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# MULTITYPE THRESHOLD GROWTH: CONVERGENCE TO POISSON–VORONOI TESSELLATIONS

BY JANKO GRAVNER AND DAVID GRIFFEATH

*University of California, Davis and University of Wisconsin*

A *Poisson–Voronoi tessellation* (PVT) is a tiling of the Euclidean plane in which centers of individual tiles constitute a Poisson field and each tile comprises the locations that are closest to a given center with respect to a prescribed norm. Many spatial systems in which rare, randomly distributed centers compete for space should be well approximated by a PVT. Examples that we can handle rigorously include *multitype threshold vote automata*, in which $\kappa$ different camps compete for voters stationed on the two-dimensional lattice. According to the deterministic, discrete-time update rule, a voter changes affiliation only to that of a unique opposing camp having more than $\theta$ representatives in the voter's neighborhood. We establish a PVT limit for such dynamics started from completely random configurations, as the number of camps becomes large, so that the density of initial "pockets of consensus" tends to 0. Our methods combine nucleation analysis, Poisson approximation, and shape theory.

**1. Introduction.** *Voronoi tessellations* have appeared frequently in many branches of science; we refer the reader to [22] for a comprehensive overview of the voluminous literature. In essence, various versions of the Voronoi tessellation arise naturally in situations where scattered centers induce a division of available space due to either competition for resources or optimization in transportation. These structures were also introduced in pure mathematics at the beginning of this century to facilitate the understanding of quotient spaces in group representations. More recently, stochastic Voronoi tessellations, in which the centers constitute a Poisson field, have been investigated extensively by researchers in random geometry. Many surprisingly detailed theorems have been proved (cf. [25], [21]), mainly for the classical isotropic ($||\cdot||_2$) case.

The static definition of a Voronoi tessellation involves a collection of centers in the Euclidean plane, each tile comprised of those locations which are closest to a given center with respect to a prescribed norm. Alternatively, one can give a dynamic definition by starting, from each of the centers, a growing ball of "controlled territory" that expands in all directions at a constant rate. These balls reach a standoff when they collide, thereby forming stable interfaces. Indeed, more than fifty years ago Johnson and Mehl [20] proposed this and various related stochastic models for crystal formation that combine Poisson-distributed nucleation centers, linear isotropic growth of droplets, and standoff interaction at the boundary. The variant, usually called the Johnson–Mehl

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model, involves continual formation of crystal seeds according to a dynamic Poisson process, whereas all nucleation for the processes we study in this paper arises from an initial field of centers.

Our interest in Poisson–Voronoi tessellations (PVT) stems from previous work on cellular automaton (CA) models for excitable media ([12], [13]), in which rare "pacemakers" emit waves that propagate until they encounter waves emanating from other sources. Typically, annihilation at the interfaces produces a locally periodic limiting configuration in which the regions slaved to different pacemakers divide the two-dimensional integer lattice into a discrete approximation to a PVT. Further computer experimentation has shown that many random cellular automata and interacting particle systems based on the threshold growth mechanism ([9], [11], [14], [15]) exhibit the same phenomenology. For simulation purposes as well, it is desirable to have *digital* algorithms which dynamically generate good approximations to Voronoi tessellations. Indeed, assuming prior knowledge of the locations of centers, such rules are not hard to concoct (see Example 0 below, or Section 4.7.1 of [1] for more sophisticated algorithms).

In this paper we consider various cellular automata which generate—via nucleation of "droplets" from widely separated centers—final states that resemble Poisson–Voronoi tessellations. Early in the project, guided by initial impressions from computer simulations, we expected to find asymptotic tilings of the classical ($l^2$) variety. However the geometry of the final tiling evidently depends on the characteristic shape with which individual droplets spread, and local, lattice-based growth seems generically to exhibit some anisotropy. Indeed, there is apparently no local, homogeneous growth model, deterministic or stochastic, which is known to grow a circle. Moreover, empirical evidence strongly suggests that all of the simplest mathematical prototypes do not. Thus, our CA rules give rise to generalized Poisson–Voronoi tessellations in the rare nucleation limit. Rather than being a drawback, we feel that this limitation contains a valuable lesson for applied modelers. As Figure 1 and computations in Section 6 suggest, visual and numerical deviations from the $l^2$ case are quite subtle in many cases. Nevertheless, one should not expect rotation-invariant crystallization patterns on a lattice. (If the goal is to simulate a classical PVT, then one can either use a large "lattice circle" as the neighbor set, obtaining the desired tiling in the limit of increasing range of interaction, or alternatively consider simple isotropic threshold growth models in Euclidean space [14].)

After initial nucleation and growth, additional complexities arise in the PVT cartoon, due to nonlinear droplet interaction and pollution of space by failed centers. These complicating factors arise naturally in models for excitable media and crystallization, and presumably also have counterparts in the modeled phenomena themselves. For instance, how does one control potential effects in the "corners" when two droplets collide? It seems worthwhile to study a class of examples which is sufficiently tractable that one can establish a rigorous theory. Even among the cellular automata we consider, it turns out that some obey the "standoff" collision rule while others do not.

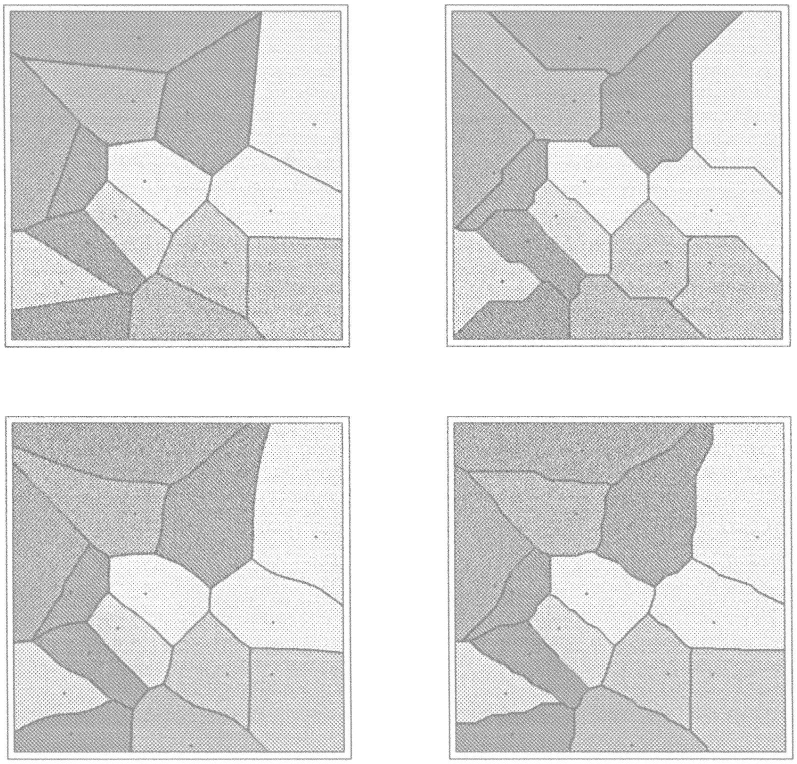

FIG. 1. *Voronoi tessellations for various norms.*

Our most basic example is provided by the multitype threshold vote automaton. In this CA rule, voters occupy sites of $\mathbf{Z}^2$. Initially, each individual adopts one of $\kappa$ opinions at random. Thereafter a voter changes affiliation to agree with the consensus of more than a threshold number of other voters in his or her neighborhood; one adds the proviso that the voter's opinion remains unchanged in case of ambiguity. If $\kappa$ is large, then local concentrations of agreement high enough to induce a change are rare, but these have a considerable advantage over their immediate surroundings, and hence are able to grow until they encounter a large region controlled by another opinion. To make this intuitive scenario precise, we need quite a few definitions and preliminary results. The rest of this introduction contains the basic theoretical set-up and statements of our main theorems, together with numerous plausible conjectures about related models. Subsequent sections address various technical preliminaries, the detailed proofs of our results, and a few applications.

The theoretical framework for Poisson–Voronoi convergence is weak convergence of random closed sets in the plane. Since the closed sets encountered—namely, tile boundaries—are inherently infinite, convergence in the Haussdorff metric will not suffice. This problem was encountered, and the requisite theory developed, by Attouch, Salinetti and Wets [4], [23], [24] in connection with certain optimization problems. All sets considered in this paper will be subsets of $\mathbf{R}^2$. Given two closed sets $A_1$ and $A_2$ and an $r > 0$, let

$$\varepsilon_r(A, B) = \inf\{\varepsilon > 0\colon A_1 \cap B(r) \subset A_2^\varepsilon \text{ and } A_2 \cap B(r) \subset A_1^\varepsilon\}.$$

Here $B(r)$ is the closed Euclidean ball of radius $r$ centered at 0, and $A^\varepsilon = A + B(\varepsilon)$ is the $\varepsilon$-fattening of a set $A$. The distance between two arbitrary closed sets $A$ and $B$ is then

$$d(A, B) = \sum_{r=1}^{\infty} 2^{-r} \varepsilon_r(A, B).$$

This is the Attouch-Wets metric [4]. We say that nonempty closed sets $A_n$ converge to a closed set $A$ if $d(A_n, A) \to 0$. Since the limit is required to be closed, uniqueness is immediate.

Note that if all sets $A_n$ are included in some fixed large ball $B(r)$, then this convergence is equivalent to convergence in the Hausdorff metric. Many equivalent ways to define the convergence are discussed in [23]. For example, $A_n \to A$ iff for any bounded open set $G$, $A \cap G \neq \varnothing$ implies $A_n \cap G \neq \varnothing$ for large enough $n$, while $A \cap cl(G) = \varnothing$ implies $A_n \cap \mathrm{cl}(G) = \varnothing$ for large enough $n$. Actually, one does not need to verify this for all open sets $G$; open balls suffice. It is also worth mentioning that the set of closed sets with this metric is compact. (It is quite possible, however, that $A_n \to \varnothing$).

By a *random closed set* we mean a mapping from a probability space $(\Omega, \mathscr{F}, P)$ to closed sets that is measurable with respect to the Attouch–Wets metric. It is natural to say that, by definition, random closed sets $A_n$ *converge in distribution* to a random closed set $A$, $A_n \to_d A$, iff

$$E(\phi(A_n)) \to E(\phi(A)),$$

for every bounded continuous functional $\phi$ from closed sets (with metric $d$) to $\mathbf{R}$. It turns out (see [24]) that $A_n \to_d A$ iff for every set $K$ which is the union of finitely many closed balls and is such that the function $P(\cdot \cap K \neq \varnothing)$ is a.s. continuous (with respect to the measure on closed sets given by $A$), it follows that $P(A_n \cap K \neq \varnothing) \to P(A \cap K \neq \varnothing)$.

Next, let us define the notion of *Voronoi tessellation*. Fix a norm $||\cdot||$ on $\mathbf{R}^2$, and also a countable set of centers $F \subset \mathbf{R}^2$ without limit points. Define $\mathscr{V}(F)$ to be the set of all points $x \in \mathbf{R}^2$ for which $\inf\{||x - z||\colon z \in F\}$ is *not* uniquely attained. One can imagine giving every point in $F$ its own color and then coloring every other point in $\mathbf{R}^2$ the same color as the closest point in $F$. Then the set of "confused" points is $\mathscr{V}(F)$; note that it is always a closed set, called the (*basic*) *Voronoi tessellation*. Furthermore, if $\zeta\colon F \to \{1, 2, \dots\}$ is a coloring of $F$, define $\mathscr{V}(F; \zeta)$ to be the set of points for which $\inf\{||x - z||\colon z \in F\}$ is attained at two or more points with different colors. Clearly, $\mathscr{V}(F; \zeta) \subset \mathscr{V}(F)$ for any $\zeta$. The connected components of $\mathscr{V}(F)^c$ or $\mathscr{V}(F; \zeta)^c$ are connected sets colored with a single color by the above convention, so accordingly are called *tiles*.

As mentioned earlier, the classical Voronoi tessellation is obtained by using the Euclidean norm $||\cdot||_2$. In this case the tiles are convex; in fact, they are polygons. We conjecture that for any other norm there exists a two-point set of centers $F$ such that the tessellation is not a line (and hence one of the tiles is nonconvex). Many results about Voronoi tessellations, especially for the classical case, together with extensive background, may be found in [25], [22] and [21].

The (basic) Poisson-Voronoi tessellation (PVT) is $\mathscr{V}(\wp)$, where $\wp$ is a Poisson point location on $\mathbf{R}^2$ with intensity 1. For any $F$, let $\mathscr{V}_\kappa(F) = \mathscr{V}(F;\zeta)$, where $\zeta$ is chosen to be uniformly distributed over $\{1,\dots,\kappa\}$ and independent for different sites of $F$. The *$\kappa$-colored PVT* is then $\mathscr{V}_\kappa(\wp)$. We identify $\mathscr{V}_\infty(\wp) = \mathscr{V}(\wp)$. Figure 1 shows how the Voronoi tessellation varies if the set of centers is kept fixed and the norms are, counterclockwise from the top right: $l^\infty$, $l^2$, $l^3$, and the one given by $||(x, y)|| = \max\{|x| + 3|y|, 3|x| + |y|\}$.

It is clear that $\mathscr{V}(\wp)$ and $\mathscr{V}_\kappa(\wp)$ are both a.s. closed subsets of $\mathbf{R}^2$. Why are they random closed subsets? We will see later that $\mathscr{V}(\cdot)$ is actually an a.s. continuous function on $\wp$, given the appropriate topology on sets of centers $F$.

Intuitively, convergence to a (basic or colored) PVT should be a ubiquitous phenomenon in spatial systems with the following features:

1. Growth proceeds from rare nucleation centers.
2. Growth is linear in time and acquires a characteristic shape.
3. Competition between regions that emanate from different centers leads to a standoff.

We shall see that this intuition is essentially correct, but important (and not immediately obvious) assumptions are needed before the heuristics can be turned into actual theorems.

