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ON THE DISTRIBUTION OF TAIL ARRAY SUMS FOR STRONGLY MIXING STATIONARY SEQUENCES[1]

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This paper concerns the asymptotic distributions of "tail array" sums of the form $\sum \psi_n(X_i - u_n)$ where $\{X_i\}$ is a strongly mixing stationary sequence, ψ_n are real functions which are constant for negative arguments, $\psi_n(x) = \psi_n(X_+)$ and $\{u_n\}$ are levels with $u_n \to \infty$. Compound Poisson limits for rapid convergence of $u_n \to \infty$ $(nP\{X_1 > u_n\} \to \tau < \infty)$ are considered briefly and a more detailed account given for normal limits applicable to slower rates $(nP(X_1 > u_n) \to \infty)$. The work is motivated by (1) the modeling of "damage" due to very high and moderately high extremes and (2) the provision of probabilistic theory for application to problems of "tail inference" for stationary sequences.

1. Introduction. "Tail array sums" $\sum \psi_n(X_i - u_n)$ for a stationary sequence $\{X_j\}$, $u_n \to \infty$, and appropriate functions $\{\psi_n\}$ with $\psi_n(x) = \psi_n(X_+)$, find application in a variety of contexts including probabilistic modeling of "damage" from high values, and statistical "tail inference." As a very simple example, the random variable

$$\zeta_n = \sum_{j=1}^{n} (X_j - u_n)_+$$

represents damage due to high X_j values as measured by the sum of the excess values above the "threshold" u_n. On the other hand, ζ_n also plays a central role in formation of statistics for the estimation of parameters in the distribution F of X_j with exponentially decreasing or regularly varying tails.

Damage modeling, quantile estimation and tail inference have been of longstanding interest in engineering science and insurance and have more recently attracted substantial interest in the statistical literature; see, for example, [1], [3], [4], [8], [9], [10], [16], [18]. Many of these papers, and procedures used in practice, assume independence. However, typical situations, such as meterological, hydrological and environmental measurement concern dependent variables, and some of the cited references also consider dependence, using special methods tailored to the problem at hand.

In the present paper we establish a simple general theory for such tail array sum problems. The limiting distributions are obtained from the results of [14]

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for (dependent) array sums under an appropriate form of strong mixing. This condition (defined precisely below) is natural and the weakest of dependence restrictions typically used in dependent central limit theory.

A basic (and often difficult) problem in dependent central limit theory is to verify Lindeberg conditions for appropriate "block sums." A final main result of this paper provides a simple approach to this verification, for the important special cases of (approximately) exponential or (by a logarithmic transformation), regularly varying tails.

Two cases are considered in which (1) $u_n \to \infty$ at a rapid rate defined by $n(1-F(u_n)) \to \tau < \infty$ and (2) $u_n \to \infty$ more slowly, with $n(1-F(u_n)) \to \infty$. These will be referred to as "high" and "moderate" levels u_n, respectively. The not unexpected compound Poisson limiting results for the former case are discussed briefly in Section 3 following the general theory and framework of Section 2.

The parameter τ can be interpreted as the asymptotic mean number of observations which exceed the level u_n. Hence in case (1) we are dealing with a finite number of exceedances. The resulting Poisson type convergence in particular provides the weak limits of the extreme upper order statistics (cf. [13], Chapter 5). However, to estimate parameters of these limiting distributions consistently, we need to rely on the weak law of large numbers and, eventually, on the central limit theorem. This can only be done if the number of "useful" observations, that is, the number of exceedances of u_n, tends to infinity. This may be achieved by (normal limit) results from case (2) for moderate levels, where the number of exceedances does tend to infinity, in important cases for which the parameters for "extreme" and "moderate" tail behavior may be related. More stringent dependence conditions are needed for case (2), as will be seen below. An early result in this direction is [20] and more recent ones are contained in [8], [9], [10], [16].

The remainder of the paper gives a more detailed discussion of normal limit behavior for the case (2) involving moderate levels u_n. Section 4 gives conditions for normal convergence including primarily a Lindeberg condition for sums of small groups of the successive terms in ζ_n. As noted above, Lindeberg conditions can be difficult to verify in dependent contexts, especially for tail arrays. However, we show in Section 6 that this may be quite simply achieved by assuming an appropriate (exponential) rate of tail decay of the distribution F.

Section 5 concerns specific cases of special interest derived from the simple functions $1_{(x>0)}$, and x_+. The normal limits obtained are also modified to provide convergence which is minimally dependent on unknown parameters by using suitable "block variance" estimates. These latter results are especially useful for statistical applications to tail estimation to be considered in [17].

Finally in this introduction, we note the precise form of the strong mixing assumption to be used throughout this paper. For a recent survey of related mixing concepts, see [5]. Let $X_1, X_2, \dots$ be a (strictly) stationary sequence, and let $\mathscr{B}_{ij}$ denote the σ-field $\sigma\{X_k\colon i \le k \le j\}$ generated by $X_i, X_{i+1}, \dots, X_j$ and for fixed n, $l < n$, let

$$\alpha_{n,l} = \sup\left\{\left|P(A \cap B) - P(A)P(B)\right|\colon A \in \mathscr{B}_{1,k}, B \in \mathscr{B}_{k+l,n}, 1 \le k \le n-l\right\}.$$

Then $\{X_j\}$ will be termed "strongly mixing $(\alpha_{n,l}, l_n)$" if $\alpha_{n,l_n} \to 0$ for some $l_n = o(n)$. It may be shown that the existence of such a sequence $\{l_n\}$ follows if $\alpha_{n,[\varepsilon n]} \to 0$ as $n \to \infty$ for each $\varepsilon > 0$.

For application in the paper, the following equivalent form of the condition will be convenient. For a complex valued random variable X, write $X \in \mathscr{B}_{ij}$ to denote $\mathscr{B}_{ij}$-measurability of X, and

$$\beta_{n,l} = \sup\big\{\big|\mathscr{E}XY - \mathscr{E}X\mathscr{E}Y\big|: X \in \mathscr{B}_{1,k}, Y \in \mathscr{B}_{k+l,n}, \\ |X| \le 1, |Y| \le 1, 1 \le k \le n-l\big\}.$$

This "array form" of strong mixing involves values of $k+l$ only up to n. It is of course implied by the standard definition (in which $k+l$ is not so restricted). For particular purposes, weaker forms of the condition may be used—replacing $\mathscr{B}_{ij}$ by the σ-field generated by the functions of $X_i, X_{i+1}, \dots, X_j$ relevant to the problem [such as $1_{(X_j > u_n)}$ or $(X_j - u_n)_+$ for given u_n], and when convenient we do so. As an example, in [20] restriction to σ-fields generated by $1_{(X_j > u_n)}$ leads to a substantial gain of generality.

