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THE WAITING TIME DISTRIBUTION FOR THE RANDOM ORDER SERVICE $M/M/1$ QUEUE

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The $M/M/1$ queue is considered in the case in which customers are served in random order. A formula is obtained for the distribution of the waiting time w in the stationary state. The formula is used to show that $P(w > t) \sim \alpha t^{-5/6} \exp(-\beta t - \gamma t^{1/3})$ as $t \to \infty$, with the constants α, β, and γ expressed as functions of the traffic intensity ρ. The distribution of w for the random order discipline is compared to that of the first in, first out discipline.

1. Introduction. In this paper we obtain a formula for the waiting time distribution of the single server queue subject to the random order service (ROS) discipline. We assume that the customer arrivals form a Poisson process with rate 1 and that the service time is exponential with mean $0 < \rho < 1$. The mean ρ is referred to as the traffic intensity. The assumption on ρ insures stability of the queue. The ROS discipline means that whenever the server becomes free, the next customer is chosen at random from the queue, each customer being equally likely to be chosen.

In much of queuing theory, the first come, first served (FIFO) discipline is the prevalent one. But, for many switching systems, the ROS discipline is the more realistic approximation. Comparing the two disciplines, it is clear that the queue lengths are the same for both. Little's law [1] then implies that the expected waiting times, in the stationary state, are also the same for the two disciplines. However, as is shown in this paper, the waiting time distributions are very different.

The waiting time w is defined to be the amount of time spent by the entering customer up to the beginning of service. We study the distribution of this quantity in the stationary state. For $n \geq 0$ and $t \geq 0$, let $G_n(t)$ be the probability that the waiting time exceeds t, given that the entering customer finds the server occupied and n customers waiting. Vaulot [7] derived differential equations for the quantities $G_n(t)$. Vaulot [7] and Riordan [5] use these equations to evaluate the moments of w. Riordan also obtains approximations to the waiting time distribution by finite sums of exponential distributions. Starting with Vaulot's system of differential equations, Kingman [3] obtains a formula for the Laplace transform of w. Further references on the above problem are found in Cohen's book on queuing theory [3].

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In this paper, we carry the analysis a step further, and produce an explicit formula for the distribution of w. In Section 5, we prove the following theorem.

THEOREM 1.1. *The distribution of w is given by*

$$P(w>t)=2(\rho^{-1}-1)\int_0^{\pi}\frac{\exp((2\phi(\theta)-\theta)\cot\theta)}{\exp(\pi\cot\theta)+1} \times\frac{\exp(-[1-2\rho^{-1/2}\cos\theta+\rho^{-1}]t)}{[1-2\rho^{-1/2}\cos\theta+\rho^{-1}]^2}\sin\theta\,d\theta, \tag{1.1}$$

where

$$\phi(\theta)=\arctan\left[\frac{\sin\theta}{\cos\theta-\rho^{1/2}}\right],\qquad 0\le\phi(\theta)\le\pi. \tag{1.2}$$

The relation $\phi=\phi(\theta)$ can be interpreted geometrically by the angles ϕ and θ of Figure 5; $\phi(\theta)$ increases continuously from 0 to π as θ increases from 0 to π. This fact follows from (1.2), and is also evident from Figure 5. For, given $t>0$, the integrand of (1.1) is a positive continuous function of θ, tending to 0 as θ tends to either 0 or π.

By analyzing the behavior of the integrand near 0, we obtain in Section 6 the behavior of $P(w>t)$ as $t\to\infty$.

THEOREM 1.2. *As $t\to\infty$,*

$$P(w>t)\sim\frac{\alpha\exp(-\beta t-\gamma t^{1/3})}{t^{5/6}}, \tag{1.3}$$

where

$$\alpha=2^{2/3}3^{-1/2}\pi^{5/6}\rho^{17/12}\frac{1+\rho^{1/2}}{(1-\rho^{1/2})^3}\exp\left(\frac{1+\rho^{1/2}}{1-\rho^{1/2}}\right)$$

$$\beta=(\rho^{-1/2}-1)^2$$

$$\gamma=3\left(\frac{\pi}{2}\right)^{2/3}\rho^{-1/6}$$

The above theorems can be used to compare the waiting time distributions, in the stationary state, for the $M/M/1$ queue governed by the FIFO and ROS disciplines.

In the former case it is known that ([3], page 195)

$$P(w>t)=\rho\exp(-(\rho^{-1}-1)t). \tag{1.4}$$

Denote $P(w>t)$ for FIFO and ROS by $g_1(t)$ and $g_2(t)$, respectively. Observe that $g_1(0)=g_2(0)$, as these quantities denote the probabilities that the FIFO and ROS queues are not empty in the stationary state. For $t > 0$, we prove in Section 7 the following.

THEOREM 1.3. *There exists a positive number* $\tau(\rho)$ *such that*

$$g_2(t) < g_1(t) \quad \text{for } 0 < t < \tau(\rho),$$
$$g_2(t) > g_1(t) \quad \text{for } t > \tau(\rho).$$

Theorem 1.3 states that it is more likely for the customer to experience both short and long waiting times under ROS than under FIFO, a result which can be backed by intuition.

The proof of Theorem 1.1 uses, for a starting point, Kingman's [4] formula for the Laplace transform $f(s)$ of w and employs methods of classical complex analysis.

This paper proceeds as follows. Section 2 reviews Kingman's derivation of the formula for $f(s)$, which represents $f(s)$ as a complex integral. Originally, $f(s)$ is defined only for $\operatorname{Re} s \geq 0$, and in Section 3 we show that Kingman's formula provides an analytic continuation of $f(s)$ to the entire s-plane minus the slit $\mathrm{I} = [-(\rho^{-1/2} + 1)^2, -(\rho^{-1/2} - 1)^2]$. Furthermore, f extends continuously to both the upper and lower sides of I, and we denote these, respectively, by f_+ and f_-. We can recover $P(w > t)$ from f by the classical inversion formula, which represents $P(w > t)$ by an integral of $f(s)$ over an infinite vertical path in the s-plane. It is difficult to obtain insight into $P(w > t)$ directly from the inversion formula. To achieve this, in Section 4 we deform the vertical path to the closed contour consisting of the slit I traversed in both directions. This contour integral can be expressed as an integral of $[f_+(x) - f_-(x)]$ over I. In Section 5 we obtain a closed expression for $[f_+(x) - f_-(x)]$ which leads to Theorem 1.1. We find it surprising that $[f_+(x) - f_-(x)]$ can be expressed in closed form, since the corresponding statement seems false for $f(s)$. In Section 6, we derive from Theorem 1.1 the asymptotics of Theorem 1.2. Finally in Section 7, we derive Theorem 1.3 from Theorems 1.1 and 1.2.

To justify the contour deformation presented in Section 4, we need estimates for f near the points ∞ and $-(\rho^{-1/2} \pm 1)^2$, the end points of I. The derivation of these estimates is intricate and is deferred to the Appendix.

Throughout this paper, we encounter repeatedly the function z^w. This function is multivalued and must be specified. For z not on the negative real axis (i.e., it is not the case that $z \leq 0$) and arbitrary w, we define

$$z^w = \exp(w \log z) \tag{1.5}$$

with

$$\log z = \log|z| + i \arg z, \tag{1.6}$$

where $\log|z|$ is real and $|\arg z| < \pi$.

Thus $\log z$ and $\arg z$ will always denote principal values. On its domain of definition, z^w is analytic in both z and w. The restriction placed on z holds in all ensuing uses of the function z^w.

From (1.5) and (1.6), we get the identities

$$z^{w} = \exp(w \log|z|)\exp(iw \arg z) = |z|^{w}\exp(iw \arg z) \tag{1.7}$$

and

$$\begin{aligned} |z^{w}| &= \exp(\operatorname{Re}(w \log z)) \\ &= \exp(\operatorname{Re} w \cdot \log|z| - \operatorname{Im} w \cdot \arg z) = |z|^{\operatorname{Re} w}\exp(-\operatorname{Im} w \cdot \arg z) \end{aligned} \tag{1.8}$$

which will be used later on.

2. The Laplace transform of w. For $n \ge 0$ and $t \ge 0$, let $G_n(t)$ be the probability that the waiting time of the arriving customer exceeds t, given that the customer finds the server occupied and n customers waiting. The probability of the latter event is given, in the stationary state, by $(1-\rho)\rho^{n+1}$. Hence

$$P(w > t) = \rho(1-\rho)\sum_{n=0}^{\infty} \rho^{n} G_n(t). \tag{2.1}$$

Let $f(s) = E(e^{-sw})$, $\operatorname{Re} s \ge 0$, be the Laplace transform of w. Then

$$f(s) = (1-\rho) + \rho(1-\rho)\sum_{n=0}^{\infty} \rho^{n} G_n^{*}(s), \qquad \operatorname{Re} s \ge 0, \tag{2.2}$$

where

$$G_n^{*}(s) = -\int_0^{\infty} e^{-st}\, dG_n(t), \qquad n \ge 0 \text{ and } \operatorname{Re} s \ge 0. \tag{2.3}$$

Vaulot [7] has shown that the random order service discipline implies

$$\rho \frac{dG_n}{dt} = \frac{n}{n+1} G_{n-1} - (1+\rho)G_n + \rho G_{n+1}, \qquad n \ge 0 \text{ and } t \ge 0, \tag{2.4}$$

where, for $n = 0$, $(n/(n+1))G_{n-1} := 0$.

Multiplying by e^{-st} and integrating over $(0,\infty)$, (2.4) converts to

$$\begin{aligned} 1 = -nG_{n-1}^{*} + (1+\rho+\rho s)(n+1)G_n^{*} - \rho(n+1)G_{n+1}^{*}, \\ n \ge 0 \text{ and } \operatorname{Re} s \ge 0, \end{aligned} \tag{2.5}$$

where, for $n = 0$, $nG_{n-1}^{*} := 0$.