Let us now prepare to introduce various models, cellular automata on $\mathbf{Z}^2$, which we will denote by $\zeta_t$. Since most of our systems, and all our rigorous results, are related to (discrete) threshold growth, it is convenient to begin with the formulation of those dynamics. Readers are referred to [14] and [15] for a more comprehensive treatment. Threshold growth has two parameters:

1. $\mathscr{N}$, a finite subset of $\mathbf{Z}^2$ which includes 0, is the *neighborhood* of the origin; $x + \mathscr{N}$ is then the neighborhood of site $x$.
2. $\theta$, a positive integer, is the *threshold*.

We call the dynamics *symmetric* if $-\mathscr{N} = \mathscr{N}$. Most of our examples will use for $\mathscr{N}$ either the range $\rho$ diamond neighborhood: $\mathscr{N} = \{x\colon ||x||_1 \le \rho\}$, or the range $\rho$ box neighborhood: $\mathscr{N} = \{x\colon ||x||_\infty \le \rho\}$.

Given $A \subset \mathbf{Z}^2$, define the transformation

$$\mathscr{T}(A) = A \cup \{x\colon |(x + \mathscr{N}) \cap A| \ge \theta\}.$$

An initial $A_0 \subset \mathbf{Z}^2$ generates *discrete threshold growth dynamics* by iteration: $A_n = \mathscr{T}^n(A_0)$. Also, let $\mathscr{T}^\infty(A_0) = \bigcup_{n\ge 0} A_n$. We say that $A_0$ *generates persistent growth* if $\mathscr{T}^\infty(A_0)$ is unbounded, that is, $A_n \ne A_{n+1}$ for every $n$.

We call the threshold growth dynamics *supercritical* if $\mathcal{T}^\infty(A_0) = \mathbf{Z}^2$ for some finite $A_0$. In this case, we define two key nucleation parameters as follows. First, let $\gamma = \gamma(\mathcal{N}, \theta)$ be the minimal number of sites needed for persistent growth, that is, the smallest $i$ for which there exists an $A_0$ which generates persistent growth and has $|A_0| = i$. Second, let $\nu = \nu(\mathcal{N}, \theta)$ be the number of sets $A_0$ with $\gamma$ sites that generate persistent growth and have their leftmost lowest sites (meaning the leftmost among the lowest sites) at the origin. We say that the growth is *voracious* if, started from any of the $\nu$ initial sets $A_0$ described above, $\mathcal{T}^\infty(A_0) = \mathbf{Z}^2$. Voracity should be automatic as soon as $\mathcal{N}$ is nice enough, but this seems quite hard to prove, even for special neighborhoods. In fact, the only substantial result is contained in recent work of Bohman [6], which establishes a much stronger condition in the case of box neighborhoods—namely, that the dynamics are *omnivorous*: $\mathcal{T}^\infty(A_0) = \mathbf{Z}^2$ for *any* finite set $A_0$ which generates persistent growth. On the other hand, in very small cases, voracity can be checked by hand, and for moderately small cases one can use a computer ([15], [16]).

Throughout most of this paper, we will assume the associated threshold growth dynamics to be symmetric, supercritical and voracious. Under these assumptions it can be proved ([15], [27]) that if a finite $A_0$ is such that $A_n \uparrow \mathbf{Z}^2$, then the closed sets $A_n/n \subset \mathbf{R}^2$ converge as $n \to \infty$ to a bounded convex neighborhood $\mathcal{L}$ of the origin. To identify $\mathcal{L}$, start by defining a version $\bar{\mathcal{T}}$ of the dynamics which has the same parameters $\mathcal{N}$ and $\theta$, but acts on sets $B \subset \mathbf{R}^2$ as follows: $\bar{\mathcal{T}}(B) = B \cup \{x \in \mathbf{R}^2: |(x+\mathcal{N}) \cap B| \geq \theta\}$. For any unit vector $u$, $\bar{\mathcal{T}}$ translates the half-space $H_u^- = \{x: \langle x, u\rangle \leq 0\}$ by a velocity $w(u) > 0$; that is, $\bar{\mathcal{T}}(H_u^-) = H_u^- + w(u)u$. If these velocities are combined into $K_{1/w} = \bigcup_u [0, 1/w(u)]u$, then $\mathcal{L}$ is given by the polar transform: $\mathcal{L} = \mathcal{L}(\mathcal{N}, \theta) = K_{1/w}^*$. In particular, $\mathcal{L}$ is independent of $A_0$. This limiting shape induces a natural norm $||\cdot||$ on $\mathbf{R}^2$ for the PVT which arise in our examples. This norm is given as the Minkowski functional of $\mathcal{L}$: $||x|| = \inf\{\lambda > 0: x \in \lambda \cdot \mathcal{L}\}$, making $\mathcal{L}$ the unit ball with respect to $||\cdot||$.

We are now ready to begin describing various multitype cellular automata and related interacting systems which give rise to Poisson–Voronoi tessellations in the rare nucleation limit. Our discussion delineates five classes of models, Examples 0–4, the first two of which will include cases that we are able to analyze rigorously. The initial example is included for its simplicity, and because it illustrates some technical issues in the main proofs to follow.

EXAMPLE 0 (Competing growth dynamics with threshold 1). Fix a small $p > 0$ and independently mark each site in $\mathbf{Z}^2$ as occupied with probability $p$. Now give every unoccupied site color 0, and each occupied site its own color from $\{1, 2, \ldots\}$. Let this configuration be $\zeta_0$. At each successive time, every 0 that has a unique nonzero color among its four closest sites assumes that color. Clearly the system fixates; denote its final state as $\zeta_\infty$. Also, let $C_p$ consist of sites $x$ which have a $y$ among their four closest neighbors such that $\zeta_\infty(y) \neq \zeta_\infty(x)$.

As $p \to 0$, it is easy to check that $\sqrt{p} \cdot \{\zeta_0 \neq 0\} \to_d \wp$. (We write $\{\zeta_0 \neq 0\}$ as shorthand for $\{x \in \mathbf{Z}^2\colon \zeta_0(x) \neq 0\}$ and use similar notation throughout.) However this property alone does *not* imply that $\sqrt{p} \cdot C_p \to_d \mathscr{V}(\wp)$, since taking a Voronoi tessellation is very far from continuous in the Attouch–Wets sense. For example, adding a new center close to an existing one is a small perturbation in the metric $d$, but has a large effect on the induced Voronoi tessellation.

Another problem is caused by the fact that the norm, which should obviously be $\|\cdot\|_1$, is not strictly convex. To see why this is an issue, take a pair of centers, put one at the origin, and move the second around. As the roving center traverses $|y| = |x|$, the induced Voronoi tessellation changes discontinuously.

As we shall see later in the paper, the easiest way to deal with the first difficulty is to require that the centers converge to $\wp$ in a slightly stronger sense. The second difficulty is nonessential, as it suffices for continuity to hold almost surely. Consequently, the following result is almost immediate.

THEOREM 1.1. *As $p \to 0$, $\sqrt{p} \cdot C_p \to_d \mathscr{V}(\wp)$, where the PVT is taken in the $l^1$ norm.*

This paper focuses on the following six-parameter family of models for "competing growth" which includes the previous example as its simplest special case.

EXAMPLE 1 (Threshold vote cellular automata). We start by describing the parameters:

(i) $\kappa \in \{2, 3, \ldots\}$ is the number of foreground colors (opinions), coded as $1, 2, \ldots, \kappa$. Any site may take on either a foreground colors or the background color 0 (e.g., an undecided opinion);
(ii) $\mathscr{N} \subset \mathbf{Z}^2$ is as in the definition of threshold growth;
(iii) $\theta_0 \in \{0, 1, 2, \ldots\}$ is the threshold required for a foreground color not to change to 0;
(iv) $\theta_1 \in \{0, 1, 2, \ldots\}$ is the threshold required for a 0 to change into a foreground color;
(v) An integer $\theta_2 \geq \theta_0$ is the threshold required for a foreground color to change into another foreground color;
(vi) $p \in [0, 1]$ is the initial density of foreground colors, uniformly distributed over the palette. In other words, $P(\zeta_0(x) = k) = p/\kappa$, for $k = 1, \ldots, \kappa$, $P(\zeta_0(x) = 0) = 1 - p$, and the colors of different sites are independent.

The competitive dynamics are now prescribed as follows:

(i) if $\zeta_t(x) = 0$ and $|\{\zeta_t = k\} \cap (x + \mathscr{N})| \geq \theta_1$ for a unique $k > 0$, then $\zeta_{t+1}(x) = k$;
(ii) if $\zeta_t(x) > 0$ and $|\{\zeta_t = k\} \cap (x + \mathscr{N})| \geq \theta_2$ for a unique $k > 0$, then $\zeta_{t+1}(x) = k$;
(iii) if $\zeta_t(x) > 0$ and $|\{\zeta_t = k\} \cap (x + \mathscr{N})| < \theta_0$ for all $k > 0$, then $\zeta_{t+1}(x) = 0$;
(iv) in all other cases $\zeta_{t+1}(x) = \zeta_t(x)$.

Different choices of the six parameters lead to different types of convergence to Poisson–Voronoi tessellations. We highlight four cases below. In the first two this convergence can be proved, while the last two elude our attempts at rigorous argument.

CASE 1 ($\theta_2 = \infty$, $\theta_0 = 0$, $p$ small). In this case foreground colors cannot change, so $\zeta_t$ fixates (since every site can change at most once). Call the final state $\zeta_\infty$.

Fix an $r > 0$ and, for $t \in [0, \infty]$, define a random set $C(r, t) \subset \mathbf{Z}^2$ of "slightly confused sites" to consist of all $x$ for which there is no color $k > 0$ in $\zeta_t$ that occupies at least 3/4 of the sites in $B_\infty(x, r)$. (Here and throughout we use the standard notation for lattice balls: $B_r(x, R) = \{y \in \mathbf{Z}^2\colon ||x - y||_r \le R\}$.) As the first of our two main results, we will establish the following theorem.

THEOREM 1.2. *Assume that the threshold vote CA given by $\mathcal{N}$ and $\theta_1$ is symmetric, supercritical and voracious. Suppose also that $K_{1/w}$ is convex. Then, for a suitably large $r$,*

$$\sqrt{\nu p^\gamma} \cdot C(r, \infty) \subset \mathbf{R}^2$$

*converges in distribution to $\mathscr{V}_\kappa(\wp)$ as $p \to 0$.*

The assumption that $K_{1/w}$ is convex does not seem necessary at first glance. However, as demonstrated in [15], the theorem *fails* without it, because if $K_{1/w}$ is not convex, then nonconvexities in same-color droplets take over the environment at an increased speed. We believe that the rescaled boundaries in Theorem 1.2 still converge, but the limiting set is hard to describe without "running a continuous movie," analogous to the one in [15] but more involved. This subtle effect is hardly noticeable since $K_{1/w}$ becomes convex in the threshold-range limit ([14]), and hence, even for moderately large neighborhoods and thresholds, the set $\sqrt{\nu p^\gamma} \cdot C(r, \infty)$ does not differ appreciably from a suitably chosen Voronoi tessellation.

There are not many instances when the above convexity hypothesis applies; one class of examples has range $\rho$ box neighborhood and $\theta = 2$ ([15]). Since $\gamma = 2$, $\nu = 4\rho(2\rho + 1)$, and voracity is easy to check, we arrive at the following corollary.

COROLLARY 1.3. *Assume $\mathcal{N}$ is the range $\rho$ box neighborhood, and $\theta = 2$. Then, for a suitably large $r$,*

$$\sqrt{4\rho(2\rho + 1)} \cdot p \cdot C(r, \infty) \to_d \mathscr{V}_\kappa(\wp)$$

*as $p \to 0$.*

In particular, if $\rho = 1$, then $\mathscr{V}_\kappa(\wp)$ above is taken with respect to the $l^1$-norm; Figure 2 provides an illustration of the dynamics at an intermediate time.

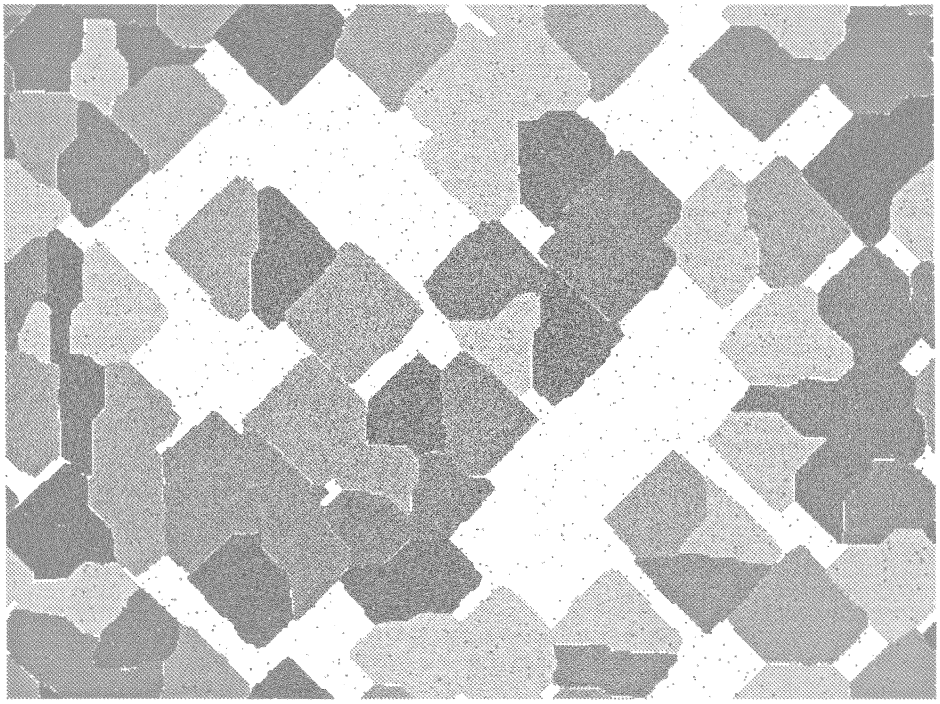

FIG. 2. *Case 1 with range 1 box neighborhood, $\theta = 2$, $\kappa = 5$, $p = 0.01$.*

CASE 2 ($\theta_0 = 0$, $p = 1$, $\kappa$ large). Now $\theta_1$ is irrelevant since 0's can never appear. Obviously, if $\theta_2 = 1$, then nothing ever changes, so all interesting cases have $\theta_2 \geq 2$.

