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POSITIVE CORRELATIONS IN THE FUZZY POTTS MODEL[1]

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The fuzzy Potts model arises by taking the q-state Potts model, then identifying r of the Potts spins with the fuzzy spin 0, and the remaining $q-r$ Potts spins with the fuzzy spin 1. Here we extend a result of Chayes by showing that the fuzzy Potts model has positive correlations. We also give an application to the percolation-theoretic behavior of the Potts model on $\mathbf{Z}^2$.

1. Introduction. The q-state *Potts model* on a finite graph $G=(V,E)$ is a random assignment of $\{1,\dots,q\}$-valued spins to the vertices of G. The Gibbs measure $\pi^G_{q,\beta}$ for the q-state Potts model on G at inverse temperature $\beta \geq 0$ is the measure on $\{1,\dots,q\}^V$ which to each $\xi \in \{1,\dots,q\}^V$ assigns probability

$$\pi^G_{q,\beta}(\xi) = \frac{1}{Z^G_{q,\beta}} \exp\left(2\beta \sum_{\langle x,y\rangle \in E} I_{\{\xi(x)=\xi(y)\}}\right). \tag{1}$$

Here $\langle x, y\rangle$ denotes the edge connecting $x,\ y\in V$, I_A is the indicator function of the event A, and $Z^G_{q,\beta}$ is a normalizing constant making $\pi^G_{q,\beta}$ a probability measure. This model is much studied in probability theory and statistical mechanics; see, for example, [9], [12], [10] and the references therein. The case $q=1$ is clearly trivial, so we will assume that $q\geq 2$. The case $q=2$ is known as the Ising model.

The $(r+s)$-state *fuzzy Potts model* is obtained by taking the q-state Potts model with $q=r+s$, then identifying r of the Potts spins with a single fuzzy spin denoted 0, and the remaining s Potts spins with another fuzzy spin denoted 1. A more precise description is as follows. Fix $\beta\geq 0$ and integers r, $s\geq 1$, set $q=r+s$ and pick a $\{1,\dots,q\}^V$-valued random object X according to the Gibbs measure $\pi^G_{q,\beta}$. Then take Y to be the $\{0,1\}^V$-valued random object obtained from X by setting

$$Y(x) = \begin{cases} 0, & \text{if } X(x)\in\{1,\dots,r\},\\ 1, & \text{if } X(x)\in\{r+1,\dots,q\}\end{cases} \tag{2}$$

for each $x\in V$. We write $\mu^G_{r,s,\beta}$ for the induced probability measure on $\{0,1\}^V$ and call it the fuzzy Potts measure with parameters r, s and β.

Maes and Vande Velde [14] studied properties of the fuzzy Potts model related to nongibbsianness and renormalization group pathologies (see, e.g.,

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[17] for a survey of that area). The fuzzy Potts model is also one of the simplest and most natural examples of a hidden Markov random field (see [13]) and may serve as a testing ground for such systems. Hidden Markov random fields have become popular, for example, in statistical image analysis, and it is obviously of some importance to investigate what properties are implicitly assumed through a specific model choice.

The main result of this paper (Theorem 1.3 below) is that the fuzzy Potts model exhibits the fundamental property of *positive correlations*, also known as the FKG property. To state the result precisely, we need a few more definitions. For any finite set T, let $\preceq$ denote the usual coordinatewise partial order on $\{0,1\}^T$, that is, for ξ, $\eta \in \{0,1\}^T$ we have $\xi \preceq \eta$ iff $\xi(x) \le \eta(x)$ for all $x \in T$. A fuction f: $\{0,1\}^T \to \mathbf{R}$ is said to be increasing if $f(\xi) \le f(\eta)$ whenever $\xi \preceq \eta$.

DEFINITION 1.1. A probability measure μ on $\{0,1\}^T$ is said to have *positive correlations* if

$$\int fg\,d\mu \ge \int f\,d\mu \int g\,d\mu$$

for all increasing f and g.

Let Y be a $\{0,1\}^T$-valued random object with distribution μ.