2. Framework and general theory. The following notation and *basic assumptions* will apply throughout. The sequence $\{X_j, j = 1, 2, \dots\}$ is stationary with marginal d.f. F, and strongly mixing $(\alpha_{n,l}, l_n)$, $\{u_n\}$ is a sequence of constants ("levels") and $\{\psi_n(x)\}$ are functions which are constant for negative arguments, $\psi_n(x) = \psi_n(X_+)$ which satisfy $\mathscr{E}\psi_n^2(X_1 - u_n) < \infty$. Further integers $l_n < r_n \to \infty$ with $r_n \le n$, $r_n = o(n)$ are chosen so that, setting $k_n = [n/r_n]$,

$$k_n(\alpha_{n,l_n} + l_n/n) \to 0. \tag{2.1}$$

Such $\{k_n\}$ will be called a *standard sequence* and the corresponding division of $(0, n]$ into intervals $J_i = ((i-1)r_n, ir_n]$, $1 \le i \le k_n$, of length r_n and a last interval $J_{k_n+1} = (k_n r_n, n]$ termed a *standard partition*. Write also

$$Y_j = Y_{nj} = (X_j - u_n)_+, \qquad 1 \le j \le n,$$
$$Z_i = Z_{ni} = \sum_{j \in J_i} \psi_n(Y_j), \qquad 1 \le i \le k_n + 1,$$

so that Z_i is the "block sum" over the ith interval.

Throughout the paper an undesignated sum $\sum$ will mean $\sum_1^n$. Our interest will focus on the (tail) array sum

$$\zeta_n = \sum_1^{k_n+1} Z_i = \sum \psi_n(X_i - u_n)\left(= \sum \psi_n(Y_i) = \sum_1^{k_n+1} Z_i\right), \tag{2.2}$$

where $u_n \to \infty$ at either a fast ("high level"), or at the somewhat slower ("moderate level") rate. The following result shows that for $\psi_n \ge 0$, ζ_n has the same asymptotic distribution as it would if the "block" sums Z_i over the partition intervals J_i were independent, and the same is true for ψ_n of unrestricted signs under a very mild asymptotic neglibility condition. This theorem, which is a variant of results of [12] and [14] facilitates the use of classical i.i.d. limit criteria in this dependent setting. For this purpose $\{\hat{Z}_i\}$ will denote the "independent sequence" associated with $\{Z_i\}$; that is, $\hat{Z}_i$ are independent with $\hat{Z}_i =_d Z_i$.

THEOREM 2.1. *Under the basic assumptions and with the above notation, suppose that*

$$Z_{k_n+1} \to_p 0, \qquad \sum_{i=1}^{k_n} Z_i^* \to_p 0, \tag{2.3}$$

where Z_i^* are i.i.d. with $Z_i^* =_d \sum_1^{l_n} \psi_n(Y_j)$. Then

$$\zeta_n = \sum \psi_n(X_i - u_n)$$

has the same limiting distribution (if any) as $\sum_{i=1}^{k_n} \hat{Z}_i$, where $\hat{Z}_i$ is the "independent sequence associated with $Z_i = \sum_{j\in J_i} \psi_n(Y_j)$"; that is, $\hat{Z}_i$ are independent with $\hat{Z}_i =_d Z_i$.

If $\psi_n(x) \geq 0$ for all $x \geq 0$, the conditions (2.3) may be omitted.

PROOF. The first statement follows from the proof (and surrounding comments) of Lemma 2.1 of [14] adapted to the discrete parameter context, defining the interval function $\zeta_n(I) = \sum_{j/n\in I} \psi_n(Y_j)$. The elimination of (2.3) from the condition when $\psi_n \geq 0$ may be simply shown as in Lemma 2.2 of [12]. □

Note that, when needed, the conditions in (2.3) are readily verified from mean and variance assumptions for Z_{k_n+1}, Z_i^* (the latter being i.i.d.) as in the following corollary.

COROLLARY 2.2. *Under the basic assumptions suppose that $\mathscr{E}\psi_n(X_1 - u_n) = 0$ and*

$$k_n \operatorname{var}\left\{\sum_1^{l_n} \psi_n(Y_j)\right\} \to 0, \qquad \operatorname{var}(Z_{k_n+1}) \to 0. \tag{2.4}$$

Then $\sum \psi_n(X_i - u_n)$ has the same limit in distribution (if any) as the i.i.d. array sum $\sum_1^{k_n} \hat{Z}_i$.

3. Very high levels—compound Poisson convergence. For purposes such as damage modeling from infrequent excesses of very high levels, it is useful to consider $u_n \to \infty$ at a fast rate, which, in this section, we take somewhat more generally to mean that

$$\limsup_n n\big(1 - F(u_n)\big) = \tau < \infty. \tag{3.1}$$

A great deal of literature is devoted to Poisson and Poisson-related properties of specific functions of excess values (in both rigorous and heuristic works). For example, the asymptotic Poisson character of exceedances of u_n (cf. [2] and references therein) has been known and widely used for a very long time. As noted, this fits the above framework with $\psi_n(x) = \psi(x) = 1_{(x>0)}$. Similarly $\psi_n(x) = x_+ = x1_{(x>0)}$ leads to the sum of exceedance values above u_n.

While our main interest is in the lower (moderate) level situation it seems worthwhile to briefly set a general class of high level cases within the framework of Theorem 2.1. For nonnegative functions ψ_n the "total damage" $\zeta_n = \sum_1^n \psi_n(X_i - u_n)$ is a special case of the random measure $\zeta_n(B) = \sum_{i/n\in B} \psi_n(X_i - u_n)$ defined for Borel sets $B \subset (0,1]$ and its limiting distribution is perhaps best considered in that framework (cf. [12]), using Laplace transform methods which are most natural for nonnegative cases. This gives a compound Poisson limiting distribution in essentially all cases of practical interest and, indeed, convergence in distribution of ζ_n as a random measure to a compound Poisson point process.

On the other hand, when ψ_n may have general sign, $\zeta_n(I)$ still has a limiting compound Poisson distribution for each interval $I \subset (0,1]$ under natural conditions as shown in the following theorem. In this CP(λ, π) will denote a compound Poisson distribution being the sum of a Poisson (mean λ) number of independent r.v.'s with d.f. π, with $\pi(0) - \pi(0-) = 0$. We further assume that X_i does not cause any damage if $X_i \le u_n$, that is, that $\psi_n(0) = 0$.