Let $G(s,z) = \sum_{n=0}^{\infty} G_n^{*}(s)z^{n}$. Since $|G_n^{*}(s)| \le 1$ for $n \ge 0$ and $\operatorname{Re} s \ge 0$, $G(s,z)$ converges for $\operatorname{Re} s \ge 0$ and $|z| < 1$. Multiplying (2.5) by z^{n} and summing over $n \ge 0$, (2.5) converts to

$$(1-z)^{-1} = (1+\rho+\rho s - z)G - \left(\rho - [1+\rho+\rho s]z + z^{2}\right)\frac{\partial G}{\partial z},$$

$$\operatorname{Re} s \ge 0 \text{ and } |z| < 1.$$

Let

$$z^{2} - (1+\rho+\rho s)z + \rho = (z - \mu(s))(z - \nu(s)) \quad \text{with } |\nu| \le |\mu|. \tag{2.6}$$

Since $\nu\mu = \rho$, we obtain $|\nu| \le \rho^{1/2} \le |\mu|$, and in Section 3 we show that inequality holds. Assume from here on that $\operatorname{Re} s > 0$, $|z| < \rho^{1/2}$ and $z \ne \nu(s)$. Equation (2.6) gives

$$\frac{\partial G}{\partial z} = \frac{\mu + \nu - z}{(z-\mu)(z-\nu)} G + \frac{1}{(z-1)(z-\mu)(z-\nu)}. \tag{2.7}$$

For fixed s, the differential equation (2.7) has a unique singularity at $z = \nu(s)$. However, G is analytic in $|z| < \rho^{1/2}$, and this forces the solution

$$\begin{aligned} G(s,z) &= \frac{1}{(z-\mu)(z-\nu)} \\ &\quad \times \int_z^\nu (1-\zeta)^{-1} \left(\frac{\zeta-\mu}{z-\mu}\right)^{-\mu/(\mu-\nu)} \left(\frac{\zeta-\nu}{z-\nu}\right)^{\nu/(\mu-\nu)} d\zeta, \end{aligned} \tag{2.8}$$

where, for simplicity, the path of integration is chosen to be $[z\nu)$, defined as the half-open line segment from z to ν which includes z and excludes ν. (For a rigorous justification of this step, we first integrate (2.7) over $[za]$, where a is interior to $[z\nu]$, and then let $a \to \nu$.) We have chosen the half-open line segment $[z\nu)$ instead of the closed line segment $[z\nu]$, because the integrand of (2.8) may have a singularity of $\zeta = \nu(s)$ [Lemma 3.2 of Section 3 insures that this singularity is integrable at $\zeta = \nu(s)$]. In accordance with the concluding remarks of the Introduction, we must show that, for $\zeta \in [z\nu]$, the quantities $(\zeta-\mu)/(z-\mu)$, $(\zeta-\nu)/(z-\nu)$ are not on the negative real axis. Observe that $\{\zeta\colon (\zeta-\mu)/(z-\mu) \le 0\}$ is the ray emanating from μ which has the same direction as $[z\mu]$. This ray lies outside the circle $|\zeta| = \rho^{1/2}$. Hence it does not meet $[z\nu]$, which lies inside the circle $|\zeta| = \rho^{1/2}$. We have thus proved the result for $(\zeta-\mu)/(z-\mu)$. For $(\zeta-\nu)/(z-\nu)$, the result follows from the fact that $(\zeta-\nu)/(z-\nu) > 0$ for $\zeta \in [z\nu)$.

If $\nu(s) = \rho$, then (2.6) implies $s = 0$. Thus $\nu(s) \ne \rho$ for $\operatorname{Re} s > 0$, so (2.8) is valid for $z = \rho$. Let $G(s) := G(s,\rho)$. From (2.2) and (2.8), we obtain the following.

THEOREM 2.1. *For* $\operatorname{Re} s > 0$, $f(s)$ *is given by*

$$f(s) = (1-\rho) + \rho(1-\rho)G(s), \tag{2.9}$$

where

$$\begin{aligned} G(s) &= \frac{1}{(\rho-\mu)(\rho-\nu)} \\ &\quad \times \int_\rho^\nu (1-\zeta)^{-1} \left(\frac{\zeta-\mu}{\rho-\mu}\right)^{-\mu/(\mu-\nu)} \left(\frac{\zeta-\nu}{\rho-\nu}\right)^{\nu/(\mu-\nu)} d\zeta. \end{aligned} \tag{2.10}$$

3. Analytic continuation of $f(s)$. We use formulas (2.9) and (2.10) to obtain the analytic continuation of $f(s)$ beyond $\operatorname{Re} s > 0$. We first discuss the domain of analyticity of the functions $\mu(s)$ and $\nu(s)$. Let

$$\mathrm{I} = \left[-(\rho^{-1/2}+1)^2, -(\rho^{-1/2}-1)^2\right] \quad \text{and} \quad \mathscr{R} = s\text{-plane minus the slit I}.$$

We show in Theorem 3.1 that $f(s)$ can be continued analytically into $\mathscr{R}$.

LEMMA 3.1. (i) *$\mu(s)$ and $\nu(s)$ are analytic in $\mathscr{R}$, and are given by*

$$\mu(s), \nu(s) = \tfrac{1}{2}\Big\{1 + \rho + \rho s \pm \rho\sqrt{\left(s + \left[\rho^{-1/2} - 1\right]^2\right)\left(s + \left[\rho^{-1/2} + 1\right]^2\right)}\Big\}, \tag{3.1}$$

the plus sign chosen for μ and the minus sign for ν. The square root function in (3.1) *is that analytic branch in $\mathscr{R}$ which is positive for $s > -[\rho^{-1/2} - 1]^2$.*

(ii) *$z = \mu(s)$ and $z = \nu(s)$ are, respectively, conformal maps from $\mathscr{R}$ onto $|z| > \rho^{1/2}$ and $0 < |z| < \rho^{1/2}$.*

(iii) *$\mu(s)$ and $\nu(s)$ extend continuously to both the upper and lower sides of* I.

We refer to the extended values of μ by μ_+ and μ_-, and those of ν by ν_+ and ν_-. Thus, if we denote the points of $\mathscr{R}$ contained, respectively, in the upper and lower parts of the s-plane by s' and s'', then for $x \in$ I,

$$\mu_+(x) := \lim_{s' \to x} \mu(s'), \qquad \mu_-(x) := \lim_{s'' \to x} \mu(s'')$$

with similar formulas for $\nu_+(x)$ and $\nu_-(x)$.

Lemma 3.1 expresses standard facts concerning the roots $\mu(s)$, $\nu(s)$ obtained by solving $Q(s, z) := z^2 - (1 + \rho + \rho s)z + \rho = 0$ for z in terms of s. The lemma becomes more lucid when solving $Q(s, z) = 0$ for s in terms of z, so that

$$s(z) = \frac{z}{\rho} + \frac{1}{z} - \left(\frac{1}{\rho} + 1\right). \tag{3.2}$$

The mapping properties of $s(z)$ are well known (see, for instance, [6], page 196). For $0 < |z| < \infty$, $s(z)$ is analytic and maps both $|z| > \rho^{1/2}$ and $0 < |z| < \rho^{1/2}$ conformally onto $\mathscr{R}$. The upper and lower parts of the circle $|z| = \rho^{1/2}$ map in continuous fashion onto I, with end points plus or minus $\rho^{1/2}$ going to $-(\rho^{-1/2} \mp 1)^2$.

The inverse to $s = s(z)$ consists of the conformal maps $z = \mu(s)$ and $z = \nu(s)$, taking $\mathscr{R}$, respectively, to $|z| > \rho^{1/2}$ and $0 < |z| < \rho^{1/2}$. Also $\mu_+(x)$ maps I to the upper part of $|z| = \rho^{1/2}$ and $\mu_-(x)$ maps I to the lower part of $|z| = \rho^{1/2}$, the reverse holding for ν_+ and ν_-. The maps $\mu(s)$ and $\nu(s)$ are depicted in Figure 1.

The terms $\mu_\pm$ and $\nu_\pm$ satisfy the following identities, which are used in Section 5:

$$\mu_+(x) = \nu_-(x), \qquad \mu_-(x) = \nu_+(x), \qquad \overline{\mu_+(x)} = \mu_-(x). \tag{3.3}$$

The identities (3.3) are illustrated in Figure 1.

We require the following lemma, which is a simple consequence of Lemma 3.1.

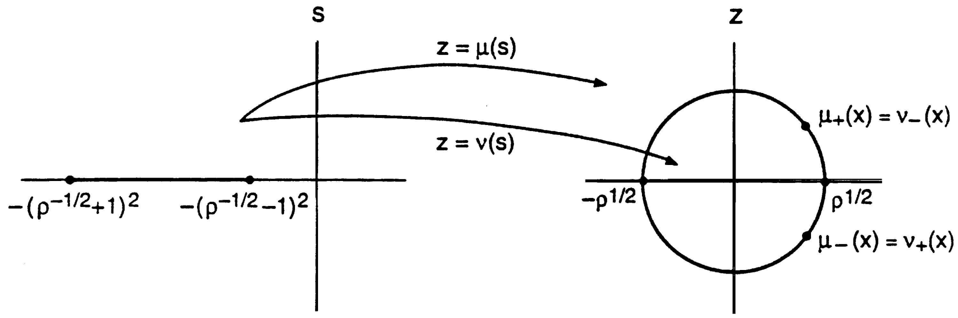

FIG. 1. *The maps* $z = \mu(s)$ *and* $z = \nu(s)$.

LEMMA 3.2. (i) *If* $s \in \mathscr{R}$, *then*

$$\text{Re}\frac{\nu}{\mu - \nu} > -\frac{1}{2}. \tag{3.4a}$$

(ii) *If* $x \in I^\circ := (-(\rho^{-1/2} + 1)^2, -(\rho^{-1/2} - 1)^2)$, *then*

$$\text{Re}\frac{\nu_+}{\mu_+ - \nu_+}, \qquad \text{Re}\frac{\nu_-}{\mu_- - \nu_-} \geq -\frac{1}{2}. \tag{3.4b}$$

PROOF. (i) From $\mu = \rho/\nu$ we get

$$\frac{\nu}{\mu - \nu} = \frac{\nu^2}{\rho - \nu^2}.$$

By Lemma 3.1, $z = \nu(s)$ maps $\mathscr{R}$ onto $0 < |z| < \rho^{1/2}$. Hence $\xi = \nu^2(s)$ maps $\mathscr{R}$ onto $0 < |\xi| < \rho$. But $\eta = \xi/(\rho - \xi)$ maps $0 < |\xi| < \rho$ onto $\{\eta\colon \text{Re}\,\eta > -\frac{1}{2}$ and $\eta \neq 0\}$, which proves Lemma 3.2(i).