DEFINITION 1.2. A probability measure μ on $\{0,1\}^T$ is said to be *monotone* if for all $x \in T$ and all $\xi, \eta \in \{0,1\}^{T\setminus\{x\}}$ such that $\xi \preceq \eta$ we have

$$\mu\big(Y(x)=1 \mid Y(T\setminus\{x\})=\xi\big) \le \mu\big(Y(x)=1 \mid Y(T\setminus\{x\})=\eta\big).$$

It is well known that μ has positive correlations if it is monotone and assigns positive probability to all elements of $\{0,1\}^T$. This is essentially the FKG inequality; see, for example, [10] for a formulation (and a proof) which fits the present situation. We are now ready for the main result.

THEOREM 1.3. *For any finite graph G, any $\beta \ge 0$ and any integers r, $s \ge 1$, we have that the fuzzy Potts measure $\mu^G_{r,s,\beta}$ is monotone. In particular, $\mu^G_{r,s,\beta}$ has positive correlations.*

The Ising case $r = s = 1$ goes back to Fortuin, Kasteleyn and Ginibre [7]. Chayes [2] extended this to $r = 1$ and arbitrary s. Our proof for arbitrary r and s is a refinement of Chayes' method, which involves the use of the Fortuin–Kasteleyn random-cluster representation [6] of the Potts model.

To argue that our result is not a trivial consequence of that of Chayes, we note that there exist q-state models with the property that identifying one of the states with 0 and the remaining $q-1$ states with 1 gives positive correlations, while identifying $r \in \{2,\dots,q-2\}$ of the states with 0 and the others with 1 does not. To see this, consider the probability measure ν on $\Omega = \{\text{North, East, South, West}\}^2$ which is simply uniform distribution over the subset of Ω obtained by disallowing configurations in which North sits next to South or East sits next to West. For a subset S of {North, East, South, West}, let μ_S be the probability measure on $\{0, 1\}^2$ obtained from ν by identifying states in S with 0 and the others with 1. It is easy to check that μ_S has positive correlations if S consists of a single spin, but not if $S =$ {East, West}.

The proof of Theorem 1.3 is given in the next section. In Section 3, we apply the result to obtain some conclusions about the percolation-theoretic behavior of the Potts model on the square lattice $\mathbf{Z}^2$.

2. Proof of main result. A key ingredient in our proof of Theorem 1.3 is the use of the Fortuin–Kasteleyn random-cluster model, which has turned out in the past decade to be an immensely useful tool for studying the Potts model. The random-cluster model involves a random assignment of 0's and 1's to the edges of G. The value 0 (resp., 1) is interpreted as the absence (resp., presence) of an edge. For $p \in [0, 1]$, $q > 0$ and a finite graph $G = (V, E)$, the *random-cluster measure* $\phi^G_{p,q}$ for G with parameters p and q is the probability measure on $\{0, 1\}^E$ which to each $\zeta \in \{0, 1\}^E$ assigns probability

$$\frac{q^{k(\zeta)}}{\hat{Z}^G_{p,q}} \prod_{e \in E} p^{\zeta(e)}(1-p)^{1-\zeta(e)},$$

where $\hat{Z}^G_{p,q}$ is a normalizing constant, and $k(\zeta)$ is the number of connected components (including isolated vertices) in the subgraph of G corresponding to ζ.

The relation between random-cluster and Potts models is best understood in terms of the following coupling P', which was implicit in the work of Swendsen and Wang [15] and made explicit by Edwards and Sokal [5]. Fix $\beta \geq 0$ and an integer $q \geq 2$, and set $p = 1 - e^{-2\beta}$. Let P be the product probability measure on $\{1, \dots, q\}^V \times \{0, 1\}^E$ corresponding to letting each vertex independently pick its spin uniformly from $\{1, \dots, q\}$, and each edge independently take value 0 or 1 with respective probabilities $1 - p$ and p. Let A be the event that each edge $\langle x, y \rangle$ linking two vertices with different spins, takes value 0. Finally, let P' be the probability measure on $\{1, \dots, q\}^V \times \{0, 1\}^E$ which arises by conditioning P on the event A. Then the spin marginal of P' equals $\pi^G_{q,\beta}$, and the edge marginal equals $\phi^G_{p,q}$. This follows from a direct counting argument.