THEOREM 3.1. *Suppose the basic assumptions and* (3.1) *hold*, $\psi_n(0) = 0$ *and* ψ_n *is either nonnegative or satisfies* (2.3) [*or the sufficient condition* (2.4)]. *Then*

$$\sum_{i=1}^{n} \psi_n(X_i - u_n) \to_d W \tag{3.2}$$

for some r.v. W *iff the d.f.'s* F_n *of* $\sum_1^{r_n} \psi_n(X_j - u_n)$ *satisfy*

$$\begin{aligned} k_n F_n(x) &\to \lambda\pi(x), \qquad x < 0, \\ k_n(1 - F_n(x)) &\to \lambda(1 - \pi(x)), \qquad x > 0, \end{aligned} \tag{3.3}$$

for some $\lambda > 0$ *and d.f.* π, *at continuity points of* $\pi(x)$. *Furthermore,* W *then has a CP*(λ, π)*-distribution. Also,* (3.2) *may be rephrased to read* $\zeta_n(I) \to_d \zeta(I)$ *where* $\zeta_n(I) = \sum_{j/n\in I} \psi_n(X_i - u_n)$, $I = (0,1]$ *and* $\zeta(I)$ *is CP*(λ, π). *The same arguments show that this holds for any interval* $I = (a, b] \subset (0,1]$ *where the limit* $\zeta(I)$ *is CP*$(\lambda(b-a), \pi)$. *Further for disjoint* $I_1, \dots, I_p$, $\zeta_n(I_1) \cdots \zeta_n(I_p)$ *are readily shown to converge jointly to the independent limits* $\zeta(I_1) \cdots \zeta(I_p)$. *This holds whether or not* ψ_n *is restricted in sign.*

PROOF. According to Theorem 2.1 (or Corollary 2.2) it is sufficient to show that the result holds with (3.2) replaced by

$$\sum_{i=1}^{k_n} \hat{Z}_i \to_d W, \tag{3.4}$$

for $\hat{Z}_i$ i.i.d. with d.f. F_n. We use [7], Theorem 4, Section 25 to prove the equivalence of (3.3) and (3.4) under the stated conditions. Now, since $\psi_n(0) = 0$ it follows from (3.1), since $r_n/n \sim 1/k_n \to 0$, that

$$\begin{aligned} P(\hat{Z}_1 \neq 0) &= P(Z_1 \neq 0) \\ &\leq r_n(1 - F(u_n)) \\ &= \frac{r_n}{n} n(1 - F(u_n)) \\ &\to 0, \qquad n \to \infty, \end{aligned}$$

so that the $\hat{Z}_i$'s are uniformly asymptotically negligible. Similarly, for $\varepsilon > 0$, using (3.1) and the fact that $k_n r_n \leq n$,

$$\begin{aligned} \limsup_n k_n \int_{|x|<\varepsilon} x^2 \, dF_n &\leq \limsup_n \varepsilon^2 k_n P(\hat{Z}_1 \neq 0) \\ &\leq \varepsilon^2 \limsup_n k_n r_n \big(1 - F(u_n)\big) \\ &\leq \varepsilon^2 \tau \to 0 \quad \text{as } \varepsilon \to 0, \end{aligned}$$

and by the same reasoning

$$\limsup_n k_n \int_{|x|<\varepsilon} x \, dF_n \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.5}$$

Hence (2) of the cited theorem is satisfied, with $\sigma^2 = 0$. Thus if (3.4) holds, then there must exist a nondecreasing function M and a nonincreasing function N with $M(-\infty) = N(\infty) = 0$ such that

$$\begin{aligned} k_n F_n(x) &\to M(x), \qquad x < 0, \\ k_n(1 - F_n(x)) &\to N(x), \qquad x > 0, \end{aligned} \tag{3.6}$$

at continuity points x of the right-hand sides. Since $M(0-) + N(0+) \leq \limsup_n k_n P(\hat{Z}_1 \neq 0) \leq \tau$, it is possible to write $M(x) = \lambda\pi(x)$, $N(x) = \lambda(1 - \pi(x))$ for some constant λ with $0 \leq \lambda \leq \tau$ and a probability measure π, so that (3.6) may be written in the form (3.3).

Conversely, if (3.3) holds, it follows from (3.6) that $k_n \int_{|x|<\tau} x \, dF_n(x) \to \int_{|x|<\tau} x\lambda \, d\pi(x)$ for any τ such that $\pm\tau$ are continuity points of π. By the remark after Theorem 4, Section 25 and (6), (8) of Section 18 of [7], it follows by easy computation that (3.4) holds and that W has the characteristic function $\exp\{\lambda \int (e^{iut} - 1) \, d\pi(x)\}$ and hence is a CP(λ, π)-distribution. □

If the restriction $\psi_n(0) = 0$ is removed, then the result still holds if the requirement that $n\psi_n(0) \to \alpha$ for some constant α, is added to (3.3), and $W - \alpha$ then has a CP(λ, π)-distribution, as can be seen by similar considerations to those in the theorem.

For $\psi_n \geq 0$, these results show that the *random measures* $\zeta_n(B)$ defined on the Borel subsets of (0, 1]) converge in distribution to a compound Poisson point process. This has the natural interpretation that for large n the damage ψ_n may be modeled by occurrence times forming a Poisson process with intensity λ, and associated independent damage magnitudes (not necessarily integer valued) with distribution π.

For a given "damage function" ψ_n, it is of clear interest for modeling to know what forms are possible for the limiting "damage distribution" π. The only (nontrivial) case of which we are aware for which this is known explicitly is the so-called "peaks over thresholds" (POT) model commonly used, for example, in hydrology, for which the damage due to a cluster of values exceeding a threshold u is the peak excess height of the cluster above u (e.g., maximum flood height). This is not precisely of the form considered here but the same arguments apply.

This POT model is discussed completely in [11], where it is shown under general conditons that π must have a "generalized Pareto" form. This confirms and makes explicit earlier more heuristic use of generalized Pareto damage distributions at Poisson times under iid assumptions.

Less explicit results are known for other damage mechanisms, where it may be necessary to resort to estimation of the distribution π, or even ad hoc fitting. The complicating feature for this high level case is the generality of the limit—involving an entire distribution π as well as the Poisson intensity λ. For the lower (moderate) levels, it will be seen (cf. next section) that more explicit results are possible since the *normal* limit involved is parametrically determined. Finally if ψ_n is not necessarily positive, the random measure interpretation does not apply but there is still joint convergence [e.g., under (2.3) of $\zeta_n(I_j)$] for disjoint I_j to independent compound Poisson limits, and the interpretation of independent damage magnitudes occurring at time points determined by a Poisson process with intensity λ still applies.