(ii) This follows by taking limits in (i). □

We obtain the analytic continuation of $f(s)$ from standard results concerning the analyticity of integrals depending on a parameter. It proves advantageous to rewrite (2.10) as (3.7). Parametrize $[\rho\nu]$ by

$$\zeta(s,t) = \nu(s) + t(\rho - \nu(s)), \qquad (s,t) \in \mathscr{R} \times [0,1] \tag{3.5}$$

and let

$$\xi(s,t) = \frac{\zeta(s,t) - \mu(s)}{\rho - \mu(s)}, \qquad (s,t) \in \mathscr{R} \times [0,1]. \tag{3.6}$$

It terms of s and t, (2.10) becomes

$$G(s) = \int_0^1 F(s,t)\,dt, \qquad \text{Re}\,s > 0, \tag{3.7}$$

where

$$F(s,t) = \frac{1}{\mu - \rho}\frac{1}{1 - \zeta}\xi^{-\mu/(\mu-\nu)}t^{\nu/(\mu-\nu)}, \qquad (s,t) \in \mathscr{R} \times (0,1]. \tag{3.8}$$

In (3.7) we imposed the restriction Re $s > 0$ since, in Section 2, $G(s)$ was only defined for Re $s > 0$. However, in (3.8), $F(s,t)$ is defined for $(s,t) \in \mathscr{R} \times (0,1]$. We use this fact to obtain the analytic continuation of G, and hence of f, into $\mathscr{R}$.

LEMMA 3.3. *The function $F(s,t)$ is continuous in (s,t) on $\mathscr{R} \times (0,1]$, and is analytic in $s \in \mathscr{R}$, for each fixed value of t.*

The lemma follows from Lemma 3.1 and (3.5), (3.6), and (3.8).

We remark that (3.5) and (3.6) actually show that the first three factors in (3.8) for $F(s,t)$ are continuous in (s,t) on $\mathscr{R} \times [0,1]$. This fact is false for the fourth factor, because $\log t$ is not continuous at $t = 0$. This slight strengthening of Lemma 3.3 is used in the proof of Theorem 3.1.

THEOREM 3.1. *The function $G(s)$ has an analytic continuation into $\mathscr{R}$, provided by the integral* (3.7). *Then $f(s)$ has an analytic continuation into $\mathscr{R}$, provided by* (2.9).

PROOF. For $0 < \varepsilon < 1$, let $G_\varepsilon(s) = \int_\varepsilon^1 F(s,t)\,dt$. From Lemma 3.3, we conclude that $G_\varepsilon(s)$ is analytic in $\mathscr{R}$ (see [6], page 99). Hence, to prove Theorem 3.1 it suffices to show

$$\lim_{\varepsilon \to 0} G_\varepsilon(s) = G(s) \quad \text{uniformly on compact subsets of } \mathscr{R}.$$

Let C be any compact subset of $\mathscr{R}$, and let $\Pi(s,t)$ be the product of the first three factors in (3.8). By the remark following the proof of Lemma 3.3, we obtain

$$|\Pi(s,t)| \le K(C) \quad \text{for } (s,t) \in \mathscr{R} \times [0,1] \tag{3.9}$$

for some constant $K(C) > 0$.

By Lemma 3.2, we get

$$|t^{\nu/(\mu-\nu)}| = t^{\operatorname{Re}(\nu/(\mu-\nu))} \le t^{-1/2} \quad \text{for } (s,t) \in \mathscr{R} \times (0,1]. \tag{3.10}$$

Hence

$$|F(s,t)| \le K(C) t^{-1/2}, \qquad (s,t) \in C \times (0,1]. \tag{3.11}$$

Since $\int_0^1 t^{-1/2}\,dt < \infty$, we conclude from (3.11) that $\lim_{\varepsilon \to 0} G_\varepsilon(s) = G(s)$, uniformly on C. $\square$

The function $G(s)$, analytic in $\mathscr{R}$, can be extended continuously to both sides of I°. Let

$$G_+(x) = \int_0^1 F_+(x,t)\,dt, \qquad G_-(x) = \int_0^1 F_-(x,t)\,dt, \qquad x \in \mathrm{I}^\circ,$$

where $F_+(x,t)$ and $F_-(x,t)$ are obtained from (3.8) by replacing, respectively, $(\mu(s), \nu(s))$ by $(\mu_+(x), \nu_+(x))$ and $(\mu_-(x), \nu_-(x))$.

THEOREM 3.2. *$G(s)$, hence $f(s)$, can be extended continuously to both the upper and lower sides of* I°; *these extensions are given, respectively, by $G_+(x)$ and $G_-(x)$.*

The proof of Theorem 3.2 differs from that of Theorem 3.1 only in minor details, and is omitted.

Letting $s \to x$ in (2.9), we obtain the relations

$$\begin{aligned} f_+(x) &= (1-\rho) + \rho(1-\rho)G_+(x), \\ f_-(x) &= (1-\rho) + \rho(1-\rho)G_-(x), \qquad x \in \mathrm{I}^\circ. \end{aligned} \tag{3.12}$$

We remark that (3.7) can be converted back to (2.10) via the variable change provided by (3.5), (2.10) now being valid for $s \in \mathscr{R}$ [an exception must be made at $s = 0$, where (2.10) is indeterminate; here we define $G(0) := \lim_{s \to 0} G(s)$]. Equation (2.10) also serves as a formula for $G_+(x)$ and $G_-(x)$, after replacing $(\mu(s), \nu(s))$, respectively, by $(\mu_+(x), \nu_+(x))$ and $(\mu_-(x), \nu_-(x))$. The advantage of (2.10) over (3.7) is the symmetry of the integrand in μ and ν, a fact that is exploited in Section 5.

4. Integral formula for $P(w > t)$. We shall prove Theorem 4.2, which gives an integral formula for $P(w > t)$. We use the following estimates for $G(s)$, which are valid in the vicinity of the points ∞ and $-(\rho^{-1/2} \pm 1)^2$. These estimates will be derived in the Appendix.

THEOREM 4.1. *$G(s)$ satisfies the estimates*

$$G(s) = O\left(\frac{1}{|s|}\right) \quad \text{as } s \to \infty, \tag{4.1}$$

$$G(s) = O\left[\log \frac{1}{\left|s + (\rho^{-1/2} \pm 1)^2\right|}\right] \quad \text{as } s \to -(\rho^{-1/2} \pm 1)^2. \tag{4.2}$$

In principle, $P(w > t)$ can be recovered from $f(s)$ by the following inversion formula (see, e.g., [8], page 70).

THEOREM 4.2. *Let $-(\rho^{-1/2} - 1)^2 < c < 0$. Then*

$$P(w > t) = -\frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{f(s)e^{st}}{s}\, ds. \tag{4.3}$$

REMARKS. (i) To take advantage of Theorem 4.1, we replace $f(s)$ in (4.3) by $f(s) - \rho = \rho(1-\rho)G(s)$. This is permissible by the familiar fact that for $c < 0$ and $t > 0$,

$$\lim_{T \to \infty} \int_{c - iT}^{c + iT} \frac{e^{ts}}{s}\, ds = 0.$$

(ii) Let $s = c + iy$. For fixed c, the integrand in (4.3) is a function of y. In the general theory, the integrand need not be absolutely integrable over

$-\infty < y < \infty$, and the integral of (4.3) must be interpreted as $\lim_{R\to\infty}\int_{-R}^{R}$. However, in our case we obtain from (4.1) the estimate

$$\frac{G(c+iy)e^{(c+iy)t}}{c+iy} = O(y^{-2}) \quad \text{as } y \to \infty,$$

guaranteeing that the integrand is absolutely integrable over $-\infty < y < \infty$.

(iii) In the general theory, the inversion formula may only hold at continuity points of $P(w > t)$. However, by (ii), the integrand in (4.3) is absolutely integrable over $-\infty < y < \infty$, and this guarantees that the right-hand side of (4.3) is continuous for all $t > 0$. A standard continuity argument then shows that (4.3) holds for all $t > 0$.

It is difficult to obtain information about $P(w > t)$ directly from the inversion formula. To do so, we convert (4.3) to a more useful integral formula.

THEOREM 4.3. *For* $t > 0$,

$$P(w > t) = \frac{1}{2\pi i}\int_{-(\rho^{-1/2}+1)^2}^{-(\rho^{-1/2}-1)^2} \frac{f_+(x) - f_-(x)}{x} e^{xt}\, dx. \tag{4.4}$$

REMARK. In (4.4) $f_+(x)$ and $f_-(x)$ are only known to exist for $x \in \mathrm{I}^\circ$. Thus the integral of (4.4) is to be interpreted as $\lim_{\varepsilon\to 0+}\int_{-(\rho^{-1/2}+1)^2+\varepsilon}^{-(\rho^{-1/2}-1)^2-\varepsilon}$. However, in Section 5 we shall obtain a formula for the integrand, which shows that it extends to a continuous function on I. Hence the integral (4.4) exists as a Riemann integral over I.

