order $m = 3$, whenever the sequence $\{\sigma_n\}$ is i.i.d. From the previous expressions for $p_k(D_0, D_1, \dots, D_{k-1})$, we obtain

$$\mathbb{E}p_2(D_0, D_1) = \frac{\mathbb{E}[\sigma_n^2]}{2}, \qquad \mathbb{E}p_3(D_0, D_1, D_2) = \frac{\mathbb{E}[\sigma_n^2]\mathbb{E}[\sigma_n]}{2}$$

and

$$\mathbb{E}p_4(D_0, D_1, D_2, D_3) = \frac{\mathbb{E}[\sigma_n^2]\mathbb{E}[\sigma_n]^2}{2},$$

so that

$$\mathbb{E}W = \lambda\frac{\mathbb{E}[\sigma_n^2]}{2} + \lambda^2\frac{\mathbb{E}[\sigma_n^2]\mathbb{E}[\sigma_n]}{2} + \lambda^3\frac{\mathbb{E}[\sigma_n^2]\mathbb{E}[\sigma_n]^2}{2} + \mathscr{O}(\lambda^4). \tag{44}$$

Of course, in this case, there are far more efficient ways of obtaining such an expansion, such as a direct use of the Pollaczek–Khinchine mean value formula, which gives

$$\mathbb{E}W = \frac{\lambda\mathbb{E}[\sigma_n^2]}{2(1-\lambda\mathbb{E}[\sigma_n])} = \frac{\mathbb{E}[\sigma_n^2]}{2}\sum_{k=1}^{\infty}\lambda^k\mathbb{E}[\sigma_n]^{k-1}, \tag{45}$$

under the sole assumption that $\mathbb{E}[\sigma_n^2] < \infty$, or Takács's recurrence formula (see [28], page 201). However, our expansion technique can be extended to the case when the sequence of service times possesses a correlation structure of finite range (cf. the remarks at the beginning of subsection 4.1). This and related topics for further characteristics of W will be the subject of future research.

4.2.2. *Tandem queues.* Consider a network of β single-server FIFO queues with infinite capacity in tandem (see Figure 2), with all queues initially empty. For this system, the matrices $A_0(n), A_1(n)$ and $B_0(n)$ have the following form:

$$A_0(n) = \begin{pmatrix} \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \sigma_n^1 & \varepsilon & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \sigma_n^2 & \cdots & \varepsilon & \varepsilon & \varepsilon \\ \vdots & & & & & \\ \varepsilon & \varepsilon & \cdots & \sigma_n^{\beta-2} & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \cdots & \varepsilon & \sigma_n^{\beta-1} & \varepsilon \end{pmatrix}, \qquad B_0(n) = \begin{pmatrix} 0 \\ \varepsilon \\ \varepsilon \\ \vdots \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix} \tag{46}$$

and

$$A_1(n) = \begin{pmatrix} \sigma_{n-1}^1 & \varepsilon & \varepsilon & \cdots & \varepsilon & \varepsilon \\ \varepsilon & \sigma_{n-1}^2 & \varepsilon & \cdots & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \sigma_{n-1}^3 & \cdots & \varepsilon & \varepsilon \\ \vdots & & & & & \\ \varepsilon & \varepsilon & \varepsilon & \cdots & \sigma_{n-1}^{\beta-1} & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \cdots & \varepsilon & \sigma_{n-1}^{\beta} \end{pmatrix}. \tag{47}$$

Therefore, the entries $(A_n)_{ij}$ of the matrix A_n are given by

$$(A_n)_{ij} = \begin{cases} \varepsilon, & \text{if } i < j, \\ \sum_{k=j}^{i-1} \sigma_{n+1}^k + \sigma_n^j, & \text{if } i \geq j \end{cases} \tag{48}$$

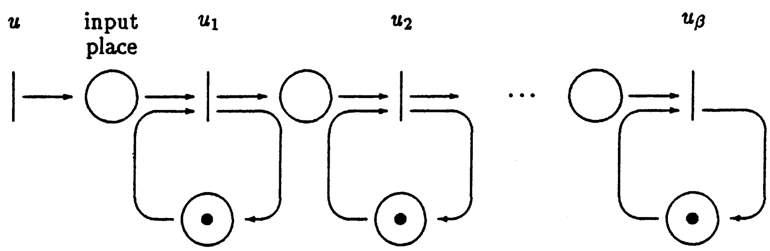

FIG. 2. *Tandem queues with infinite capacity.*

and

$$B_n = \begin{pmatrix} 0 \\ \sigma_n^1 \\ \sigma_n^1 + \sigma_n^2 \\ \sigma_n^1 + \sigma_n^2 + \sigma_n^3 \\ \vdots \\ \sum_{k=1}^{\beta-1} \sigma_n^k \end{pmatrix}. \tag{49}$$

Then the ith component W^i of the random vector W given by (9) describes the stationary waiting time of a randomly chosen customer until the beginning of service on server i, where the matrices D_n in (10) have the following form. From (48) we get that

$$(A_{-1} \otimes A_{-2})_{ij} = \begin{cases} \varepsilon, & \text{if } i < j, \\ \max_{j \le l \le i} \left\{ \sum_{k=l}^{i-1} \sigma_0^k + \sum_{k=j}^{l} \sigma_{-1}^k + \sigma_{-2}^j \right\}, & \text{if } i \ge j \end{cases}$$

and, for general $n \ge 1$, the entries $(A_{-1} \otimes A_{-2} \otimes \cdots \otimes A_{-n})_{ij}$ are equal to

$$\begin{cases} \varepsilon, & \text{if } i < j, \\ \max_{j \le l_{n-1} \le \cdots \le l_1 \le i} \left\{ \sum_{k=l_1}^{i-1} \sigma_0^k + \sum_{k=l_2}^{l_1} \sigma_{-1}^k + \cdots + \sum_{k=l_{n-1}}^{l_{n-2}} \sigma_{-n+2}^k \right. & \\ \qquad\qquad \left. + \sum_{k=j}^{l_{n-1}} \sigma_{-n+1}^k + \sigma_{-n}^j \right\}, & \text{if } i \ge j. \end{cases}$$

Thus, using (49), we have

$$\begin{aligned} D_n^i &= (A_{-1} \otimes A_{-2} \otimes \cdots \otimes A_{-n} \otimes B_{-n})^i \\ &= \max_{1 \le l_n \le \cdots \le l_1 \le i} \left\{ \sum_{k=l_1}^{i-1} \sigma_0^k + \sum_{k=l_2}^{l_1} \sigma_{-1}^k + \cdots + \sum_{k=l_n}^{l_{n-1}} \sigma_{-n+1}^k + \sum_{k=1}^{l_n} \sigma_{-n}^k \right\} \end{aligned} \tag{50}$$

(see [5] for more details on this formula). Consider now the particular case when the service times are deterministic. By σ^i we denote the service time in queue $i \in \{1, \ldots, \beta\}$. Without loss of generality, we can and will assume that $\sigma^1 \le \sigma^2 \le \cdots \le \sigma^\beta$. In the other case, say $\sigma^i > \sigma^{i+1}$ for some $i < \beta$, we can consider the ith queue and the $(i+1)$th queue as *one* single-server queue with service time $\sigma^i + \sigma^{i+1}$ because in front of the $(i+1)$th server the waiting room is always empty. By this assumption on service times, we get from (50)

$$D_n = \begin{pmatrix} n\sigma^1 \\ \sigma^1 + n\sigma^2 \\ \sigma^1 + \sigma^2 + n\sigma^3 \\ \sigma^1 + \sigma^2 + \sigma^3 + n\sigma^4 \\ \vdots \\ \sum_{k=1}^{\beta-1} \sigma^k + n\sigma^\beta \end{pmatrix}. \tag{51}$$

For systems with deterministic service times, we will use the following abbreviated notation, which is consistent with the (max, +)-setting. Namely, we write i^k instead of $k\sigma^i$, and $i^k j^l$ instead of $k\sigma^i + l\sigma^j$. With this notation

$$D_n = \begin{pmatrix} 1^n \\ 12^n \\ 123^n \\ 1234^n \\ \vdots \\ 123\cdots\beta^n \end{pmatrix}. \tag{52}$$

Expansions: deterministic service times case. From (18) and (19) and from (51), we get the following series expansion for the expected stationary waiting time $\mathbb{E}W^i$ which an arbitrarily chosen customer has to spend in the network until the beginning of his service in queue i, where $c_1 = \sum_{l=1}^{i-1} \sigma^l$, $c_2 = \sigma^i$:

$$\mathbb{E}W^i = c_1 + \lambda \frac{c_2^2}{2} + \lambda^2 \frac{c_2^3}{2} + \lambda^3 \frac{c_2^4}{2} + \cdots. \tag{53}$$

Note that the coefficient $c_2^2/2$ of the linear term of this expansion is equal to the expected stationary residual service time in queue i, whereas the coefficients of the second-order and third-order terms seem to be less intuitive. At first glance, it looks surprising that the coefficients of *all* orders depend on $c_2 = \sigma^i$ only, but *not* on the service times $\sigma^1, \sigma^2, \dots, \sigma^{i-1}$ in the earlier stations. However, the fact that

$$\mathbb{E}W^i = c_1 + \lambda \frac{c_2^2}{2} \left(\frac{1}{1 - \lambda c_2} \right)$$

is well known in queueing theory. For example, it can easily be concluded from an invariance property derived in [22] for tandem queues with infinite buffers and constant service times.

Expansions: random service times case. Assume that

$$\mathbb{E}(\sigma_n^i)^{m+3} < \infty \quad \text{for every } i \in \{1, \dots, \beta\}. \tag{54}$$

It is easy to see that the integrability condition (17) is then satisfied. Moreover, from (20) and (50), we get the following expression for the absolute term $\mathbb{E}p_1(D_0^i)$ in the expansion (18), of $\mathbb{E}W^i$:

$$\mathbb{E}p_1(D_0^i) = \mathbb{E}D_0^i = \sum_{k=1}^{i-1} \mathbb{E}\sigma_0^k.$$

In the same way we get the following expression for the coefficient $\mathbb{E}p_2(D_0^i, D_1^i)$ of the linear term:

$$\mathbb{E}p_2(D_0^i, D_1^i) = \tfrac{1}{2}\mathbb{E}(D_1^i - D_0^i)^2$$

and for the coefficient $\mathbb{E}p_3(D_0^i, D_1^i, D_2^i)$ of the quadratic term

$$\begin{aligned}&\mathbb{E}p_3(D_0^i, D_1^i, D_2^i)\\ &\quad = \tfrac{1}{6}\{\mathbb{E}(D_1^i - D_0^i)^3 + \mathbb{E}(D_2^i - D_0^i)^3 - 3\mathbb{E}(D_1^i - D_0^i)^2(D_2^i - D_0^i)\},\end{aligned}$$

where

$$D_1^i - D_0^i = \max_{1\le l\le i}\left\{\sigma_{-1}^l + \sum_{k=1}^{l-1}(\sigma_{-1}^k - \sigma_0^k)\right\}$$

and

$$D_2^i - D_0^i = \max_{1\le l_2\le l_1\le i}\left\{\sigma_{-2}^{l_2} + \sum_{k=1}^{l_2-1}(\sigma_{-2}^k - \sigma_0^k) + \sigma_{-1}^{l_1} + \sum_{k=l_2}^{l_1-1}(\sigma_{-1}^k - \sigma_0^k)\right\}.$$

Let us now consider a few special cases:

1. Assume that $\beta = i = 2$ and $\sigma_n^1 = c$ is deterministic. Then we have

$$D_1^2 - D_0^2 = \max\{c, \sigma_{-1}^2\} \quad\text{and}\quad D_2^2 - D_0^2 = \max\{2c, c + \sigma_{-1}^2, \sigma_{-2}^2 + \sigma_{-1}^2\}.$$

With the notation $G(x) = P(\sigma_n^2 \le x)$, this gives

$$\mathbb{E}p_2(D_0^2, D_1^2) = \tfrac{1}{2}\left\{c^2G(c) + \int_c^\infty x^2\,dG(x)\right\}$$

and

$$\begin{aligned}&\mathbb{E}p_3(D_0^2, D_1^2, D_2^2)\\ &\quad = \tfrac{1}{6}\bigg\{c^3G(c) + \int_c^\infty x^3\,dG(x) + 8c^3[G(c)]^2 + G(c)\int_c^\infty (c+x)^3\,dG(x)\\ &\qquad + \int_c^\infty\int_c^\infty (x+y)^3\,dG(y)\,dG(x) - 6c^3[G(c)]^2\\ &\qquad - 3G(c)\int_c^\infty x^2(c+x)\,dG(x) - 3\int_c^\infty\int_c^\infty x^2(x+y)\,dG(y)\,dG(x)\bigg\}\end{aligned}$$

$$= \tfrac{1}{6}\{c^2 G(c)[c + 1 + G(c)(2c - 1) + 3\tilde{G}_1(c)] \\ + 3\tilde{G}_1(c)\tilde{G}_2(c) - 2G(x)\tilde{G}_3(c)\},$$

where $\tilde{G}_j(c) = \int_c^\infty x^j \, dG(x)$.