Consequently, a spin configuration $X \in \{1, \dots, q\}^V$ distributed according to the Gibbs measure $\pi^G_{q,\beta}$ can be obtained as follows. First pick an edge configuration $W \in \{0, 1\}^E$ according to the random-cluster measure $\phi^G_{p,q}$, and then obtain X from W by assigning independent spins, uniformly distributed on $\{1, \dots, q\}$, to the connected components of Y.

Warmup. We are now ready for a simple proof that the fuzzy Potts measure $\mu^G_{r,s,\beta}$ has positive correlations for the special case where f and g are the fuzzy spins of two given vertices $x, y \in V$; that is, $f(\eta) = \eta(x)$ and $g(\eta) = \eta(y)$ for all $\eta \in \{0,1\}^V$. By the symmetry between different spin values in the Potts model, we have

$$\int f\, d\mu^G_{r,s,\beta} = \pi^G_{q,\beta}\big(X(x) \in \{r+1,\dots,q\}\big) = \frac{s}{q}$$

and similarly for g, so that

$$\int f\, d\mu^G_{r,s,\beta} \int g\, d\mu^G_{r,s,\beta} = \left(\frac{s}{q}\right)^2.$$

To calculate $\int fg\, d\mu^G_{r,s,\beta}$, we take the configuration of Potts spins to be generated from the random-cluster model according to the procedure described above. Let α be the $\phi^G_{p,q}$-probability that x and y are in the same connected component. Then

$$\begin{aligned}\int fg\, d\mu^G_{r,s,\beta} &= \pi^G_{q,\beta}\big(X(x) \in \{r+1,\dots,q\}, X(y) \in \{r+1,\dots,q\}\big)\\ &= \alpha\frac{s}{q} + (1-\alpha)\left(\frac{s}{q}\right)^2\end{aligned}$$

and since $\alpha(s/q) + (1-\alpha)(s/q)^2 \geq (s/q)^2$ we have the desired correlation inequality. □

We now proceed with the more involved task of proving the full statement of Theorem 1.3. What we need to show is that for $x \in V$ and $\eta, \eta' \in \{0,1\}^{V\setminus\{x\}}$ such that $\eta \preceq \eta'$, we have

$$\begin{aligned}&\mu^G_{r,s,\beta}\big(Y(x) = 1 \mid Y(V\setminus\{x\}) = \eta\big)\\ &\qquad \leq \mu^G_{r,s,\beta}\big(Y(x) = 1 \mid Y(V\setminus\{x\}) = \eta'\big).\end{aligned} \tag{3}$$

By considering a sequence $\eta = \eta_1 \preceq \eta_2 \preceq \cdots \preceq \eta_n = \eta'$ such that, for each $i \in \{1,\dots,n-1\}$, η_i and η_{i+1} differ only at a single vertex, we have reduced the problem to showing that (3) holds for all $\eta \preceq \eta'$ which differ only at a single vertex $y \in V$. This is the same as saying that $Y(x)$ and $Y(y)$ are conditionally positively correlated given $Y(V\setminus\{x,y\})$. Hence *Theorem* 1.3 *follows once the following proposition has been established.*

PROPOSITION 2.1. *For any $x, y \in V$ and any $\eta \in \{0,1\}^{V\setminus\{x,y\}}$, we have*

$$\begin{aligned}&\mu^G_{r,s,\beta}\big(Y(x) = 1, Y(y) = 1 \mid Y(V\setminus\{x,y\}) = \eta\big)\\ &\quad \geq \mu^G_{r,s,\beta}\big(Y(x) = 1 \mid Y(V\setminus\{x,y\}) = \eta\big)\\ &\quad\quad \times \mu^G_{r,s,\beta}\big(Y(y) = 1 \mid Y(V\setminus\{x,y\}) = \eta\big).\end{aligned}$$