4. Moderate levels—normal convergence. As noted, for both modeling and tail inference it is important to consider levels $u_n \to \infty$ at a slower rate and specifically such that (3.1) is replaced by

$$c_n = n(1 - F(u_n)) \to \infty, \tag{4.1}$$

to be assumed in this and subsequent sections. Further to the basic assumptions it will be assumed that $\psi_n(X_1 - u_n)$ has been standardized so that

$$\mathscr{E}\psi_n(X_j - u_n) = 0, \qquad \operatorname{var}\left\{\sum_{j=1}^{r_n} \psi_n(X_j - u_n)\right\} = 1/k_n \tag{4.2}$$

so that $\mathscr{E}Z_i = 0$, $\operatorname{var} Z_i = 1/k_n$ with Z_i as in Section 2 and hence $\operatorname{var}\{\sum_1^{k_n} \hat{Z}_i\} = k_n \operatorname{var}\{\hat{Z}_1\} = 1$, where $\{\hat{Z}_i\}$ is the i.i.d. sequence associated with $\{Z_i\}$. In particular, these normalizations will be automatic for the class of applications defined by (4.5) and (4.6) below. The following criterion for normal convergence then holds.

THEOREM 4.1. *Suppose that the basic assumptions and* (2.3) *hold and that* ψ_n *satisfies* (4.2). *Then*

$$\zeta_n = \sum \psi_n(X_j - u_n) \to_d N(0,1) \tag{4.3}$$

if and only if the Lindeberg condition

$$k_n \mathscr{E}\{Z_1^2 1_{(|Z_1|>\varepsilon)}\} \to 0 \quad \text{as } n \to \infty, \text{ each } \varepsilon > 0, \tag{4.4}$$

is satisfied, where $Z_1 = \sum_1^{r_n} \psi_n(X_j - u_n)$, *as before*.

PROOF. This is immediate from the classical normal convergence criterion (cf. [15]) since by Theorem 2.1, ζ_n has the same limiting distribution as $\sum_1^{k_n} \hat{Z}_i$ where $\hat{Z}_i$ are independent, $\hat{Z}_i =_d Z_1$ and $k_n \operatorname{var} Z_1 = 1$. □

Commonly the (normalized) ψ_n are obtained from unnormalized functions $\phi_n x$ with $\phi_n(x) = 0$ for $x < 0$ and with $\mathscr{E}\phi_n^2(X_1 - u_n) < \infty$ by writing

$$\psi_n(x) = \sigma_n^{-1}[\phi_n(x) - \mathscr{E}\phi_n(X_1 - u_n)], \tag{4.5}$$

where

$$\sigma_n^2 = k_n \operatorname{var}\left\{\sum_{j=1}^{r_n} \phi_n(X_j - u_n)\right\}, \tag{4.6}$$

the negligibility conditions (2.4) becoming

$$k_n \sigma_n^{-2} \operatorname{var}\left\{\sum_{j=1}^{l_n} \phi_n(Y_j)\right\} \to 0, \qquad \sigma_n^{-2} \operatorname{var}\left\{\sum_{j=1}^{n-r_n k_n} \phi_n(Y_j)\right\} \to 0. \tag{4.7}$$

With the established notation, the tail sum considered in (4.3) is

$$\zeta_n = \sum \psi_n(X_i - u_n) = \sigma_n^{-1} \sum [\phi_n(X_j - u_n) - \mathscr{E}\phi_n(X_1 - u_n)]. \tag{4.8}$$

Note that (4.2) holds for ψ_n defined by (4.5) by virtue of (4.6).

Theorem 4.1 then applies to the function ψ_n defined by (4.5), specifically stated as follows.

COROLLARY 4.2. *Let* ϕ_n *be a function on* $[0,\infty)$ *with* $\mathscr{E}\phi_n^2(X_1 - u_n) < \infty$, *and* σ_n *defined by* (4.6) *and suppose that the basic assumptions and* (2.1) *and* (4.7) *hold. Then*

$$\zeta_n = \sigma_n^{-1}\left[\sum_1^n \phi_n(X_j - u_n) - n\mathscr{E}\phi_n(X_1 - u_n)\right] \to_d N(0,1) \tag{4.9}$$

if and only if the Lindeberg condition (4.4), $k_n \mathscr{E}\{Z_1^2 1_{|Z_1|>\varepsilon)}\} \to 0$, *holds where*

$$Z_1 = \sigma_n^{-1} \sum_1^{r_n} [\phi_n(X_j - u_n) - \mathscr{E}\phi_n(X_1 - u_n)]. \tag{4.10}$$

Particular cases (cf. Section 5) are given by specializing ϕ_n to functions $\phi_n = \phi$, independent of n [such as $1_{(x>0)}$, x_+, for which the ζ_n are, respectively, the number N_n of exceedances of u_n, and the sum of exceedance values $\sum(X_i - u_n)_+$]. It is also of interest to consider linear combinations of such ζ_n, as in Example 5.5.

As noted earlier, verification of Lindeberg conditions for array sums may require very strong local dependence restrictions. For tail sums as here considered, the technical details of verification may be even more complicated. However, simple distributional tail conditions may sometimes be used to avoid the restrictive dependence assumptions. This is shown in Section 6 (Theorem 6.2) for exponential decay rates, which are of particular importance in tail estimation problems (cf. [17]).

The further conditions (2.3) and (4.7) in Theorem 4.1 and Corollary 4.2 will also be verified in Section 6 for exponential decay rates. However, the following result shows that (4.7) may be simply verified when $\phi(x) \geq 0$ without even assuming tail conditions, for levels u_n which do not increase too slowly, that is, such that $c_n = n(1 - F(u_n)) = o(k_n)$.

LEMMA 4.3. *If $\psi_n(x)$ is defined by* (4.5), *with $\phi_n(x) \geq 0$ all x, $\phi_n(0) = 0$ and $c_n = o(k_n)$ and the basic assumptions hold*, *then*

$$\sigma_n^2 \sim k_n \mathscr{E}\left(\sum_{j=1}^{r_n} \phi_n(Y_j)\right)^2 \tag{4.11}$$

and (4.7) *holds*.

PROOF. For K denoting a generic constant,

$$\begin{aligned}\left(\mathscr{E}\left\{\sum_1^{r_n} \phi_n(Y_j)\right\}\right)^2 &= r_n^2(\mathscr{E}\phi_n(Y_1))^2 = r_n^2(\mathscr{E}(\phi_n(Y_1)1_{(Y_1>0)}))^2 \\ &\leq r_n^2 \mathscr{E}\phi_n^2(Y_1)P\{Y_1 > 0\} = r_n^2(1 - F(u_n))\mathscr{E}\phi_n^2(Y_1) \\ &\leq K r_n \frac{c_n}{n}\mathscr{E}\left(\sum_1^{r_n} \phi_n(Y_1)\right)^2\end{aligned}$$

since $1 - F(u_n) = c_n/n$ and $\phi_n(Y_i) \geq 0$, each i. Since $k_n r_n \sim n$ and $c_n = o(k_n)$, it thus follows that

$$\left(\mathscr{E}\sum_1^{r_n} \phi_n(Y_j)\right)^2 = o\left\{\mathscr{E}\left(\sum_1^{r_n} \phi_n(Y_j)\right)^2\right\},$$

which yields (4.11).