2. Assume now that $\beta = 3$ and $\sigma^i_{-n} = \sigma(n)$ for all i, a model which was, for instance, considered in [15]. In this case, we obtain the following expressions:

$$D_0^3 = 2\sigma(0),$$
$$D_1^3 = \sigma(1) + 2\max\{\sigma(0), \sigma(1)\},$$
$$D_2^3 = \sigma(1) + \sigma(2) + 2\max\{\sigma(0), \sigma(1), \sigma(2)\},$$

so that

$$\mathbb{E}[W^3] = 2\mathbb{E}[\sigma(0)] + \lambda\tfrac{1}{2}(\mathbb{E}[\sigma(0)] + 2\mathbb{E}[(\sigma(1) - \sigma(0))^+]) \\ + \lambda^2 \tfrac{1}{6}(2\mathbb{E}[\sigma(0)] + 2\mathbb{E}[\max\{(\sigma(1) - \sigma(0))^+, \sigma(2)\sigma(0)\}]) + \mathscr{O}(\lambda^3),$$

where $x^+ = \max\{x, 0\}$.

4.2.3. *Blocking queues in tandem.* Consider a system of four single-server FIFO queues in tandem depicted by the Petri net of Figure 3. The first station, which is fed by the arrival point process, has an infinite capacity buffer, whereas all other stations have no buffering capacity. Here, the mechanism is that of "blocking after service"; that is, in each station, a customer can always start its service but once its service is completed, the customer can only proceed to the downstream station whenever this one is empty (this is also called *manufacturing blocking*). In Figure 3, the places of type p_1 represent the recycling of the servers, the places of type p_2 represent the servers and the places of type p_3 are used to enforce the blocking. The transition that precedes place p_2 in station 1 has a constant firing time equal to σ^1, whereas the transition which follows this place has a firing time equal to 0. A similar structure is repeated in all stations, the only difference being in the value

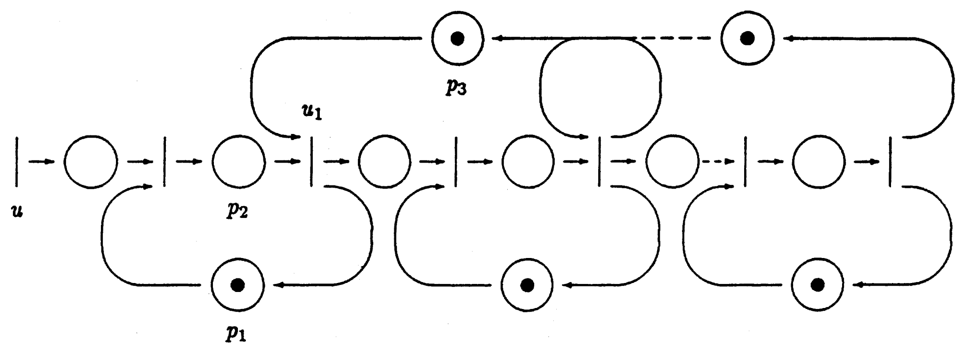

FIG. 3. *Blocking after service.*

of the service times which are equal to σ^i in station i. In the initial state, all stations are empty. Let us take as state variables the variables x_n^i where x_n^i gives the time when customer n leaves station i (or equivalently the time when the transition which follows the place of type p_2 in station i starts its nth firing).

When eliminating the state variables corresponding to the transitions that precede the places of type p_2, we obtain that the resulting state vectors x_n satisfy a recurrence equation of type (26) with

$$A_0(n) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 3 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 4 & \varepsilon \end{pmatrix}, \qquad A_1(n) = \begin{pmatrix} 1 & 0 & \varepsilon & \varepsilon \\ \varepsilon & 2 & 0 & \varepsilon \\ \varepsilon & \varepsilon & 3 & 0 \\ \varepsilon & \varepsilon & \varepsilon & 4 \end{pmatrix}, \qquad B_0(n) = \begin{pmatrix} 1 \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix}.$$

Let us assume that $\sigma^1 \le \sigma^2 \le \sigma^3 \le \sigma^4$. Then

$$A = A_0^*(0) \otimes A_1(0) = \begin{pmatrix} 1 & 0 & \varepsilon & \varepsilon \\ 12 & 2 & 0 & \varepsilon \\ 123 & 23 & 3 & 0 \\ 1234 & 234 & 34 & 4 \end{pmatrix},$$

$$B = A_0^*(0) \otimes B_0(0) = \begin{pmatrix} 1 \\ 12 \\ 123 \\ 1234 \end{pmatrix}.$$

Finally, the matrices D_n defined in (10) are given by

$$D_0 = \begin{pmatrix} 1 \\ 12 \\ 123 \\ 1234 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 12 \\ 123 \\ 1234 \\ 1234^2 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 123 \\ 1234 \\ 1234^2 \\ 1234^3 \end{pmatrix} \tag{55}$$

and, for $n \ge 3$,

$$D_n = \begin{pmatrix} 1234^{n-2} \\ 1234^{n-1} \\ 1234^{n} \\ 1234^{n+1} \end{pmatrix}. \tag{56}$$

The results are easily generalized to a general dimension β, and the vectors $W_n = (W_n^1, \dots, W_n^\beta)$ satisfy the recurrence equation

$$W_{n+1} = A \otimes C(\tau_n) \otimes W_n \oplus B,$$

with

$$A = \begin{pmatrix} 1 & 0 & \varepsilon & \varepsilon & \varepsilon & \cdots \\ 12 & 2 & 0 & \varepsilon & \varepsilon & \cdots \\ 123 & 23 & 3 & 0 & \varepsilon & \cdots \\ \vdots & & & & & \\ 123\cdots\beta & 23\cdots\beta & \cdots & & \beta & \end{pmatrix}$$

and

$$B = \begin{pmatrix} 1 \\ 12 \\ 123 \\ \vdots \\ 123\cdots\beta \end{pmatrix}.$$

Using the fact that $\sigma^1 < \sigma^2 < \cdots < \sigma^\beta$, we get that, for $n \leq \beta$,

$$A^n = \begin{pmatrix} 123\cdots n & 23\cdots n & \cdots & n & 0 & \varepsilon & \varepsilon & \cdots \\ 123\cdots n+1 & 23\cdots n+1 & \cdots & \cdots & n+1 & 0 & \varepsilon & \cdots \\ \vdots & & & & & & & \\ 123\cdots\beta & 23\cdots\beta & \cdots & & & & & \beta \\ 123\cdots\beta^2 & 23\cdots\beta^2 & \cdots & & & & & \beta^2 \\ 123\cdots\beta^n & 23\cdots\beta^n & \cdots & & & & & \beta^n \end{pmatrix},$$

whereas, for $n \geq \beta$,

$$A^n = \begin{pmatrix} 12\cdots\beta^{n-\beta} & 2\cdots\beta^{n-\beta} & \cdots & \beta^{n-\beta} \\ 12\cdots\beta^{n+1-\beta} & 2\cdots\beta^{n+1-\beta} & \cdots & \beta^{n+1-\beta} \\ \vdots & & & \\ 12\cdots\beta^n & 2\cdots\beta^n & \cdots & \beta^n \end{pmatrix}.$$

This gives

$$D_n = A^n \otimes B = \begin{pmatrix} \displaystyle\sum_{k=1}^{(n+1)\wedge\beta} \sigma^k + (n+1-\beta)^+\sigma^\beta \\ \displaystyle\sum_{k=1}^{(n+2)\wedge\beta} \sigma^k + (n+2-\beta)^+\sigma^\beta \\ \vdots \\ \displaystyle\sum_{k=1}^{\beta} \sigma^k + n\sigma^\beta \end{pmatrix}.$$

Expansions. Using the same approach as before, we obtain

$$\begin{aligned}\mathbb{E}W^i = d_0 + \lambda\frac{(d_1-d_0)^2}{2} &+ \lambda^2\frac{1}{6}[(d_1-d_0)^3 + (d_2-d_0)^3 \\ &\qquad - 3(d_1-d_0)^2(d_2-d_0)] \\ + \lambda^3\frac{1}{24}\big[&(d_1-d_0)^4 + (d_2-d_0)^4 + (d_3-d_0)^4 \\ &- 4((d_1-d_0)^3(d_2-d_0) + (d_2-d_0)^3(d_3-d_0)) \\ &- 6(d_1-d_0)^2(d_3-d_0)^2 + 12(d_1-d_0)^2 \\ &\quad + 12(d_1-d_0)^2(d_2-d_0)(d_3-d_0)\big] + \mathscr{O}(\lambda^4),\end{aligned}$$

where

$$d_n(=D_n^i) = \sum_{j=1}^{(n+i)\wedge\beta} \sigma^j + (n+i-\beta)^+\sigma^\beta.$$

Take $i = 1$ and assume that $\beta \geq 4$. Then, from the above expansion, we get

$$\mathbb{E}W^1 = \sigma^1 + \lambda\frac{(\sigma^2)^2}{2} + \lambda^2\frac{(\sigma^3)^3 - (\sigma^2)^3 + 3\sigma^2(\sigma^3)^2}{6} + \mathscr{O}(\lambda^3).$$

Observe that, in the above expansion of $\mathbb{E}W^i$, the coefficient of the kth-order term depends only on the service times $\sigma^2, \sigma^3, \ldots, \sigma^{k+i}$ of the first $k+i$ servers. In particular, this coefficient does not depend on β provided that the total number of queues is sufficiently large.

What about blocking before service? Consider a system of $\beta = 5$ single-server FIFO queues in tandem with "blocking before service"; that is, in each station, a customer can only start its service whenever the downstream station is empty (this is also called *communication blocking*). The Petri net description is given in Figure 4. The places of type p_1 represent the recycling of the

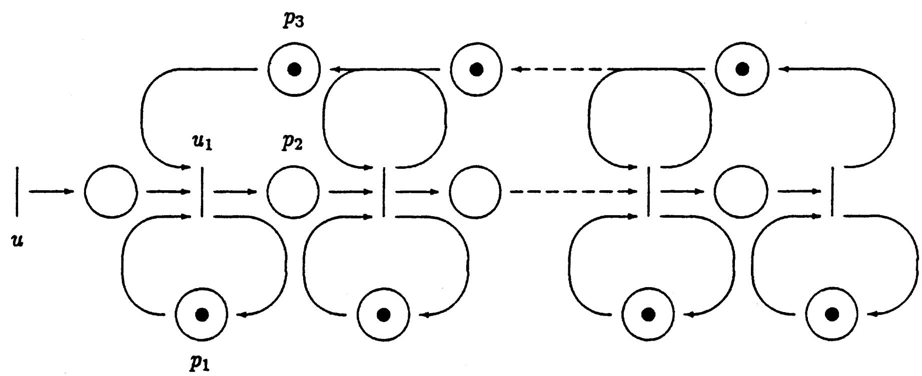

FIG. 4. *Blocking before service.*

servers. A sequence of random variables σ_n^i is associated with server i, indicating the successive service times there. The places of type p_2 represent the servers and the places of type p_3 are used to enforce the blocking. Notice that the number of tokens in circuits containing places of types p_2 and p_3 represents the maximal capacity for each station (in service and in the queue), which is 1 in this example. Let σ^i be the (deterministic) service time in station $i \in \{1, \dots, 5\}$. We assume that $\sigma^1 \le \sigma^2 \le \cdots \le \sigma^5$. Then, for x_n^i defined as the time when the service of customer n is started in station i, the framework of subsection 4.1 applies with

$$A_0(n) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 3 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 & \varepsilon \end{pmatrix}, \quad A_1(n) = \begin{pmatrix} 1 & 2 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 2 & 3 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 3 & 4 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 4 & 5 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 5 \end{pmatrix}, \quad B_0(n) = \begin{pmatrix} 0 \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix},$$

and so

$$A = A_0^*(0) \otimes A_1(0) = \begin{pmatrix} 1 & 2 & \varepsilon & \varepsilon & \varepsilon \\ 1^2 & 12 & 3 & \varepsilon & \varepsilon \\ 1^2 2 & 12^2 & 23 & 4 & \varepsilon \\ 1^2 23 & 12^2 3 & 23^2 & 34 & 5 \\ 1^2 234 & 12^2 34 & 23^2 4 & 34^2 & 45 \end{pmatrix},$$

$$B = A_0^*(0) \otimes B_0(0) = \begin{pmatrix} 0 \\ 1 \\ 12 \\ 123 \\ 1234 \end{pmatrix}.$$