It is not clear whether $\zeta_t$ fixates in this case, although we suspect it does. Regardless, with the definition of $C(r, t)$ as in Case 1, we do not need fixation in order to state and prove our second main result.

THEOREM 1.4. *Assume that the threshold vote CA given by $\mathscr{N}$ and $\theta_2$ is symmetric, supercritical and voracious, and that $\theta_2 \geq 2$. Then, for a suitably large $r$, and any $t \geq \kappa^\gamma$,*

$$\sqrt{\frac{\nu}{\kappa^{\gamma-1}}} \cdot C(r, t)$$

*converges in distribution to $\mathscr{V}(\wp)$ as $\kappa \to \infty$.*

In this case the convexity assumption on $K_{1/w}$ is unnecessary since same-color droplets do not interact on the relevant scale. As an illustration with an interesting power-law convergence, take $\mathscr{N}$ to be the range 2 diamond neighborhood, and $\theta = 4$. Then $\gamma = 4$, $\nu = 398$, and we have checked voracity

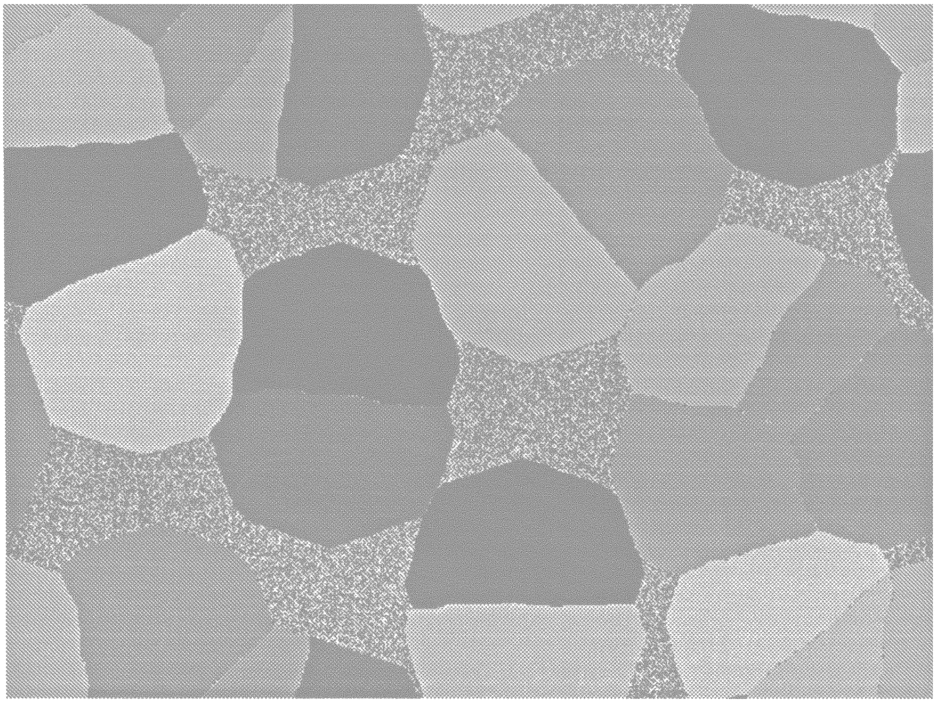

FIG. 3. *Case 2 with range 2 diamond neighborhood,* $\theta = 4$, $\kappa = 200$.

([15], [16]), so

$$\sqrt{398} \cdot \kappa^{-3/2} \cdot C \to_d \mathscr{V}(\wp),$$

where the norm is given by $||(x, y)|| = \max\{|x| + 3|y|, 3|x| + |y|\}$. The mechanism by which the dynamics achieve their final state is indicated in Figure 3.

CASE 3 ($\mathscr{N} = B_\infty(0, \rho)$, $\theta_0 = \theta_1 = a|\mathscr{N}|$, $\theta_2 = \infty$). Let $C(t)$ be the set of all sites which have more than one color in their neighborhood at time $t$. Also, call a nonnegative real function $f$ defined on an unbounded subset of $[0, \infty)$ regularly exponentially decaying if there is a constant $\beta > 0$ so that $f(u) = \phi(u)e^{-\beta u}$ for some function $\phi$ that is regularly varying at infinity.

Based on simulations, we offer a conjecture.

CONJECTURE 1.5. Assume that $a \in (p/\kappa, 1/2)$. Then there exists a regularly exponentially decaying function $f$ (depending on $p$, $a$ and $\kappa$) such that for $t \geq 1/f(\rho^2)^2$,

$$f(\rho^2) \cdot C(t) \to_d \mathscr{V}_\kappa(\wp) \quad \text{as } \rho \to \infty.$$

Here, the Voronoi tessellation is taken with respect to the norm obtained from Euclidean threshold growth dynamics $\mathscr{T}_E$ with $\mathscr{N}_E = B_\infty(0, 1) \subset \mathbf{R}^2$

and $\theta_E = 4a$; these dynamics map a measurable set $B \subset \mathbf{R}^2$ into $\mathscr{T}_E(B) = \{x \in \mathbf{R}^2\colon \text{area}((x + \mathscr{N}_E) \cap B) \geq \theta_E\}$ (see [14], [16]). Convexity of $K_{1/w}$ is then automatic [14].

In this case, most foreground colors change to 0 since a site typically sees about $(p/\kappa)\,|\mathscr{N}|$ other sites of each foreground color. Hence a local concentration of some foreground color exceeding $a|\mathscr{N}|$ only occurs at a large deviation rate, but such concentrations take over the surrounding background-covered space ($a < 1/2$ ensures supercriticality) until they encounter other spreading foreground clusters.

CASE 4 ($\mathscr{N} = B_\infty(0, \rho)$, $\theta_0 = 0$, $p = 1$, $\theta_2 = a|\mathscr{N}|$). Defining $C(t)$ as in Case 3, we offer the following.

CONJECTURE 1.6. Assume that $a \in (1/2\kappa, (\kappa + 1/2\kappa)) \setminus \{1/\kappa, 1/2\}$. Then there exists a regularly exponentially decaying function $f$ (depending on $a$ and $\kappa$) such that for any $t \geq 1/f(\rho^2)^2$,

$$f(\rho^2) \cdot C(t) \to_d \mathscr{V}_\kappa(\wp) \quad \text{as } \rho \to \infty.$$

A number of interesting issues arise in this case, which we discuss only briefly here. See [9] for some related interacting particle systems. One instance of Case 4 is shown in Figure 4.

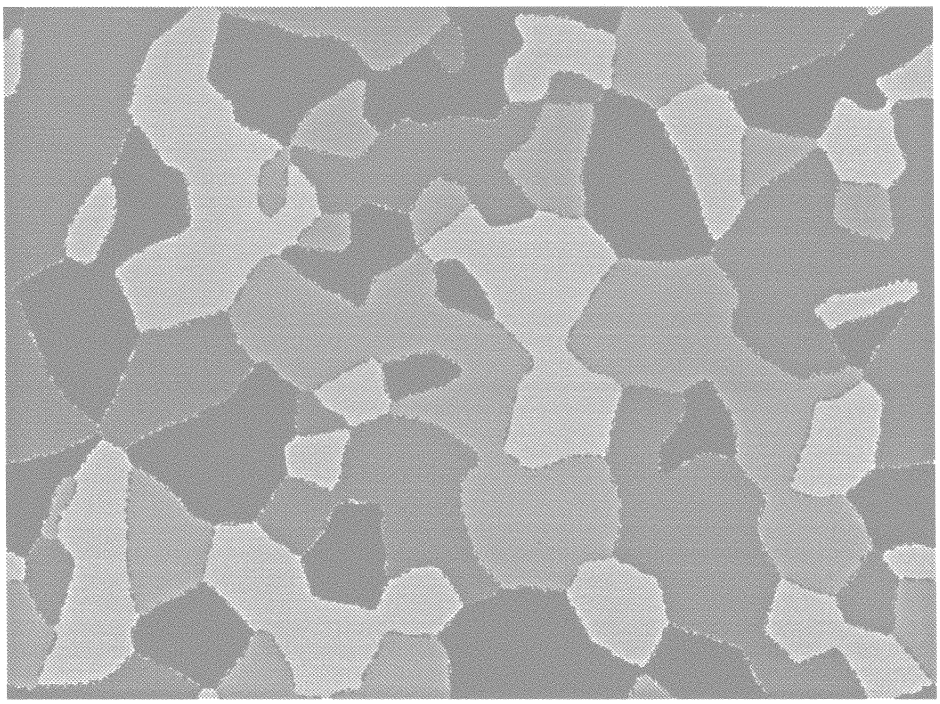

FIG. 4. *A fixated state in Case 4 with* $\kappa = 4$, $\rho = 10$, $\theta_2 = 140$.

Again, the Voronoi tessellation is taken with respect to the norm obtained from Euclidean threshold growth in $\mathbf{R}^2$, with $\mathscr{N}_E = B_\infty(0, 1)$. If $a > 1/\kappa$, then $\theta_E = 4(\kappa a - 1)/\kappa - 1$. This threshold is obtained by noting that in the threshold–range limit, a point becomes occupied if the area $\alpha$ of the intersection of its neighborhood with the occupied set satisfies $\alpha + (1/\kappa)(4 - \alpha) \geq 4a$. Similarly, if $a < 1/\kappa$, then $\theta_E = 4(1 - \kappa a)$.

One has to assume $a < (\kappa + 1)/2\kappa$ because $a$ must be small enough that sites on the boundary between a large solid region and noise see more than $\theta$ sites of the region's color. On the other hand, $a$ must be large enough that the same sites do not see more than $\theta$ representatives of two distinct colors, which gives $a > 1/2\kappa$.

One must also exclude $a = 1/\kappa$ in order to ensure rare nucleation. To understand this, note that if $a = 1/\kappa$ then the probability that a fixed site $x$ sees more than $\theta_2$ of a unique color converges as $\rho \to \infty$ to $\kappa \cdot P(X_1 \geq 0, X_2 \leq 0, \dots, X_\kappa \leq 0)$, where $(X_1, \dots, X_\kappa)$ is a degenerate normal vector with zero expectation, $E(X_k^2) = \kappa^{-1} - \kappa^{-2}$, and $E(X_k X_l) = -\kappa^{-2}$ for every $k \neq l$. Therefore, since $\kappa$ is fixed, the nucleation density does not converge to 0. However it does become smaller and smaller for larger and larger $\kappa$. Sending first $\kappa$ and then $\rho$ to infinity, and noting that the limit is threshold 0 growth in this scheme, the boundaries should tend to $\mathscr{V}(\wp)$ taken in the $l^\infty$ norm.

Finally, $a = 1/2$ must be excluded because boundaries of the Voronoi tessellation are not stable in this case. Rather, they evolve by surface tension. This is not a problem either for $a > 1/2$ (solid regions with small curvature at the boundary cannot grow into each other) or for $a < 1/2$ (a protective layer of noise remains between solid regions). See [17] for further discussion of the $a = 1/2$, $\kappa = 2$ majority vote rule. When $a = 1/2$ and $\kappa > 2$ (multitype absolute majority dynamics) the behavior is much trickier to understand, and the evolution presumably fixates in a peculiar configuration that has little to do with Voronoi tessellations (see September 4–10, 1995 Recipe at [18]).

Our remaining examples of convergence to Poisson–Voronoi tiles introduce new levels of complexity. For the most part, their rigorous mathematical analysis will require new methods beyond the scope of currently available technology. Nevertheless, we formulate additional conjectures for these examples in the hope that they will encourage further research.

EXAMPLE 2 (Threshold voter automata). This is an elaboration of a model from [11]. Here $\zeta_t(x) \in \{0, 1\}$, $\mathscr{N} = B_\infty(0, \rho)$, and $\theta = a|\mathscr{N}|$ for an $a \in (\frac{1}{4}, \frac{3}{4}) \setminus \{\frac{1}{2}\}$. Start from a product measure with equal densities of 0's and 1's. At each update the site at $x$ changes color iff the number of sites in $x + \mathscr{N}$ with the opposite color exceeds $\theta$.

The presumed behavior is just as in Conjecture 1.6 with $\kappa = 2$. As a matter of fact, when $a > 1/2$ the two dynamics actually coincide. For $a < 1/2$, however, the dynamics differ: in this case the boundaries of the tessellation stabilize by developing "periodic teeth." See [11] for more discussion about threshold voter automata and corresponding threshold voter models with random dynamics.

EXAMPLE 3 (Nearest-neighbor Greenberg–Hastings models). The dynamics of the nearest neighbor Greenberg–Hastings model $\zeta_t$ with $\kappa$ colors, $0, 1, \dots, \kappa - 1$ (cf. [13]) are as follows: at each update the changes $1 \to 2, 2 \to 3, \dots,$ and $\kappa - 1 \to 0$ occur automatically, while a 0 only changes to 1 if at least one 1 is present among its four nearest neighbors (otherwise remaining 0). Assume that the initial state $\zeta_0$ is uniform product measure over the available colors. If $\kappa$ is very large, then most sites quickly become quiescent (turn into 0's). However, rare wave-emitting centers also emerge and proceed to compete for control of space. We believe that no strictly local definition of boundaries between areas controlled by different centers is possible because of occasional random disturbances. Instead, we propose the following slightly involved formulation.