PROOF. Fix x, y and η as in the proposition, and set $V_0 = \{z \in V \setminus \{x,y\}: \eta(z) = 0\}$ and $V_1 = \{z \in V\setminus\{x,y\}: \eta(z) = 1\}$. Let B be the event that $X(z) \in \{1,\dots,r\}$ for all $z \in V_0$ and $X(z) \in \{r+1,\dots,q\}$ for all $z \in V_1$. Let P'' be the probability measure on $\{1,\dots,q\}^V \times \{0,1\}^E$ obtained by conditioning

P' on the event B. The spin marginal of P'' (i.e., the projection of P'' on $\{1,\dots,q\}^V$) is then equal to the measure π'' obtained by conditioning $\pi^G_{q,\beta}$ on B. Note that the assertion of the proposition is the same as saying that

$$\begin{aligned}\pi''\bigl(X(x)\in\{r+1,\dots,q\},X(y)\in\{r+1,\dots,q\}\bigr)\\ \geq \pi''\bigl(X(x)\in\{r+1,\dots,q\}\bigr)\pi''\bigl(X(y)\in\{r+1,\dots,q\}\bigr),\end{aligned} \tag{4}$$

so this is what we proceed to show. We consider separately the cases when x and y are linked by an edge in G (Case 2) and when they are not (Case 1).

Consider first Case 1. Upon noting that P'' alternatively can be described as P conditioned on the event $(A\cap B)$, we can calculate the projection ϕ'' of P'' on $\{0,1\}^E$ by counting, for each $\zeta\in\{0,1\}^E$, the number of elements of $\{1,\dots,q\}^V$ that ζ can be paired with to produce a spin-edge configuration in $(A\cap B)$. We get that ϕ'', to each $\zeta\in\{0,1\}^E$, assigns probability

$$\phi''(\zeta)=\frac{1}{Z''}r^{k_0(\zeta)}s^{k_1(\zeta)}q^{k_x(\zeta)+k_y(\zeta)}I_D\prod_{e\in E}p^{\zeta(e)}(1-p)^{1-\zeta(e)}. \tag{5}$$

Here Z'' is a normalizing constant, $k_0(\zeta)$ [resp., $k_1(\zeta)$] is the number of connected components intersecting V_0 (resp., V_1), $k_x(\zeta)$ is 1 if x is in a singleton connected component and 0 otherwise, $k_y(\zeta)$ is defined analogously, and D is the event that no connected component in ζ intersects both V_0 and V_1.

The coupling P'' shows that a spin configuration $X\in\{1,\dots,q\}^V$ with distribution π'' can be obtained as follows. First pick an edge configuration $W\in\{0,1\}^E$ according to ϕ'', and then obtain X from W by assigning independent spins to the connected components of W, in such a way that the spin of a connected component $\mathscr{C}$ is taken according to uniform distribution on

$$\begin{aligned}&\{1,\dots,r\}, &&\text{if } \mathscr{C} \text{ intersects } V_0,\\ &\{r+1,\dots,q\}, &&\text{if } \mathscr{C} \text{ intersects } V_1,\\ &\{1,\dots,q\}, &&\text{if } \mathscr{C} \text{ is a singleton vertex } x \text{ or } y.\end{aligned}$$

Next, define the function $f_x\colon\{0,1\}^E\to\mathbf{R}$ as

$$f_x(\zeta)=\begin{cases}0, & \text{if } \mathscr{C}_x \text{ intersects } V_0,\\ \dfrac{s}{q}, & \text{if } \mathscr{C}_x \text{ is a singleton},\\ 1, & \text{otherwise},\end{cases}$$

where $\mathscr{C}_x$ is the connected component of ζ containing x. Define f_y analogously. The significance of f_x and f_y is that $f_x(W)$ is the conditional probability that $X(x)\in\{r+1,\dots,q\}$ given W, and similarly for $f_y(W)$.