Finally, for the matrices D_n defined in (10), we have, with the notation $\gamma = \sigma^4 + \sigma^5 = 45$,

$$\begin{aligned} &D_0 = \begin{pmatrix} 0 \\ 1 \\ 12 \\ 123 \\ 1234 \end{pmatrix}, \qquad D_1 = \begin{pmatrix} 12 \\ 123 \\ 1234 \\ 123\gamma \\ 1234\gamma \end{pmatrix}, \\ &D_2 = \begin{pmatrix} 12^2 3 \\ 123^2 4 \\ 1234\gamma \\ 123\gamma^2 \\ 1234\gamma^2 \end{pmatrix}, \qquad D_3 = \begin{pmatrix} 12^2 3^2 4 \\ 123^2 4\gamma \\ 1234\gamma^2 \\ 123\gamma^3 \\ 1234\gamma^3 \end{pmatrix} \end{aligned} \tag{57}$$

and, for $n \geq 4$,

$$D_n = \begin{pmatrix} 12^2 3^2 4\gamma^{n-3} \\ 123^2 4\gamma^{n-2} \\ 1234\gamma^{n-1} \\ 123\gamma^{n} \\ 1234\gamma^{n} \end{pmatrix}. \tag{58}$$

Using the same type of techniques as in the previous example, we obtain the following expansion for the waiting time W^1 in the first buffer:

$$\begin{aligned} \mathbb{E}W^1 = &\frac{\lambda}{2}(\sigma^1+\sigma^2)^2 \\ &+\frac{\lambda^2}{6}[(\sigma^1+\sigma^2)^3+(\sigma^1+2\sigma^2+\sigma^3)^3-3(\sigma^1+\sigma^2)^2(\sigma^1+2\sigma^2+\sigma^3)] \\ &+\mathscr{O}(\lambda^3). \end{aligned}$$

4.2.4. *Kanban system.* Let us consider the Kanban system with two stages given in Figure 5. For more details on this type of manufacturing systems, see [17]. Let us just mention that Kanban lines describe ways to operate multi-stage production lines, and that each stage describes the environment of one machine. In this picture, stage 1 corresponds to the set of places p_1 to p_5: place p_1 is the input buffer of the machine of stage 1, place p_2 is the machine itself (the cycle containing places p_2 and p_4 being present to translate the fact that only one object can be manufactured by machine 1 at a given time) and

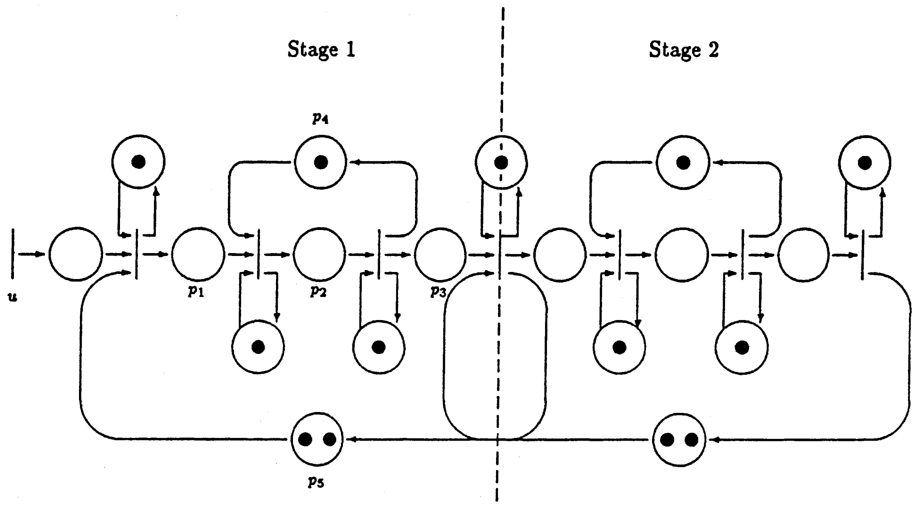

FIG. 5. *A two-stage Kanban system.*

place p_3 represents the output buffer of machine 1. The cycle which contains place p_5 translates the maximum total buffer capacity within stage 1. For this example, this total capacity is 2 (i.e., the total number of objects in the environment of machine 1, be it in the input buffer, being processed by the machine or in the output buffer, is at most 2), which translates into the fact that there are two tokens in place p_5. As in our previous examples, whenever the total capacity of the downstream stage (corresponding to the environment of machine 2) is reached, no object can move from the output buffer of machine 1 (p_3) to the input buffer of machine 2.

In this example, the only nonzero firing times in stage 1 are those associated with the transition which precedes place p_2, and we will denote σ_n^1 the duration of its nth firing (with a similar notation for the corresponding transition of stage 2). Of course, this firing time is just the manufacturing time of the nth object manufactured by machine i. We have added loops on all transitions in order to fulfill our assumption on diagonal terms of the A matrices. Adding them is of no importance since all other transitions have deterministic firing times all equal to 0. So we have a stochastic event graph with $\beta = 7$ internal transitions and $M = 2$. By numbering the transitions from 1 to 7 from left to right (excluding u), we obtain the following characteristics, for $x_n = (x_n^1, \ldots, x_n^7)$,

$$A_0(n) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \sigma_n^1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \sigma_n^2 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \end{pmatrix}, \qquad A_1(n) = \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \sigma_{n-1}^1 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \sigma_{n-1}^2 & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix}$$

and

$$A_2(n) = \begin{pmatrix} \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon \end{pmatrix}, \qquad B_0(n) = \begin{pmatrix} 0 \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \\ \varepsilon \end{pmatrix}.$$

We can easily reduce the dimension by eliminating the variables x^2 and x^5, considering only the vector $x'_n = (x_n^1, x_n^3, x_n^4, x_n^6, x_n^7)$. The elimination of x^2

goes as follows, departing from the initial seven-dimensional system:

$$
\begin{aligned}
x_n^1 &= x_{n-1}^1 \oplus x_{n-2}^4 \oplus u_n,\\
x_n^2 &= x_n^1 \oplus (x_{n-2}^2 \otimes \sigma_{n-1}^1) \oplus x_{n-1}^3,\\
x_n^3 &= (x_n^2 \otimes \sigma_n^1) \oplus x_{n-1}^3,\\
x_n^4 &= x_n^3 \oplus x_{n-1}^4 \oplus x_{n-2}^7,\\
x_n^5 &= x_n^4 \oplus (x_{n-1}^5 \otimes \sigma_{n-1}^2) \oplus x_{n-1}^6,\\
x_n^6 &= (x_n^5 \otimes \sigma_n^2) \oplus x_{n-1}^6,\\
x_n^7 &= x_n^6 \oplus x_{n-1}^7.
\end{aligned}
\tag{59}
$$

Using the second line of (59), we get

$$
\begin{aligned}
x_n^3 &= (x_n^1 \otimes \sigma_n^1) \oplus (x_{n-1}^2 \otimes \sigma_{n-1}^1 \otimes \sigma_n^1) \oplus (x_{n-1}^3 \otimes \sigma_n^1) \oplus x_{n-1}^3\\
&= (x_n^1 \otimes \sigma_n^1) \oplus (x_{n-1}^2 \otimes \sigma_{n-1}^1 \otimes \sigma_n^1) \oplus (x_{n-1}^3 \otimes \sigma_n^1)\\
&= (x_n^1 \otimes \sigma_n^1) \oplus (x_{n-1}^3 \otimes \sigma_n^1),
\end{aligned}
$$

where the reason for the last equality is due to the third line of (59); that is,

$$x_{n-1}^3 \geq x_{n-1}^2 \otimes \sigma_{n-1}^1$$

and therefore

$$x_{n-1}^3 \otimes \sigma_n^1 \geq x_{n-1}^2 \otimes \sigma_{n-1}^1 \otimes \sigma_n^1.$$

The elimination of x^5 is similar.

The reduced vector $x_n' = (x_n^1, x_n^3, x_n^4, x_n^6, x_n^7)$ satisfies the same recursion of order 2, but with the five-dimensional matrices:

$$
\tilde{A}_0(n) = \begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon\\
\sigma_n^1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon\\
\varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon\\
\varepsilon & \varepsilon & \sigma_n^2 & \varepsilon & \varepsilon\\
\varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon
\end{pmatrix}, \qquad
\tilde{A}_1(n) = \begin{pmatrix}
0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon\\
\varepsilon & \sigma_n^1 & \varepsilon & \varepsilon & \varepsilon\\
\varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon\\
\varepsilon & \varepsilon & \varepsilon & \sigma_n^2 & \varepsilon\\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 0
\end{pmatrix}
$$

and

$$
\tilde{A}_2(n) = \begin{pmatrix}
\varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon\\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon\\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & 0\\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon\\
\varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon
\end{pmatrix}, \qquad
\tilde{B}_0(n) = \begin{pmatrix}
0\\ \varepsilon\\ \varepsilon\\ \varepsilon\\ \varepsilon
\end{pmatrix}.
$$

In this reduced dimension, the $A_0^*(n)$ matrix (we drop the $\sim$ from now on, as we will only work with the five-dimensional system) is given by

$$A_0^*(n) = \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \sigma_n^1 & 0 & \varepsilon & \varepsilon & \varepsilon \\ \sigma_n^1 & 0 & 0 & \varepsilon & \varepsilon \\ \sigma_n^1 + \sigma_n^2 & \sigma_n^2 & \sigma_n^2 & 0 & \varepsilon \\ \sigma_n^1 + \sigma_n^2 & \sigma_n^2 & \sigma_n^2 & 0 & 0 \end{pmatrix}.$$

From this, we can compute the matrices $\overline{A}_1(n)$ and $\overline{A}_2(n)$ and the vector $\overline{B}_0(n)$:

$$\overline{A}_1(n) = \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \sigma_n^1 & \sigma_n^1 & \varepsilon & \varepsilon & \varepsilon \\ \sigma_n^1 & \sigma_n^1 & 0 & \varepsilon & \varepsilon \\ \sigma_n^1 + \sigma_n^2 & \sigma_n^1 + \sigma_n^2 & \sigma_n^2 & \sigma_n^2 & \varepsilon \\ \sigma_n^1 + \sigma_n^2 & \sigma_n^1 + \sigma_n^2 & \sigma_n^2 & \sigma_n^2 & 0 \end{pmatrix},$$

$$\overline{A}_2(n) = \begin{pmatrix} \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \sigma_n^1 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \sigma_n^1 & \varepsilon & 0 \\ \varepsilon & \varepsilon & \sigma_n^1 + \sigma_n^2 & \varepsilon & \sigma_n^2 \\ \varepsilon & \varepsilon & \sigma_n^1 + \sigma_n^2 & \varepsilon & \sigma_n^2 \end{pmatrix}, \qquad \overline{B}_0(n) = \begin{pmatrix} 0 \\ \sigma_n^1 \\ \sigma_n^1 \\ \sigma_n^1 + \sigma_n^2 \\ \sigma_n^1 + \sigma_n^2 \end{pmatrix}.$$