Since clearly $\sum_1^{r_n} \phi_n(Y_j) \ge \sum_{i=1}^{[r_n/l_n]} V_{ni}$ where $V_{ni} =_d V_n = \sum_{j=1}^{l_n} \phi_n(Y_j)$, it follows that $\mathscr{E}(\sum_1^{r_n} \phi_n(Y_j))^2 \ge [r_n/l_n]\mathscr{E}V_n^2$ and hence

$$\begin{aligned} \sigma_n^{-2}k_n \operatorname{var}\left\{\sum_1^{l_n} \phi_n(Y_j)\right\} &= \sigma_n^{-2}k_n \operatorname{var} V_n \\ &\le \sigma_n^{-2}k_n\mathscr{E}V_n^2 \le K\frac{l_n}{r_n}\sigma_n^{-2}k_n\mathscr{E}\left(\sum_1^{r_n}\phi_n(Y_j)\right)^2 \qquad (4.12) \\ &\sim K\frac{l_n}{r_n} \end{aligned}$$

by (4.6) and (4.11). But $l_n/r_n \to 0$ so that $k_n \operatorname{var}\{\sum_1^{l_n} \psi_n(Y_j)\} \to 0$. Thus the first limit in (4.7) holds and the second follows similarly. □

5. Practical issues and special cases. The asymptotic normality in (4.9) involves the (usually) unknown normalizing constants σ_n, and it is clearly important for some applications to replace them by functions of the sequence values X_i (e.g., an appropriately consistent estimator) for inference purposes such as the construction of confidence intervals. Lemma 5.1 gives mild conditions under which σ_n^2 may in fact be replaced in (4.9) by an estimate based on "independent" blocks:

$$s_n^2 = \sum_{i=1}^{k_n}(B_{n,i} - \overline{B}_n)^2 = \sum_{i=1}^{k_n} B_{n,i}^2 - k_n\overline{B}_n^2 \qquad (5.1)$$

(see Theorem 5.2) where $B_{n,i}$ is the ith block sum $\sum_{j\in J_i} \phi_n(X_j - u_n)$ and $\overline{B}_n = k_n^{-1}\sum_1^{k_n} B_{n,i}$.

Now write J_i' for the first $r_n - l_n$ integers of J_i, $U_i = \sum_{j\in J_i'} \psi_n(Y_j)$, $V_i = Z_i - U_i = \sum_{j\in J_i - J_i'} \psi_n(Y_j)$ for ψ_n given by (4.5).

LEMMA 5.1. *With the notation above suppose that the basic assumptions*, (2.3), *and* (4.2) *hold and in addition suppose that* $k_n \operatorname{var} Z_1^2 \to 0$. *Then*

$$\begin{aligned} &\text{(i)} \quad \sum_{i=1}^{k_n} Z_i^2 \to_p 1, \\ &\text{(ii)} \quad k_n^{-1/2}\sum_{i=1}^{k_n} Z_i \to_p 0. \end{aligned} \qquad (5.2)$$

PROOF. This follows a similar pattern to that of Lemma 2.1 of [14] and hence will be sketched only. First note that $\sum_1^{k_n} V_i^2 \to_p 0$ since $\mathscr{E}\sum V_i^2 = k_n\mathscr{E}V_1^2 = k_n \operatorname{var}\{\sum_1^{l_n} \phi_n(Y_j)\} \to 0$ by (4.7) and that

$$\left|\sum V_iZ_i\right| \le \left(\sum V_i^2\right)^{1/2}\left(\sum Z_i^2\right)^{1/2} \to_p 0$$

since $k_n\mathscr{E}Z_1^2 = k_n \operatorname{var} Z_1 = 1$, so that $\sum_1^{k_n} Z_i^2$ is tight.

Hence $\sum_1^{k_n} Z_i^2 - \sum_1^{k_n} U_i^2 = 2\sum_1^{k_n} V_i Z_i + \sum_1^{k_n} V_i^2 \to_p 0$ and thus

$$\left|\mathscr{E}\exp\left(it\sum U_j^2\right) - \mathscr{E}\exp\left(it\sum Z_j^2\right)\right| \le \mathscr{E}\left|1 - \exp\left(it\sum(Z_j^2 - U_j^2)\right)\right| \to 0 \tag{5.3}$$

by dominated convergence. Further, by an obvious induction on the strong mixing (see Section 1 and [14, 19]),

$$\left|\mathscr{E}\exp\left(it\sum U_j^2\right) - \prod \mathscr{E}\exp(itU_j^2)\right| \le 16k_n\alpha_{n,l_n} \to 0. \tag{5.4}$$

Now let $(\hat{Z}_j, \hat{U}_j)$ be independent in j but $(\hat{Z}_j, \hat{U}_j) =_d (Z_j, U_j)$. Then

$$\begin{aligned}\left|\prod \mathscr{E}\exp(itU_j^2) - \prod \mathscr{E}\exp(itZ_j^2)\right| &= \left|\mathscr{E}\exp\left(it\sum \hat{U}_j^2\right) - \mathscr{E}\exp\left(it\sum \hat{Z}_j^2\right)\right| \\ &\le \mathscr{E}\left|1 - \exp\left(it\sum(\hat{Z}_j^2 - \hat{U}_j^2)\right)\right| \to 0\end{aligned} \tag{5.5}$$

by repeating the argument giving (5.3), for $\hat{Z}_j$, $\hat{U}_j$, $\hat{V}_j = \hat{Z}_j - \hat{U}_j$.
Combining (5.3)–(5.5) gives

$$\mathscr{E}\exp\left(it\sum Z_j^2\right) - \prod \mathscr{E}\exp(itZ_j^2) \to 0 \tag{5.6}$$

so that it is sufficient to show $\sum Z_j^2 \to_p 1$ assuming independence of the Z_j. But with this assumption, as above

$$\mathscr{E}\left(\sum_1^{k_n} Z_i^2\right) = k_n \mathscr{E} Z_1^2 = 1,$$

whereas, by assumption,

$$\operatorname{var}\left(\sum_1^{k_n} Z_i^2\right) = k_n \operatorname{var} Z_1^2 \to 0$$

giving (i) of (5.2). The proof of (ii) is even simpler, noting that it holds under assumed independence of Z_i, (since then $k_n^{-1/2}\sum_1^{k_n} Z_i$ has zero mean and variance $k_n^{-1} \to 0$) and hence holds in general. □

As a result of this the following holds.

