The Annals of Applied Probability
1998, Vol. 8, No. 3, 676–707

OPTION REPLICATION WITH TRANSACTION COSTS: GENERAL DIFFUSION LIMITS

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Transaction costs preclude the construction of hedging strategies for general contingent claims. Leland introduced the concept of diffusion limits of hedging strategies in a small transaction cost limit. This paper establishes such diffusion limits for very general hedging strategies and also establishes leading order asymptotic expressions for the replication error. In addition to subsuming previously considered temporal strategies, the results in this paper yield new results, namely expressions for replication errors of stock price strategies and a variety of "renewal" strategies. Most importantly, this paper provides a unified methodology for calculating hedging strategies and replication errors in the small transaction cost limit. This is an essential component of optimization methods, when, for example one is trying to minimize replication error for a given initial portfolio value.

1. Introduction. The Black–Scholes framework for option pricing and replication is predicated on, among other things, the absence of transaction costs. In the presence of any source of transaction costs, such as those arising from a bid–ask spread, there no longer exist hedging strategies which achieve perfect replication for a general payoff. That is to say, the system is intrinsically incomplete, and arbitrage pricing theory cannot be used to price contingent claims.

The development of a mathematical framework to deal with transaction costs has proceeded along several different lines. The results in this paper are related to the pioneering work of Leland (1985), in which a replicating portfolio is rebalanced at equal time intervals. By increasing the frequency of rebalancings while letting transaction costs vanish at an appropriate rate, a cost of replication is obtained as a solution to a Black–Scholes partial differential equation with an enhanced volatility.

Subsequently, Henrotte (1991) extended this concept of diffusion limits of replicating portfolios to hedging strategies based on, for example, stock price changes. Furthermore, Henrotte clearly articulates the need to consider the asymptotic replication error, since any implementation of a hedging strategy in the presence of transaction costs will necessarily involve discrete (as opposed to continuous) rebalancing. Henrotte then compares the performance

Received August 1995; revised June 1997.
AMS 1991 *subject classifications*. 90A09, 60H30, 60G44.
Key words and phrases. Transaction costs, hedging strategies.

of the hedging strategy in which rebalancings occur at equally spaced times and a strategy in which rebalancings occur when the underlying price changes by prescribed amounts.

Grannan and Swindle (1994) expanded further upon the use of limiting hedging strategies by optimizing over classes of strategies. Specifically, they considered an expanded set of time-interval strategies which allow for varying time intervals. Exact expression for the limiting hedging strategies and replication errors were established and then used to construct and solve various optimization problems. For example, strategies were obtained which minimize the replication error given an initial portfolio value, or which minimize a weighted sum of initial portfolio value and replication error. It was shown that the resulting strategies significantly outperform the standard constant time-interval strategies.

Avelleneda and Paras (1994) also utilize diffusion limits, and show that in the case where transaction costs are large (in a limiting sense) and when payoffs are nonconvex, then, if option replication is replaced with dominance of the option payoff, the associated hedging strategy is prescribed by a solution to a partial differential equation (PDE) with free boundaries. This work is essentially a continuous time version of Bensaid, Lesner, Pages and Scheinkman (1992), in which dominating strategies are calculated in the presence of transaction costs in the setting of Cox, Ross and Rubinstein (1979).

Currently, the primary difficulty in considering more general classes of limiting hedging strategies is that one is confronted with the chore of establishing the form of the associated "Black–Scholes" PDE, in addition to the much more daunting task of calculating the replication error to leading order in the transaction cost parameter. The purpose of this paper is to dispose of these difficulties once and for all by calculating the PDE and the associated replication error for limiting hedging strategies in considerable generality. This will be done for a broad class of diffusions describing the price of the underlying, and the conditions which are imposed on the hedging strategies are stated solely in terms of the asymptotics of the moments of time and price changes between rebalancings and regularity properties of the resulting PDE. In short, all of the theory will be done here, and further efforts at optimization over classes of hedging strategies will require only that the user be able to calculate the required moments and check the conditions of the theorems.