So, the matrix A_n is given by

$$A_n = \begin{pmatrix} 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \sigma_{n+1}^1 & \sigma_{n+1}^1 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \sigma_{n+1}^1 & \varepsilon & \varepsilon \\ \sigma_{n+1}^1 & \sigma_{n+1}^1 & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & \sigma_{n+1}^1 & \varepsilon & 0 \\ \sigma_{n+1}^1 + \sigma_{n+1}^2 & \sigma_{n+1}^1 + \sigma_{n+1}^2 & \sigma_{n+1}^2 & \sigma_{n+1}^2 & \varepsilon & \varepsilon & \varepsilon & \sigma_{n+1}^1 + \sigma_{n+1}^2 & \varepsilon & \sigma_{n+1}^2 \\ \sigma_{n+1}^1 + \sigma_{n+1}^2 & \sigma_{n+1}^1 + \sigma_{n+1}^2 & \sigma_{n+1}^2 & \sigma_{n+1}^2 & 0 & \varepsilon & \varepsilon & \sigma_{n+1}^1 + \sigma_{n+1}^2 & \varepsilon & \sigma_{n+1}^2 \\ 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon \\ \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 & \varepsilon & \varepsilon & \varepsilon & \varepsilon & 0 \end{pmatrix}.$$

From this, we obtain the following expressions:

$$
\begin{aligned}
&D_0^1 = 0,\\
&D_0^2 = D_0^3 = \sigma_0^1,\\
&D_0^4 = D_0^5 = \sigma_0^1 + \sigma_0^2,\\
&D_1^1 = 0,\\
&D_1^2 = D_1^3 = \sigma_{-1}^1 + \sigma_0^1,\\
&D_1^4 = D_1^5 = \max\{\sigma_{-1}^1 + \sigma_0^1 + \sigma_0^2, \sigma_{-1}^1 + \sigma_{-1}^2 + \sigma_0^2\},\\
&D_2^1 = \sigma_{-2}^1,\\
&D_2^2 = \sigma_{-2}^1 + \sigma_{-1}^1 + \sigma_0^1,\\
&D_2^3 = \max\{\sigma_{-2}^1 + \sigma_{-1}^1 + \sigma_0^1, \sigma_{-2}^1 + \sigma_{-2}^2\},\\
&D_2^4 = D_2^5 = \max\big\{\sigma_{-2}^1 + \sigma_{-1}^1 + \sigma_0^1 + \sigma_0^2, \sigma_{-2}^1 + \sigma_{-1}^1 + \sigma_{-1}^2 + \sigma_0^2,\\
&\qquad\qquad\qquad \sigma_{-2}^1 + \sigma_{-2}^2 + \sigma_{-1}^2 + \sigma_0^2\big\}.
\end{aligned}
$$

Expansions: deterministic case. Consider the deterministic case. From what precedes, $D_0^5 = \sigma^1 + \sigma^2$, $D_1^5 = D_0^5 + \max\{\sigma^1, \sigma^2\}$ and $D_2^5 = D_1^5 + \max\{\sigma^1, \sigma^2\}$. Therefore, we obtain the following expansion for the stationary total system time S, which coincides with coordinate W^7 of the seven-dimensional system, or equivalently with W^5 of the five-dimensional one:

$$\mathbb{E}S = \sigma^1 + \sigma^2 + \lambda \frac{[\max\{\sigma^1, \sigma^2\}]^2}{2} + \lambda^2 \frac{[\max\{\sigma^1, \sigma^2\}]^3}{2} + \mathscr{O}(\lambda^3), \tag{60}$$

under the stability condition $\rho < 1$ of Theorem 1 which here takes the form $\lambda \max\{\sigma^1, \sigma^2\} < 1$.

Expansions: stochastic case. Consider the time which elapses between the arrival of an object and the time it leaves machine 1, that is, variable W^3 of the five-dimensional system. From what precedes, whenever the system is stable and under the assumption that the random variables σ_n^1 and σ_n^2, with distribution functions G_1 and G_2, respectively, are independent and have moments of order 5, we obtain

$$\mathbb{E}W^3 = \mathbb{E}[\sigma^1] + \lambda \frac{c_1}{2} + \lambda^2 \frac{c_2}{6} + \mathscr{O}(\lambda^3). \tag{61}$$

The coefficients are given by the following integrals:

$$c_1 = \int_{\mathbb{R}_+} x^2 G_1(dx), \tag{62}$$

an expression which does not depend on G_2, whereas

$$c_2 = \int_{\mathbb{R}_+^4} h(x_0, x_1, x_2, y) G_1(dx_0) G_1(dx_1) G_1(dx_2) G_2(dy), \tag{63}$$

with

$$h(x_0, x_1, x_2, y) = x_1^3 + (\max\{x_1 + x_2, x_2 + y - x_0\})^3 \\ - 3x_1^2 \max\{x_1 + x_2, x_2 + y - x_0\}.$$

A similar expansion can be derived for S, involving a six-dimensional integral for the computation of the coefficient of λ^2 and so on.

4.3. *Remarks on the computation of the coefficients.* As we have seen, the computation of the coefficients reduces to the computation of certain d-dimensional integrals such as (63), for instance.

In case of exponentially distributed firing times (or more generally of firing times with rational Laplace transforms), such integrals can always be reduced to the integration of polynomial–exponential functions (functions involving products of two types of functions:

1. exponentials of linear functions of $x_0, x_1, \ldots$
2. polynomials in $x_0, x_1, \ldots$)

over polyhedrons, which leads to closed-form expressions.

5. Factorial moment expansion. In order to prove the series expansion stated in subsection 3.1, we will use a general idea which consists of expanding the expectation of vector-valued functionals of marked point processes. More precisely, we use a formula which expresses this expectation by a sum of integrals of much simpler functionals w.r.t. higher-order factorial moment measures of the underlying point process, with a remainder term which is the integral of a functional with respect to a higher-order Palm measure. For univariate (unmarked) point processes, this concept has been developed in [10] starting from a corresponding first-order expansion obtained in [3]. Related higher-order expansions for functionals of independently marked Poisson processes have been considered in [21], [30] and [34], and for more general marked point processes in [11] and [13]; see also the survey given in [14]. In the present paper we will concentrate on higher-order expansions for the expectation of vector-valued functionals of weakly independently marked Poisson processes.

5.1. *Expansion kernels.* For any given natural number α, let ψ be an $\mathbb{R}^\alpha$-valued functional of a marked point process, that is, a measurable mapping ψ: $\mathscr{M} \times \mathscr{K}^\infty \to \mathbb{R}^\alpha$, where $\mathscr{M}$ is the space of all realizations of the point process $\{T_n\}$ and $\mathscr{K}^\infty$ is the space of all sequences $Z = \{Z_n\}$ of potential marks. We assume that the *mark space* $\mathscr{K}$ is a complete separable metric space. Note that the sequence $\{T_n\}$ of points may be infinite, finite or empty, whereas the sequence $Z = \{Z_n\}$ of potential marks is always two-sided infinite. Let Z_n denote the mark of point T_n.

As in subsection 2.1, we represent a realization $\{t_n\}$ of the point process $\{T_n\}$ by the counting measure $\mu = \sum_n \delta_{t_n}$. Then $\mathscr{M}$ is the set of all counting measures μ which are locally finite and such that $\mu(\{s\})$ is either 0 or 1 for all $s \in \mathbb{R}$. By o we denote the null measure, representing an input with no arrivals [i.e., $o(\mathbb{R}) = 0$].

For every $s \in \mathbb{R}$, let the restriction $\mu|^s$ of $\mu \in \mathscr{F}$ be defined by

$$\mu|^s(D) = \mu(D \cap (s, \infty)).$$

Furthermore, for any $s \in \mathbb{R}$ and $z \in \mathscr{K}^\infty$, let

$$\psi_s(\mu, z) = \psi(\mu|^s + \delta_s, z) - \psi(\mu|^s, z). \tag{64}$$

Let $k \geq 1$ be an arbitrary, but fixed integer. For any $s_1, \dots, s_k \in \mathbb{R}$, let $\psi_{s_1,\dots,s_k}$ be defined by iteration of the mapping $\psi \to \psi_s$, that is,

$$\psi_{s_1,\dots,s_k}(\mu, z) = (\dots(\psi_{s_1})_{s_2}\dots)_{s_k}(\mu, z).$$

Note that the functional $\psi_{s_1,\dots,s_k}$ can be written in the form

$$\psi_{s_1,\dots,s_k}(\mu, z) = \begin{cases} \displaystyle\sum_{j=0}^{k}(-1)^{k-j} \sum_{\pi \in K_{k,j}} \psi\Big(\mu|^{s_k} + \sum_{i \in \pi} \delta_{s_i}, z\Big), & \text{for } s_1 < \dots < s_k, \\ 0, & \text{otherwise,} \end{cases} \tag{65}$$

where $K_{k,j}$ denotes the collection of all the subsets of $\{1, \dots, k\}$ containing j elements. Following [10], we call the functional ψ *continuous at* ∞ if

$$\lim_{s \to \infty} \psi(\mu|^s + \nu, z) = \psi(\nu, z), \qquad \lim_{s \to -\infty} \psi(\mu|^s, z) = \psi(\mu, z) \tag{66}$$

for all $\mu, \nu \in \mathscr{M}$, $z \in \mathscr{K}^\infty$ with $\nu(\mathbb{R}) < \infty$.

5.2. *General representation formula.* For the stationary Poisson process $\{T_n\}$ with intensity λ and for the stationary sequence $Z = \{Z_n\}$ of $\mathscr{K}$-valued random variables which is independent of $\{T_n\}$, let P_λ denote the distribution of $\{T_n\}$, and Q the distribution of Z.

A slight variant of the following result is given in [30].

THEOREM 5. *Let $m \geq 1$ be a fixed integer. If the functional ψ is continuous at ∞, then*

$$\int_{\mathbb{R}^k} \int_{\mathscr{K}^\infty} \int_{\mathscr{M}} \big|\psi^i_{s_1,\dots,s_k}(\mu, z)\big| P_\lambda(d\mu)\, Q(dz)\, ds_1 \cdots ds_k < \infty \tag{67}$$

for all $k = 1, \dots, m$ and if

$$\limsup_{\lambda \to 0} \int_{\mathbb{R}^{m+1}} \int_{\mathscr{K}^\infty} \int_{\mathscr{M}} \big|\psi^i_{s_1,\dots,s_{m+1}}(\mu, z)\big| P_\lambda(d\mu)\, Q(dz)\, ds_1 \cdots ds_{m+1} < \infty \tag{68}$$

for the ith component ψ^i of ψ, then

$$\begin{aligned} &\mathbb{E}\psi^i(\{T_n, Z_n\}) \\ &\quad = \mathbb{E}\psi^i(o, \{Z_n\}) + \sum_{k=1}^{m} \lambda^k \int_{\mathbb{R}^k} \mathbb{E}\psi^i_{s_1,\dots,s_k}(o, \{Z_n\})\, ds_1 \cdots ds_k + \mathscr{O}(\lambda^{m+1}). \end{aligned} \tag{69}$$

5.3. *Expansion of* $\mathbb{E}W^i$. In this section we return to the stationary state variable W given in (9) assuming again that the stationary sequence $\{Z_n\} = \{A_n, B_n\}$ of random matrices possesses the monotonicity, boundedness and independence properties formulated in subsections 3.1 and 3.2. This means, in particular, that $\{T_n, A_n, B_n\}$ is a so-called *weakly independently marked* Poisson process where the mark space $\mathscr{K}$ is the product of two matrix spaces (of $\alpha \times \alpha$- and $\alpha \times 1$-dimensional matrices, respectively). Our goal is to use Theorem 5 in order to show that an expansion of $\mathbb{E}W^i$ of the form (18) exists. For doing this we consider the following functional ψ given by

$$\begin{aligned} \psi(\mu, z) &= b_0 \oplus \bigoplus_{n=1}^{\mu((-\infty,0))} a_{-1} \otimes \cdots \otimes a_{-n} \otimes C(-t_{-n}) \otimes b_{-n} \\ &= d_0 \oplus (t_{-1} \otimes d_1) \oplus (t_{-2} \otimes d_2) \oplus \cdots, \end{aligned} \tag{70}$$

where $\mu = \sum_n \delta_{t_n}$ and $z = \{a_n, b_n\}$ and where a_n and b_n denote the realizations of the random matrices A_n and B_n, respectively. In the last expression, the meaning of the products $t \otimes d$, where t is a real number and d is a vector, is the same as in the conventional algebra; that is, each coordinate of d is "multiplied" by t.