PROOF. Let Γ be the contour depicted in Figure 2. Thus Γ consists of the vertical line segment $l = [c - iR, c + iR]$, the semicircle C of radius R

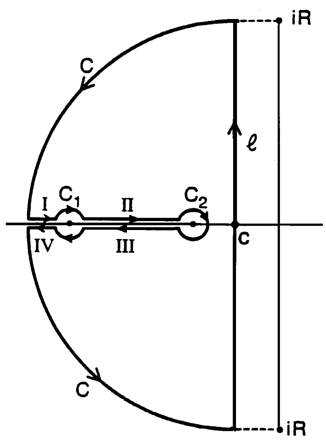

FIG. 2. *The contour* Γ.

centered at c and to the left of l, the two circles C_1 and C_2 of radius ε centered, respectively, at $-(\rho^{-1/2}+1)^2$ and $-(\rho^{-1/2}-1)^2$, and the cross-cuts I–IV joining these circles. As $c<0$, we conclude that for all $t>0$, $g(s) := ((f(s)-\rho)/s)e^{st}$ is analytic on Γ and its interior. Hence, by Cauchy's theorem,

$$\int_l + \int_C + \int_{C_1} + \int_{C_2} + \int_{\mathrm{II}} + \int_{\mathrm{III}} = 0. \tag{4.5}$$

Observe that $\int_{\mathrm{I}} + \int_{\mathrm{IV}} = 0$, so these integrals do not appear in (4.5). However, $\int_{\mathrm{II}}$ and $\int_{\mathrm{III}}$ do appear in (4.5) as $g(x)$ assumes the distinct values $((f_+(x)-\rho)/x)e^{xt}$ and $((f_-(x)-\rho)/x)e^{xt}$ along II and III.

From Theorem 4.1, we obtain

$$\int_C = O\left(\frac{1}{R}\right) \quad \text{as } R \to \infty, \tag{4.6}$$

$$\int_{C_j} = O\left(\varepsilon \log\frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \to 0 \text{ for } j = 1,2. \tag{4.7}$$

We now let $\varepsilon \to 0$ and $R \to \infty$ in (4.5). We conclude from (4.6) and (4.7) that

$$\int_{c-i\infty}^{c+i\infty} \frac{f(s)e^{st}}{s}\,ds + \int_{-(\rho^{-1/2}+1)^2}^{-(\rho^{-1/2}-1)^2} \frac{f_+(x)-f_-(x)}{x} e^{xt}\,dx = 0. \tag{4.8}$$

Equations (4.3) and (4.8) give Theorem 4.2. □

5. Proof of Theorem 1.1. We derive Theorem 1.1 from Theorem 4.2. To do this, we first obtain an integral formula for $[G_+(x) - G_-(x)]$ for $x \in \mathrm{I}^\circ$, which we then evaluate in closed form.

5.1. *Integral formula for* $[G_+(x) - G_-(x)]$. Let $\nu = \rho^{1/2}e^{i\theta}$, $0<|\theta|<\pi$ and $(\zeta-\nu)/(\rho-\nu) \neq t$ where $t \le 0$. Geometrically, the latter means that $\zeta \notin r(\nu)$, where $r(\nu)$ is the ray emanating from ν which has the same direction as $[\rho\nu]$. The term $r(\nu)$ is depicted in Figure 3, where $\nu = \nu_+, \mu_+$.

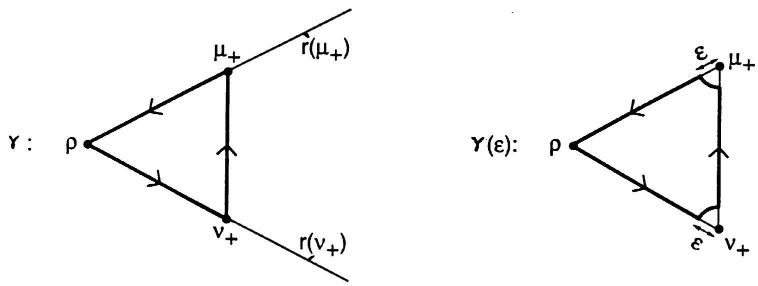

FIG. 3. *The contours* γ *and* $\gamma(\varepsilon)$.

For these values of ν and ζ, define

$$K(\nu, \zeta) := \left(\frac{\zeta - \nu}{\rho - \nu} \right)^{\nu^2/(\rho - \nu^2)}. \tag{5.1}$$

LEMMA 5.1. *For given* ν, $K(\nu, \zeta)$ *is analytic in the* ζ*-plane minus* $r(\nu)$, *and*

$$|K(\nu, \zeta)| \le \left| \frac{\zeta - \nu}{\rho - \nu} \right|^{-1/2} \exp\left(\frac{\pi}{2} |\cot \theta| \right). \tag{5.2}$$

PROOF. The analyticity follows from (5.1). As $\nu = \rho^{1/2} e^{i\theta}$, we have

$$\frac{\nu^2}{\rho - \nu^2} = \frac{e^{i\theta}}{e^{-i\theta} - e^{i\theta}} = -\frac{1}{2} + \frac{i}{2} \cot \theta. \tag{5.3}$$

Equations (1.8), (5.1), and (5.3) give

$$\begin{aligned} |K(\nu, \zeta)| &= \left| \frac{\zeta - \nu}{\rho - \nu} \right|^{-1/2} \exp\left(-\frac{\cot \theta}{2} \arg \frac{\zeta - z}{\rho - \nu} \right) \\ &\le \left| \frac{\zeta - \nu}{\rho - \nu} \right|^{-1/2} \exp\left(\frac{\pi}{2} |\cot \theta| \right) \end{aligned}$$

□

THEOREM 5.1. *For* $x \in \mathrm{I}^\circ$,

$$\begin{aligned} &G_+(x) - G_-(x) \\ &\quad = \frac{1}{|\rho - \nu_+(x)|^2} \int_{\mu_+(x)}^{\nu_+(x)} (1 - \zeta)^{-1} K(\mu_+(x), \zeta) K(\nu_+(x), \zeta) \, d\zeta, \end{aligned} \tag{5.4}$$

where $\int_{\mu_+(x)}^{\nu_+(x)}$ *denotes the integral over the vertical line segment* $[\mu_+(x), \nu_+(x)]$.

PROOF. For $x \in \mathrm{I}^\circ$, rewrite (2.10) as

$$\begin{aligned} G_+(x) &= \frac{1}{(\rho - \mu_+(x))(\rho - \nu_+(x))} \\ &\quad \times \int_\rho^{\nu_+(x)} (1 - \zeta)^{-1} K(\mu_+(x), \zeta) K(\nu_+(x), \zeta) \, d\zeta \end{aligned} \tag{5.5}$$

with a similar formula for $G_-(x)$, replacing the plus sign by the minus sign.

By (3.3), the expression for $G_-(x)$ is identical with the one for $G_+(x)$, except that the upper limit $\nu_+(x)$ appearing in $\int_\rho^{\nu_+(x)}$ is to be replaced by $\nu_-(x)$. Furthermore, since $\mu_+(x)$ and $\nu_+(x)$ are conjugate, we have

$$(\rho - \mu_+(x))(\rho - \nu_+(x)) = |\rho - \nu_+(x)|^2.$$

We conclude that

$$G_+(x) - G_-(x) = \frac{1}{|\rho - \nu_+(x)|^2} \times \left[\int_\rho^{\nu_+(x)} L(x,\zeta)\,d\zeta - \int_\rho^{\nu_-(x)} L(x,\zeta)\,d\zeta \right], \tag{5.6}$$

where

$$L(x,\zeta) = (1-\zeta)^{-1} K(\mu_+(x),\zeta) K(\nu_+(x),\zeta). \tag{5.7}$$

Let Δ be the closed triangle with vertices ρ, $\mu_+(x)$, $\nu_+(x)$. See Δ depicted in Figure 3.

The function $L(x,\zeta)$ is analytic in Δ, except at the points $\mu_+(x)$ and $\nu_+(x)$. Near the latter, we obtain from (5.2) the estimates

$$L(x,\zeta) = O\left(|\zeta - \mu_+(x)|^{-1/2}\right) \quad \text{as } \zeta \to \mu_+(x), \tag{5.8a}$$

$$L(x,\zeta) = O\left(|\zeta - \nu_+(x)|^{-1/2}\right) \quad \text{as } \zeta \to \nu_+(x). \tag{5.8b}$$

The estimates (5.8) imply that Cauchy's theorem applies to γ, the perimeter of Δ. That is, $\int_\gamma L(x,\zeta)\,d\zeta = 0$. For a formal justification of this formula, we indent γ near $\mu_+(x)$ and $\nu_+(x)$ with circular arcs of radius ε, as indicated in Figure 3. Let $\gamma(\varepsilon)$ be the resulting curve. By Cauchy's theorem $\int_{\gamma(\varepsilon)} L(x,\zeta)\,d\zeta = 0$. Letting $\varepsilon \to 0$ and using (5.8), we get $\int_\gamma L(x,\zeta)\,d\xi = 0$. Rewrite the latter as

$$\int_\rho^{\nu_+(x)} L(x,\zeta)\,d\zeta - \int_\rho^{\nu_-(x)} L(x,\zeta)\,d\zeta = \int_{\nu_-(x)}^{\nu_+(x)} L(x,\zeta)\,d\zeta. \tag{5.9}$$

Theorem 5.1 follows from (5.6) and (5.9). $\square$

We parametrize $[\nu_+\mu_+]$ by

$$\zeta = \nu_+ + (\mu_+ - \nu_+)t, \qquad 0 \le t \le 1. \tag{5.10}$$

Theorem 5.2 restates Theorem 5.1 in terms of parameter t.

For $x \in \mathrm{I}^\circ$, let

$$\nu_+(x) = \rho^{1/2} e^{-i\theta}, \qquad \mu_+(x) = \rho^{1/2} e^{i\theta}, \qquad 0 < \theta < \pi. \tag{5.11}$$

Also, let

$$\phi := \arg[\mu_+(x) - \rho], \qquad 0 \le \phi \le \pi. \tag{5.12}$$

The angle ϕ is depicted in Figure 4.

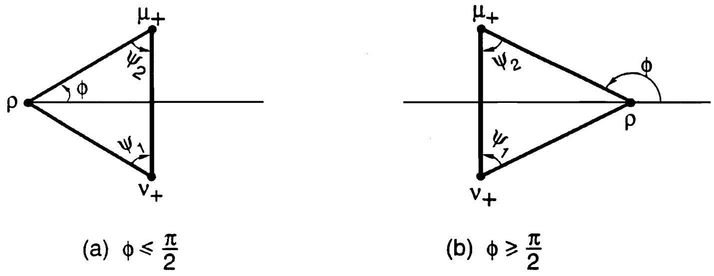

FIG. 4. *The angles* ψ_1, ψ_2 *and* ϕ.