Start by defining $\zeta_\infty(x) = \lim_{t\to\infty} \zeta_{\kappa t}(x)$. (This limit exists; see [13].) Let $\mathscr{N}$ be the range 1 diamond neighborhood. Set

$$\begin{aligned} C_1 &= \big\{x\colon |\{\zeta_\infty = \zeta_\infty(x)\} \cap (x + \mathscr{N})| \geq 2\big\} \\ &\quad \cup \big\{x\colon |\{\zeta_\infty = (\zeta_\infty(x) + 1) \bmod \kappa\} \cap (x + \mathscr{N})| \geq 3\big\}, \\ C_2 &= \big\{x\colon |\{\zeta_\infty = (\zeta_\infty(x) - 1) \bmod \kappa\} \cap (x + \mathscr{N})| \geq 3\big\}. \end{aligned}$$

The set $C_1$ captures two different ways in which stable boundaries are established. A boundary site $x$ of a region controlled by one center must have all its nearest neighbor bonds open (i.e., the color difference across the bond is, modulo $\kappa$, 0 or $\pm 1$; see [13]). Such an $x$ may either have a $y \in x + \mathscr{N}$ of the same color but controlled by another center, or a $z \in x + \mathscr{N}$ which is controlled by two centers. When taking the value 0, such a $z$ must see at least three 1's, as some case checking shows. Unfortunately, the second condition in the definition of $C_1$ is also satisfied by sites belonging to "glitches" in the waves which arise because nucleation typically creates rather twisted centers. However such glitches meander left and right, so they can be eliminated by making use of the set $C_2$.

CONJECTURE 1.7. There exists a regularly exponentially decaying function $f$ and a sufficiently large $c > 0$ so that

$$f(\kappa) \cdot (C_1 \setminus (C_2 + B_\infty(0, c\kappa))) \to_d \mathscr{V}(\wp),$$

where the norm is $||\cdot||_1$.

This claim seems very difficult to prove; some partial results in the right direction may be found in [13]). For more general Greenberg–Hastings models [12] it seems hard to even devise a corresponding conjecture.

EXAMPLE 4 (Random growth models). Finally, we note that various monotone multitype growth models with random dynamics should converge to Poisson–Voronoi tessellations in the rare nucleation limit. For additive two-type systems such as Richardson's model there is a very well-developed shape and interaction theory (cf. [8]). Thus, one can state and prove a multitype stochastic counterpart to our Theorem 1.1, though we will not do so here. Similarly, Markovian counterparts to the models in our Examples 1–3 should exhibit corresponding phenomenology, but for these nonadditive interacting particle systems even the essential shape theory is currently beyond the reach of rigorous mathematics.

## 2. Preliminaries.

2.1. *Continuity of* $\mathscr{V}(\cdot)$. Denote by $\mathscr{L} = \{x \in \mathbf{R}^2\colon ||x|| \le 1\}$ the unit ball in the norm $||\cdot||$. Let us start by analyzing the case of only two centers.

LEMMA 2.1. *Let* $F = \{x, y\}$, $x \neq y$.

(a) *The tile of* $F$ *that contains* $x$ *is star shaped with respect to* $x$.

(b) *Assume that the line segment* $[x, y]$ *is not parallel to any line segment in* $\partial\mathscr{L}$. *Then* $z \in \mathscr{V}(F)$ *implies that* $y + \alpha(z-y) \notin \mathscr{V}(F)$ *for any* $\alpha \in (0, 1)$.

Note that (a) immediately implies that any tile in any Voronoi tessellation is star shaped, since it is an intersection of star-shaped sets.

PROOF. We can assume that $y = 0$. For (a), we need to prove that $||z|| < ||z - x||$ and $\alpha \in (0, 1)$ implies $||\alpha z|| < ||\alpha z - x||$, which follows since

$$||z|| < ||z-x|| = ||(1-\alpha)z + (\alpha z - x)|| \le (1-\alpha)||z|| + ||\alpha z - x||. \tag{2.1}$$

For (b), assume that $||z|| = ||z - x||$ and $||\alpha z|| = ||\alpha z - x||$, for some $\alpha \in (0, 1)$. Then equality holds throughout in (2.1). Hence $||(1-\alpha)z/||z|| + \alpha(\alpha z - x)/(\alpha||z||)|| = 1$ implying that the entire line segment

$$[z/||z||, z/||z|| - x/(\alpha||z||)] \subset \partial\mathscr{L}.$$

This line segment is clearly parallel to $[0, x]$, a contradiction. □

LEMMA 2.2. *Assume that* $F = \{x, y\}$, $x \neq y$, *and* $[x, y]$ *is not parallel to any line segment in* $\partial\mathscr{L}$. *Assume that* $x_n \to x$. *Set* $F_n = \{x_n, y\}$. *Then* $\mathscr{V}(F_n) \to \mathscr{V}(F)$ (*in the metric* $d$).

PROOF. Assume again that $y = 0$, and fix any $\varepsilon > 0$ and $r > 0$.

First we show that $\mathscr{V}(F_n) \cap B(r) \subset \mathscr{V}(F)^\varepsilon$ for large $n$. Otherwise, there are $z_n \in \mathscr{V}(F_n) \cap B(r)$ such that $z_n \notin \mathscr{V}(F)^\varepsilon$ and $z_n \to z \in B(r)$. But $||z_n - x_n|| = ||z_n||$ implies $||z-x|| = ||z||$, so $z \in \mathscr{V}(F)$. On the other hand $\mathrm{dist}(z, \mathscr{V}(F)) \ge \varepsilon$, a contradiction.

To prove that $\mathscr{V}(F) \cap B(r) \subset \mathscr{V}(F_n)^\varepsilon$ eventually, assume that $z_n \in (\mathscr{V}(F) \cap B(r)) \setminus \mathscr{V}(F_n)^\varepsilon$ converge to $z \in \mathscr{V}(F) \cap B(r)$ and that $\mathrm{dist}(z, \mathscr{V}(F_n)) \ge \varepsilon/2$ for every $n$. Assume, in addition, that $||z|| < ||z - x_n||$ for infinitely many, and then by discarding, for all $n$. The case where the opposite inequality holds infinitely often can be handled similarly. Choose $\alpha \in (0, 1)$ such that $(1-\alpha)||z-x|| < \varepsilon/2$.

By Lemma 2.1, $||\alpha(z-x)|| < ||x+\alpha(z-x)||$, and therefore $||\alpha(z-x_n)|| < ||x_n+\alpha(z-x_n)||$ for sufficiently large $n$. Since the opposite inequality holds if $\alpha$ is replaced by 1, continuity implies that there exist $\alpha_n \in (\alpha, 1)$ such that $||\alpha_n(z-x_n)|| = ||x+\alpha_n(z-x_n)||$. Therefore $x_n+\alpha_n(z-x_n) \in \mathscr{V}(F_n)$. But

$$||z-(x_n+\alpha_n(z-x_n))|| = (1-\alpha_n)||z-x_n|| \leq (1-\alpha)||z-x|| + ||x-x_n|| < \varepsilon/2,$$

if $n$ is large enough, implying that $\operatorname{dist}(z, \mathscr{V}(F_n)) < \varepsilon/2$, a contradiction. □

LEMMA 2.3. *Assume that no line segment between any pair of the three distinct points $x_1, x_2, x_3$ is parallel to any line segment in $\partial\mathscr{L}$. Then there is at most one $z \in \mathbf{R}^2$ such that $\{x_1, x_2, x_3\} \subset z+\partial\mathscr{L}$.*

The proof is left as a geometric exercise for the reader, who can also consult [3]. The authors wish to thank Tom Sallee for the reference and a very slick proof of this fact.

We are now ready to prove a continuity theorem which is the main objective of this section.

THEOREM 2.4. *Let $F = \{x^1, \dots, x^m\}$ be a set of distinct points such that none of the line segments $[x^i, x^j]$, $i \neq j$, is parallel to a line segment in $\partial\mathscr{L}$. Let $x_n^i$ converge to $x^i$ as $n \to \infty$ for every $i = 1, \dots, m$, and denote $F_n = \{x_n^1, \dots, x_n^m\}$. Then $\mathscr{V}(F_n) \to \mathscr{V}(F)$ as $n \to \infty$ (in the metric $d$).*

PROOF. We begin by proving the result for $m = 2$. In this case, by Lemma 2.2,

$$\mathscr{V}\{x_n^1, x_n^2\} = \mathscr{V}\{0, x_n^2 - x_n^1\} + x_n^1 \to \mathscr{V}\{0, x^2 - x^1\} + x^1 = \mathscr{V}\{x^1, x^2\}.$$

In the general case, assume first that $z \in \mathscr{V}(F)$. Lemma 2.3 implies that there are only finitely many points which are equidistant to three or more points in $F$. Therefore we can assume, by replacing $z$ with a nearby point if necessary, that $||z-x^1|| = ||z-x^2|| < ||z-x^k||$ for $k > 2$. Lemma 2.2 implies that there are $z_n \in \mathscr{V}\{x_n^1, x_n^2\}$ such that $z_n \to z$. For a large enough $n$, then, $||z_n - x_n^1|| = ||z_n - x_n^2|| < ||z_n - x_n^k||$, and hence $z_n \in \mathscr{V}(F_n)$.

Conversely, assume that $z_n \in \mathscr{V}(F_n)$ converge to $z$. Then by considering a subsequence, relabeling the centers $x^k$ if necessary, and passing to the limit, we conclude that there exists $m' \in \{2, \dots, m\}$ such that $||z-x^1|| = ||z-x^k|| < ||z-x^l||$ for $k \leq m' < l$. Therefore, $z \in \mathscr{V}(F)$. □

COROLLARY 2.5. *Let $F$ and $F_n$ be as above, and suppose $\zeta$ is any coloring of $F$. Define the coloring $\zeta_n$ of $F_n$ by $\zeta_n(x_n^k) = \zeta(x^k)$. Then $\mathscr{V}(F_n; \zeta_n) \to \mathscr{V}(F; \zeta)$ as $n \to \infty$.*

The proof is a straightforward modification of the argument in the preceding theorem.

2.2. *Convergence of centers.* Let $\mathscr{C} = \{$countable subsets of $\mathbf{R}^2$ without limit points$\}$. What does it mean for $F_n \in \mathscr{C}$ to converge to $F \in \mathscr{C}$? We explained in Example 0 that the Attouch–Wets sense is not the most appropriate for our purposes.

Start with two finite sets $F_1, F_2 \subset B(r)$. To define the distance between them, assume that $F_1 = \{x_1, \dots, x_m\}$, $F_2 = \{y_1, \dots, y_m\}$, where the points listed are distinct, and without loss of generality $m > n$. Set

$$\bar{d}_r(F_1, F_2) = \min_{\pi \in \Pi_m} \max\left\{ \max_{k=1,\dots,n} \|x_{\pi(k)} - y_k\|, \max_{k=n+1,\dots,m} \operatorname{dist}(x_{\pi(k)}, B(r)^c) \right\},$$

where $\Pi_m$ is the set of permutations of $(1, \dots, m)$ and $\operatorname{dist}(x, B) = \inf\{\|x - y\|_2, y \in B\}$. [If $F_1 = F_2 = \varnothing$, then $\bar{d}_r(F_1, F_2) = 0$.]

For arbitrary $F_1, F_2 \in \mathscr{C}$, define

$$\bar{d}(F_1, F_2) = \sum_{r=1}^{\infty} \bar{d}_r(F_1 \cap B(r), F_2 \cap B(r)) 2^{-r}.$$

It is easy to check that this defines a metric in $\mathscr{C}$, under which $\mathscr{C}$ is separable (finite subsets of $\mathbf{Q}^2$ are dense), but *not* compact (a pair of points each converging to 0 has no convergent subsequence). We note that this metric is equivalent to the one given on page 627 in [7] if one views the set of centers as a counting measure.

In this metric, $F_n \to F$ iff for every $B(r)$ such that $F \cap \partial B(r) = \varnothing$, $F_n \cap B(r)$ and $F \cap B(r)$ eventually have the same cardinality and it is possible to pair points between the two sets so that points in $F_n \cap B(r)$ converge to those in $F \cap B(r)$.

For every $r > 0$ and $C \in \mathscr{C}$, define $\delta_r(C) = \inf\{\|x - y\|_2 \colon x, y \in C,\ \|x\|_2 < r, \|y\|_2 < r\}$ and

$$\Delta(C) = \sum_{r=1}^{\infty} \left( \frac{1}{\delta_r(C)} \wedge r \right) 2^{-r}.$$

It is not hard to see that $\Delta\colon (\mathscr{C}, \bar{d}) \to \mathbf{R}$ is continuous and therefore $\{C \colon \Delta(C) \le x\}$ is a closed subset of $\mathscr{C}$ for every $x \ge 0$.

LEMMA 2.6. *The following are equivalent*:

(i) $\bar{d}(F_n, F) \to 0$;

(ii) $d(F_n, F) \to 0$ *and for every* $r$ *there is a* $\eta_r > 0$ *so that* $\delta_r(F_n) \ge \eta_r$ *for every* $n$;

(iii) $d(F_n, F) \to 0$ *and there is an* $M > 0$ *so that* $\Delta(F_n) \le M$ *for every* $n$;

(iv) *For every bounded open set* $G \subset \mathbf{R}^2$ *such that* $F \cap \partial G = \varnothing$, $|F_n \cap G| \to |F \cap G|$.

PROOF.
(i) $\Rightarrow$ (ii). Observe that

$$\varepsilon_{r-\bar{d}_r(F, F')}(F, F') \leq \bar{d}_r(F, F').$$

Assume that $\delta = \bar{d}(F, F')$ is small and let $k = \max\{r\colon 2^{-r} \geq \delta\}$. Then $r - 2^r\delta \geq r - 1$, $r = 1, \dots, k$, and therefore [since $\bar{d}_r(F, F') \leq 2^r\delta$],

$$\varepsilon_{r-1}(F, F') \leq \bar{d}_r(F, F'), \; r = 2, \dots, k.$$

Hence

$$d(F, F') \leq \sum_{r=1}^{k-1} 2^{-r}\bar{d}_{r+1}(F, F') + \sum_{r=k}^{\infty} r2^{-r} \leq 2\delta + 6k2^{-k} \leq 8\sqrt{\delta}.$$

Also, if $\bar{d}_{r+1}(F, F') < 1$, then

$$||\Delta(F \cap B(r)) - \Delta(F' \cap B(r))|| \leq 2\bar{d}_{r+1}(F, F').$$

(ii) $\Rightarrow$ (i). Fix $r$, and let $\varepsilon < 1$, $\varepsilon < \delta_r/2$. Assume that $\varepsilon_r(F, F_n) < \varepsilon$. If $x \in F \cap B(r - \varepsilon)$, then there is a unique $x_n \in F_n \cap B(x, \varepsilon)$ and these exhaust $F_n \cap B(r - \varepsilon)$. Hence $\bar{d}_r(F, F_n) \leq \varepsilon$.