THEOREM 5.2. *Suppose the conditions of Lemma* 5.1 *with* ψ_n *given by* (4.5) *and* (4.6) *hold*; *that is*, ζ_n *is asymptotically normal* (*e.g., by Corollary* 4.2). *Then*

$$\zeta_n^* = s_n^{-1}\sum_1^n [\phi_n(X_j - u_n) - \mu_n] \to_d N(0,1) \tag{5.7}$$

[s_n *being defined by* (5.1) *and* $\mu_n = \mathscr{E}\phi_n(X_1 - u_n)$].

PROOF. This will follow from (4.9) by showing that $s_n/\sigma_n \to_p 1$, or equivalently that $s_n/s'_n \to_p 1$ where

$$s'^2_n = \sum_{i=1}^{k_n}\left(\sum_{j\in J_i}\phi_n(X_j - u_n) - r_n\mu_n\right)^2 \tag{5.8}$$

since it follows from (5.2)(i) that $s'_n/\sigma_n \to_p 1$. But it is readily checked that

$$s'^2_n - s^2_n = k_n(\bar{B}_n - r_n\mu_n)^2,$$

giving

$$\begin{aligned} 1 - \frac{s^2_n}{s'^2_n} &= \frac{\sigma^2_n}{s'^2_n} k_n \sigma_n^{-2}(\bar{B}_n - r_n\mu_n)^2 \\ &= \left(\frac{\sigma_n}{s'_n}\right)^2 k_n^{-1}\left(\sum_1^{k_n} Z_i\right)^2 \to_p 0 \end{aligned}$$

by (5.2)(ii) and the fact that $\sigma_n/s'_n \to_p 1$. □

We conclude this section with three examples. In these we assume that the basic assumptions and (2.3) are satisfied.

EXAMPLE 5.3. Let $\phi_n(x) = 1_{(x>0)}$ and write $N_n = \sum 1_{(x_i > u_n)}$. In this case

$$\mathscr{E}\{\phi_n(X_1 - u_n)\} = P\{X_1 > u_n\} = 1 - F(u_n) = c_n/n.$$

Write $N_{n,i} = \sum_{j\in J_i} 1_{\{X_i > u_n\}}$, the number of exceedances of u_n by X_j for $j \in J_i$, that is, the ith block sum. Then $Z_1 = \sigma_n^{-1}(N_{n,1} - c_n)$ where $\sigma^2_n = k_n \operatorname{var} N_{n,1}$ and if Z_1 satisfies the Lindeberg condition (4.4), Theorem 4.1 gives

$$\sigma_n^{-1}(N_n - c_n) \to_d N(0,1). \tag{5.9}$$

The special case of a normal sequence is proved in [20] under suitable (polynomial) bounds on the rate of decay of the covariance function.

If further, the conditions of Theorem 5.2 hold, then

$$s_n^{-1}(N_n - c_n) \to_d N(0,1) \tag{5.10}$$

with $s^2_n = \sum_1^{k_n} N^2_{n,i} - k_n\overline{N}^2_n$, $\overline{N}_n = \sum_{i=1}^{k_n} N_{n,i}/k_n$.

EXAMPLE 5.4. Here $\phi_n(x) = x_+$. Write $S_n = \sum(X_j - u_n)_+$, $S_{n,i}$ for the ith block sum $\sum_{j\in J_i}(X_j - u_n)_+$, and

$$\mu_n = \mathscr{E}(X_1 - u_n)_+, \qquad \sigma^2_n = k_n \operatorname{var} S_{n,1}. \tag{5.11}$$

Thus if $Z_1 = \sigma_n^{-1}(S_{n,1} - r_n\mu_n)$ satisfies (4.4), Theorem 4.1 gives

$$\sigma_n^{-1}(S_n - n\mu_n) \to_d N(0,1) \tag{5.12}$$

and if the conditions of Theorem 5.2 hold, then also

$$s_n^{-1}(S_n - n\mu_n) \to_d N(0,1) \tag{5.13}$$

with

$$s_n^2 = \sum_{i=1}^{k_n} S_{n,i}^2 - k_n \overline{S}_n^2 \tag{5.14}$$

where $\overline{S}_n = k_n^{-1} \sum_{i=1}^{k_n} S_{n,i}$. □

In both "high level" probabilistic modeling and tail inference it is sometimes of interest to combine "damage measures." This is illustrated by the following useful combination of the above two cases, N_n, S_n being defined in Examples 5.3 and 5.4.

EXAMPLE 5.5. Here $\phi^{(1)}(x) = 1_{(x>0)}$, $\phi^{(2)}(x) = x_+$, $\phi_n(x) = \phi^{(2)}(x) + a_n\phi^{(1)}(x)$ for some constants a_n, so that

$$\zeta_n = \sigma_n^{-1}\{S_n - n\mu_n^{(1)} + a_n(N_n - c_n)\} \tag{5.15}$$

in which $\mu_n^{(1)} = E(X_1 - u_n)_+$ and

$$\sigma_n^2 = k_n \operatorname{var}(S_{n,1} + a_n N_{n,1}). \tag{5.16}$$

Under the conditions of Theorem 4.1, $\zeta_n \to_d N(0,1)$. This is useful for high level damage modeling, in which typically $a_n > 0$. For tail inference, a useful special choice is $a_n = -\beta_n$ for $\beta_n = n\mu_n^{(1)}/c_n$ (e.g., if F is an exponential d.f., $F(x) = 1 - \exp\{-x/\beta\}$, then $\beta_n = \beta$), which gives

$$\sigma_n^{-1}\{S_n - \beta_n N_n\} \to_d N(0,1). \tag{5.17}$$

As in the above cases a more "data based" form of (5.16) is available under the conditions of Theorem 5.2 in replacing σ_n by s_n' where here [as in (5.8) with $\mu_n = 0$],

$$(s_n')^2 = \sum_{i=1}^{k_n}\{S_{n,i} - \beta_n N_{n,i}\}^2. \tag{5.18}$$

This expression for s_n' still contains the unknown quantity β_n. While our focus here is not on inference applications, it seems useful for completeness (e.g., as a basis for construction of confidence intervals) to give conditions under which β_n in (5.18) may be replaced by the random variables

$$\hat{\beta}_n = N_n^{-1} \sum (X_j - u_n)_+, \tag{5.19}$$

which form a basis for tail parameter estimation. Let s_n denote the modification of s'_n in (5.18) obtained by replacing β_n by $\hat{\beta}_n$. The replacement of σ_n by s_n leaves (5.12) unaltered if $s_n/\sigma_n \to_p 1$ which will occur if

$$(\hat{\beta}_n - \beta_n)^2 \sum_1^{k_n} N_{n,i}^2/\sigma_n^2 \to_p 0, \tag{5.20}$$

since writing $\xi_i = \sum_{j\in J_i}(X_j - u_n)_+$ gives

$$(s_n/\sigma_n)^2 = \sum_{i=1}^{k_n}\big[\big(\xi_i - \beta_n N_{n,i}\big)/\sigma_n + (\beta_n - \hat{\beta}_n)N_n(J_i)/\sigma_n\big]^2$$

and $\sum_{i=1}^{k_n}[(\xi_i - \beta_n N_n(J_i))/\sigma_n]^2 \to_p 1$ by (5.2)(i).