5.3.1. *Integrability.* First we show that the expectation $\mathbb{E}W^i$ exists for each $i \in \{1, \ldots, \alpha'\}$. This follows from a corresponding result for the expectation of the maximum of a random walk with negative drift.

Let r be as defined in subsection 3.2, and let

$$h_n = \bigoplus_i \{ (a_{-(rn+1)} \otimes a_{-(rn+2)} \otimes \cdots \otimes a_{-r(n+1)} \otimes (b_{-r(n+1)} \oplus O))^i \} \tag{71}$$

denote the realizations of H_n defined in (16). Using (70) and the monotonicity assumption on the sequence d_n^i, we obtain the following bounds:

$$d_l^i + t_{-l} \le d^i_{(n+1)r} + t_{-(nr+1)}, \quad \forall\, nr < l \le (n+1)r,$$

for all $1 \le i \le \alpha'$. It is easy to check that, in addition,

$$d^i_{(n+1)r} \le h_0 + h_1 + \cdots + h_n. \tag{72}$$

Therefore, for all $1 \le i \le \alpha'$,

$$\begin{aligned} \psi^i(\mu, z) &\le \max_j \{b_0^j\} + \sup_{n \ge 0} \{(h_0 + \cdots + h_n) + t_{-(rn+1)}\}^+ \\ &\le \max_j \{b_0^j\} + h_0 + \sup_{n \ge 1} \left\{ \sum_{k=1}^n (h_k + (t_{-(rk+1)} - t_{-(r(k-1)+1)})) \right\}^+ \\ &\le \max_j \{b_0^j\} + \varphi(\mu, z), \end{aligned} \tag{73}$$

where

$$\varphi(\mu, z) = h_0 + \sup_{n \geq 1}\left\{\sum_{k=1}^{n}(h_{2k} + (t_{-(r2k+1)} - t_{-(r(2k-1)+1)}))\right\}^+ \\ + \sup_{n \geq 1}\left\{\sum_{k=1}^{n}(h_{2k-1} + (t_{-(r(2k-1)+1)} - t_{-(r(2k-2)+1)}))\right\}^+. \tag{74}$$

Because $\{H_n\}$ is a sequence of 1-dependent random variables, the random variables $H_2, H_4, \ldots$ are i.i.d. and independent of the i.i.d. random variables $-(T_{-(2r+1)} - T_{-(r+1)}), -(T_{-(4r+1)} - T_{-(3r+1)}), \ldots$ which are Erlang distributed with expectation $r\lambda^{-1}$. Since the sequences $H_1, H_3, \ldots$ and $-(T_{-(r+1)} - T_{-1}), -(T_{-(3r+1)} - T_{-(2r+1)}), \ldots$ have the same properties, the finiteness of $\mathbb{E}W^i$ now follows from the well-known fact that, under condition (15), the random walks, the realizations of which are considered in (74), have negative drifts and that, under (17), the expectations of their maxima are finite (see, e.g., Theorem VIII.2.1 in [1]).

5.3.2. *Conditions for the expansion.* It is easily checked that the functional ψ given by (70) is a.s. finite and a.s. continuous at ∞ whenever $\rho < 1$ (both properties follow directly from the backward monotone construction that is used for proving the existence of a solution of (7) of Theorem 1; see Chapter 7 of [6]). We now show that the conditions (67) and (68) are fulfilled.

Let us first prove that (67) holds for $k = 1$. For all $l \in \{0, 1, \ldots\}$ and $s \in [t_{-(l+1)}, t_{-l})$, we have

$$\psi(\mu|^s, z) = d_0 \oplus \bigoplus_{n=1}^{l}(d_n \otimes t_{-n}),$$

$$\psi(\mu|^s + \delta_s, z) = d_0 \oplus \bigoplus_{n=1}^{l}(d_n \otimes t_{-n}) \oplus (d_{l+1} \otimes s) = \psi(\mu|^s, z) \oplus (s \otimes d_{l+1}).$$

Note that $t_{-l} + d^i_{l+1} < 0$ implies $s \otimes d^i_{l+1} < 0$, which, in turn, implies $\psi^i_s(\mu, z) = 0$. Thus,

$$\begin{aligned} \left|\psi^i_s(\mu, z)\right| &\leq \left[d^i_0 \oplus d^i_1 \oplus \bigoplus_{n=2}^{\infty}(d^i_n \otimes t_{-(n-1)})\right] \times \mathbf{1}\big(t_{-l} + d^i_{l+1} > 0\big) \\ &\leq \Big[\max_j\{d^j_0 + d^j_1\} + \max_{i,j}\{(a_{-1})_{i,j}\} + \varphi(\mu, z \circ \theta^{-1})\Big] \\ &\quad \times \mathbf{1}\big(t_{-l} + d^i_{l+1} > 0\big), \end{aligned} \tag{75}$$

where $\varphi(\mu, z)$ was defined in (74).

LEMMA 4. *If* $\mathbb{E}[(H_n)^{q+1}] < \infty$, *then*

$$\mathbb{E}[\varphi^q(\{T_n\}, \{Z_n\})] < \infty, \tag{76}$$

where the mapping φ *is given in* (74).

PROOF. See Theorem VIII.2.1 in [1].

LEMMA 5. *There exists a random variable* $\rho(\mu, z)$ *such that*

$$\mathbf{1}\big(t_{-l} + d^i_{l+1} > 0\big) \le \mathbf{1}(-s < \rho(\mu, z)), \tag{77}$$

and if $\mathbb{E}[(H_n)^{q+1}] < \infty$, *then*

$$\mathbb{E}[\rho^q(\{T_n\}, \{Z_n\})] < \infty. \tag{78}$$

The proof is subdivided into two parts.

Definition of the upper bound. For deriving the upper bound (77), we use an estimate which is similar to that constructed in (74). However, instead of considering the differences between H_n and r interarrival times, we will compare H_n to $r-1$ interarrival times.

First, the fact that $s \in [t_{-(l+1)}, t_{-l})$ implies that

$$\mathbf{1}\big(t_{-l} + d^i_{l+1} > 0\big) \le \mathbf{1}(-s < \beta(\mu, z)),$$

where

$$\beta(\mu, z) = \sup_{p \ge 0}\big\{-t_{-(p+1)}\colon\ d^i_{-(p+1)} + t_{-p} > 0\big\}.$$

In addition, we have

$$\begin{aligned}\beta(\mu, z) &= \sup_{m=0,\dots,r-2}\ \sup_{q\ge 0}\big\{-t_{-(q(r-1)+m+1)}\colon\ d^i_{q(r-1)+m+1} + t_{-(q(r-1)+m)} > 0\big\}\\ &\le (-t_{-2r^2}) + \gamma(\mu, z),\end{aligned}$$

with

$$\begin{aligned}\gamma(\mu, z) &= \sup_{m=0,\dots,r-2}\ \sup_{q\ge r}\big\{-t_{-(q+1)(r-1)}\colon\ d^i_{q(r-1)+m+1} + t_{-(q(r-1)+m)} > 0\big\}\\ &\le \sup_{q\ge r}\big\{-t_{-(q+1)(r-1)}\colon\ d^i_{qr} + t_{-q(r-1)} > 0\big\}\\ &\le \sup_{q\ge 1}\big\{-t_{-(q+1)(r-1)}\colon\ h_0 + \dots + h_{q-1} + t_{-q(r-1)} > 0\big\},\end{aligned}$$

where we used (72). So, we have

$$\gamma(\mu, z) \le \sup_{q\ge 1}\left\{-t_{-(q+1)(r-1)}\colon\ \sum_{p=0}^{q-1} h_p + (t_{-(p+1)(r-1)} - t_{-p(r-1)}) > 0\right\}. \tag{79}$$

Let us introduce the following notation:

$$\xi^0_q = \sum_{\{j\colon\ 0\le 2j\le q-1\}} (h_{2j} + t_{-(2j+1)(r-1)} - t_{-2j(r-1)})$$

and

$$\xi^1_q = \sum_{\{j\colon\ 0\le 2j-1\le q-1\}} (h_{2j-1} + t_{-2j(r-1)} - t_{-(2j-1)(r-1)}).$$

Then the expression on the right-hand side of (79) is equal to

$$\sup_{q\geq 1}\{-t_{-(q+1)(r-1)}\colon\ \xi_q^0+\xi_q^1>0\},$$

so that

$$\begin{aligned}\gamma(\mu,z)&\leq\sup_{q\geq 1}\{-t_{-(q+1)(r-1)}\colon\ \xi_q^0>0 \text{ or } \xi_q^1>0\}\\&\leq\sup_{q\geq 1}\{-t_{-(q+1)(r-1)}\colon\ \xi_q^0>0\}+\sup_{q\geq 1}\{-t_{-(q+1)(r-1)}\colon\ \xi_q^1>0\}.\end{aligned}$$

Now, each of the terms in this last sum can be bounded from above as follows (we consider the first term only, the second one can be handled analogously):

$$\sup_{q\geq 1}\{-t_{-(q+1)(r-1)}\colon\ \xi_q^0>0\}\leq\zeta^{1,0}(\mu,z)+\zeta^{0,0}(\mu,z),$$

with

$$\zeta^{1,0}(\mu,z)=\sup_{q\geq 1}\left\{-\sum_{\{j\colon\ 0\leq 2j+1\leq q+1\}}(t_{-(2j+1)(r-1)}-t_{-2j(r-1)})\colon\ \xi_q^0>0\right\}\tag{80}$$

and

$$\zeta^{0,0}(\mu,z)=\sup_{q\geq 1}\left\{-\sum_{\{j\colon\ 0\leq 2j\leq q+1\}}(t_{-2j(r-1)}-t_{-(2j-1)(r-1)})\colon\ \xi_q^0>0\right\}.\tag{81}$$

We now define

$$\rho(\mu,z)=(-t_{-2r^2})+\zeta^{1,0}(\mu,z)+\zeta^{0,0}(\mu,z)+\zeta^{1,1}(\mu,z)+\zeta^{0,1}(\mu,z),\tag{82}$$

where the last two terms are defined in a symmetrical way.

Integrability. In order to prove that $\rho(\{T_n\},\{Z_n\})$ admits a finite mth moment, it is enough to prove that $\zeta^{1,0}(\{T_n\},\{Z_n\})$ and $\zeta^{0,0}(\{T_n\},\{Z_n\})$ do.

LEMMA 6. *The mth moment of $\zeta^{1,0}$ is finite provided that $\mathbb{E}[(H_n)^{m+1}]<\infty$.*

PROOF. The variable $\zeta^{1,0}(\mu,z)$ is related to the realization of a last exit time of the random walk with increments $h_{2p}+t_{-(2p+1)(r-1)}-t_{-2p(r-1)}$; this random walk has a negative drift when r is large enough. This follows from (14) and (15) which imply that

$$\lambda<(r-1)[\mathbb{E}H_n]^{-1}\tag{83}$$

for $r\in\mathbb{N}$ large enough. More precisely, $\zeta^{1,0}(\{T_n\},\{Z_n\})$ can be bounded from above by the sum of two random variables:

1. V, the customer–Palm age variable of the busy period of a stable E_{r-1}/GI/1 queue:

$$V = \sup\Bigg\{-\sum_{q=0}^{n}(T_{-(2q+1)(r-1)} - T_{-2q(r-1)}):\\ \sum_{q=0}^{n}(H_{2q} + T_{-(2q+1)(r-1)} - T_{-2q(r-1)}) > 0\Bigg\}.$$

2. C, the length of one full busy cycle (here the busy cycle which precedes the busy period containing customer 0) in this queue.

In order to prove the desired integrability result, we use Corollary 1a of [36], which states that, under the assumption $\mathbb{E}[(H_n)^{m+1}] < \infty$, both the busy cycle–Palm distributions of busy periods and busy cycles have $(m+1)$th moments.

So, the random variable C has a finite $(m+1)$th moment.