THEOREM 5.2. *For* $x \in \mathrm{I}^\circ$,

$$G_+(x) - G_-(x) = \frac{-i\exp((\phi - \pi/2)\cot\theta)}{|\rho - \nu_+(x)|} \times \int_0^1 \frac{t^{-1/2-(i/2)\cot\theta}(1-t)^{-1/2+(i/2)\cot\theta}}{1 - \nu_+ - (\mu_+ - \nu_+)t}\,dt. \tag{5.13}$$

PROOF. For $\zeta \in [\nu_+ \mu_+]$, $\arg((\zeta - \nu_+)/(\rho - \nu_+))$ and $\arg((\zeta - \mu_+)/(\rho - \mu_+))$ are constant in ζ and given respectively by

$$\psi_1 := \arg\frac{\mu_+ - \nu_+}{\rho - \nu_+}, \qquad \psi_2 := \arg\frac{\nu_+ - \mu_+}{\rho - \mu_+}. \tag{5.14}$$

To relate ψ_1 and ψ_2 to ϕ, consider separately the cases $0 \le \phi \le \pi/2$ and $\pi/2 \le \phi \le \pi$ depicted in Figure 4(a) and 4(b). If $\phi \le \pi/2$, then Figure 4(a) shows that $\psi_2 = -\psi_1 \ge 0$, and that ψ_2 is complementary to ϕ. Hence

$$\psi_1 = -\psi_2 = \phi - \frac{\pi}{2}. \tag{5.15}$$

If $\phi \ge \pi/2$, then Figure 4(b) shows that $\psi_1 = -\psi_2 \ge 0$ and that ψ_1 is complementary to $\pi - \phi$, leading again to (5.15).

Equations (1.7), (5.1), (5.3) and (5.15) give

$$K(\nu_+(x), \zeta) = \left|\frac{\zeta - \nu_+(x)}{\rho - \nu_+(x)}\right|^{-1/2-(i/2)\cot\theta} \times \exp\left\{\frac{1}{2}(\cot\theta - i)\left(\phi - \frac{\pi}{2}\right)\right\} \tag{5.16a}$$

$$K(\mu_+(x), \zeta) = \left|\frac{\zeta - \mu_+(x)}{\rho - \mu_+(x)}\right|^{-1/2+(i/2)\cot\theta} \times \exp\left\{\frac{1}{2}(\cot\theta + i)\left(\phi - \frac{\pi}{2}\right)\right\}. \tag{5.16b}$$

Equations (5.4) and (5.16), together with $|\rho-\mu^+|=|\rho-\nu^+|$, give

$$G_+(x)-G_-(x) = \frac{\exp\left(\left(\phi-\frac{\pi}{2}\right)\cot\theta\right)}{|\rho-\nu_+(x)|} \times \int_{\mu_+}^{\nu_+} \frac{|\zeta-\nu_+|^{-1/2-(i/2)\cot\theta}|\zeta-\mu_+|^{-1/2+(i/2)\cot\theta}}{1-\zeta}\,d\zeta. \tag{5.17}$$

From (5.10) we get

$$|\zeta-\nu_+|=|\mu_+-\nu_+|t, \qquad |\zeta-\mu_+|=|\mu_+-\nu_+|(1-t). \tag{5.18}$$

Theorem 5.2 follows from (5.17), (5.18) and the variable change (5.10). □

5.2. $[G_+(x)-G_-(x)]$ *in closed form.* Theorem 5.3 evaluates $[G_+-G_-]$ in closed form. We obtain it from Theorem 5.2 via a series of lemmas.

LEMMA 5.2. *Let* $-1<\operatorname{Re}\alpha<0$ *and* $z\in\mathscr{R}_1=$ *z-plane minus* $[1,\infty)$. *Then*

$$\int_0^1 \frac{t^\alpha(1-t)^{-(1+\alpha)}}{1-zt}\,dt=(1-z)^{-(\alpha+1)}\frac{\pi}{\sin\pi(1+\alpha)}. \tag{5.19}$$

PROOF. Let $I_\alpha(z)$ be the integral in (5.19). For fixed α, $I_\alpha(z)$ is analytic in $\mathscr{R}_1$. The right-hand side of (5.19) is also analytic in $\mathscr{R}_1$. Hence it suffices to prove Lemma 5.2 for $0<z<1$, (5.19) following for all other z by analytic continuation.

Rewrite $I_\alpha(z)$ as

$$I_\alpha(z)=\int_0^1\left(\frac{t}{1-t}\right)^\alpha\frac{1}{1-t}\,\frac{1}{(1-zt)}\,dt. \tag{5.20}$$

Letting $u=(1-z)(t/(1-t))$, (5.20) converts to

$$I_\alpha(z)=(1-z)^{-(\alpha+1)}\int_0^\infty\frac{u^\alpha}{1+u}\,du=(1-z)^{-(\alpha+1)}\frac{\pi}{\sin\pi(1+\alpha)}. \tag{5.21}$$

For the evaluation of $\int_0^\infty(u^\alpha/(1+u))\,du$, see [6], page 105. □

Now let

$$\alpha=-\frac{1}{2}-\frac{i}{2}\cot\theta,\qquad z=\frac{\mu_+-\nu_+}{1-\nu_+}.$$

Let $\mathscr{T}$ be the integral on the right-hand side of (5.13). We obtain from Lemma 5.2,

$$\mathscr{T}=\frac{\pi}{\sin((\pi/2)-(i\pi/2)\cot\theta)}\,\frac{1}{1-\nu_+}\left(\frac{1-\mu_+}{1-\nu_+}\right)^{-1/2+(i/2)\cot\theta} \tag{5.22}$$

But

$$\sin\left(\frac{\pi}{2} - \frac{i\pi}{2}\cot\theta\right) = \cos\left(\frac{i\pi}{2}\cot\theta\right) = \frac{\exp(\pi\cot\theta) + 1}{2\exp\left(\frac{\pi}{2}\cot\theta\right)}. \tag{5.23}$$

From (5.22) and (5.23), we obtain the following.

LEMMA 5.3. *The quantity* $\mathscr{T}$ *is given by*

$$\mathscr{T} = \frac{2\pi\exp\left(\frac{\pi}{2}\cot\theta\right)}{\exp(\pi\cot\theta) + 1}\frac{1}{1-\nu_+}\left(\frac{1-\mu_+}{1-\nu_+}\right)^{-1/2+(i/2)\cot\theta} \tag{5.24}$$

We simplify (5.24).

LEMMA 5.4. *The quantities* μ_+ *and* ν_+ *satisfy the identity*

$$\frac{1}{1-\nu_+}\left(\frac{1-\mu_+}{1-\nu_+}\right)^{-1/2+(i/2)\cot\theta} = \frac{\rho^{1/2}}{|\rho-\nu_+|}\exp((\phi-\theta)\cot\theta). \tag{5.25}$$

PROOF. Let $\psi = \arg(1-\nu_+)$. In Figure 5, ψ is depicted as an angle with vertex at ν^+.

We have

$$1-\nu_+ = |1-\nu_+|e^{i\psi}, \qquad |1-\mu_+| = |1-\nu_+|e^{-i\psi} \tag{5.26}$$

so that

$$\frac{1-\mu_+}{1-\nu_+} = e^{-2i\psi}, \qquad |-2\psi| < \pi. \tag{5.27}$$

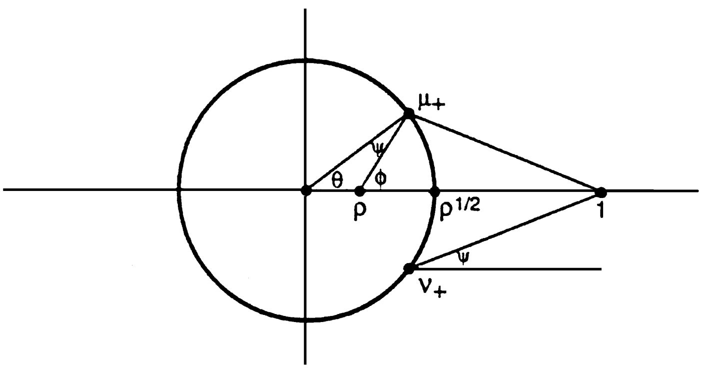

FIG. 5. *The angles* ψ, ϕ *and* θ.

Equation (5.27) gives

$$(5.28)\qquad \left(\frac{1-\mu_+}{1-\nu_+}\right)^{-1/2+(i/2)\cot\theta} = \exp(\psi\cot\theta + \psi i).$$

Equations (5.26) and (5.28) give

$$(5.29)\qquad \frac{1}{1-\nu_+}\left(\frac{1-\mu_+}{1-\nu_+}\right)^{-1/2+(i/2)\cot\theta} = \frac{1}{|1-\nu_+|}\exp(\psi\cot\theta).$$

From $\nu_+ = \rho/\mu_+$, we obtain

$$(5.30)\qquad |1-\nu_+| = \left|\frac{\mu_+ - \rho}{\mu_+}\right| = \frac{|\nu_+ - \rho|}{\rho^{1/2}}$$

and

$$(5.31)\qquad \psi = \arg(1-\nu_+) = \arg\frac{\rho-\mu_+}{0-\mu_+}.$$

Equation (5.31) implies that ψ is also an angle with vertex at μ_+ (see Figure 5). Thus θ, ψ, $\pi-\phi$ are the three internal angles of the triangle with vertices 0, μ_+, ρ. As these sum to π, we have

$$(5.32)\qquad \psi = \phi - \theta.$$

Lemma 5.4 follows from (5.29), (5.30) and (5.32). □

Combining Theorem 5.2 and Lemmas 5.3 and 5.4, we obtain:

THEOREM 5.3. *For* $x \in \mathrm{I}^\circ$,

$$(5.33)\qquad G_+(x) - G_-(x) = \frac{-2\pi i\rho^{1/2}}{|\rho-\nu_+(x)|^2}\,\frac{\exp((2\phi-\theta)\cot\theta)}{\exp(\pi\cot\theta)+1}.$$

PROOF OF THEOREM 1.1. From Theorems 2.1, 4.3 and 5.3, we get

$$(5.34)\qquad \begin{aligned} P(w>t) &= -\rho^{3/2}(1-\rho) \\ &\quad\times\int_{-[\rho^{-1/2}+1]^2}^{-[\rho^{-1/2}-1]^2}\frac{\exp((2\phi-\theta)\cot\theta)}{\exp(\pi\cot\theta)+1}\,\frac{\exp(xt)}{|\rho-\nu_+(x)|^2 x}\,dx.\end{aligned}$$

The map

$$(5.35)\qquad x = -\left[1-2\rho^{-1/2}\cos\theta+\rho^{-1}\right] = \frac{-1}{\rho^2}|\rho-\nu_+|^2$$

takes $[0,\pi]$ to I. Theorem 1.1 follows from (5.34) by performing the variable change (5.35). □

6. Proof of Theorem 1.2. We prove Theorem 1.2 giving the asymptotic behavior of $P(w>t)$ as $t\to\infty$. The method used is known as Laplace's method (see [2], page 249). The basic idea is to show that, for large t, the behavior of the integral in (1.1) depends only on the behavior of the integrand near $\theta = 0$. We first make the variable change $\theta = \theta(y)$, $0 \le y \le 4\rho^{-1/2}$, where $\theta(y)$ is inverse to

$$y = 2\rho^{-1/2}(1 - \cos\theta), \qquad 0 \le \theta \le \pi. \tag{6.1}$$

Then (1.1) converts to

$$P(w > t) = \exp\Big(-\big(\rho^{-1/2} - 1\big)^2 t\Big)\int_0^{4\rho^{-1/2}} h(\theta(y))\exp(-ty)\,dy, \tag{6.2}$$

where

$$h(\theta) = \frac{(1-\rho)\rho^{-1/2}}{\left[1 - 2\rho^{-1/2}\cos\theta + \rho^{-1}\right]^2}\,\frac{\exp((2\phi - \theta)\cot\theta)}{\exp(\pi\cot\theta) + 1}. \tag{6.3}$$

Lemma 6.1. *If* $\theta \to 0$, *then*

$$h(\theta) \sim c_0 \exp(-\pi/\theta), \tag{6.4}$$

where

$$c_0 = \frac{\rho^{3/2}(1 + \rho^{1/2})}{(1 - \rho^{1/2})^3}\exp\left(\frac{1 + \rho^{1/2}}{1 - \rho^{1/2}}\right). \tag{6.5}$$

Proof. We have

$$\cot\theta = \frac{1}{\theta} - \frac{\theta}{3} + \cdots, \qquad \theta \text{ small}.$$

From Figure 5 we get

$$\phi = \arctan\left(\frac{\sin\theta}{\cos\theta - \rho^{1/2}}\right), \qquad 0 \le \phi \le \pi,$$

which implies

$$\phi \sim \frac{\theta}{1 - \rho^{1/2}} \quad \text{as } \theta \to 0.$$

Hence, as $\theta \to 0$,

$$\begin{aligned} \frac{\exp((2\phi - \theta)\cot\theta)}{\exp(\pi\cot\theta) + 1} &\sim \exp\left(\left(2\frac{\phi}{\theta} - 1\right)\theta\cot\theta - \pi\cot\theta\right) \\ &\sim \exp\left(\frac{1 + \rho^{1/2}}{1 - \rho^{1/2}}\right)\exp(-\pi/\theta). \end{aligned} \tag{6.6}$$

Lemma 6.1 follows from (6.3) and (6.6). □

Lemma 6.2. *If* $y \to 0$, *then*

$$h(\theta(y)) \sim c_0 \exp(-\pi\rho^{-1/4}y^{-1/2}). \tag{6.7}$$

PROOF. Expanding $\cos\theta$ into powers of θ^2 and inverting (6.1), we obtain

$$\theta = \rho^{1/4} y^{1/2}\left[1 + \sum_{n=1}^{\infty} a_n y^n\right], \quad \text{for } y \text{ small} \tag{6.8}$$

for certain coefficients $a_1, a_2, \dots$.

Lemma 6.2 follows from (6.4) and (6.8).

Let

$$k(y) = \frac{h(\theta(y))}{c_0}\exp(\pi\rho^{-1/4}y^{-1/2}), \qquad 0 < y \le 4\rho^{-1/2}. \tag{6.9}$$

From (6.7) we get

$$k(0) := \lim_{y\to 0} k(y) = 1. \tag{6.10}$$

Equations (6.2) and (6.9) give

$$\begin{aligned} P(w > t) &= c_0 \exp\left(-(\rho^{-1/2} - 1)^2 t\right) \\ &\quad\times \int_0^{4\rho^{-1/2}} \exp\left(-\left[\pi\rho^{-1/4}y^{-1/2} + ty\right]\right)k(y)\,dy. \end{aligned} \tag{6.11}$$

Let

$$J(t) := \int_0^a \exp\left(-(c_1 y^{-1/2} + ty)\right)dy \quad \text{where } a, c_1 > 0. \tag{6.12}$$

To prove Theorem 1.2, we obtain the asymptotic behavior of $J(t)$ as $t \to \infty$ (it does not depend on a), and show that the asymptotic behavior of the integral of (6.11) is unaffected when $k(y)$ is replaced by 1.

LEMMA 6.3. *For* $t > 0$,

$$J(t) \sim \left(\frac{4\pi}{3}\right)^{1/2}\left(\frac{c_1}{2}\right)^{1/3} t^{-5/6}\exp\left(-3\left(\frac{c_1}{2}\right)^{2/3} t^{1/3}\right). \tag{6.13}$$

PROOF. The change of variable $v = t^{2/3}y$ converts (6.12) to

$$J(t) = t^{-2/3}\int_0^{at^{2/3}} \exp\left(-t^{1/3}g(v)\right)dv, \tag{6.14}$$

where

$$g(v) = c_1 v^{-1/2} + v, \qquad 0 < v < \infty. \tag{6.15}$$

Differentiation gives

$$g'(v) = -\tfrac{1}{2}c_1 v^{-3/2} + 1, \qquad g''(v) = \tfrac{3}{4}c_1 v^{-5/2}. \tag{6.16}$$

Equation (6.16) shows that $g(v)$ is a positive convex function for $0 < v < \infty$, and has an absolute minimum at $v_0 = (c_1/2)^{2/3}$, where $g'(v_0) = 0$. Assume that $v_0 < at^{2/3}$ and split the integral of (6.14) into the three parts:

$$\int_0^{at^{2/3}} = \int_0^{v_0-\delta} + \int_{v_0-\delta}^{v_0+\delta} + \int_{v_0+\delta}^{at^{2/3}}, \tag{6.17}$$

where $0 < \delta < v_0,\ at^{2/3} - v_0$. We estimate these parts, beginning with $\int_{v_0-\delta}^{v_0+\delta}$. The Taylor expansion for $g(v)$ is

$$(6.18)\quad g(v) = g(v_0) + \frac{g''(v_0)}{2}(v - v_0)^2 + O\big(|v - v_0|^3\big) \quad \text{as } v \to v_0.$$

From (6.16) we get

$$(6.19)\qquad g(v_0) = 3\left(\frac{c_1}{2}\right)^{2/3}, \qquad g''(v_0) = \frac{3}{2}\left(\frac{c_1}{2}\right)^{-2/3}.$$

For $0 < \varepsilon < (g''(v_0)/2)$, choose $0 < \delta(\varepsilon) < v_0$, so that (6.18) gives the inequalities

$$(6.20)\qquad \begin{aligned} g(v_0) &+ \left(\frac{g''(v_0)}{2} - \varepsilon\right)(v - v_0)^2 \\ &\le g(v) \le g(v_0) + \left(\frac{g''(v_0)}{2} + \varepsilon\right)(v - v_0)^2 \\ &\qquad\qquad \text{for } |v - v_0| < \delta(\varepsilon). \end{aligned}$$

The variable change $z = t^{1/6}(v - v_0)$ and (6.20) give the estimates

$$(6.21)\qquad \begin{aligned} t^{-1/6} &\exp\big(-g(v_0)t^{1/3}\big)\int_{-\delta t^{1/6}}^{\delta t^{1/6}} \exp\left(-\left[\frac{g''(v_0)}{2} + \varepsilon\right]z^2\right) dz \\ &\le \int_{v_0-\delta}^{v_0+\delta} \\ &\le t^{-1/6}\exp\big(-g(v_0)t^{1/3}\big)\int_{-\delta t^{1/6}}^{\delta t^{1/6}} \exp\left(-\left[\frac{g''(v_0)}{2} - \varepsilon\right]z^2\right) dz. \end{aligned}$$

Next, we estimate $\int_0^{v_0-\delta}$ and $\int_{v_0+\delta}^{at^{2/3}}$. Since $g(v)$ is decreasing over $[0, v_0 - \delta]$,

$$(6.22)\qquad 0 \le \int_0^{v_0-\delta} \le v_0 \exp\big(-g(v_0 - \delta)t^{1/3}\big).$$

As $g'(v)$ is positive and increasing over $[v_0, \infty)$, the variable change $w = g(v)$ gives

$$(6.23)\qquad \begin{aligned} 0 \le \int_{v_0+\delta}^{at^{2/3}} &\le \int_{g(v_0+\delta)}^{\infty} \frac{\exp(-t^{1/3}w)}{g'(v)}\, dw \\ &\le \frac{t^{-1/3}}{g'(v_0 + \delta)}\exp\big(-g(v_0 + \delta)t^{1/3}\big). \end{aligned}$$

Let

$$S(t) = \frac{J(t)}{\sqrt{2\pi/(g''(v_0))}\, t^{-5/6}\exp\big(-g(v_0)t^{1/3}\big)}.$$