We claim that the events $X(x)\in\{r+1,\dots,q\}$ and $X(y)\in\{r+1,\dots,q\}$ are conditionally independent given W. To see this, suppose first that $f_x(W)=f_y(W)=s/r$. Then x and y have to be in different connected components of W, implying the asserted conditional independence. Otherwise at least one of $f_x(W)$ and $f_y(W)$ is equal to 0 or 1, in which case the conditional independence is automatic. The left-hand side of (4) therefore equals $\int f_x f_y \, d\phi''$. Since the right-hand side equals $\int f_x \, d\phi'' \int f_y \, d\phi''$, we are done with Case 1 if we can show that

$$\int f_x f_y \, d\phi'' \geq \int f_x \, d\phi'' \int f_y \, d\phi''. \tag{6}$$

To this end, let us investigate the single-edge conditional probabilities in ϕ''. Partition E into seven sets E_0, E_1, E_{01}, E_{0x}, E_{0y}, E_{1x}, E_{1y} as follows. E_0 (resp., E_1) is the set of edges connecting two vertices in V_0 (resp., V_1). E_{01} is the set of edges with one endpoint in V_0 and the other in V_1. E_{0x} contains those edges which have x as one endpoint and the other in V_0, and E_{0y}, E_{1x}, E_{1y} are defined analogously. Since all edges in E_{01} are absent with π''-probability 1, we view ϕ'' as a probability measure on $\{0,1\}^{E \setminus E_{01}}$ rather than on $\{0,1\}^E$. A direct application of (5) gives the single-edge conditional probabilities,

$$\phi''(W(e) = 1 \mid W(E \setminus \{e\}) = \zeta) = \begin{cases} p, & \text{if there is a path in } \zeta \text{ between the endpoints of } e, \\ \dfrac{p}{p + (1-p)r}, & \text{otherwise,} \end{cases}$$

if $e \in E_0$;

$$\phi''\big(W(e) = 1 \mid W(E \setminus \{e\}) = \zeta\big) = \begin{cases} p, & \text{if there is a path in } \zeta \text{ between the endpoints of } e, \\ \dfrac{p}{p + (1-p)s}, & \text{otherwise,} \end{cases}$$

if $e \in E_1$;

$$\phi''\big(W(e) = 1 \mid W(E \setminus \{e\}) = \zeta\big) = \begin{cases} p, & \text{if there is a path in } \zeta \text{ between the endpoints of } e, \\ \dfrac{p}{p + (1-p)r}, & \text{if no such path exists, but some } e' \in E_{0x} \setminus \{e\} \text{ is present in } \zeta, \\ \dfrac{p}{p + (1-p)q}, & \text{if } x \text{ is a singleton in } \zeta, \\ 0, & \text{if some } e' \in E_{1x} \text{ is present in } \zeta, \end{cases}$$

if $e \in E_{0x}$;

$$
\phi''\big(W(e) = 1 \mid W(E \setminus \{e\}) = \zeta\big)
$$

$$
= \begin{cases} p, & \text{if there is a path in } \zeta \text{ between the endpoints of } e, \\[2ex] \dfrac{p}{p + (1-p)s}, & \text{if no such path exists, but some } e' \in E_{1x} \setminus \{e\} \text{ is present in } \zeta, \\[2ex] \dfrac{p}{p + (1-p)q}, & \text{if } x \text{ is a singleton in } \zeta, \\[2ex] 0, & \text{if some } e' \in E_{0x} \text{ is present in } \zeta, \end{cases}
$$

if $e \in E_{1x}$; and similarly for $e \in E_{0y}$ and $e \in E_{1y}$.

We see that ϕ'' is far from being monotone in the sense of Definition 1.2. However, an inspection of the above single-edge conditional probabilities show that these have the following monotonicity-like property. For $e \in (E_0 \cup E_{0x} \cup E_{0y})$ the conditional probability $\phi''(W(e) = 1 \mid W(E \setminus \{e\}) = \zeta)$ increases as edges in $(E_0 \cup E_{0x} \cup E_{0y})$ are added and edges in $(E_1 \cup E_{1x} \cup E_{1y})$ are deleted, whereas for $e \in (E_1 \cup E_{1x} \cup E_{1y})$, the same conditional probability increases as edges in $(E_1 \cup E_{1x} \cup E_{1y})$ are added and edges in $(E_0 \cup E_{0x} \cup E_{0y})$ are deleted. This means that if we define the auxiliary $\{0,1\}^{E \setminus E_{01}}$-valued random object $\tilde{W}$ by setting