In order to prove a similar property for the mth moment of V, we first derive a bound on this variable. Let $G(x) = P(T_0 - T_{-(r-1)} \le x)$ and let $\tilde{\tau}$ be a random variable with distribution function $\tilde{G}(x)$ given by

$$\tilde{G}(x) = \frac{\int_0^x (1 - G(u))\,du}{\int_0^\infty (1 - G(u))\,du}.$$

Because $T_0 - T_{-(r-1)}$ is Erlang distributed, we have $G(x) \le \tilde{G}(x)$ for all x. This means that we can assume that $T_0 - T_{-(r-1)} \ge \tilde{\tau}$ with probability 1. Thus,

$$\begin{aligned} V &= \sup\Bigg\{-\sum_{q=0}^{n}(T_{-(2q+1)(r-1)} - T_{-2q(r-1)}):\\ &\qquad \sum_{q=0}^{n}(H_{2q} + T_{-(2q+1)(r-1)} - T_{-2q(r-1)}) > 0\Bigg\}\\ &\le \sup\Bigg\{-\sum_{q=0}^{n}(T_{-(2q+1)(r-1)} - T_{-2q(r-1)}):\\ &\qquad H_0 - \tilde{\tau} + \sum_{q=1}^{n}(H_{2q} + T_{-(2q+1)(r-1)} - T_{-2q(r-1)}) > 0\Bigg\}\\ &\le \tilde{V} + (T_0 - T_{-(r-1)}), \end{aligned}$$

where $\tilde{V}$ has the same distribution as the (continuous-time) stationary age variable of the busy period of an E_{r-1}/GI/1 queue.

But for $\mathbb{E}\tilde{V}^m < \infty$ to hold, it is enough that the $(m+1)$th moment of full busy periods be finite. □

LEMMA 7. *The mth moment of $\zeta^{0,0}$ is finite provided that $\mathbb{E}[(H_n)^{m+1}] < \infty$.*

PROOF. The differences $t_{-2j(r-1)} - t_{-(2j-1)(r-1)}$ considered before the colon in (81) and the differences $t_{-(2j+1)(r-1)} - t_{-2j(r-1)}$ considered in the definition of ξ_q^0 are realizations of *independent* sequences of random variables. Thus, for proving the finiteness of the moment of order m of $\zeta^{0,0}$, it suffices to prove the finiteness of the moment of order m for the random variable

$$N = \sum_{q \geq 1} \mathbf{1}(\xi_q^0 > 0),$$

which counts the number of customers served in the busy period which contains customer 0.

But this again follows from Corollary 1a of [36], where it is shown that the number of customers served per busy period has a finite moment of order $m+1$ if the service times have a finite moment of the same order. □

From (75) and Lemma 5, we finally get that

$$\begin{aligned}
\mathbb{E} \int_{\mathbb{R}} &|\psi_s^i(\{T_n\}, \{Z_n\})|\, ds \\
&\leq \mathbb{E}\Big\{\Big[\max_i \{B_0^i + (A_{-1} \otimes B_{-1})^i\} + \max_{i,j}\{(A_{-1})_{i,j}\} + \varphi(\{T_n\}, \{Z_{n+1}\})\Big] \\
&\qquad\qquad \times [\rho(\{T_n\}, \{Z_n\})]\Big\} \\
&\leq 3\mathbb{E}\Big\{\Big[\max_i \{B_0^i + (A_{-1} \otimes B_{-1})^i\}\Big]^2 + \Big[\max_{i,j}\{(A_{-1})_{i,j}\}\Big]^2 \\
&\qquad\qquad + [\varphi(\{T_n\}, \{Z_{n+1}\})]^2 + [\rho(\{T_n\}, \{Z_n\})]^2\Big\},
\end{aligned}$$

where the obvious inequality $xy \leq x^2 + y^2$ was used. Thus, (76) and (78) give

$$\int_{\mathbb{R}} \int_{\mathscr{K}^\infty} \int_{\mathscr{M}} \big|\psi_s^i(\mu, z)\big| P_\lambda(d\mu)\, Q(dz)\, ds < \infty,$$

provided that $\mathbb{E}[(H_n)^3] < \infty$. This completes the proof of (67) for $k = 1$.

We now prove (67), for all $k = 1, \ldots, m$, and (68), under the assumption that $\mathbb{E}[(H_n)^{m+3}] < \infty$.

For this, we take $r > k$ such that

$$\lambda < (r-k)\Big[\mathbb{E} \max_i \{(A_{-1} \otimes A_{-2} \otimes \cdots \otimes A_{-r} \otimes (B_{-r} \oplus O))^i\}\Big]^{-1}$$

[this is possible in view of (14) and (15)]. For r as above, we have

$$\big|\psi^i_{s_1,\ldots,s_k}(\mu, z)\big| \leq 2^k \varphi_k(\mu, z) \prod_{j=1}^k \mathbf{1}(\beta_k(\mu, z) > -s_j), \tag{84}$$

where

$$\varphi_k(\mu, z) = \max_j \left\{\sum_{n=0}^k d_n^j\right\} + \max_{i,j}\{(a_{-1} \otimes \cdots \otimes a_{-k})_{i,j}\} + \varphi(\mu, z \circ \theta^{-k}) \tag{85}$$

and

$$\beta_k(\mu, z) = \sup_{p\geq 0}\{-t_{-(p+1)}\colon\ d^i_{p+k} + t_{-p} > 0\}. \tag{86}$$

The proof of (84) is similar to that of (75). For this, we use the representation formula (65) for $\psi_{s_1,\dots,s_k}(\mu, z)$.

First, observe that in this sum there are at most 2^k terms, each bounded from above by $\varphi_k(\mu, z)$.

Next, observe that these terms can be grouped into pairs of terms with different signs, where the corresponding subsets π and π' differ only in one element, say s_j. Take $s_k \in [t_{-(l+1)}, t_{-l})$ and $s_1 < \cdots < s_k$. If $t_{-l} + d^i_{l+k} < 0$, then $s_k + d^i_{l+k} < 0$, which, in turn, implies that $s_j + d^i_{l+k-(j-1)} < 0$ for all $j = 1, \dots, k$, so that all these pairs actually compensate. So, it is enough to consider $s_1, \dots, s_k$ such that $-s_j < \beta_k(\mu, z)$ for all $j = 1, \dots, k$. This leads to the factor $\prod_{j=1}^k \mathbf{1}(\beta_k(\mu, z) > -s_j)$.

Thus, we get

$$\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^k}\left|\psi^i_{s_1,\dots,s_k}(\{T_n\}, \{Z_n\})\right| ds_1\cdots ds_k \\
&\qquad \leq 2^k\mathbb{E}(\varphi_k(\{T_n\}, \{Z_n\}))^{k+1}\mathbb{E}(\beta_k(\{T_n\}, \{Z_n\}))^{k+1},
\end{aligned}$$

using the inequality $(xy)^k \leq x^{k+1}+y^{k+1}$. The finiteness of $\mathbb{E}(\varphi_k(\{T_n\}, \{Z_n\}))^{k+1}$ easily follows from our assumptions.

In order to get an upper bound on $\beta_k(\mu, z)$, we can proceed as in the case $k = 1$, by considering the differences between H_n and $r - k$ interarrival times. This gives

$$\begin{aligned}
\beta_k(\mu, z) &= \sup_{m=0,\dots,r-(k+1)}\ \sup_{q\geq 0}\{-t_{-(q(r-k)+m+1)}\colon\ d^i_{q(r-k)+m+k} + t_{-(q(r-k)+m)} > 0\} \\
&\leq (-t_{-2r^2}) + \gamma_k(\mu, z),
\end{aligned}$$

with

$$\begin{aligned}
\gamma_k(\mu, z) &= \sup_{m=0,\dots,r-(k+1)}\ \sup_{q\geq r}\{-t_{-(q+1)(r-k)}\colon\ d^i_{q(r-k)+m+k} + t_{-(q(r-k)+m)} > 0\} \\
&\leq \sup_{q\geq r}\{-t_{-(q+1)(r-k)}\colon\ d^i_{qr} + t_{-q(r-k)} > 0\} \\
&\leq \sup_{q\geq r}\{-t_{-(q+1)(r-k)}\colon\ h_0 + \cdots + h_{q-1} + t_{-q(r-k)} > 0\}.
\end{aligned}$$

The rest of the proof of (67) for all $k = 1, \dots, m$ is based on arguments similar to those in the case $k = 1$.

In order to show (68), observe that the random variables $\varphi(\{T_n\}, \{Z_n\})$ and, consequently, also the random variables $\varphi_k(\{T_n\}, \{Z_n\})$ are stochastically decreasing as $\lambda \to 0$. Moreover, one can even construct a probability space such that this monotonicity property holds pathwise. This follows from the well-known fact that a stationary Poisson process with a smaller intensity can be obtained by thinning from a stationary Poisson process with a larger intensity. Another consequence of this is that, for each fixed $s_1 < 0$, the function $l_{s_1}(\lambda) = \sup\{l\colon\ s_1 < t_{-l}(\lambda)\}$ is pathwise decreasing. Since d_l^i is increasing in l, this means that the set of those s_1, for which $\psi_{s_1,\dots,s_k}$ is not equal to 0, is a decreasing function of λ. Thus, (68) holds because, from the above proof for (67), it follows now that the integral in (68) is uniformly bounded as $\lambda \to 0$.

6. Calculation of coefficients.

6.1. *Recursion formula.* It turns out that, for the functional ψ given by (70) and under the monotonicity assumption (13), the coefficients of λ^k in the series expansion (69) can be determined recursively. Because of (12), (70) can be rewritten in the form

$$\psi(\mu, z) = d_0 \oplus \bigoplus_{n=1}^{\mu((-\infty,0))} C(-t_{-n}) \otimes d_n, \tag{87}$$

where $d_0, d_1, \dots$ denotes the realizations of the sequence of α-dimensional random vectors $D_0, D_1, \dots$ defined in (12); that is, $d_k = (\bigoplus_{n=1}^{k} a_{-n}) \otimes b_{-k}$ whose ith components d_k^i satisfy

$$d_0^i \le d_1^i \le \cdots \tag{88}$$

for every $i \in \{1, \dots, \alpha'\}$. This gives

$$\psi^i(\mu, z) = d_0^i \oplus \bigoplus_{n=1}^{\mu((-\infty,0))} (d_n^i + t_{-n}). \tag{89}$$

Using the notation

$$p_{k+1}(d_0^i, d_1^i, \dots, d_k^i) = \int_{\mathbb{R}^k} \psi^i_{s_1,\dots,s_k}(o, z)\, ds_1 \cdots ds_k \tag{90}$$

for the coefficients of λ^k in (69), we obtain the following result.

THEOREM 6. *For each $k \ge 1$ and $i \in \{1, \dots, \alpha'\}$, it holds that*

$$\begin{aligned} &p_{k+1}(d_0^i, d_1^i, \dots, d_k^i) \\ &\quad = \sum_{n=0}^{k-1} \int_{d_n^i - d_0^i}^{d_{n+1}^i - d_0^i} \Big[p_k(\underbrace{d_0^i, \dots, d_0^i}_{n}, d_{n+1}^i - u, \dots, d_k^i - u) \\ &\qquad\qquad - p_k(\underbrace{d_0^i, \dots, d_0^i}_{n+1}, d_{n+1}^i - u, \dots, d_{k-1}^i - u) \Big]\, du. \end{aligned} \tag{91}$$

PROOF. From (65) and (89) we get

$$
\begin{aligned}
p_{k+1}&(d_0^i, d_1^i, \ldots, d_k^i) \\
&= \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{k-1}}^\infty \psi^i_{-s_1,\ldots,-s_k}(o,z)\, ds_k \cdots ds_1 \\
&= \int_0^\infty \int_{s_1}^\infty \cdots \int_{s_{k-1}}^\infty \sum_{l=0}^{k} (-1)^{k-l} \sum_{\pi \in K_{k,l}} d_0^i \oplus \bigoplus_{j=1}^{l} \bigl(d_j^i - s_{\pi(j)}\bigr)\, ds_k \cdots ds_1,
\end{aligned}
$$

where $(s_{\pi(1)}, \ldots, s_{\pi(l)})$ with $s_{\pi(1)} < \cdots < s_{\pi(l)}$ is the subset of those l components selected from $(s_1, \ldots, s_k)$ by $\pi \in K_{k,l}$. We decompose the outer integral in the following way:

$$
\int_0^\infty \cdots \quad = \sum_{n=0}^{k-1} \int_{d_n^i - d_0^i}^{d_{n+1}^i - d_0^i} \cdots .
$$

Next, for each of these summands, we decompose the inner sum:

$$
\sum_{\pi \in K_{k,l}} \cdots \quad = \sum_{\pi \in K_{k,l},\ \pi \ni 1} \cdots \quad + \sum_{\pi \in K_{k,l},\ \pi \not\ni 1} \cdots .
$$