From (6.21), (6.22) and (6.23) and the inequalities

$$g(v_0) < g(v_0 \pm \delta),$$

we obtain

$$\sqrt{\frac{g''(v_0)}{g''(v_0) + 2\varepsilon}} \le \liminf_{t\to\infty} S(t) \le \limsup_{t\to\infty} S(t) \le \sqrt{\frac{g''(v_0)}{g''(v_0) - 2\varepsilon}}. \tag{6.24}$$

Letting $\varepsilon \to 0$ in (6.24), we conclude that $\lim_{t\to\infty} S(t) = 1$; that is,

$$J(t) \sim \sqrt{\frac{2\pi}{g''(v_0)}}\, t^{-5/6} \exp\bigl(-g(v_0)t^{1/3}\bigr) \quad \text{as } t \to \infty. \tag{6.25}$$

Equation (6.25) becomes (6.13) after inserting (6.19). □

PROOF OF THEOREM 1.2. For given $\varepsilon > 0$, choose $0 < a(\varepsilon) < 4\pi^{-1/2}$ so that

$$1 - \varepsilon < k(y) < 1 + \varepsilon \quad \text{for } 0 \le y \le a(\varepsilon). \tag{6.26}$$

Split the integral of (6.11) into

$$\int_0^{4\pi^{-1/2}} = \int_0^{a(\varepsilon)} + \int_{a(\varepsilon)}^{4\rho^{-1/2}}. \tag{6.27}$$

We estimate the integrals on the right-hand side of (6.27). Let $M = \max_{0 \le y \le 4\rho^{-1/2}} k(y)$. Then

$$\int_{a(\varepsilon)}^{4\rho^{-1/2}} \le M \int_{a(\varepsilon)}^{\infty} e^{-ty}\, dy = \frac{Me^{-a(\varepsilon)t}}{t}. \tag{6.28}$$

Let

$$R(t) = \frac{\int_0^{4\rho^{-1/2}} \exp\bigl(-\bigl(c_1 y^{-1/2} + ty\bigr)k(y)\bigr)\, dy}{(4\pi/3)^{1/2}(c_1/2)^{1/3} t^{-5/6} \exp\bigl(-3(c_1/2)^{2/3} t^{1/3}\bigr)},$$

where $c_1 = \pi\rho^{-1/4}$. From (6.13), (6.26) and (6.28), we obtain

$$1 - \varepsilon \le \liminf_{t\to\infty} R(t) \le \limsup_{t\to\infty} R(t) \le 1 + \varepsilon. \tag{6.29}$$

Letting $\varepsilon \to 0$, we get

$$\lim_{t\to\infty} R(t) = 1. \tag{6.30}$$

Theorem 1.2 follows from (6.5), (6.11) and (6.30). □

7. Proof of Theorem 1.3. We prove Theorem 1.3, which compares the distribution of the waiting time w of a stable $M/M/1$ queue, in the stationary state, for FIFO and ROS. For these disciplines, denote w, respectively, by w_1 and w_2 and let

$$g_i(t) = P(w_i > t) \quad \text{for } t \geq 0,\ i = 1, 2.$$

LEMMA 7.1. *At* $t = 0$, *we have*

$$g_2'(0) < g_1'(0).$$

PROOF. The variable change

$$x = 1 - 2\rho^{-1/2}\cos\theta + \rho^{-1}$$

converts (1.1) to

$$g_2(t) = \int_c^d H(x) e^{-xt}\,dx, \qquad t \geq 0, \tag{7.1}$$

where

$$c = (\rho^{-1/2} - 1)^2, \qquad d = (\rho^{-1/2} + 1)^2$$

and $H(x)$ is continuous on $[c, d]$, positive on (c, d), and 0 at c and d. From (7.1), we conclude that $g_2(t)$ is infinitely differentiable for $t \geq 0$. From (1.4) and (7.1) we obtain

$$g_1'(0) = -(1 - \rho), \tag{7.2a}$$

$$g_2'(0) = -\int_c^d xH(x)\,dx. \tag{7.2b}$$

Let $E_i = E(w_i)$, $i = 1, 2$. From (1.4) and (7.1),

$$E_1 = \int_0^\infty g_1(t)\,dt = \frac{\rho^2}{1 - \rho}, \tag{7.3a}$$

$$E_2 = \int_0^\infty g_2(t)\,dt = \int_c^d H(x)\left[\int_0^\infty e^{-xt}\,dt\right] dx = \int_c^d \frac{H(x)}{x}\,dx. \tag{7.3b}$$

The expected length of the queue is the same for FIFO and ROS. Hence, by Little's law [1],

$$E_1 = E_2. \tag{7.4}$$

Equations (7.3) and (7.4) give

$$\int_c^d \frac{H(x)}{x}\,dx = \frac{\rho^2}{1 - \rho}. \tag{7.5}$$

The probability that the queue is empty in the stationary state is $g_2(0)$, which equals ρ. Hence we obtain from (7.1)

$$\int_c^d H(x)\,dx = g_2(0) = \rho. \tag{7.6}$$

Equations (7.5), (7.6) and the Schwarz inequality yield

$$(7.7)\quad \rho^2 = \left[\int_c^d x^{1/2}x^{-1/2}H(x)\,dx\right]^2 < \left[\int_c^d xH(x)\,dx\right]\left[\int_c^d \frac{H(x)}{x}\,dx\right] = \left[\int_c^d xH(x)\,dx\right]\left[\frac{\rho^2}{1-\rho}\right]$$

so that

$$(7.8)\qquad \int_c^d xH(x)\,dx > 1-\rho.$$

Lemma 7.1 follows from (7.2) and (7.8). □

PROOF OF THEOREM 7.1. Let $r(t) = g_2(t)/g_1(t)$. From (4.1) and (7.1),

$$(7.9)\qquad r(t) = \frac{1}{\rho}\int_c^d H(x)\exp\bigl(-(x-\rho^{-1}+1)t\bigr)\,dx.$$

Differentiating, we get

$$(7.10)\quad r''(t) = \frac{1}{\rho}\int_c^d (x-\rho^{-1}+1)^2 H(x)\exp\bigl(-(x-\rho^{-1}+1)t\bigr)\,dt.$$

Equation (7.10) shows that $r''(t) > 0$ for $t \ge 0$. Hence $r(t)$ is a convex positive function for $t \ge 0$. We have $g_1(0) = g_2(0) = \rho$, so we conclude from Lemma 7.1 that $g_2(t) < g_1(t)$ or $r(t) < 1$ for small $t > 0$. On the other hand, we conclude from (1.3) and (1.4) that $r(t) > 1$ for large $t > 0$. Theorem 1.3 then follows from the convexity of $r(t)$ and the intermediate value property of continuous functions. □

APPENDIX

We prove Theorem 4.1, stated in Section 4. We treat separately the cases $s \to \infty$ and $s \to -(\rho^{-1/2} \pm 1)^2$.

THEOREM 4.1a. *For $s \to \infty$ we have*

$$(A.1)\qquad G(s) = O\left(\frac{1}{|s|}\right).$$

PROOF. From (3.1) we obtain, for $s \to \infty$,

$$(A.2a)\qquad \mu(s) = 1 + \rho + \rho s + O\left(\frac{1}{|s|}\right),$$

$$(A.2b)\qquad \nu(s) = O\left(\frac{1}{|s|}\right).$$

We estimate the factors in (3.8) for $F(s,t)$. From (3.5), (3.6) and (A.2), we obtain

$$\lim_{s\to\infty}\zeta(s,t)=\rho t,\qquad \lim_{s\to\infty}\xi(s,t)^{-\mu(s)/(\mu(s)-\nu(s))}=1 \tag{A.3}$$

uniformly for $0\le t\le 1$.

Also, by Lemma 3.2,

$$|t^{\nu/(\mu-\nu)}|=t^{\mathrm{Re}(\nu/(\mu-\nu))}\le t^{-1/2}\quad \text{for } (s,t)\in\mathscr{R}\times(0,1]. \tag{A.4}$$

From (A.2)–(A.4) and (3.8), we get

$$|F(s,t)|=O\left(\frac{t^{-1/2}}{|s|}\right)\quad \text{as } s\to\infty, \text{ uniformly for } 0\le t\le 1. \tag{A.5}$$

Equation (A.5) gives

$$G(s)=\int_0^1 F(s,t)\,dt=O\left(\frac{1}{|s|}\right)\quad \text{as } s\to\infty.$$

□

THEOREM 4.1b. *For* $s\to-(\rho^{-1/2}\pm 1)^2$, *we have*

$$G(s)=O\left[\log\frac{1}{\left|s+(\rho^{-1/2}\pm 1)^2\right|}\right]. \tag{A.6}$$

Theorem 4.1b will be derived from the following estimate for $F(s,t)$.

THEOREM A.1. *For* $(s,t)\in\mathscr{R}\times[0,1]$ *and* s *sufficiently close to* $-(\rho^{-1/2}\pm 1)^2$,

$$|F(s,t)|\le\frac{t^{-1/2}|\xi|^{-1/2}}{\rho^{1/2}(1-\rho^{1/2})^2}. \tag{A.7}$$

To prove (A.7), we rewrite (3.8) as

$$F(s,t)=\frac{1}{\mu-\rho}\,\frac{1}{1-\zeta}\,\frac{1}{\xi}\left(\frac{t}{\xi}\right)^{\nu/(\mu-\nu)} \tag{A.8}$$

and estimate each factor in (A.8). Only the last factor proves difficult, and we establish some preliminary lemmas which give an upper bound for it.

Divide $\mathscr{R}$ into four quarters by means of the real axis and the line $\mathrm{Re}\, s=-(\rho^{-1}+1)$, which is the perpendicular bisector of the slit I. We label these quarters in the counterclockwise manner by I–IV, with I labelling the northeast quarter.

LEMMA A.1.