This paper generalizes the Leland limit to what we believe is a maximal and nontrivial extension. Not only is the associated PDE identified for very general hedging strategies, but the replication error is calculated to leading order. Prior to this paper the only class of strategies for which a replication error has been obtained are fixed time interval strategies [see Henrotte (1991) and Grannan and Swindle (1994)]. These results are also established for general payoffs, in particular wthout the restrictive assumption of convexity which has pervaded earlier results. The proofs are substantially more difficult as a consequence. When the payoff is not convex we must also assume some regularity of the solution to the resulting nonlinear PDE which we believe to be generic; however, we do not yet have conditions in terms of the payoff function and coefficients which guarantee this regularity.

Most natural strategies fall under the umbrella of the results of this paper. Examples include (1) time interval strategies which incorporate stock level, so that the duration to the next portfolio rebalancing depends upon the current stock price (this subsumes the constant time interval strategies); (2) price change strategies, in which rebalancings occur when the stock price changes by prescribed amounts; (3) renewal strategies in which price–time boundaries are set upon each rebalancing with the subsequent rebalancing occuring when the price trajectory hits the boundary [this is effectively a hybrid of classes (1) and (2)]; (4) Delta-strategies based upon deviations of the portfolio Δ, which were also discussed in Henrotte (1991). To actually optimize within a given class of strategies, one must evaluate the moments appearing in the results. This task, which can be very difficult, is the limiting factor in broadening the classes of strategies in the construction of effective hedging methods.

Before proceeding, we should point out that a variety of other avenues have been exploited to illuminate the effect of transaction costs in option pricing and replication. In discrete time settings, numerous authors, in addition to Bensaid, Lesne, Pages, and Scheinkman, (1992) have addressed the issue of option replication [see, e.g., Boyle and Vorst (1992), Constantinides and Zariphopoulou (1995) and Edirisighe (1993)]. Another approach developed in Davis, Panas and Zariphopoulu (1993) adopts a control perspective and proceeds by assigning an appropriate utility function to the writer of an option and then obtaining the minimum price at which the writer would be willing to sell an option via the solution of two optimal control problems. Further developments along these lines involve bounding option prices over classes of utility functions as done in Constantinides and Zariphopoulou (1995).

This paper is structured as follows. In Section 2 we will discuss the basic set-up and touch upon the nature of the results to follow. Section 3 contains the two central theorems of the paper. Theorem 3.1 establishes the PDE which prescribes the limiting replicating strategy, and Theorem 3.2 provides the replication error to leading order in the transaction cost parameter. The third section also contains the proof of Theorem 3.1 as well as an illustrative example of these results for strategies based on price changes. This example is particularly useful in showing that the list of conditions required for Theorem 3.2 are in fact natural, and it also yields results on the price level strategies which have only been conjectured to date. The rather lengthy proof of Theorem 3.2 is given in Section 4. In addition, Section 5 contains remarks about potential applications of these results.

2. Limiting hedging strategies. In this section we will describe the essential feature of the construction of replicating strategies in the limits of small transaction costs as first done in Leland (1985). However, we will do this in the context of very general rebalancing criteria and with the spot price driven by a general diffusion.

The price of the underlying will be taken to be of the form

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dB_t, \tag{1}$$

where B_t denotes a standard Brownian motion and where we assume that μ, σ, σ_t, σ_x and σ_{xx} are continuous and satisfy a polynomial growth condition. In addition, we assume that the drift and volatility are such that the hitting times of zero and infinity are infinite almost surely and that all moments of X_t exist. For the case of homogeneous diffusions (time-independent coefficients), the hitting time conditions are well studied [see, e.g., Karlin and Taylor (1981)] and a linear growth bound on the drift and volatility suffices to control the moments. Finally, we need a nondegeneracy condition on the diffusion; $\sigma(t, X_t)$ uniformly positive on compact subsets of $(0, \infty)$ is adequate. For ease of exposition we will take interest rates to zero. This has the benefit of reducing the length of the equations considerably and results in no loss of generality, as this situation can be realized by a change of variables corresponding to measuring currency in risk free bonds. In the remainder of the paper, Ω denotes the space of continuous functions from $[0, T] \to \mathbb{R}$ (i.e., the path space of X_t), $P(\cdot)$ denotes the associated probability measure on Ω and $E(\cdot)$ denotes expectation with respect to this measure.

The market is not frictionless—each trade incurs transaction costs. In this paper we will consider proportional transaction costs, where the cost associated with a change in the spot position of value ΔS is $\kappa|\Delta S|$: κ is an exogeneously prescribed constant of proportionality that indicates the level of transaction costs in the market. Forms of transaction costs other than proportional could also be considered, although different scaling limits would result.