$$
\tilde{W}(e) = \begin{cases} 1 - W(e), & \text{for } e \in E_0 \cup E_{0x} \cup E_{0y}, \\ W(e), & \text{for } e \in E_1 \cup E_{1x} \cup E_{1y}, \end{cases}
$$

and write $\tilde{\phi}''$ for the distribution of $\tilde{W}$, then $\tilde{\phi}''$ is monotone in the sense of Definition 1.2. We wish to apply the FKG inequality to deduce that $\tilde{\phi}''$ has positive correlations. Some care is needed, because $\tilde{\phi}''$ does *not* assign positive probability to all $\zeta \in \{0,1\}^{E \setminus E_{01}}$. On the other hand, $\tilde{\phi}''$ is irreducible in the sense that for any $\zeta, \zeta' \in \{0,1\}^{E \setminus E_{01}}$, we can move from ζ to ζ' via successive single-edge flips without passing through configurations with zero $\tilde{\phi}''$-probability. This property is in fact enough, in conjunction with monotonicity, to be able to apply the FKG inequality (see [10]). Hence $\tilde{\phi}''$ has positive correlations.

To conclude Case 1, note that $f_x(W)$ and $f_y(W)$ are increasing functions of the auxiliary configuration $\tilde{W}$, so that the positive correlations property of $\tilde{\phi}''$ implies (6).

We continue with Case 2, where x and y are linked by an edge $\langle x, y\rangle$ in G. If $\langle x, y\rangle$ is removed from G, then we are back in Case 1, where we have already established (4), which may be rewritten as

$$
\begin{aligned}
(7)\qquad &\sum_{\substack{\xi\in B\\ \xi(x)\in\{r+1,\dots,q\},\,\xi(y)\in\{r+1,\dots,q\}}}\pi^G_{q,\beta}(\xi)\sum_{\substack{\xi\in B\\ \xi(x)\in\{1,\dots,r\},\,\xi(y)\in\{1,\dots,r\}}}\pi^G_{q,\beta}(\xi)\\
&\quad\geq\sum_{\substack{\xi\in B\\ \xi(x)\in\{r+1,\dots,q\},\,\xi(y)\in\{1,\dots,r\}}}\pi^G_{q,\beta}(\xi)\sum_{\substack{\xi\in B\\ \xi(x)\in\{1,\dots,r\},\,\xi(y)\in\{1+r,\dots,q\}}}\pi^G_{q,\beta}(\xi).
\end{aligned}
$$

Reinserting the edge $\langle x, y\rangle$ into G has two effects on the above expressions. First, the normalizing constant in $\mu^G_{r,s,\beta}$ changes, but this has no influence on the inequality due to cancellation. Second, $\mu^G_{r,s,\beta}(\xi)$ is multiplied with $\exp(2\beta)$ for all ξ such that $\xi(x) = \xi(y)$. However, all such ξ appear on the left-hand side of (7), so the inequality remains, and Case 2 is taken care of. □

This concludes the proof of Theorem 1.3. The result (and the proof) goes through in the somewhat greater generality where the interaction in the underlying Potts model may be of different strength at different edges; this amounts to having a positive constant J_{xy} in each term of the sum in (1). The corresponding random-cluster representation then has different values of p for different edges. In fact, Chayes [2] formulated his proof for $r = 1$ in that generality.

Another direction of generalization is the following. The fuzzy Potts model with parameters r, s and β can be obtained directly from the random-cluster model with parameters $p = 1-e^{-2\beta}$ and $q = r+s$ instead of going through the Potts model. To do this, one first picks an edge configuration according to the random-cluster measure $\phi^G_{p,q}$, and then assigns spin 0 or 1 independently to each connected component with respective probabilities r/q and s/q. This procedure does not require r and s to be integers, and by taking them to be positive reals rather than just integers we obtain the *fractional fuzzy Potts model*. Our proof of Theorem 1.3 works also in this setting, as long as $r \geq 1$ and $s \geq 1$. We do not expect the fractional fuzzy Potts model to have any simple description as a hidden Markov random field, but the model may still be of some mathematical interest; some motivation for this is given at the end of the next section.