(i) $$\mathrm{Im}\frac{\nu}{\mu-\nu}\ge 0\quad \textit{if } s\in \mathrm{II}\cup\mathrm{IV},$$

(ii) $$\mathrm{Im}\frac{\nu}{\mu-\nu}\le 0\quad \textit{if } s\in \mathrm{I}\cup\mathrm{III}.$$

PROOF. (i) $\mu = \rho/\nu$ gives $\nu/(\mu - \nu) = \nu^2/(\rho - \nu^2)$, which we rewrite as

$$\frac{\nu}{\mu - \nu} = \eta \circ \xi \circ z, \tag{A.9}$$

where $z = \nu(s)$, $\xi = z^2$, $\eta = \xi/(\rho - \xi)$. Respectively, z maps II, IV to the portions of $|z| < \rho^{1/2}$ contained in the third and first quadrant of the z-plane. The two portions are mapped by ξ to the upper half of the disk $|\xi| < \rho$ and η maps the latter to the upper half of the half-plane $\operatorname{Re}\eta > -\frac{1}{2}$. We conclude from (A.9) that $\operatorname{Im}(\nu/(\mu - \nu)) \geq 0$ if $s \in \mathrm{II} \cup \mathrm{IV}$.

The proof of (ii) proceeds in a similar manner. □

LEMMA A.2. *For $s \in \mathscr{R}$ and sufficiently close to $-(\rho^{-1/2} \pm 1)^2$, we have*

$$\text{(i)} \qquad \operatorname{Im}\frac{\mu - \nu}{\mu - \rho} \geq 0 \quad \text{if } s \in \mathrm{I} \cup \mathrm{III},$$

$$\text{(ii)} \qquad \operatorname{Im}\frac{\mu - \nu}{\mu - \rho} \leq 0 \quad \text{if } s \in \mathrm{II} \cup \mathrm{IV}.$$

PROOF. (i) From $\mu = \rho/\nu$, we get

$$\frac{\mu - \nu}{\mu - \rho} = \frac{\rho - \nu^2}{\rho(1 - \nu)} = \frac{|\nu|^2\nu - \rho\bar{\nu} + \rho - \nu^2}{\rho|1 - \nu|^2}. \tag{A.10}$$

Equating imaginary parts, we get

$$\begin{aligned} \operatorname{Im}\frac{\mu - \nu}{\mu - \rho} &= \frac{\left(|\nu|^2 + \rho\right)\operatorname{Im}\nu - \operatorname{Im}\nu^2}{\rho|1 - \nu|^2} \\ &= \frac{\left(|\nu|^2 + \rho - 2\operatorname{Re}\nu\right)\operatorname{Im}\nu}{\rho|1 - \nu|^2}. \end{aligned} \tag{A.11}$$

Rewrite (A.11) as

$$\operatorname{Im}\frac{\mu - \nu}{\mu - \rho} = \frac{\left[|1 - \nu|^2 - (1 - \rho)\right]\operatorname{Im}\nu}{\rho|1 - \nu|^2}. \tag{A.12}$$

Let $s \in \mathrm{I}$ and s close to $-(\rho^{-1/2} - 1)^2$. Then $\operatorname{Im}\nu(s) \leq 0$, and $\nu(s)$ is close to $\rho^{1/2}$. As $|1 - \rho^{1/2}|^2 - (1 - \rho) = 2(\rho - \rho^{1/2}) < 0$, we obtain $|1 - \nu|^2 - (1 - \rho) < 0$, and conclude from (A.12) that $\operatorname{Im}((\mu - \nu)/(\mu - \rho)) \geq 0$.

Let $s \in \mathrm{III}$ and s close to $-(\rho^{-1/2} + 1)^2$. Then $\operatorname{Im}\nu(s) \geq 0$ and $\nu(s)$ is close to $-\rho^{1/2}$. As $|1 + \rho^{1/2}|^2 - (1 - \rho) = 2(\rho + \rho^{1/2}) > 0$, we obtain $|1 - \nu|^2 - (1 - \rho) > 0$, and conclude from (A.12) that $\operatorname{Im}((\mu - \nu)/(\mu - \rho)) \geq 0$, thus proving (i).

The proof of (ii) proceeds in a similar manner. □

LEMMA A.3. *If $s \in \mathscr{R}$, then*

$$\operatorname{Re}\frac{\mu-\nu}{\mu-\rho} > 0.$$

PROOF. Equating real parts in (A.10), we get

$$\operatorname{Re}\frac{\mu-\nu}{\mu-\rho} = \frac{1}{\rho|1-\nu|^2}\left[\rho - \operatorname{Re}\nu^2 - \left(\rho - |\nu|^2\right)\operatorname{Re}\nu\right]. \tag{A.13}$$

Since $\operatorname{Re}\nu^2 \le |\nu|^2$, (A.13) gives

$$\operatorname{Re}\frac{\mu-\nu}{\mu-\rho} \ge \frac{1}{\rho|1-\nu|^2}\left(\rho - |\nu|^2\right)(1-\operatorname{Re}\nu). \tag{A.14}$$

By Lemma 3.1, $|\nu| < \rho^{1/2}$ for $s \in \mathscr{R}$, hence Lemma A.3 follows from A.14). □

We now give the following proof.

PROOF OF THEOREM A.1. We upper bound the factors in (A.8). Use

$$|\zeta(s,t)| < \rho^{1/2} < |\mu(s)|$$

to obtain the bounds

$$\left|\frac{1}{\mu-\rho}\right| < \frac{1}{\rho^{1/2}-\rho}, \qquad \left|\frac{1}{1-\zeta}\right| < \frac{1}{1-\rho^{1/2}}. \tag{A.15}$$

To upper bound $(t/\xi)^{\nu/(\mu-\nu)}$ use

$$\begin{aligned}\left|\left(\frac{t}{\xi}\right)^{\nu/(\mu-\nu)}\right| &= \left|\frac{t}{\xi}\right|^{\operatorname{Re}(\nu/(\mu-\nu))}\exp\left(-\operatorname{Im}\frac{\nu}{\mu-\nu}\arg\left(\frac{t}{\xi}\right)\right)\\ &= \left|\frac{t}{\xi}\right|^{\operatorname{Re}(\nu(\mu-\nu))}\exp\left(\operatorname{Im}\frac{\nu}{\mu-\nu}\arg\xi\right)\end{aligned} \tag{A.16}$$

and rewrite (3.6) as

$$\xi = t + (1-t)\frac{\mu-\nu}{\mu-\rho}. \tag{A.17}$$

By Lemmas A.1 and A.2, we get

$$\operatorname{Im}\frac{\nu}{\mu-\nu}\operatorname{Im}\xi = (1-t)\operatorname{Im}\frac{\nu}{\mu-\nu}\operatorname{Im}\frac{\mu-\nu}{\mu-\rho} \le 0 \quad \text{for } s \text{ sufficiently close to } -\left(\rho^{-1/2}\pm 1\right)^2. \tag{A.18}$$

Since $(\operatorname{Im}\xi)(\arg\xi) \ge 0$, we obtain from (A.18),

$$\operatorname{Im}\frac{\nu}{\mu-\nu}\arg\xi \le 0 \quad \text{for } s \text{ sufficiently close to } -\left(\rho^{-1/2}\pm 1\right)^2. \tag{A.19}$$

By Lemma (A.3) and (A.17), we get

$$|\xi| \geq \operatorname{Re} \xi \geq t \tag{A.20}$$

so that (A.16), (A.19), (A.20) and Lemma 3.2 give

$$\left|\left(\frac{t}{\xi}\right)^{\nu/(\mu-\nu)}\right| \leq \left|\frac{t}{\xi}\right|^{-1/2} \quad \text{for } s \text{ sufficiently close to } -(\rho^{-1/2} \pm 1)^2. \tag{A.21}$$

Theorem A.1 then follows from (A.8), (A.15) and (A.21). □

PROOF OF THEOREM 4.1b. Let $z = (\nu - \mu)/(\rho - \mu)$. By Lemma A.3, $\operatorname{Re} z > 0$, and from (3.1),

$$z = O\left(\left|s + (\rho^{-1/2} \pm 1)^2\right|^{1/2}\right) \quad \text{as } s \to -(\rho^{-1/2} \pm 1)^2. \tag{A.22}$$

Let

$$A(z) = \int_0^1 t^{-1/2} |t + (1-t)z|^{-1/2}\, dt.$$

From Theorem A.1 and (3.7),

$$|G(s)| \leq \frac{A(z)}{\rho^{1/2}(1 - \rho^{1/2})^2} \quad \text{for } s \text{ sufficiently close to } -(\rho^{-1/2} \pm 1)^2. \tag{A.23}$$

We upper bound $A(z)$. We have

$$|t + z| \geq \operatorname{Re}(t + z) = t + \operatorname{Re} z > t. \tag{A.24}$$

Hence

$$|t + (1-t)z| = \left|(t+z)\left(1 - \frac{zt}{t+z}\right)\right| \geq |t+z|(1 - |z|). \tag{A.25}$$

Also

$$|t + z|^2 = t^2 + |z|^2 + 2t \operatorname{Re} z \geq t^2 + |z|^2. \tag{A.26}$$

For $|z| < 1$, (A.25) and (A.26) give

$$|t + (1-t)z|^{-1/2} \leq (t^2 + |z|^2)^{-1/4} (1 - |z|)^{-1/2}. \tag{A.27}$$

Hence

$$A(z) \leq (1 - |z|)^{-1/2} \int_0^1 t^{-1/2} (t^2 + |z|^2)^{-1/4}\, dt. \tag{A.28}$$

The variable change $t = |z| r$ converts (A.28) to

$$A(z) \leq (1 - |z|)^{-1/2} \int_0^{1/|z|} \frac{dr}{r^{1/2}(1 + r^2)^{1/4}}. \tag{A.29}$$

Let $I(z)$ be the integral appearing in (A.29). As

$$r^{1/2}(1+r^2)^{1/4} \sim r \quad \text{as } r \to \infty$$

we obtain

$$I(z) \sim \log\frac{1}{|z|} \quad \text{as } z \to 0$$

and we conclude from (A.29) that

$$A(z) = O\left(\log\frac{1}{|z|}\right) \quad \text{as } z \to 0. \tag{A.30}$$

Theorem 4.1b follows from (A.22), (A.23) and (A.30). □

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