(ii) $\Leftrightarrow$ (iii). This is straightforward.

(i) $\Rightarrow$ (iv). Obvious, since $|\cdot \cap G|$ is continuous at $F$.

(iv) $\Rightarrow$ (ii). By [23], $d(F_n, F) \to 0$. To prove (2.2), take an $r$ so that $F \cap \partial B(r) = \varnothing$ and choose $\varepsilon$ small enough that $\{B(x, \varepsilon)\colon x \in F \cap B(r)\}$ is a disjoint collection of balls included in $B(r)$. Then, for large enough $n$, $F_n \cap B(r)$ must consist of exactly one representative from every ball $B(x, \varepsilon)$. $\square$

COROLLARY 2.7. *For every $M > 0$, $\mathscr{C}_M = \{C \in \mathscr{C}\colon \Delta(C) \leq M\}$, with metric $\bar{d}$, is a compact* (*and hence complete*) *separable metric space*.

PROOF. Assume $F_n$ is a sequence in $\mathscr{C}_M$. Then $F_n$ has a convergent subsequence in the Attouch–Wets metric. Therefore, (3) from Lemma 2.6 holds for that subsequence. $\square$

Thus, $\mathscr{C}$ is a direct limit of compact metric spaces.

Now let $F$ be a *random* collection of centers, that is, a measurable mapping from a probability space $(\Omega, \mathscr{F}, P)$ to $(\mathscr{C}, \bar{d})$. In fact, this is equivalent to requiring that for every open set $G \subset \mathbf{R}^2$, $\{|F \cap G| = 1\} \in \mathscr{F}$. This rather technical point is not really needed, so the proof is left to the reader. It is, however, important to note that any random collection of centers is a random closed set.

THEOREM 2.8. *A sequence $F_n$ of random elements in $\mathscr{C}$ converges weakly to a random element $F$ in $\mathscr{C}$ if any of the following equivalent conditions holds*:

(i) $E(\Phi(F_n)) \to E(\Phi(F))$ *for every bounded continuous* $\Phi\colon \mathscr{C} \to \mathbf{R}$;

(ii) $F_n \to_d F$ (*as random closed sets*) *and for every* $r > 0$,

$$\lim_{\delta \to 0} \sup_n P(\delta_r(A_n) < \delta) = 0;$$

(iii) *for every bounded open set* $G \subset \mathbf{R}^2$ *with* $P(\partial G \cap F = \varnothing) = 1$, $|F_n \cap G| \to |F \cap G|$ *in distribution* (*as random variables*).

PROOF. Notice first that (i)⇒(iii) is clear.

(ii) ⇒ (i). We can assume that $F_n$ take values in some $\mathscr{C}_M$ and contains a weakly convergent subsequence. But then Attouch–Wets theory (which guarantees uniqueness) finishes off the argument.

(iii) ⇒ (ii). The first part of (ii) is straightforward, while convergence in the Attouch–Wets sense follows by [24]. □

In fact, the $G$ in (iii) can be assumed to be "nice" sets such as finite collections of open balls.

The most important point here is that $\wp$ is a random element in $\mathscr{C}$ and standard convergence techniques such as the Chen-Stein method (see, e.g., [2]) imply weak convergence in $\mathscr{C}$. To be more precise, assume first that $p > 0$ and that $F_p$ is a random collection of points in $\mathbf{Z}^2$ obtained by independently choosing each site with probability $p$. Then $\sqrt{p} \cdot F_p$ is an element from $\mathscr{C}$ and it is easy to see that (iii) from Theorem 2.8 is satisfied, so $\sqrt{p} \cdot F_p \to \wp$ as $p \to 0$. The Chen–Stein method allows us to replace $F_p$ by another set $F'_p \subset \mathbf{Z}^2$ of mildly dependent occupied sites with density $p$. Details will be given in Section 3.

Let us make another remark, not used in the sequel, but interesting since it gives a natural definition of weak convergence to $\mathscr{V}(\wp)$. By using Lemma 2.1 one can show that $\mathscr{V}(\wp)$ has probability 0 of being tangent to any given ball $B(x, r)$ (i.e., intersecting its boundary without intersecting its interior). Hence $P(\mathscr{V}(\wp) \cap B(x, r) \neq \varnothing)$ is an a.s. continuous function and the same is true if we replace $B(x, r)$ with a finite collection of balls. Therefore, random closed sets $A_n$ converge weakly to $\mathscr{V}(\wp)$ if and only if for every finite collection $G$ of open balls,

$$P(A_n \cap G \neq \varnothing) \to P(\mathscr{V}(\wp) \cap G \neq \varnothing).$$

2.3. *Sufficient conditions for convergence to a Poisson–Voronoi tessellation.*

PROPOSITION 2.9. *Let $Q_n$ be random elements in $\mathscr{C}$, and let random closed sets $C_n$ be defined on the same probability space. Assume that the following hold*:

(a) $Q_n \to \wp$ *weakly in* $\mathscr{C}$;

(b) $\varepsilon_r(C_n, \mathscr{V}(Q_n)) \to 0$ *in probability for each* $r > 0$.

*Then* $C_n \to_d \mathscr{V}(\wp)$ *in the Attouch–Wets sense as* $n \to \infty$.

PROOF. As a first step, we claim that the mapping $\mathscr{V}$ from $(\mathscr{C}, \bar{d})$ into closed subsets of $\mathbf{R}^2$ equipped with the metric $d$, given by $F \mapsto \mathscr{V}(F)$, is an a.s. continuous function on $\wp$. To see this, note that there are only countably many line segments in $\partial\mathscr{L} = \{\|\cdot\| = 1\}$, so a.s. there are no two centers $x, y \in \wp$ such that $[x, y]$ is parallel to a line segment in $\partial\mathscr{L}$. Moreover, there exists a constant $c > 0$ (depending on the norm $\|\cdot\|$) such that $\mathscr{V}(\wp) \cap B(r)$ depends only on centers in $\wp \cap B(cr)$, given that $\wp \cap B(r) \neq \varnothing$. If $F$ is such a configuration of centers, then it follows from Theorem 2.4 that $F_n \to F$ implies $\mathscr{V}(F_n) \to \mathscr{V}(F)$.

The second step is to choose a continuous function $\phi$ on closed sets. Such a function is uniformly continuous and bounded (since the Attouch–Wets metric is compact). Let $M = \sup_A |\phi(A)|$. Fix an $\varepsilon > 0$. Choose $\delta > 0$ small and $r$ large so that $|\phi(A) - \phi(B)| < \varepsilon$ as soon as $\varepsilon_r(A, B) < \delta$. Then

$$\begin{aligned}
&|E(\phi(C_n)) - E(\phi(\mathscr{V}(\wp)))| \\
&\quad \leq E(|\phi(C_n)) - \phi(\mathscr{V}(Q_n))|) + |E(\phi(\mathscr{V}(Q_n))) - E(\phi(\mathscr{V}(\wp)))| \\
&\quad \leq \varepsilon + 2M \cdot P(\varepsilon_r(C_n, \mathscr{V}(Q_n)) > \delta) + |E(\phi(\mathscr{V}(Q_n))) - E(\phi(\mathscr{V}(\wp)))|.
\end{aligned}$$

The second and third terms above go to 0 by the hypotheses and continuity of $\mathscr{V}(\cdot)$. □

PROPOSITION 2.10. *Let $Q_n$ be random elements in $\mathscr{C}$ and let random closed sets $C_n$ be defined on the same probability space. Fix an integer $\kappa \geq 2$, and color the points in $Q_n$ independently and uniformly with $\kappa$ colors. Assume the following*:

(a) *$Q_n$ converge weakly in $\mathscr{C}$ to $\wp$;*
(b) *$\varepsilon_r(C_n, \mathscr{V}_\kappa(Q_n)) \to 0$ in probability for each $r > 0$.*

*Then $C_n \to_d \mathscr{V}_\kappa(\wp)$, in the Attouch–Wets sense, as $n \to \infty$.*

The proof is very similar to that for Proposition 2.9, so we omit it.

For the sake of brevity, we will present a complete proof of our convergence result in the $\kappa = \infty$ case (Theorem 1.4), but give only a sketch in the case of finite $\kappa$ (Theorem 1.2). The arguments for both results are long and involved, so let us take a brief intermission here, and illustrate some key ideas by proving Theorem 1.1.

PROOF OF THEOREM 1.1. As already mentioned, condition (a) of Proposition 2.9 is easy. In fact, one can couple $\wp$ and $\{\zeta_0 \neq 0\}$ by declaring that $\zeta_0(x) \neq 0$ iff $B_\infty(\sqrt{p}x, \sqrt{p}/2) \cap \wp \neq 0$; this coupling has the property that $\bar{d}(\{\zeta_0 \neq 0\}, \wp) \to 0$ a.s. as $p \to 0$. Moreover, coupling can also be used to check condition (b). If $z \in \mathscr{V}(\wp) \cap B(r)$ and $\varepsilon > 0$ is small, then by Lemmas 2.1 and 2.3, there are integer sites $z_1, z_2$ such that $\sqrt{p}z_1, \sqrt{p}z_2 \in B(z, \varepsilon)$, and points $x_1, x_2 \in \wp$ so that $\|\sqrt{p}z_1 - x_1\|_1 < \|\sqrt{p}z_1 - x\|_1 - \varepsilon/2$ for any $x \in \wp \setminus \{x_1\}$, and similarly for $z_2$ and $x_2$. Hence $\zeta_\infty(z_1)$ has the color of $x_1$ and $\zeta_\infty(z_2)$ the color of $x_2$, so $\zeta_\infty(z_1) \neq \zeta_\infty(z_2)$. Connect $z_1$ and $z_2$ with some nearest-neighbor path which stays in $B(z, \varepsilon)$. Two nearest neighbors on this path must have different colors, implying that they are in $C_p$. Therefore,

$\mathscr{V}(\wp)\cap B(r) \subset \sqrt{p}C_p + B(\varepsilon)$. For the other direction, choose $z_1 \in C_p \cap B(p^{-1/2}r)$ and $\varepsilon > 0$. If $\mathrm{dist}_2(\sqrt{p}z_1, \mathscr{V}(\wp)) > \varepsilon$, then it is easy to see that $z_1$ and all its neighbors must have the same color as the closest point in $\wp$. This contradiction concludes the proof. $\square$

**3. Convergence of nuclei to the Poisson point location.** This section takes care of condition (a) in Propositions 2.9 and 2.10. Our argument is very similar to the corresponding one in [15]. In the context of Theorem 1.4, call a site $x \in \mathbf{Z}^2$ a *nucleus* if there exists a set $A \subset \mathbf{Z}^2$ with the following properties:

1. the lowest leftmost point of $A$ is $x$;
2. $|A| = \gamma$;
3. $A$ generates persistent growth;
4. all sites of $A$ have the same color in $\zeta_0$.

Let $Q_\kappa$ be the random set of all nuclei, and let $\Pi(p)$ be the random subset of $\mathbf{Z}^2$ in which every site is included independently with probability $p$ .

LEMMA 3.1. *Fix an $R > 0$ and let $B = B_\infty(0, R)$. Then there exists a constant $C > 0$, depending only on $R$, such that the total variation distance between $Q_\kappa \cap (\kappa^{\gamma-1} \cdot B)$ and $\Pi(\nu/\kappa^{\gamma-1}) \cap (\kappa^{\gamma-1} \cdot B)$ is bounded above by $C/\kappa$.*

PROOF. Let $G_x$ be the event that there is a center at $x$, $p_x = P(G_x)$, and $p_{xy} = P(G_x \cap G_y)$. Choose $M$ large enough that $\mathscr{N} \subset B_\infty(0, M)$. Increase $M$ if necessary so that the growth model started from any set $A_0$ with $|A_0| < \gamma$ never leaves $A_0 + B_\infty(0, M)$. Then any nucleus $A$ satisfying the four properties above must be included in $B_\infty(0, 2\gamma M)$. Set $B_x = B_\infty(x, (4\gamma+1)M)$. Then $y \notin B_x$ implies that $G_x$ and $G_y$ are independent. Also, put $I = \sqrt{\kappa^{\gamma-1}} \cdot B$. It follows that

$$p_x = \nu\kappa^{-\gamma+1} + \mathscr{O}(\kappa^\gamma),$$

$$b_1 = \sum_{x\in I}\sum_{y\in B_x} p_x p_y = (8\gamma M+1)^2(2R\sqrt{\kappa^{\gamma-1}}+1)^2 p_x^2 = \mathscr{O}(\kappa^{-\gamma+1}),$$

$$b_2 = \sum_{x\in I}\sum_{y\in B_x\setminus\{x\}} p_{xy} \le |I||B_x|P\big(\text{one color has at least } \gamma+1 \text{ representatives in } B_x\big) = \mathscr{O}(\kappa^{-1}).$$

Now Theorem 2 from [2] implies that the total variation distance between $Q_\kappa \cap I$ and $\Pi(\nu/\kappa^{\gamma-1}) \cap I$ is bounded above by $4(b_1 + b_2)$. $\square$

LEMMA 3.2. *As $\kappa \to \infty$, $\sqrt{\nu/\kappa^{\gamma-1}}Q_\kappa$ converges weakly (in $\mathscr{C}$) to $\wp$.*

The proof follows immediately from Theorem 2.8 and Lemma 3.1.

In the context of Theorem 1.2, fix a constant $c$ and call site $x$ a *nucleus* if:

1. the lowest leftmost point of $A$ is $x$;
2. $|A| = \gamma$;
3. $A$ generates persistent growth;

4. all sites in $A$ have the same foreground color in $\zeta_0$;
5. the number of sites in $B_\infty(x, c)$ with any foreground color is at most $\gamma$.