3. The Potts model on $\mathbf{Z}^2$.

We consider the Potts model on the square lattice $\mathbf{Z}^2$ with edges connecting (Euclidean) nearest neighbors. Gibbs measures are defined in the usual DLR sense: a probability measure π on $\{1,\dots,q\}^{\mathbf{Z}^2}$ is said to be a Gibbs measure for the q-state Potts model at inverse temperature β if it admits conditional probabilities such that for all finite regions $\Lambda \in \mathbf{Z}^2$, all $\xi \in \{1,\dots,q\}^\Lambda$ and all $\xi' \in \{1,\dots,q\}^{\mathbf{Z}^2\setminus\Lambda}$, we have

$$
\begin{aligned}
&\mu\big(X(\Lambda)=\xi \mid X(\mathbf{Z}^2\setminus\Lambda)=\xi'\big)\\
&\quad=\frac{1}{Z^{\Lambda,\xi'}_{q,\beta}}\exp\Bigg(2\beta\sum_{\substack{\langle x,y\rangle\\ x,y\in\Lambda}} I_{\{\xi(x)=\xi(y)\}}+2\beta\sum_{\substack{\langle x,y\rangle\\ x\in\Lambda,\,y\in\mathbf{Z}^2\setminus\Lambda}} I_{\{\xi(x)=\xi'(y)\}}\Bigg)
\end{aligned}
$$

where the normalizing constant $Z^{\Lambda,\xi'}_{q,\beta}$ may depend on ξ' but not on ξ.

It is well known that there is a critical value $\beta_c = \beta_c(q) > 0$ such that for $\beta < \beta_c$ there is a unique Gibbs measure, while for $\beta > \beta_c$ there are multiple Gibbs measures. It has been conjectured, based on nonrigorous random-cluster arguments, that $\beta_c(q) = \frac{1}{2}\log(1+\sqrt{q})$, but this is so far only known for $d = 2$ and for d sufficiently large (see [18] and [11]). In the nonuniqueness regime of the parameter space, there is, for each $j \in \{1, \dots, q\}$, a particular Gibbs measure $\pi^j_{q,\beta}$ which arises as a limit of finite volume Gibbs measures with boundary condition "all j" (see, e.g., [9], [12] or [10] for the details of this construction). In the uniqueness regime, we write simply $\pi_{q,\beta}$ for the unique Gibbs measure.

We are interested in percolation-theoretic properties of these Gibbs measures. A subset S of $\{1, \dots, q\}$ is said to percolate in the Gibbs measure π if π assigns positive probability to the existence of an infinite self-avoiding path in $\mathbf{Z}^2$, all of whose vertices take spin values in S. Coniglio, Nappi, Peruggi and Russo [4] considered the Ising case $q = 2$, and characterized the Gibbs multiplicity regime in terms of percolation as follows: In the uniqueness regime, neither of the two spins percolate in $\pi_{2,\beta}$, whereas in the nonuniqueness regime, spin 1 (but not spin 2) percolates in $\pi^1_{2,\beta}$. Chayes [2] extended this picture to arbitrary q by showing that no single spin percolates in $\pi_{q,\beta}$ in the uniqueness regime, that 1's percolate in $\pi^1_{q,\beta}$ in the nonuniqueness regime, and moreover that the latter percolation is so "dominant" that even the combined efforts of the spins $\{2, \dots, q\}$ fail to percolate. (Related results for the so-called Ashkin–Teller model on $\mathbf{Z}^2$ appear in [3].) The particular geometry of $\mathbf{Z}^2$ is crucial for this kind of sharp dichotomy, as already in three dimensions a single spin may percolate in the uniqueness regime, as shown in [1].

When a single spin fails to percolate, it is natural to ask how many spins are needed to percolate. We offer the following bound, which says that strictly more than half of the spins are needed.