To give convenient sufficient conditions for (5.20), write $\sigma_n^{(1)}$, $\sigma_n^{(2)}$ for the respective variances defined in Examples 5.3 and 5.4 and assume that the conditions in Examples 5.3 and 5.4 hold, including (4.4). In addition, assume that $\{\beta_n\}$ is bounded and

$$\sigma_n^{(k)} = O(\sigma_n), \qquad k = 1, 2, \tag{5.21}$$

$$c_n/\sigma_n \to \infty, \qquad c_n/\sigma_n = O\big(k_n^{1/2}\big). \tag{5.22}$$

It then follows from the asymptotic normality in Example 5.3 that $N_n/c_n \to_p 1$ and from Example 5.4 that $\hat{\beta}_n - \beta_n \to_p 0$ as $n \to \infty$.

An easy computation [using (5.22)] shows that $\sum_1^{k_n} N_{n,i}^2/\sigma_n^2$ have uniformly bounded means and thus are tight so that (5.20) holds.

Hence σ_n may be replaced by s_n in (5.17) under the above conditions which while numerous are relatively simple aside from the Lindeberg condition. The latter is in any case required for the validity of (5.17) itself and, as noted, an example of its verification will be given in Section 6.

Note finally that $\hat{\beta}_n$ itself is central to certain tail estimation problems and has a limiting normal distribution

$$s_n^{-1}N_n(\hat{\beta}_n - \beta_n) \to N(0,1), \tag{5.23}$$

which follows at once from (5.17).

6. The Lindeberg condition—exponentially decreasing tails. As noted earlier, the verification of the Lindeberg condition for general array sums has typically required stringent mixing assumptions. Our purpose in this section is twofold: (1) to show that for *tail* array sums the use of the standard strong mixing augmented by the tail conditions on the marginal d.f. can suffice and (2) to provide basic results which are relevant to tail estimation problems [17].

The calculations assume the important special case of an exponentially decreasing tail for the marginal d.f., and should be regarded as a prototype type analysis for other types of tail behavior. For tail estimation the analysis already applies more widely since, for example, a logarithmic transformation leads to regularly varying tails.

Specifically, it will be assumed throughout this section that ϕ_n defined on $[0, \infty)$ is left continuous, $\mathscr{E}\phi_n^2(Y_1) < \infty$ and that the d.f. F of the X_i's has the exponential type of decay rate in the sense that

$$[1 - F(t+x)]/(1 - F(t)) \to e^{-x/\beta}, \qquad t \to \infty \text{ all } x > 0,\ \text{some } \beta > 0. \tag{6.1}$$

To give simple sufficient conditions for the Lindeberg criterion, it is convenient to truncate $\phi(Y_j) = \phi(X_j - u_n)$ as follows. For constants w_n to be specified, define

$$Y'_j = Y_j 1_{(Y_j \le w_n)} + w_n 1_{(Y_j > w_n)}, \qquad 1 \le j \le n,$$
$$Z'_i = \sigma_n^{-1} \sum_{j \in J_i} \big(\phi_n(Y'_j) - \mathscr{E}\phi_n(Y'_1)\big),$$

with σ_n as in (4.6).

LEMMA 6.1. *Suppose the basic assumptions and* (6.1) *hold, and that* $\phi_n(x)$ *is nonnegative and nondecreasing.* (i) *If* w_n *for some* $0 < \varepsilon < 1$ *satisfies*

$$\frac{c_n r_n}{\sigma_n^2} \int_{w_n}^{\infty} \exp\{(\varepsilon - \beta^{-1})x\}\, d\phi_n^2(x) \to 0 \quad \text{as } n \to \infty, \tag{6.2}$$

then $k_n \mathscr{E}(Z_1 - Z'_1)^2 \to 0$ *as* $n \to \infty$.

(ii) *For integers* h_n, *and with* ε *as in* (i), *setting* $\alpha_n = \int_{w_n}^{\infty} \exp\{(\varepsilon - \beta^{-1})x\}\, d\phi_n^2(x)$,

$$\operatorname{var}\Big\{\sum_{j=1}^{h_n} \phi_n(Y_j)\Big\} = O\big(h_n^2 c_n n^{-1}(\phi_n^2(w_n) + \alpha_n)\big).$$

PROOF. (i) Note, using Minkowski's inequality for the second step below and integration by parts for the fifth, that for n large

$$\begin{aligned}
r_n^{-2}\sigma_n^2 \mathscr{E}(Z_1 - Z'_1)^2 &= r_n^{-2}\mathscr{E}\Big\{\sum_{j=1}^{r_n} \big(\phi_n(Y_j) - \phi_n(w_n)\big) 1_{(Y_j > w_n)}\Big\}^2 \\
&\le \mathscr{E}\big\{(\phi_n(Y_1) - \phi_n(w_n))^2 1_{(Y_1 > w_n)}\big\} \\
&\le \mathscr{E}\big\{(\phi_n^2(Y_1) - \phi_n^2(w_n)) 1_{(Y_1 > w_n)}\big\} \\
&= \int_{u_n + w_n}^{\infty} \big(\phi_n^2(x - u_n) - \phi_n^2(w_n)\big)\, dF(x) \\
&= \int_{w_n}^{\infty} \big(1 - F(y + u_n)\big)\, d\phi_n^2(y) \\
&\le \big(1 + \varepsilon)(1 - F(u_n)\big) \int_{w_n}^{\infty} \exp\{(\varepsilon - \beta^{-1})x\}\, d\phi_n^2(x)
\end{aligned}$$

by Proposition 1.7 of [6]. Hence the desired conclusion follows by (6.2) since $1 - F(u_n) = c_n/n$.