Now let $Q_p$ be the set of all nuclei. Then the following result can be proved in essentially the same way as Lemma 3.2.

LEMMA 3.3. *There exists a $c$ large enough that the colors of all nuclei are independent and uniform (from among the $\kappa$ available foreground colors), and $\sqrt{\nu p^\gamma}\, Q_p$ converges weakly (in $\mathscr{C}$) to $\wp$ as $p \to 0$.*

**4. Growth in a polluted environment.** The results of Section 3 deal with the nucleation phase of threshold vote dynamics: they explain the positions of droplets that do grow. However, those droplets now face an environment that contains "crud" left over from unsuccessful centers. Fortunately, the density of such troublesome sites is very low, so to complete the proof of Theorem 1.4, we need only show that they do not affect the growth on the relevant scale. This argument is naturally divided into two parts. The first shows that the polluted environment does not impede the growth. The second shows that crud does not boost growth. We needed to deal with this latter problem in [14], so the necessary lemmas from that paper are applied. We begin, however, with a lemma indicating that we can restrict consideration to independent crud with finite range effects.

LEMMA 4.1. *Fix an integer $r > 0$. Let $\zeta_0$ be the independent and uniform coloring of sites in $\mathbf{Z}^2$ with $\kappa$ colors (as in Case 2 of Example 1). Then $\zeta_0$ and $\Pi(2(2r+1)^2/\kappa)$ can be coupled so that*

$$\begin{aligned} &\{x\colon \zeta_0(x) = \zeta_0(y) \text{ for some } y \in B_\infty(x, r) \setminus \{x\}\} \\ &\qquad \subset \Pi(2(2r+1)^2/\kappa) + B_\infty(0, 2r), \end{aligned} \tag{4.1}$$

*as soon as $\kappa \ge 2(2r+1)^2$.*

PROOF. Let $\Xi$ be the set on the left-hand side of (4.1). Set $p = 2(2r+1)^2/\kappa$ and note that for every $x \in \mathbf{Z}^2$, $P(x \in \Xi) = 1 - (1 - 1/\kappa)^{(2r+1)^2} < p$. Also note that any two sites $x$ and $y$ with $\|x - y\|_\infty > 2r$ are in $\Xi$ independently.

Order all sites in $\mathbf{Z}^2$: $x_1, x_2, x_3, \ldots$; also let $\mathscr{D} = \mathbf{Z}^2$. Choose the color of $x_1$, and then put it in $\Xi$ with probability $1 - (1 - 1/\kappa)^{(2r+1)^2} < p$. If it turns out to be in $\Xi$, remove from $\mathscr{D}$ all sites in $B_\infty(x_1, 2r)$. If it is not in $\Xi$ [this event changes the measure on colors in $B_\infty(x_1, r)$ in the obvious way], choose the first site $x_{k_1}$ still in $\mathscr{D}$. By summing over all possible color configurations in $B_\infty(x_{k_1}, r) \setminus \{x_{k_1}\}$, one sees that this site will be in $\Xi$ with probability $p' \le (2r+1)^2/(\kappa - (2r+1)^2) \le p$. Choose its color by the appropriate conditional distribution, and make it in $\Xi$ with probability $p'$.

At every step, consider the next $x_k$ which is still in $\mathscr{D}$. The probability of it being occupied is less than $p$, as the color configuration in $B_\infty(x_k, r)$ is only influenced by points that are *not* in $\Xi$. Choose its color, and assign the point to

$\Xi$ by the appropriate conditional probability. If it turns out to be in $\Xi$, remove from $\mathscr{D}$ all points in $B_\infty(x_k, 2r)$.

It is clear that there exists a coupling such that all points in $\Xi$ which are never excluded from $\mathscr{D}$ are included in a set of sites on $\mathbf{Z}^2$ independently open with probability $p$. After this is done, determine colors for points that were removed from $\mathscr{D}$, by appropriate conditional probabilities. Note that any such point $y$ is within $\|\cdot\|_\infty$-distance $2r$ of a site in $\Xi$ which was never excluded from $\mathscr{D}$. Therefore $y$ is at distance $\le 2r$ from a site in $\Pi(p)$. □

Our next lemma is the first step in controlling growth in a restricted environment. For a subset $S \subset \mathbf{Z}^2$, we define restricted threshold growth $\mathscr{T}|S$ by

$$(\mathscr{T}|S)(A) = A \cup \{x \in S\colon |A \cap (x + \mathscr{N})| \ge \theta\}.$$

LEMMA 4.2. *Assume that threshold growth given by $\mathscr{N}$ and $\theta$ is symmetric, voracious and supercritical and fix an initial $A_0$ which generates persistent growth. Fix also a unit vector $u$ and let $u^\perp$ be a unit vector orthogonal to $u$. Also fix a large $M > 0$ and set $S_M = \{x \in \mathbf{Z}^2\colon |\langle x, u^\perp\rangle| \le M, \langle x, u\rangle \ge -M\}$. Then $(\mathscr{T}|S_M)^n(A_0)/n \to [0, w_M(u)]u$ as $n \to \infty$ and*

$$\lim_{M\to\infty} w_M(u) = w(u), \tag{4.2}$$

*uniformly in $u$.*

PROOF. Note that convergence of $(\mathscr{T}|S_M)^n(A_0)/n$ as $n \to \infty$ follows easily from subadditivity. We now recall a fact about unrestricted growth from [14]. Given $\varepsilon > 0$, one can approximate $\mathscr{L}$ by a convex smooth set $\mathscr{L}_\varepsilon$ which is within Hausdorff distance $\varepsilon$ of $\mathscr{L}$ and is such that for large enough $n$, $(n+1-\varepsilon)\mathscr{L}_\varepsilon \subset \mathscr{T}(n\mathscr{L}_\varepsilon)$. Form a new set $\mathscr{L}'_m$ which is obtained from $n\mathscr{L}_\varepsilon$ in the following way: $\mathscr{L}'_m = \bigcup_{\lambda\in[0,m]}(n\mathscr{L}_\varepsilon + \lambda u)$. It is clear that $\mathscr{L}'_{m+1-\varepsilon} \subset \mathscr{T}(\mathscr{L}'_m)$, which implies (4.2). □

We now describe the setting for the next two percolation lemmas. Call sites in $\Pi(p)$ *open* (think of $p$ as very small). Let $r > 0$ be fixed. A cluster is any $l^\infty$–connected component of $\Pi(p) + B_\infty(0, r)$. Also, for every $x$, $\mathrm{Clus}(x)$ is the cluster containing $x$ (possibly $\varnothing$).

LEMMA 4.3. *Fix an $\varepsilon > 0$. Then for every $p > 0$ there exists a constant $\Gamma = \Gamma_\varepsilon(p) \ge 0$ such that $\Gamma_\varepsilon(p) \to \infty$ as $p \to 0$, and for every* (*deterministic*) *set $A$ of $n$ points*

$$P\left(\left|\bigcup_{x\in A}\mathrm{Clus}(x)\right| \ge \varepsilon n\right) \le e^{-\Gamma n}. \tag{4.3}$$

PROOF. Call $x \in \mathbf{Z}^2$ *$r$-open* if $B_\infty((2r+1)x, 2r)$ includes at least one open site. If $x$ and $y$ are not range 1 box neighbors, then the events $\{x\ r\text{-open}\}$ and $\{y\ r\text{-open}\}$ are independent; if they are range 1 box neighbors, then the two events are positively correlated. It is also clear that

$$\Pi(p) + B_\infty(0, r) \subset \bigcup_{x\ r\text{-open}} B_\infty((2r+1)x, r),$$

so it is enough to show (4.3) with $\mathrm{Clus}(x)$ replaced by $\mathrm{Clus}'(x)$, the $l^\infty$-connected component of $r$-open sites which includes $x$.

To do this, notice first that $p_r = P(0\ r\text{-open}) = 1 - (1-p)^{(4r+1)^2+1}$. Since the number of $l^\infty$-connected sets with $n$ sites, one of which is 0, is bounded above by $3(13)^n$ (cf. [19], page 75),

$$P(|\mathrm{Clus}'(0)| \geq n) \leq 3(13)^n p_r^{n/9}. \tag{4.4}$$

Moreover, we claim that if $Z_1, \dots, Z_n$ are i.i.d. distributed as $|\mathrm{Clus}'(0)|$, then for every finite $m$,

$$P\left(\left|\bigcup_{x\in A} \mathrm{Clus}'(x)\right| \geq m\right) \leq P\left(\sum_{i=1}^n Z_i \geq m\right). \tag{4.5}$$

This will be proved by induction on $n$, the cardinality of $A$. Let $A = \{a_1, \dots, a_n\}$ and fix a deterministic $l^\infty$ connected set $B$, such that $a_1 \in B$. Assume that $a_1, \dots, a_k \in B$, but $a_{k+1}, \dots, a_n \notin B$, for some $k \geq 1$. Denote $B' = B + B_\infty(0,1)$ (obtained by adding to $B$ all sites outside $B$ that have an $l^\infty$ neighbor inside $B$). Then

$$\begin{aligned}
&P\left(\left|\bigcup_{i=1}^n \mathrm{Clus}'(a_i)\right| \geq m \mid \mathrm{Clus}'(a_1) = B\right)\\
&\qquad = P\left(\left|\bigcup_{i=k+1}^n \mathrm{Clus}'(a_i)\right| \geq m - |B| \mid \text{no site in } B' \text{ open}\right)\\
&\qquad \leq P\left(\left|\bigcup_{i=k+1}^n \mathrm{Clus}'(a_i)\right| \geq m - |B|\right)\\
&\qquad \leq P(Z_{2+k} + \dots + Z_n \geq m - |B|)\\
&\qquad \leq P(Z_2 + \dots + Z_n \geq m - |B|).
\end{aligned} \tag{4.6}$$

The first inequality in (4.6) follows from the FKG inequality (cf. [19]), the second by the induction hypothesis. Therefore

$$\begin{aligned}
P\left(\left|\bigcup_{i=1}^n \mathrm{Clus}'(a_i)\right| \geq m\right) &\leq \sum_{i=0}^\infty P(Z_2 + \dots + Z_n \geq m - i) P(Z_1 = i)\\
&= P(Z_1 + \dots + Z_n \geq m),
\end{aligned}$$

proving (4.5). This, together with (4.4) and Lemma 6.1 in [15], completes the proof. □

LEMMA 4.4. *For any $n$ there exists a sufficiently large constant $K$ so that*

$$P(\text{a cluster of size } \geq K \text{ intersects } B_\infty(0, p^{-n})) \to 0, \tag{4.7}$$

*as $p \to 0$.*

PROOF. Inequality (4.4) and the bound $p_r \le 30r^2 p$ imply that the probability in (4.7) is at most

$$\sum_{k\ge K} 3(13\times 30r^2)^k p^{(k/9)-n},$$

which goes to 0 as $p\to 0$ as soon as $K > 10n$. □

At this point, the next lemma about growth on the complement of a small density of excluded points is harder to state than to prove.

LEMMA 4.5. *Fix an* $r>0$ *and assume that* $p>0$ *is small. Let* $S=S_{p,r}$ *be the infinite* $l^1$*-connected component of*

$$\mathbf{Z}^2 \setminus (\Pi(p)+B_\infty(0,2r)).$$

*Fix an* $\varepsilon>0$. *Then for a small enough* $p$ *there exist a* $\Gamma>0$ *and an* $R>0$ *so that for every* $A_0\subset S$ *which generates persistent growth and satisfies* $\mathit{dist}_\infty(A_0,S^c)\ge R$,

$$P(\textit{there is a site } x\in n(1-\varepsilon)\mathscr{L} \textit{ with } \mathit{dist}_\infty(x,(\mathscr{T}|S)^n(A_0)\cup S^c)\ge R)\le e^{-\Gamma n},$$

*for large* $n$.

PROOF. Start by choosing $M$ from Lemma 4.2 so that $w_M(u)>w(u)-\varepsilon$ for all $u$. Also make $M$ large enough that the dynamics can "turn corners" on $M\times M$ squares. This means that if three squares are arranged in an L, then an occupied $M/2\times M/2$ square in the middle of one of the two extremal squares will create a copy of this square in the middle of the other extremal square. The fact that this is possible also follows from Lemma 4.2. Finally, make $M$ large enough so the following is true. Take an $M\times M$ square and restrict the dynamics to it. Then every $A_0$ which generates persistent growth and is included in an $M/2\times M/2$ square in the middle of the $M\times M$ square fills the $M/2\times M/2$ square completely. This property is also ensured, for large $M$, by the rather useful Lemma 4.2.

Now fix $c_1$ and $M$. For every unit vector $u$, there are only $\mathscr{O}(n)$ sites in $S'=S'(u;M,c_1)=\{x\colon \|x\|_\infty\le c_1 n, |\langle x,u\rangle|\le M\}$, and therefore at most $e^{c_2 n}$ sets of the form $S'=S'(u;M,c_1)$. Hence, by Lemma 4.3, with $r+2M$ instead of $r$, the probability that one of the sets $S'$ is intersected by clusters of total size $\varepsilon n$ is smaller than $e^{-\Gamma n}$, for some $\Gamma>0$.

Thus, the statement in the lemma holds with $R=2M$. □

Let us now turn to growth in a slightly helpful environment. Again, fix $r>0$, and a small $p>0$. Define

$$\begin{aligned}\bar{\mathscr{T}}_p(A) = A &\cup \{x\in\mathbf{Z}^2\colon |(x+\mathscr{N})\cap A|\ge\theta\}\\ &\cup\{x\in\mathbf{R}^2\colon \mathrm{dist}_\infty(x,A)\le r \text{ and } \mathrm{dist}_\infty(x,\Pi(p))\le r\}.\end{aligned}$$

In these dynamics the sites of $\Pi(p)$ lie dormant until the growth reaches them. The idea is that if we pretend that no sets in $\Pi(p)$ grow, then $\bar{\mathcal{T}}_p$ is an upper bound for the growth dynamics in the polluted environment. The result below is a restatement of Lemma 6.4 from [15].