A trader will now enter the market with $h(0, X_0)$ dollars. The goal of the trader is to engage in a trading strategy which replicates a contingent claim $u(X_T)$ as closely as possible in the sense of minimal expected square error. Here T is the time of expiry of the claim, and we restrict our attention to European claims, which can be exercised only at time T. A trading strategy will consist of both a sequence of stopping times τ_i with $i \in \{1, 2, \dots\}$ and a prescription of the amount of underlying the trader is to hold at time τ_i. By analogy with the Black–Scholes theory, the number of shares held will be given by $h_x(\tau_i, X_{\tau_i})$ for some (yet to be determined) function $h(t, x)$.

The resulting values of such a hedging strategy at expiry T, denoted by $\hat{u}(X)$, is given by the following bookkeeping formula:

$$\begin{aligned}\hat{u}(X) = h(0, X_0) &+ \sum_{\tau_i < T} h_x(\tau_i, X_{\tau_i})\big(X_{\tau_{i+1}} - X_{\tau_i}\big)\\ &- \kappa \sum_{\tau_i < T} \big|h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i})\big| X_{\tau_{i+1}}.\end{aligned} \tag{2}$$

The first term on the right-hand side is the initial value of the replicating portfolio, the second term corresponds to gains and losses sustained in the underlying position (there is no comparable term for the risk free asset as interest rates are zero) and the third term describes transaction cost losses.

While perfectly general, this equation as it stands is rather useless. The question of a selection of a "good" hedging strategy has yet to be addressed, and a direct assault on (2) for a given sequence of stopping times is complicated significantly by the transaction cost term. Leland (1985) introduced an ingenious way to circumvent this difficulty by taking a limit in which times between rebalancings vanish. This essentially involves considering a sequence of stopping times τ_i^δ in which loosely speaking $\tau_{i+1}^\delta - \tau_i^\delta \sim \delta$. Taking the limit of continuous rebalancing, $\delta \to 0$, the scaling properties of the Brownian motion suggest that the transaction cost term in (2) should scale in this way: $\kappa\delta^{-1}\delta^{1/2} = \kappa\delta^{-1/2}$. Therefore, to keep the transaction cost term nondegenerate, one must take $\kappa \sim \delta^{1/2}$. Leland carried out this program for the case of uniform time intervals $\tau_i^\delta = i\delta$.

REMARK 2.1. Taking the limit $\kappa \sim \delta^{1/2}$ yields a nondegenerate limit from which one can extract useful information about replicating strategies for small but positive values of δ. In other words, if the exogeneously prescribed κ is small, then $\delta = \kappa^2$ is small, and one expects any results obtained from the limit to be approximately correct. Of course, one of the tasks of this paper is to calculate the errors of this approximation.

The results in this paper, which will be stated precisely and proved in the remaining sections, are twofold. First, aside from technical conditions, for a sequence of rebalancing times τ_i^δ if

$$E\left(\tau_{i+1}^\delta - \tau_i^\delta | \mathscr{F}_{\tau_i^\delta}\right) = \theta\left(\tau_i^\delta, X_{\tau_i^\delta}\right)\delta + o(\delta^{3/2}), \tag{3}$$

$$E\left(|X_{\tau_{i+1}^\delta} - X_{\tau_i^\delta}| \,|\, \mathscr{F}_{\tau_i^\delta}\right) = \lambda\left(\tau_i^\delta, X_{\tau_i^\delta}\right)\delta^{1/2} + o(\delta) \tag{4}$$

[see Definition 3.1 and the conditions (T) in Section 3 for details], then hedging according to (2) with $h(t, x)$ given by the solution to

$$h_t(t, x) + \frac{1}{2}\sigma^2(t, x)h_{xx}(t, x) + x\frac{\lambda}{\theta}(t, x)|h_{xx}|(t, x) = 0 \tag{5}$$

with

$$h(T, x) = u(x) \tag{6}$$

results in perfect replication in the limit $\delta \to 0$ (i.e., $E[(\hat{u}(X) - u(X_T))^2] \to 0$). In addition, expressions for the expected square replication error are calculated to leading order in δ. In tandem, these two results are essential to carrying out optimization along the lines of that done in Grannan and Swindle (1994) for broader classes of strategies.