Furthermore,

$$
\begin{aligned}
&\int_{d_n^i - d_0^i}^{d_{n+1}^i - d_0^i} \int_{s_1}^\infty \cdots \int_{s_{k-1}}^\infty \sum_{l=1}^{k} (-1)^{k-l} \sum_{\pi \in K_{k,l},\ \pi \ni 1} d_0^i \oplus \bigoplus_{j=1}^{l} \bigl(d_j^i - s_{\pi(j)}\bigr)\, ds_k \cdots ds_1 \\
&\quad = \int_{d_n^i - d_0^i}^{d_{n+1}^i - d_0^i} \int_{s_1}^\infty \cdots \int_{s_{k-1}}^\infty \sum_{l=1}^{k} (-1)^{k-l} \sum_{\pi \in K_{k,l},\ \pi \ni 1} d_0^i \oplus \bigoplus_{j=1}^{\min(n,l)} \bigl(d_j^i - s_{\pi(j)}\bigr) \\
&\qquad \oplus \bigoplus_{j=\min(n,l)+1}^{l} \bigl(d_j^i - s_{\pi(j)}\bigr)\, ds_k \cdots ds_1
\end{aligned}
$$

because, from $d_n^i - d_0^i \le s_1$, it follows that

$$
d_j^i - s_{\pi(j)} \le d_j^i - s_1 \le d_n^i - s_1 \le d_0^i
$$

for all $j \le \min(n,l)$. By the substitution $s_{\pi(j)} \to s_{\pi(j)} - s_1$, this gives the plus term in (91). The minus term in (91) follows analogously. □

6.2. *Polynomial solution.* Now we derive the more explicit expression for the coefficients $\mathbb{E} p_{k+1}(D_0^i, D_1^i, \ldots, D_k^i)$ of λ^k in the series expansion of $\mathbb{E} W^i$ as stated in Theorem 2. For each nondecreasing sequence $x_0, x_1, \ldots$ of nonnegative numbers (i.e., $0 \le x_0 \le x_1 \le \cdots$), we consider a sequence of numbers

$$
p_1(x_0),\ p_2(x_0, x_1), \ldots,\ p_k(x_0, x_1, \ldots, x_{k-1}), \ldots,
$$

which satisfy the integral recursion formula (25), with

$$
p_1(x_0) = x_0, \qquad p_2(x_0, x_1) = \tfrac{1}{2}\bigl[x_0^2 + x_1^2 - 2x_0 x_1\bigr]. \tag{92}
$$

It is easy to see that the functions p_1, p_2 given in (92) satisfy (25) for $k = 1$, because

$$\int_0^{x_1-x_0}[(x_1-u)-x_0]\,du = \tfrac{1}{2}(x_1^2-x_0^2) - x_0(x_1-x_0)$$
$$= \tfrac{1}{2}\big[x_0^2+x_1^2-2x_0x_1\big] = p_2(x_0,x_1).$$

THEOREM 7. *The functions* p_k *being the solution of* (25) *and* (92) *coincide with the polynomials of* (19) *in Theorem* 2.

PROOF. By induction with respect to $k \geq 2$, from (25) we easily get that p_k is translation invariant in the sense that

$$p_k(x_0+u, x_1+u, \ldots, x_{k-1}+u) = p_k(x_0, x_1, \ldots, x_{k-1}) \tag{93}$$

for each $u \geq 0$ and $k = 2, 3, \ldots$. Using (93), we can rewrite the recursion formula (25) as follows:

$$\begin{aligned} &p_{k+1}(x_0, x_1, \ldots, x_k) \\ &\quad = \sum_{n=0}^{k-1}\int_{x_n-x_0}^{x_{n+1}-x_0}\Big[p_k(\underbrace{x_0+u, \ldots, x_0+u}_{n}, x_{n+1}, \ldots, x_k) \\ &\qquad\qquad - p_k(\underbrace{x_0+u, \ldots, x_0+u}_{n+1}, x_{n+1}, \ldots, x_{k-1})\Big]\,du. \end{aligned}$$

Thus, we have

$$\begin{aligned} p_{k+1}(x_0, x_1, \ldots, x_k) = \sum_{n=0}^{k-1}\int_{x_n}^{x_{n+1}}\Big[&p_k(\underbrace{v, \ldots, v}_{n}, x_{n+1}, \ldots, x_k) \\ &- p_k(\underbrace{v, \ldots, v}_{n+1}, x_{n+1}, \ldots, x_{k-1})\Big]\,dv. \end{aligned} \tag{94}$$

Clearly, for $k = 1, 2$, formulas (92) and (19) coincide. Assuming now that (19) is true for some natural k, we show that it also holds for $k+1$. By inserting (19) into the right-hand side of (94), we get

$$\begin{aligned} &p_{k+1}(x_0, x_1, \ldots, x_k) \\ &\quad = \sum_{n=0}^{k-1}\sum_{(i_0, i_1, \ldots, i_{k-1})\in S_k}(-1)^{q_k(i_0, i_1, \ldots, i_{k-1})}\int_{x_n}^{x_{n+1}}\Bigg[\frac{v^{i_0+\cdots+i_{n-1}}}{i_0!\cdots i_{n-1}!}\frac{x_{n+1}^{i_n}}{i_n!}\cdots\frac{x_k^{i_{k-1}}}{i_{k-1}!} \\ &\qquad\qquad - \frac{v^{i_0+\cdots+i_n}}{i_0!\cdots i_n!}\frac{x_{n+1}^{i_{n+1}}}{i_{n+1}!}\cdots\frac{x_{k-1}^{i_{k-1}}}{i_{k-1}!}\Bigg]\,dv \end{aligned}$$

$$= \sum_{n=0}^{k-1} \sum_{(i_0, i_1, \dots, i_{k-1}) \in S_k} (-1)^{q_k(i_0, i_1, \dots, i_{k-1})}$$

$$\times \left[\frac{x_{n+1}^{i_0+\cdots+i_n+1}}{(i_0+\cdots+i_{n-1}+1)i_0!\cdots i_n!} \frac{x_{n+2}^{i_{n+1}}}{i_{n+1}!} \cdots \frac{x_k^{i_{k-1}}}{i_{k-1}!} \right.$$

$$- \frac{x_n^{i_0+\cdots+i_{n-1}+1}}{(i_0+\cdots+i_{n-1}+1)i_0!\cdots i_{n-1}!} \frac{x_{n+1}^{i_n}}{i_n!} \cdots \frac{x_k^{i_{k-1}}}{i_{k-1}!}$$

$$- \frac{x_{n+1}^{i_0+\cdots+i_{n+1}+1}}{(i_0+\cdots+i_n+1)i_0!\cdots i_{n+1}!} \frac{x_{n+2}^{i_{n+2}}}{i_{n+2}!} \cdots \frac{x_{k-1}^{i_{k-1}}}{i_{k-1}!}$$

$$\left. + \frac{x_n^{i_0+\cdots+i_n+1}}{(i_0+\cdots+i_n+1)i_0!\cdots i_n!} \frac{x_{n+1}^{i_{n+1}}}{i_{n+1}!} \cdots \frac{x_{k-1}^{i_{k-1}}}{i_{k-1}!} \right].$$

Next we reorder the summands of the last expression. This gives

$$p_{k+1}(x_0, x_1, \dots, x_k) = I_0 + \sum_{n=1}^{k-1} I_n + I_k,$$

where

$$I_0 = \sum_{(i_0, i_1, \dots, i_{k-1}) \in S_k} (-1)^{q_k(i_0, i_1, \dots, i_{k-1})} \left[\frac{x_0^{i_0+1}}{(i_0+1)i_0!} \frac{x_1^{i_1}}{i_1!} \cdots \frac{x_{k-1}^{i_{k-1}}}{i_{k-1}!} \frac{x_0^1}{1!} \frac{x_1^{i_0}}{i_0!} \cdots \frac{x_k^{i_{k-1}}}{i_{k-1}!} \right],$$

$$I_k = \sum_{(i_0, i_1, \dots, i_{k-1}) \in S_k} (-1)^{q_k(i_0, i_1, \dots, i_{k-1})} \left[\frac{1}{i_0+\cdots+i_{k-2}+1} - \frac{1}{i_0+\cdots+i_{k-1}+1} \right] \times \frac{x_k^{i_0+\cdots+i_{k-1}+1}}{i_0!\cdots i_{k-1}!}$$

and, for $1 \le n \le k-1$,

$$I_n = \sum_{(i_0, i_1, \dots, i_{k-1}) \in S_k} (-1)^{q_k(i_0, i_1, \dots, i_{k-1})}$$

$$\times \left[\left(\frac{1}{i_0+\cdots+i_{n-2}+1} - \frac{1}{i_0+\cdots+i_{n-1}+1} \right) \frac{x_n^{i_0+\cdots+i_{n-1}+1}}{i_0!\cdots i_{n-1}!} \frac{x_{n+1}^{i_n}}{i_n!} \cdots \frac{x_k^{i_{k-1}}}{i_{k-1}!} \right.$$

$$\left. - \left(\frac{1}{i_0+\cdots+i_{n-1}+1} - \frac{1}{i_0+\cdots+i_n+1} \right) \frac{x_n^{i_0+\cdots+i_n+1}}{i_0!\cdots i_n!} \frac{x_{n+1}^{i_{n+1}}}{i_{n+1}!} \cdots \frac{x_{k-1}^{i_{k-1}}}{i_{k-1}!} \right].$$

Thus, the proof of (19) will be finished if we show that, for each $n = 0, 1, \dots, k$,

$$I_n = \sum_{\{(i_0, i_1, \dots, i_k) \in S_{k+1}:\ i_0 = \cdots = i_{n-1} = 0,\ i_n \ge n+1\}} (-1)^{q_{k+1}(i_0, i_1, \dots, i_k)} \frac{x_0^{i_0}}{i_0!} \frac{x_1^{i_1}}{i_1!} \cdots \frac{x_k^{i_k}}{i_k!}. \tag{95}$$

It can easily be seen that (95) holds for $n = 0$. Namely, in the sum which defines I_0, each nonnegative term in the brackets with $i_0 = 0$ appears once more as a minus term (with $i_{k-1} = 0$). Consequently, these terms cancel each other out. Furthermore,

$$q_k(i_0, i_1, \ldots, i_{k-1}) = q_{k+1}(i_0 + 1, i_1, \ldots, i_{k-1}, 0)$$

and

$$q_k(i_0, i_1, \ldots, i_{k-1}) + 1 = q_{k+1}(1, i_0, i_1, \ldots, i_{k-1}).$$

This gives (95) for $n = 0$. In order to prove (95) for $n = 1, 2, \ldots, k-1$, we proceed in the following way. Observe that, in the brackets of the definition of I_n for $1 \le n \le k-1$, the nonnegative terms with $i_{n-1} = 0$ and the minus terms with $i_n = 0$ vanish. Furthermore, we have $i_0 + \cdots + i_{n-1} \ge n$ if $i_{n-1} > 0$ and, analogously, $i_0 + \cdots + i_n \ge n+1$ if $i_n > 0$. Thus, it suffices to use the following fact: for each $j \in \{2, 3, \ldots\}$ and $n \in \{0, 1, \ldots, j-2\}$, the sum

$$\begin{aligned} I_j^{(n)} = &\sum_{(i_0, i_1, \ldots, i_n) \in S_{j-1}^{(n)}} (-1)^{q_n(i_0, i_1, \ldots, i_n)} \\ &\times \left(\frac{1}{i_0 + \cdots + i_{n-1} + 1} - \frac{1}{i_0 + \cdots + i_n + 1} \right) \frac{1}{i_0! \cdots i_n!}, \end{aligned}$$

where

$$\begin{aligned} S_{j-1}^{(n)} = \big\{ &(i_0, i_1, \ldots, i_n) \in \{0, 1, \ldots\}^{n+1}\colon\ i_0 + i_1 + \cdots + i_n = j-1\ i_n > 0 \\ &\text{and if } i_s = l > 1, \text{ then } i_{s-1} = i_{s-2} = \cdots = i_{(s-l+1)_+} = 0 \big\} \end{aligned}$$

simplifies to

$$I_j^{(n)} = \frac{j - (n+1)}{j!}. \tag{96}$$

This gives (95) for $n = 1, 2, \ldots, k-1$. Moreover, from (96), putting $j = k+1$ and $n = k-1$, we get (95) for $n = k$. Finally, we show how (96) follows from standard combinatorial formulas. From the definition of $S_{j-1}^{(n)}$ we get that, in the sum defining $I_j^{(n)}$, the variable i_n either equals $j-1$ or belongs to $\{1, \ldots, n\}$. With the notation $i = i_n$, this gives

$$I_j^{(n)} = \frac{j-1}{j!} - \sum_{i=1}^{n} \frac{i}{i!(j-i)j} \sum_{(i_0, i_1, \ldots, i_{n-i}) \in S_{j-i-1}^{(n-i)}} (-1)^{q_{n-i}(i_0, i_1, \ldots, i_{n-i})} \frac{1}{i_0! \cdots i_{n-i}!}.$$