LEMMA 4.6. *Fix an $\varepsilon > 0$, and $A_0 \subset \mathbf{Z}^2$. Then for sufficiently small $p$ there exists a $\Gamma > 0$, so that*

$$P(\bar{\mathcal{T}}_p^n(A_0) \not\subset (1+\varepsilon)n\mathcal{L})) \le e^{-\Gamma n},$$

*for $n$ large.*

The last lemma deals with the threshold growth model started from a complement of a convex wedge, defined for two unit vectors $u_1$ and $u_2$ by $Q = Q_{u_1,u_2} = H^-_{u_1} \cup H^-_{u_2}$. Note that the additive dynamics given by $\mathcal{T}_a(A) = A + \mathcal{L}$ merely translate the cone, that is, $\mathcal{T}_a(Q) = Q + v$, where $v$ is a vector determined by the speed of the two half–spaces (see [15]).

LEMMA 4.7. *Assume that $K_{1/w}$ is convex. Then for every $\varepsilon > 0$, there exists a small enough $p$ and a $\Gamma > 0$ such that*

$$P(\bar{\mathcal{T}}_p^n(Q) \not\subset Q + n(1+\varepsilon)v) \le e^{-\Gamma n},$$

*for large $n$.*

For the proof, see [15].

**5. Convergence to Poisson–Voronoi tessellations.** In this section we prove Theorems 1.2 and 1.4. Let us begin by slightly reformulating Theorem 1.4. Recall that the state space is $\{1, \dots, \kappa\}^{\mathbf{Z}^2}$ and the dynamics start from a random configuration $\zeta_0$ distributed according to uniform product measure on $\kappa$ colors. Then $x$ changes at time $t+1$ to the unique color $k$ such that $|(x+\mathcal{N}) \cap \{\zeta_t = k\}| \ge \theta_2$; $x$ keeps the color it had at time $t$ if there is no such $k$.

Now fix an $\alpha \in (1/2, 1)$, a positive integer $r$, and a time $t$, and define

$$C(r, t; \alpha) = \left\{ x \in \mathbf{Z}^2\colon \frac{|B_\infty(x, r) \cap \{\zeta_t = k\}|}{|B_\infty(x, r)|} \le \alpha \right.$$
$$\left. \text{for every } k \in \{1, \dots, \kappa\} \right\}. \tag{5.1}$$

THEOREM 5.1. *Assume that threshold growth given by $\mathcal{N}$ and $\theta_2$ is symmetric, supercritical and voracious, and let $\gamma$ and $\nu$ be the associated nucleation parameters. Then for every $\alpha \in (1/2, 1)$ and $\gamma' > \gamma - 1$, there exists a constant $r_0$ so that for every $r \ge r_0$ and every $t \in [k^{\gamma'/2}, \infty)$,*

$$\sqrt{\frac{n_0}{\kappa^{\gamma-1}}}\, C(r, t; \alpha) \to_d \mathcal{V}(\wp)$$

*as $\kappa \to \infty$.*

To explain the statement, we remark that $C(r, t; \alpha)$ effectively "cleans the crud" left by failed centers. This crud consists of isolated pieces with possibly large, but finite radius. Once the crud is cleaned, cluster boundaries (which are now a bit fuzzier) resemble a Poisson–Voronoi tessellation.

It is expedient to introduce here a notion that will be used several times later in the paper. If $x = x_0, x_1, \dots, x_n = y$ is an $l^1$-path connecting $x$ and $y$, then call $\bigcup_i B_\infty(x_i, r)$ an $r$-highway connecting $x$ and $y$.

PROOF. We apply Proposition 2.9. The set of all nuclei $Q_\kappa$ satisfies condition (a) by Lemma 3.2. To show (b), choose a small $\varepsilon > 0$. Now pick $R$ and $\kappa_0$ so that if $\kappa \geq \kappa_0$ then, with probability at least $1 - \varepsilon$, $B' = B_\infty(0, R\kappa^{(\gamma-1)/2})$ will include a nucleus. Note that $R$ and $\kappa_0$ will depend on $\varepsilon$. Also, let $B'' = B_\infty(0, R'\kappa^{(\gamma-1)/2})$, where $R' = c \cdot R$ and $c$ (which depends only on the norm and not on $\varepsilon$) is large enough that the Voronoi tessellation on $B'$ depends only on centers in $B''$.

Next, take $M > 10 \cdot \mathrm{diam}_\infty(\mathcal{N})$ so large that we have the following:

(i) The growth model started from any $A_0$ with $|A_0| \leq \gamma$ which does *not* generate persistent growth, does not exit $A_0 + B_\infty(0, M)$.

(ii) For any site $x$ inside the discrete circle $B(0, M/10) \cap \mathbf{Z}_2$ there are at least $\theta_2$ points in $B(0, M/10) \cap \mathbf{Z}_2 \cap (x + \mathcal{N})$.

To see why (ii) holds, see the proof of Proposition 2.3 in [15]. Note that the discrete circle in (ii), once it is covered entirely with one color, can never be changed by the threshold vote dynamics, no matter what happens outside.

Having chosen $M$, take $\kappa$ large enough that each of the events described in (v)–(vii) below is overwhelmingly likely (o.l.), that is, has probability greater than $1 - \varepsilon$.

(iii) Draw the Voronoi tessellation $\mathcal{V}'$ centered at the nuclei and with norm $\|\cdot\|$. Any site in $B'$ which is at $l^\infty$-distance at least $M$ from $\mathcal{V}'$ can be reached from its center by a straight line $\Lambda_x \subset \mathbf{R}^2$ such that $\Lambda_x + B_\infty(0, M)$ does not intersect the boundaries.

This event is o.l. by Lemmas 3.1 and 2.1.

(iv) Declare a site $x \in \mathbf{Z}^2$ $M$-occupied if $\zeta_0(x) = \zeta_0(y)$ for some $y \in B_\infty(x, M)$. Any $l^\infty$ connected set of $M$-occupied sites intersecting $B'$ has size at most $M^2$.

This event is o.l. by Lemmas 4.1 and 4.4.

(v) No two nuclei in $B''$ have the same color, and $B_\infty(x, 2M)$ does not contain $\gamma + 1$ sites of the same color for any $x \in B''$.

The probability of this event is $1 - \mathcal{O}(1/\kappa)$.

(vi) Fix any nucleus $q \in B''$. Assume that $\zeta_0(q) = k$, and let $A_0 = \{\zeta_0 = k\}$. At any time $t$, let $F_t(q)$ be the $l^\infty$-component of $\mathscr{T}(\mathscr{N}, \theta_2)^t(A_0) + B_\infty(0, M)$ which includes $q$. Then, for $t \le R' t^{(\gamma-1)/2}$, $F_t(q) \subset q + (1+\varepsilon)t\mathscr{L}$.

This event is o.l. by Lemmas 4.1 and 4.6.

(vii) Fix any nucleus $q \in B''$ and let $A_0$ be the $\gamma$ points of this nucleus. Write $S = \mathbf{Z}^2 \setminus \{M\text{-occupied points}\}$. At any time $t$, let $G_t(q)$ denote the sites in $(\mathscr{T}|S)^t(A_0)$ that can be connected to $q$ by an $M$-highway consisting of sites in $(\mathscr{T}|S)^t(A_0)$. For $t \le R'' \kappa^{(\gamma-1)/2}$, all points in $(q + (1-\varepsilon)t\mathscr{L}) \cap \mathbf{Z}^2$, within $l^\infty$ distance $M^2$ of $M$-occupied points, are in $G_t(q)$.

This event is o.l. by Lemmas 4.1 and 4.3.

It follows from (i)–(vii) that at time $\kappa^{\gamma'/2}$ [indeed, even by time $R'\kappa^{(\gamma-1)/2}$], there exists a constant $M_1$ (dependent on $M$) such that every site $x \in B'$ at $l^\infty$ distance at least $\varepsilon\kappa^{(\gamma-1)/2}$ from $\mathscr{V}'$ has a site $y$ such that the following holds.

(viii) $||x - y||_\infty \le M_1$ and $B_\infty(y, M)$ is completely filled with the color of the closest nucleus, which will never change again.

This implies that, for $r$ large enough and $t \ge \kappa^{\gamma'}$, $C(r, t; \alpha) \cap B' \subset \mathscr{V}' + B_\infty(0, \varepsilon\kappa^{\gamma-1})$.

On the other hand, for a point $z \in \mathscr{V}' \cap B'$ we can find $x_1$ and $x_2$ which are both at $l^\infty$ distance at least $\varepsilon\kappa^{(\gamma-1)/2}$ from $\mathscr{V}'$, and at most $2\varepsilon\kappa^{(\gamma-1)/2}$ from $z$. Moreover, $x_1$ and $x_2$ can be chosen to be from different tiles of $\mathscr{V}'$. Then there exist $y_1$ and $y_2$ such that (viii) holds if $x$, $y$ are replaced either by $x_1, y_1$ or $x_2, y_2$. If $y_1$ and $y_2$ are connected by an $l^1$–path, some point $z'$ on this path must be in $C(r, t; \alpha)$. Hence, $z \in C(r, t; \alpha) + B_\infty(0, 2\varepsilon\kappa^{\gamma-1})$. This completes the proof. □

To reformulate Theorem 1.2, recall that now the state space is $\{0, 1, \dots, \kappa\}^{\mathbf{Z}^2}$, and we start from a product measure with density $1 - p$ of 0's and $p/\kappa$ of each of the foreground colors. Then only 0's may change, and a 0 at $x$ at time $t$ will change into $k$ iff $k$ is the unique color such that $|(x + \mathscr{N}) \cap \{\zeta_t = k\}| \ge \theta_1$. Define $C(r, t; \alpha)$ as in (5.1). We can include $t = \infty$ since $\zeta_\infty$ is defined in this case.

THEOREM 5.2. *Assume that threshold growth given by $\mathscr{N}$ and $\theta_1$ is symmetric, supercritical, voracious and such that $K_{1/w}$ is convex, and let $\gamma$ and $\nu$ be the associated nucleation parameters.*

*For every $\alpha \in (1/2, 1)$ and $\gamma' > \gamma$, there exists a constant $r_0$ such that for every $r \ge r_0$, and every $t \in [p^{-\gamma'/2}, \infty]$,*

$$\sqrt{\nu p^\gamma}\, C(r, t; \alpha) \to_d \mathscr{V}_\kappa(\wp)$$

*as $\kappa \to \infty$.*

The proof is very similar to that of Theorem 5.1, except that one also needs Lemma 4.7 to justify an analogue of (6).

**6. An application: the tile containing the origin.** Of all quantities in a Poisson–Voronoi tessellation, perhaps the most basic is the average area of the tiles. Rather remarkably, the answer turns out to be 1, independently of the norm. This follows from the translation invariance of $\wp$ and $\mathscr{V}(\wp)$: all tiles have the same expected area and every tile includes exactly one point in $\wp$.

To delve deeper, it is therefore natural to ask about the expected area of the tile that includes a single given point, say the origin. We denote this tile by $T_0$. The answer which, due to size bias, should be a little larger than 1, is given by the triple integral below. Numerical computations show that in the Euclidean case the result to three significant digits is 1.280. This has been known for a long time; see [25], [21]). In the $l^1$ (or $l^\infty$) case it is 1.271, and in the case of the norm in the example after Theorem 1.4 the three significant digits are again 1.280. Evidently $E(T_0)$ does not vary greatly with the norm.

As always, let $\mathscr{L}$ be the unit ball in the norm $||\cdot||$. We now write $e_\theta = (\cos\theta, \sin\theta)$ for a unit vector in direction $\theta$, and set

$$\begin{aligned}\Phi(r_1, \theta_1, r_2, \theta_2) &= \text{Area}[(r_1 e_{\theta_1} + r_1||e_{\theta_1}||\mathscr{L}) \cup (r_2 e_{\theta_2} + r_2||e_{\theta_2}||\mathscr{L})],\\ A(r, \theta_1, \theta_2) &= \Phi(1, \theta_1, r, \theta_2).\end{aligned}$$

Note that $A(1/r, \theta_1, \theta_2) = A(r, \theta_1, \theta_2)/r^2$.

PROPOSITION 6.1. *The expected area of $T_0$ in $\mathscr{V}(\wp)$ is*

$$\int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^1 \frac{r}{A(r, \theta_1, \theta_2)^2}\, dr.$$

PROOF. Let $\mu_r$ be the arc-length measure on $\partial(r\mathscr{L})$. Then the expected value is given by

$$\begin{aligned}&\int_{x\in\mathbf{R}^2} P(x \text{ in the same tile as } 0)\, dx\\ &\quad= \int_{x\in\mathbf{R}^2} dx \int_0^\infty dr \int_{\xi\in\partial(r\mathscr{L})} \frac{1}{\mu_r(\partial(r\mathscr{L}))} \frac{d}{dr}(\text{Area}[r\mathscr{L}]) \exp(-\text{Area}[r\mathscr{L}])\\ &\qquad\qquad \times \exp(-\text{Area}[(x + ||x - r\xi||\mathscr{L}) \setminus (r\mathscr{L})])\, d\mu_r(\xi)\\ &\quad= \int_{x\in\mathbf{R}^2} dx \int_{y\in\mathbf{R}^2} \exp(-\text{Area}[(x + ||x-y||\mathscr{L}) \cup (||y||\mathscr{L})])\, dy\\ &\quad= \int_{y\in\mathbf{R}^2} dy \int_{x\in\mathbf{R}^2} \exp(-\text{Area}[(x - y + ||x-y||\mathscr{L}) \cup (-y + ||y||\mathscr{L})])\, dx\\ &\quad= \int_{y\in\mathbf{R}^2} dy \int_{x\in\mathbf{R}^2} \exp(-\text{Area}[(x + ||x||\mathscr{L}) \cup (y + ||y||\mathscr{L})])\, dx\end{aligned}$$

Note that the last line is the formula for the expected square of the area of a tile. Switching to polar coordinates, this gives us

$$\begin{aligned}
&\int_0^\infty r_1\,dr_1 \int_0^{2\pi} d\theta_1 \int_0^\infty r_2\,dr_2 \int_0^{2\pi} \exp(-\Phi(r_1,\theta_1,r_2,\theta_2))\,d\theta_2\\
&\quad= \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^\infty r_1\,dr_1 \int_0^\infty \exp(-r_1^2 A(r_2/r_1,\theta_1,\theta_2)) r_2\,dr_2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (r_2 = r_1 r)\\
&\quad= \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^\infty r\,dr \int_0^\infty \exp(-r_1^2 A(r,\theta_1,\theta_2)) r_1^3\,dr_1\\
&\quad= \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \int_0^\infty \frac{r}{2A(r,\theta_1,\theta_2)^2}\,dr\\
&\quad= \int_0^{2\pi} d\theta_1 \int_0^{2\pi} d\theta_2 \biggl(\int_0^1 \frac{r}{2A(r,\theta_1,\theta_2)^2}\,dr + \int_1^\infty \frac{r}{2A(1/r,\theta_1,\theta_2)^2}\frac{1}{r^4}\,dr\biggr),
\end{aligned}$$

which, after substituting $r$ for $1/r$ in the last integral, gives the result. □

We now proceed to determine the asymptotic behavior for the expectation of the quantity $G_0 = G_0(r,t) = \{x$: there exists an $l^1$-path $0 = z_0, z_1, \dots, z_n = x$ such that $\zeta_t(z) = \zeta_t(0)$, for every $z \in \bigcup_{i=1}^n B_\infty(z_i, r)\}$. In words, $G_0(r,t)$ consists of points connected to 0 by an $r$-highway that goes entirely through points of the same color. Our next result is conceivably true already for $r = 0$. Standing in the way of this improvement is the possibility of "percolation through boundaries," an unlikely scenario that we cannot rule out.