(ii) Using Minkowski's inequality for the second step and in the third step that ϕ_n is nondecreasing, that $F(u_n+w_n)-F(u_n) \le 1-F(u_n)$ and $1-F(u_n+w_n) \le 1 - F(u_n)$, we have

$$\begin{aligned}\operatorname{var}\left\{\sum_{j=1}^{h_n} \phi_n(Y_j)\right\} &\le \mathscr{E}\left\{\left(\sum_{j=1}^{h_n} \phi_n(Y_j)\right)^2\right\} \le h_n^2 \mathscr{E}(\phi_n^2(Y_1)) \\ &\le h_n^2 \biggl\{ 2(1 - F(u_n))\phi_n^2(w_n) \\ &\qquad + \int_{u_n+w_n}^{\infty} \bigl(\phi_n^2(x-u_n) - \phi_n^2(w_n)\bigr)\, dF(x)\biggr\}.\end{aligned}$$

Part (ii) now follows by estimating the second term as in part (i) and using $1 - F(u_n) = c_n/n$. □

The following results are now simply obtained.

THEOREM 6.2. *Suppose the basic assumptions*, (6.1) *and* (2.4) *hold*, w_n *satisfies* (6.2) *and* $\phi_n(x)$ *is nonnegative and nondecreasing and*

$$r_n \sigma_n^{-1} \phi(w_n) \to 0. \tag{6.3}$$

Then the Lindeberg condition (4.4) *holds so that by Theorem* 4.1,

$$\zeta_n = \sigma_n^{-1} \sum_{j=1}^{n} \bigl[\phi(X_j - u_n) - \mathscr{E}\phi(X_j - u_n)\bigr] \to_d N(0,1).$$

PROOF. It is readily checked that for any X, Y,

$$(X+Y)^2 1_{(|X+Y| \ge \varepsilon)} \le 4\bigl(X^2 1_{(|X| \ge \varepsilon/2)} + Y^2 1_{(|Y| \ge \varepsilon/2)}\bigr) \tag{6.4}$$

from which it follows that

$$\begin{aligned}k_n \mathscr{E}\bigl\{(Z_1 - \mathscr{E}Z_1)^2 1_{(|Z_1 - \mathscr{E}Z_1| > \varepsilon)}\bigr\} &\le 4k_n \mathscr{E}\bigl\{(Z_1' - \mathscr{E}Z_1')^2 1_{(|Z_1' - \mathscr{E}Z_1'| > \varepsilon/2)}\bigr\} \\ &\quad + 4k_n \mathscr{E}(Z_1 - Z_1')^2.\end{aligned}$$

The first term on the right tends to zero trivially since by (6.3),

$$|Z_1'| \le \sigma_n^{-1} r_n \phi(w_n) \to 0$$

(so that the indicator is zero for large n), and the last term tends to zero by Lemma 6.1 so that the Lindeberg condition (4.4) holds, as desired. □

According to Lemma 4.3, (4.7) and then (2.4) hold if ϕ is nonnegative and $c_n = n(1 - F(u_n)) = o(k_n)$. This may also be obtained from a strengthened version of (6.3).

COROLLARY 6.3. *Suppose the basic assumptions and* (6.1) *hold*, w_n *satisfies* (6.2) *and* $\phi_n(x)$ *is nonnegative and nondecreasing and*

$$(l_n^2 r_n^{-1} c_n + r_n^2)\sigma_n^{-2}\phi_n^2(w_n) \to 0. \tag{6.5}$$

Then

$$\zeta_n = \sigma_n^{-1} \sum_{j=1}^{n} \big[\phi(X_j - u_n) - \mathscr{E}\phi(X_j - u_n)\big] \to_d N(0,1).$$

PROOF. By assumption, all the conditions of Theorem 6.2 except (2.4) are satisfied. To prove (2.4) it is sufficient to show that (4.7) holds. Now, since $k_n = [n/r_n]$,

$$\begin{aligned} k_n \sigma_n^{-2} l_n^2 c_n n^{-1}(\phi_n^2(w_n) + \alpha_n)) &\sim l_n^2 r_n^{-1} c_n \sigma_n^{-2}(\phi_n^2(w_n) + \alpha_n)) \\ &\to 0, \end{aligned}$$

by (6.2) together with $l_n/r_n \to 0$ and (6.5), which by Lemma 6.1(ii) proves the first part of (4.7). Further, $n - r_n k_n < r_n$, and

$$\sigma_n^{-2} r_n^2 c_n n^{-1}\big(\phi_n^2(w_n) + \alpha_n\big) \to 0,$$

by (6.2), (6.5), and $r_n/n \to 0$, $c_n/n \to 0$, respectively. The second part of (4.7) then again follows from Lemma 6.1(ii). □

In the final result of this section we generalize Theorem 6.2 to include functions $\phi(x)$ such as those in Example 5.5.

THEOREM 6.4. *Suppose the assumptions of Theorem* 6.2 *are satisfied for each of the functions* $\phi_1(x)$, $\phi_2(x)$ *and write* $\phi(x) = \alpha_1\phi_1(x) + \alpha_2\phi_2(x)$, α_1, α_2 (*positive or negative*) *constants. Let* $\sigma_n^{(1)}$, $\sigma_n^{(2)}$, σ_n *be defined as in* (4.6) *relative to* ϕ_1, ϕ_2, ϕ, *respectively, and suppose that* $\sigma_n^{(k)} \le K\sigma_n$, $k = 1, 2$, $n = 1, 2, 3, \ldots$ *for some constant* $K \ge 0$. *Then* (4.9) *holds*; *that is*,

$$\zeta_n = \sigma_n^{-1} \sum_{j=1}^{n} \big[\phi(X_j - u_n) - \mathscr{E}\phi(X_1 - u_n)\big] \to_d N(0,1).$$

PROOF. If $\zeta_n^{(k)}$ is defined as ζ_n above with ϕ_k for ϕ and $\sigma_n^{(k)}$ for σ_n, $k = 1, 2$, Theorem 6.2 shows that $\zeta_n^{(k)} \to_d N(0,1)$ and hence by Theorem 4.1 the Lindeberg conditions

$$k_n \mathscr{E}\left\{(Z_{n,1}^{(k)})^2 1_{\{|Z_{n,1}^{(k)} - \mathscr{E}Z_{n,1}^{(k)}| \ge \varepsilon\}}\right\} \to 0 \qquad \text{as } n \to \infty, \text{each } \varepsilon > 0$$

hold, where $Z_{n,1}^{(k)} = (\sigma^{(k)_n})^{-1} \sum_{j \in J_1} \phi_k(Y_j)$, $k = 1, 2$.

Since $\sigma_n^{(k)} \le K\sigma_n$, this Lindeberg condition continues to hold for each $k = 1, 2$ if $\sigma_n^{(k)}$ is replaced by σ_n in the definition of $Z_{n,1}^{(k)}$ and hence it holds for $\alpha_1 Z_{n,1}^{(1)} + \alpha_2 Z_{n,1}^{(2)}$ by the inequality (6.4). The remaining conditions of Theorem 4.1 regarding ϕ are readily checked, giving the stated result. □

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