With the notation $s = \min\{l\colon\ i_l > 0\}$, the inner sum of the last expression can be written in the following form:

$$\begin{aligned}
&\sum_{(i_0, i_1,\dots,i_{n-i})\in S^{(n-i)}_{j-i-1}} (-1)^{q_{n-i}(i_0,i_1,\dots,i_{n-i})} \frac{1}{i_0!\cdots i_{n-i}!}\\
&\quad= \sum_{s=0}^{n-i} \frac{1}{(j-1-n+s)!} \sum_{r=1}^{n-i-s} (-1)^r \sum_{\{i_1+\cdots+i_r=n-i-s\colon\ i_l>0\}} \frac{1}{i_1!\cdots i_r!}\\
&\quad= \sum_{s=0}^{n-i} \frac{(-1)^{n-i-s}}{(j-1-n+s)!(n-i-s)!}.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
I^{(n)}_j &= \frac{j-1}{j!} - \sum_{i=1}^{n} \frac{1}{(i-1)!(j-i)j} \sum_{s=0}^{n-i} \frac{(-1)^{n-i-s}}{(j-1-n+s)!(n-i-s)!}\\
&= \frac{j-1}{j!} - \sum_{s=0}^{n-1}(-1)^s \frac{1}{j\ s!} \sum_{i=1}^{n-s} \frac{1}{(i-1)!(j-i)(j-1-i-s)!}.
\end{aligned}$$

Now it remains to show that

$$\sum_{s=0}^{n-1}(-1)^s \frac{1}{js!} \sum_{i=1}^{n-s} \frac{1}{(i-1)!(j-i)(j-1-i-s)!} = \frac{n}{j!}. \tag{97}$$

It is easy to see that (97) holds for $n = 1, 2$. Assuming that (97) is true for some $n \le j-3$, we show that it also holds for $n+1$. Namely,

$$\begin{aligned}
&\sum_{s=0}^{n}(-1)^s \frac{1}{js!} \sum_{i=1}^{n+1-s} \frac{1}{(i-1)!(j-i)(j-1-i-s)!}\\
&\quad= \frac{n}{j!} + \sum_{s=0}^{n}(-1)^s \frac{1}{js!} \frac{1}{(n-s)!(j+s-n-1)(j-n-2)!}\\
&\quad= \frac{n}{j!} + \frac{1}{j(j-n-2)!} \sum_{s=0}^{n}(-1)^s \frac{1}{s!(n-s)!(j+s-n-1)}\\
&\quad= \frac{n}{j!} + \frac{1}{j(j-n-2)!} \frac{1}{(j-n-1)(j-n)\cdots(j-1)}\\
&\quad= \frac{n}{j!} + \frac{1}{j!} = \frac{n+1}{j!}.
\end{aligned}$$

Thus, (97) is proved. This completes the proof of Theorem 7. □

7. Differentiability, admissibility. In general, the conditions of Theorem 5, which were checked for (max, +)-linear systems in subsection 5.3, are not sufficient to ensure any differentiability property.

The aim of the present section is to show that, under certain tail conditions on the random variables D^i_n defined in (10) (which boil down to Cramér-type conditions on the firing times in the Petri net case), the functions $f^i(\lambda) = \mathbb{E}W^i$, $i = 1, \ldots, \alpha'$, are actually infinitely differentiable in λ in a right neighborhood of 0, and that, for all $n \geq 1$,

$$\lim_{\lambda \downarrow 0} (f^i(\lambda))^{(n)} = \mathbb{E}p_{n+1}(D_0^i, D_1^i, \ldots, D_n^i), \tag{98}$$

where the polynomials p_k and the random variables D_k^i are those defined in Theorem 2.

In that sense, the expansion given in Theorem 2 is a Taylor-type expansion indeed.

The proof of this property is based on the notion of *admissibility* defined in [34], which is sufficient to grant the above differentiability property as well as the formula

$$\lim_{\lambda \downarrow 0} (f^i(\lambda))^{(n)} = \int_{\mathbb{R}^k} \mathbb{E}\psi^i_{s_1, \ldots, s_k}(o, \{Z_n\})\, ds_1 \ldots ds_k \tag{99}$$

(see Theorem 5 for the notation), which, in turn, implies (98) in view of Theorem 7.

THEOREM 8. *Let* $F_n = \oplus_{i=1}^{\alpha} \{(A_{-1} \otimes \cdots \otimes A_{-n} \otimes (B_{-n} \oplus 0))_i\}$. *If there exists* $\theta^* > 0$ *such that, for all* $\theta \in [0, \theta^*)$,

$$\mathbb{E}e^{\theta} F_n \leq L_\theta(\phi(\theta))^n \tag{100}$$

for some finite functions $\phi(\theta) > 1$ *and* L_θ, *then the functional* ψ *defined in* (70) *is admissible.*

PROOF. In order to prove the admissibility of ψ, we have to show that there exist constants $K, N < \infty$ and $1 < a < \infty$, $\theta > 0$, such that, for all $s < t < 0$,

$$\int\int \left|\psi(\mu|^t, z) - \psi(\mu|^s, z)\right| \, P_\lambda(d\mu | \, \mathscr{C}(l, j)) Q(dz) \leq K(j + l)^N a^{j+l} e^{-\theta t}, \tag{101}$$

where $\mathscr{C}(l, j) = \{\mu' \colon \mu'([s, t)) = l, \mu'([t, 0)) = j\}$. Let $\mathscr{B}^i(l, j)$ be the event

$$\mathscr{B}^i(l, j) = \left\{ \bigcap_{n=j+1}^{j+l} \{d_n^i - t_{-n} < 0\} \right\}. \tag{102}$$

On $\mathscr{B}^i(l, j) \cap (\mathscr{C}(l, j) \times \mathscr{K}^\infty)$, we have $\psi(\mu|^t, z) = \psi(\mu|^s, z)$, so that

$$\begin{aligned} &\int\int \left|\psi(\mu|^t, z) - \psi(\mu|^s, z)\right| \, P_\lambda(d\mu \mid \mathscr{C}(l, j)) Q(dz) \\ &\quad = \int\int \left|\psi(\mu|^t, z) - \psi(\mu|^s, z)\right| \mathbf{1}_{\overline{\mathscr{B}^i(l, j)}} \, P_\lambda(d\mu \mid \mathscr{C}(l, j)) Q(dz) \leq f^{1/2} g^{1/2}, \end{aligned}$$

where

$$f = \int\int (\psi(\mu|^t, z) - \psi(\mu|^s, z))^2 \, P_\lambda(d\mu \mid \mathscr{C}(l, j)) Q(dz) \tag{103}$$

and

$$g = P\Big(\overline{\mathscr{B}^i(l, j)} \mid \mathscr{C}(l, j) \times \mathscr{K}^\infty\Big). \tag{104}$$

Using now the special form of ψ and the independence assumptions, we obtain

$$f \le \mathbb{E}\big[\big(D^i_{l+j}\big)^2 \mid \mathscr{C}(l, j) \times \mathscr{K}^\infty\big] = \mathbb{E}\big[\big(D^i_{l+j}\big)^2\big]. \tag{105}$$

Let $\chi_n = \max_{i,\, j}\{(A_{-n})_{i,\, j} + (B_{-n})_j \oplus 0\}$. We have, for all $n \ge 1$,

$$(D^i_n)^2 \le \left(\sum_{p=1}^n \chi_p\right)^2 \le n \sum_{p=1}^n (\chi_p)^2$$

and so $f \le n^2\kappa$, with $\kappa = \mathbb{E}(\chi_1)^2 = \mathbb{E}(\max_i D^i_1)^2 < \infty$.

As for g, we have

$$\begin{aligned} g &= P\left(\bigcup_{n=j+1}^{j+l} \{D^i_n - T_{-n} > 0\} \mid \mathscr{C}(l, j) \times \mathscr{K}^\infty\right) \\ &\le P\big(D^i_{j+l} > t \mid \mathscr{C}(l, j) \times \mathscr{K}^\infty\big) = P\big(D^i_{j+l} > t\big) \\ &= P(\exp(uD^i_{j+l}) > e^{ut}) \le \mathbb{E}[\exp(uD^i_{j+l})]e^{-ut} \\ &\le L_u \phi(u)^{j+l} e^{-ut}, \end{aligned}$$

where u is any real number in the interval $(0, \theta^*)$, and where we used Chebyshev's inequality and assumption (100) in order to derive the last two inequalities. Finally, admissibility is proved with $\theta = u/2$, $a = (\phi(u))^{1/2}$, $N = 1$ and $K = \sqrt{\kappa L_u}$. $\square$

Remark. In a stochastic event graph, (100) is satisfied whenever the firing times $\{\sigma^k_n,\ k = 1, \dots, \beta\}$ are i.i.d. in n and satisfy the following Cramér-type property:

$$\mathbb{E}\left[\exp\left(\theta \sum_{k=1}^{\beta} \sigma^k_n\right)\right] \le \xi(\theta), \tag{106}$$

where $\xi(\theta)$ is some finite function on the interval $[0, \theta^*)$ for some $\theta^* > 0$.

In order to see this, we can write

$$\begin{aligned} &\mathbb{E}[\exp(\theta D^i_n)] \\ &\quad\le \mathbb{E}[\exp(\theta \max_i D^i_n)] \\ &\quad= \mathbb{E}\Big[\exp\Big(\theta\Big\{\max_{i_0,\, i_1, \dots, i_n \in [1, \dots, \alpha]} (A_{-1})_{i_0,\, i_1} + \cdots + (A_{-n})_{i_{n-1},\, i_n} + (B_{-n})_{i_n}\Big\}\Big)\Big] \\ &\quad= \mathbb{E}\Big[\max_{i_0,\, i_1, \dots, i_n \in [1, \dots, \alpha]} \exp(\theta\{(A_{-1})_{i_0,\, i_1} + \cdots + (A_{-n})_{i_{n-1},\, i_n} + (B_{-n})_{i_n}\})\Big] \\ &\quad\le \sum_{i_0,\, i_1, \dots, i_n \in [1, \dots, \alpha]} \mathbb{E}\big[\exp(\theta\{(A_{-1})_{i_0,\, i_1} + \cdots + (A_{-n})_{i_{n-1},\, i_n} + (B_{-n})_{i_n}\})\big]. \end{aligned}$$

The following uniform bound follows from the definition of A_n and B_n in (30) and (31):

$$(A_{-1})_{i_0,\,i_1} + \cdots + (A_{-n})_{i_{n-1},\,i_n} + (B_{-n})_{i_n} \le \sum_{k=1}^{\beta} \sum_{p=0}^{n} \sigma_{-p}^{k}.$$

Thus, using (106) and the independence assumptions, we get

$$\mathbb{E}\big[\exp(\theta D_n^i)\big] \le \alpha^n \xi(\theta)^{n+1}, \tag{107}$$

which concludes the proof of (100) in this case.

8. Future research. Future research will bear on the extension of this computational point process approach to the derivation of expansions for Laplace transforms and higher moments and eventually to the case of networks with non-Poisson input processes.

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INRIA–SOPHIA ANTIPOLIS
2004 ROUTE DES LUCIOLES
B.P. 93
06902 SOPHIA ANTIPOLIS
CEDEX
FRANCE
E-mail: Francois.Baccelli@inria.fr

ABTEILUNG STOCHASTIK
UNIVERSITÄT ULM
D-89069 ULM
GERMANY
E-mail: volker.schmidt@mathematik.uni-ulm.de