THEOREM 6.2. *Let $\zeta_t$ be as in Theorem* 5.1. *For $t \in [\kappa^{\gamma'}, \infty)$ and $r$ sufficiently large*,

$$E(|G_0(r,t)|) \sim \frac{E(\mathrm{Area}(T_0))}{\nu}\kappa^{\gamma-1} \quad \text{as } \kappa \to \infty.$$

Three lemmas are needed for the proof.

LEMMA 6.3. *Fix an $r > 0$ and a small $p > 0$. Given an integer $R > 0$, let $H(R)$ be all the points in $B_\infty(0,R)$ that are connected to 0 by an $l^1$–path of sites in $B_\infty(0,R) \cap (\Pi(p) + B_\infty(0,r))^c$. Then, as $p \to 0$,*

$$\sup_{R>0} \frac{E(|H(R)|)}{R^2} \to 1. \tag{6.1}$$

To understand the way this lemma will be used, note that it provides a uniform bound on the expected number of sites in $B_\infty(0,R)$ that are surrounded by an $l^\infty$-contour of sites in $\Pi(p) + B_\infty(0,r)$.

PROOF. Clearly, (6.1) is established once we prove that

$$\inf_{x\in\mathbf{Z}^2} P\big(x \text{ connected to } 0 \text{ by an } l^1\text{-path contained in } (\Pi(p)+B_\infty(0,r))^c \cap B_\infty(0,\|x\|_\infty)\big) \to 1 \tag{6.2}$$

as $p \to 0$. If the event in (6.2) does not hold, $x$ and 0 have to be separated by an $l^\infty$-path in $\Pi(p)+B_\infty(0,r)$ which either encircles one of them or else hits the boundary. The probability that this happens is bounded above by

$$2 \sum_{1\le i\le \|x\|_\infty/2} p^{i/(2r+1)^2} 9^i \le 18 p^{1/(2r+1)^2}/(1-9p^{1/(2r+1)^2}),$$

which proves (6.2). □

Fix a $\delta \ge 0$. For a closed subset $A \subset \mathbf{R}^2$, let $\Psi_\delta(A)$ be the connected component of $(A+B(0,\delta))^c$ which includes the origin. The next lemma is needed because Area$[\Psi_0]$ is *not* continuous.

LEMMA 6.4. *Fix a realization of $\mathscr{V}(\wp)$. There exists a $\delta_0 > 0$ (dependent on the realization) such that* Area$[\Psi_\delta]$ *is continuous for any $\delta \le \delta_0$.*

PROOF. Let $x_0 \in \wp \cap T_0$ and define $\delta_0 = 2\,\mathrm{dist}(x_0, \mathscr{V}(\wp))$. Then the proof follows immediately from Lemma 2.1 and the a.s. finiteness of $T_0$. □

Before stating the next lemma, we introduce some additional notation. The random sets $D = D(r,t)$ and $C' = C'(r,t)$ are defined by

$$D = \{x \in \mathbf{Z}^2 \colon x \text{ is connected to a nucleus by an } r\text{-highway of the same color at time } t\},$$

$$C' = \{x \in \mathbf{Z}^2 \colon \text{either } x \in D \text{ or } x \text{ is surrounded by an } r\text{-highway loop of sites in } D\}^c.$$

LEMMA 6.5. *For $t \in [\kappa^{\gamma'}, \infty)$ and $r$ large,*

$$\sup_\kappa \frac{1}{\kappa^{2(\gamma-1)}} E(|C'(r,t)|^2) < \infty. \tag{6.3}$$

PROOF. Let $q_0$ be the nucleus closest to 0 (in the $\|\cdot\|$ norm—with some arbitrary convention in the case of ties) and call its color 1 (without loss of generality). Note that for $M$ sufficiently large, the event $\{q_0 = z\}$ is independent of the configuration outside $B_\infty(0, M\cdot(\|z\|_\infty+1))$.

Fix $R$ and $R'$ as in the proof of Theorem 5.1. Declare a site $x \in \mathbf{Z}^2$ closed if (i)–(vii) in the proof of that theorem are satisfied with all the boxes translated by the vector $2R\kappa^{(\gamma-1)/2}x$, and, in addition, no nucleus in $2R\kappa^{(\gamma-1)/2}x + B''$ has color 1. Then $P(x \text{ closed}) \ge 1-\varepsilon$ for $\kappa$ large enough. Moreover, there exists a constant $c$ (independent of $\varepsilon$) so that sites $x$ and $y$ are closed independently as soon as $\|x-y\|_\infty \ge c$ (see the proof of Theorem 5.1).

Moreover, we claim that for $r$ large enough, $2R\kappa^{(\gamma-1)/2}x + B'$ will not intersect $C'(r,t)$ provided $x$ is closed. This follows from the fact that a color moving from a nucleus $q$ on a wide highway which avoids all failed centers cannot be stopped unless that highway is reached by growth from another nucleus $q'$. More precisely, at time $t$, for every site $y \in 2R\kappa^{(\gamma-1)/2}x + B'$ there exists a path $y_1, \dots y_n$ of sites of the same color, such that $\|y_i - y_{i-1}\|_\infty \le r$, $y_n$ is a nucleus, and $\|y - y_1\|_\infty \le r$.

By virtue of finite range dependence, we can choose $\varepsilon$ small enough so that there exists a $\Gamma > 0$ so that for positive integers $k$ and $n$,

$$P(\text{an } l^\infty \text{ cluster of size greater than or equal to } n, \text{ consisting of nonclosed sites, intersects } B_\infty(0,k)) \le (2k+1)^2 e^{-\Gamma n}.$$

It follows that for some constant $c$

$$P(|C'(r,t)| \ge n\kappa^{\gamma-1} + ck^2 \mid \|q_0\|_\infty = k) \le c\frac{k^2}{\kappa^{\gamma-1}} e^{-\Gamma n},$$

and therefore, for a different $c$,

$$E\left[\frac{|C'(r,t)|^2}{\kappa^{2(\gamma-1)}} \,\middle|\, \|q_0\|_\infty = k\right] \le c\left(\frac{k^4}{\kappa^{2(\gamma-1)}} + \frac{k^2}{\kappa^{\gamma-1}}\right).$$

Multiplying by $P(\|q_0\|_\infty = k)$ and summing over $k$ gives

$$E\left[\frac{|C'(r,t)|^2}{\kappa^{2(\gamma-1)}}\right] \le c\left(\frac{1}{\kappa^{2(\gamma-1)}} E(\|q_0\|_\infty^4) + \frac{1}{\kappa^{\gamma-1}} E(\|q_0\|_\infty^2)\right).$$

Now, since $\|q_0\|_\infty$ is of order $\kappa^{(\gamma-1)/2}$, a routine argument establishes (6.3). □

PROOF OF THEOREM 6.2. First we observe that, as its proof shows, Theorem 5.1 still holds if $C$ is replaced by $C'$. (This seems like a more elegant statement than Theorem 5.1, but it depends on knowing where the nuclei are.) Denote $\mathscr{V}'' = \nu\kappa^{-\gamma-1}(C' + B_\infty(0, 1/2)) \subset \mathbf{R}^2$ and let $T_0(\kappa) = \Psi_0(\mathscr{V}'')$. A priori, this set is possibly $\varnothing$, or unbounded. What we need to show is that $E[\text{Area}[T_0(\kappa)]] \to E[\text{Area}[T_0]]$ as $\kappa \to \infty$. By virtue of (6.3), it is enough to show that for every $R > 0$,

$$\text{(6.4)} \quad E(\text{Area}[T_0(\kappa) \cap B_\infty(0,R)]) \to E(\text{Area}[T_0 \cap B_\infty(0,R)]) \quad \text{as } \kappa \to \infty.$$

Fix an $\varepsilon > 0$. Then Lemma 6.3, Lemma 6.4 and Theorem 5.1 imply that we can choose $\delta > 0$ small enough so that

$$\text{(6.5)} \quad |E(\text{Area}[\Psi_\delta(\mathscr{V}'') \cap B_\infty(0,R)]) - E(\text{Area}[\Psi_\delta(\mathscr{V}) \cap B_\infty(0,R)])| \le \delta$$

for large enough $\kappa$. It is easy to see that

$$\text{(6.6)} \quad E(\text{Area}[\Psi_\delta(\mathscr{V}) \cap B_\infty(0,R)]) \to E(\text{Area}[T_0 \cap B_\infty(0,R)]) \quad \text{as } \delta \to 0.$$

Finally, the proof of Theorem 5.1 implies that, for $\delta > 0$ small enough,

$$\text{(6.7)} \quad P(\text{Area}[(\Psi_0(\mathscr{V}'') \cap \Psi_\delta(\mathscr{V}'')^c] < \varepsilon) > 1 - \varepsilon$$

for all large $\kappa$. Together, (6.5)–(6.7) and standard arguments establish (6.4) and complete the proof. □

One can obtain a corresponding limit theorem in the setting of suitably large fixed $\kappa$ as $p \to 0$. Let $P_\kappa$ and $E_\kappa$ be the probability and expectation associated with the $\kappa$-colored Voronoi tessellation. The following extra ingredient is needed.

LEMMA 6.6. *There exists a* $\kappa_0$ *so that for* $\kappa \geq \kappa_0$, $\mathrm{Area}(T_0)$ *has exponential tails with respect to* $P_\kappa$, *and hence* $E_\kappa(\mathrm{Area}(T_0)) < \infty$. *Moreover,* $E_\kappa(\mathrm{Area}(T_0)) \to E(\mathrm{Area}(T_0))$ *as* $\kappa \to \infty$.

It seems clear that $\kappa_0 = 3$ works here, and that $E_2(\mathrm{Area}(T_0)) = \infty$. More generally, color points in $\wp$ independently with only two colors, choosing color 1 with probability $p$ and color 0 with probability $1 - p$. Let $P_p$ and $E_p$ be the induced measure and expectation. Call $T_0(1)$ the connected set of 1's which includes 0 (possibly $\emptyset$) in the resulting Voronoi tessellation. Then we conjecture that $P(T_0(1)$ is unbounded$) > 0$ iff $p > 1/2$ and $E(\mathrm{Area}(T_0(1))) < \infty$ iff $p < 1/2$. This would then be a natural example of percolation on the Euclidean plane with critical density 1/2. To date, the conjecture has not been proved, although Benjamini and Schramm [5] have recently established a kind of conformal invariance for this model. See also [26] for a different percolation problem associated with Voronoi tessellations.

PROOF. Let $d_r$ be the diameter of an $r \times r$ square in the norm $||\cdot||$; then $d_r = cr$ for some constant $c > 0$. Call a point $x \in \mathbf{Z}^2$ closed if the set $\wp \cap B_\infty(xr, r/2) \subset \mathbf{R}^2$ is nonempty and the set $\wp \cap (B_\infty(xr, r/2) + d_r\mathscr{L}) \subset \mathbf{R}^2$ has no points with color 1. This ensures that no points in $B_\infty(xr, r/2) \subset \mathbf{R}^2$ have color 1.

Clearly, for any $\varepsilon > 0$, first $r$ and then $\kappa$ can be chosen large enough that $P(x$ is closed$) \geq 1 - \varepsilon$. Moreover, any $x$ and $y$ sufficiently far from each other (namely, points with $y - x \notin 4c\mathscr{L}$) are closed independently. It follows by a standard path counting argument that there is a constant $\Gamma > 0$ such that $P(0$ has color 1 and $\mathrm{Area}(T_0) \geq n) \leq e^{-\Gamma n}$. This proves the first assertion. The second follows from monotonicity and dominated convergence. □

The following theorem can now be proved, much in the same way as Theorem 6.2, but with a few more technical difficulties, so we omit the proof.

THEOREM 6.7. *Let* $\zeta_t$ *be as in Theorem* 5.2. *There exists a* $\kappa_0$ *so that if* $\kappa \geq \kappa_0$, *r is sufficiently large*, *and* $t \in [p^{-\gamma'}, \infty]$, *then*

$$E(|G_0(r,t)|) \sim \frac{E_\kappa(\mathrm{Area}(T_0))}{\nu p^\gamma} \quad \text{as } p \to 0.$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
DAVIS, CALIFORNIA 95616
E-MAIL: gravner@feller.ucdavis.edu

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WISCONSIN
MADISON, WISCONSIN 53706
E-MAIL: griffeat@math.wisc.edu