THEOREM 3.1. *Consider the q-state Potts model on $\mathbf{Z}^2$ at inverse temperature $\beta < \beta_c$, and set $S = \{1, \dots, r\}$. If $r \le q/2$, then S does not percolate in $\pi_{q,\beta}$.*

PROOF. Pick $X \in \{1, \dots, q\}^{\mathbf{Z}^2}$ according to $\pi_{q,\beta}$, and obtain the fuzzy Potts configuration $Y \in \{0, 1\}^{\mathbf{Z}^2}$ from X using (2) with $r \le q/2$. Write $\mu_{r,s,\beta}$ for the distribution of Y. We need to show that 0's do not percolate in $\mu_{r,s,\beta}$.

Following the approach of [2], we make use of a theorem of Gandolfi, Keane and Russo [8], stating that if μ is a probability measure on $\{0, 1\}^{\mathbf{Z}^2}$ which:

(i) is invariant under translations and axis reflections,
(ii) is ergodic under horizontal and vertical translations (separately), and
(iii) has positive correlations,

then 0's and 1's cannot both percolate in μ.

Suppose now for contradiction that 0's do percolate in $\mu_{r,s,\beta}$. Since $r \le q/2$, we have $r \le s$, and since $\pi_{q,\beta}$ is symmetric under permutations of $\{1, \dots, q\}$, we have that the set of 0's is stochastically dominated by the set of 1's in $\mu_{r,s,q}$. Hence 1's percolate as well, so in order to get the desired contradiction we just need to verify that $\mu_{r,s,\beta}$ satisfies properties (i)–(iii) above. Properties (i) and (ii) are immediate from the corresponding properties of $\pi_{q,\beta}$, which are well known and easy to check. To see that $\mu_{r,s,\beta}$ also satisfies (iii), we recall that $\pi_{q,\beta}$ arises as a weak limit of measures $\pi_{q,\beta}^{\Lambda_i}$, where $(\Lambda_1, \Lambda_2, \ldots)$ is any increasing sequence of finite subgraphs of $\mathbf{Z}^2$ converging to $\mathbf{Z}^2$ in the usual way. It follows that $\mu_{r,s,\beta}$ is a weak limit of $\mu_{r,s,\beta}^{\Lambda_i}$ as $i \to \infty$. The measures $\mu_{r,s,\beta}^{\Lambda_i}$ have positive correlations by Theorem 1.3, and since the positive correlations property is preserved under weak limits we are done. $\square$

If we set $r_c(q,\beta)$ to be the minimal number of spins needed to percolate in $\pi_{q,\beta}$, we thus have

$$r_c(q,\beta) > \frac{q}{2}. \tag{8}$$

It would be interesting to gain a better understanding of how $r_c(q,\beta)$ behaves as a function of β on $[0,\beta_c)$. It is not clear whether it should be increasing or decreasing (or neither). For β sufficiently close to 0, one may argue as in the final section of [1] to deduce that r spins suffice (resp., do not suffice) if $r/q > p_c$ (resp., $r/q < p_c$), where p_c is the critical value for independent site percolation on $\mathbf{Z}^2$. The value of p_c is believed to be around 0.59, and the currently best rigorous bounds are $0.556 < p_c < 0.680$, due to [16] and [19]. The upper bound thus barely fails to show that $r = 2$ suffices to percolate when $q = 3$ and β is small. Is $r = 2$ sufficient for percolation throughout $[0,\beta_c)$ when $q = 3$? More generally, is $r = q - 1$ sufficient throughout $[0,\beta_c)$?

As with the case of finite graphs, we can again consider also the fractional fuzzy Potts model. For $\beta < \beta_c$, the fractional fuzzy Potts measure $\mu_{r,s,\beta}$ is obtained by picking an edge configuration for $\mathbf{Z}^2$ according to the (unique) infinite-volume random-cluster measure $\phi_{p,q}$ with $q = r + s$ and $p = 1 - e^{-2\beta}$ (see [11]), and then assigning i.i.d. spins (0 or 1 with respective probabilities r/q and s/q) to the connected components. We can then define $r_c(q,\beta)$ to be a real-valued infimum rather than an integer-valued minimum. The inequality in (8) then still holds provided $q \geq 2$, except that we can no longer prove that the inequality is strict.

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