3. The main results. The first task is to specify the hedging strategies to be considered in this paper. These strategies are indexed by δ, which will later be taken to zero with $\kappa = \delta^{1/2}$. The assumptions below are stated in terms of moments of the relevant stopping times and stopped underlying process. While there are a rather overwhelming number of conditions on the asymptotic behavior of various moments, they are natural in that they are easily shown to be satisfied by the strategies discussed previously, such as for time interval and stock price level strategies. In addition, required smoothness properties of the associated Black–Scholes PDE will be stated.

A few remarks about notation are in order. We will write τ_i for the stopping times τ_i^δ, leaving the δ dependence implicit. The underlying filtration $(\mathscr{F}_t)_{0\le t\le T}$ is right continuous and contains all the null sets. The expressions $O(\delta)$ and $o(\delta)$ have their usual meanings—the former is bounded by $C\delta$ for some constant C, and the latter converges to zero when divided by δ as $\delta \to 0$. We will also need the following extension of this notation.

DEFINITION 3.1. The symbol $O_i(\delta)$ denotes a sequence of expressions bounded in magnitude by $\delta Z_i(\delta)$ where $\{Z_i(\delta)\}$ is a sequence of positive $\mathscr{F}_{\tau_i}$-measurable random variables such that $\sup_i Z_i(\delta) \in L^8(\Omega)$ (uniformly in δ). If, in addition to the above conditions, $\sup_i E[|Z_i(\delta)|^4] \to 0$ as $\delta \to 0$ then the expression is considered to be $o_i(\delta)$.

With this notation at our disposal, we are now in a position to define the class of hedging strategies that we consider.

Stopping time assumptions. Rebalancing times $0 = \tau_0 < \tau_1 < \cdots$ are typically articulated without reference to the time of expiration T. In order to avoid ambiguities, we define $N = \min\{i \ge 1: \tau_i \ge T\}$, the number of τ_i's up to T, and we will always take τ_i to be $\tau_i \wedge T$. As a result, $\tau_N = T$. The sequence of rebalancing times $\{\tau_i^\delta\}$ must have the following asymptotic properties as $\delta \to 0$:

$$\text{(T1)}\qquad E\big[\tau_{i+1} - \tau_i|\mathscr{F}_{\tau_i}\big] = \theta(\tau_i, X_{\tau_i})\delta + o_i(\delta^{3/2}) \ge \theta_0\delta > 0,$$

$$\text{(T2)}\qquad E\Big[|\tau_{i+1} - \tau_i|^2\Big|\mathscr{F}_{\tau_i}\Big] = \eta(\tau_i, X_{\tau_i})\delta^2 + o_i(\delta^2),$$

$$\text{(T3)}\qquad E\big[|X_{\tau_{i+1}} - X_{\tau_i}|\big|\mathscr{F}_{\tau_i}\big] = \lambda(\tau_i, X_{\tau_i})\delta^{1/2} + o_i(\delta),$$

$$\text{(T4)}\qquad E\Big[|\tau_{i+1} - \tau_i|^4\Big|\mathscr{F}_{\tau_i}\Big] = O_i(\delta^3),$$

$$\text{(T5)}\qquad E\big[\operatorname{sign}(X_{\tau_{i+1}} - X_{\tau_i})(\tau_{i+1} - \tau_i)|\mathscr{F}_{\tau_i}\big] = o_i(\delta),$$

$$\text{(T6)}\qquad E\big[(X_{\tau_{i+1}} - X_{\tau_i})(\tau_{i+1} - \tau_i)|\mathscr{F}_{\tau_i}\big] = o_i(\delta^{3/2}),$$

$$\text{(T7)}\qquad E\big[(X_{\tau_{i+1}} - X_{\tau_i})|X_{\tau_{i+1}} - X_{\tau_i}|\big|\mathscr{F}_{\tau_i}\big] = o_i(\delta),$$

where θ, η and λ are continuous and have a polynomial growth order in x.

REMARK 3.1. Conditions (T1), (T2) and (T3) yield coefficients which appear in both PDE (5) and the expression for the replication error which will be discussed later. Condition (T4) will be used for controlling the replication error uniformly over the rebalancing intervals. Correlations of sign $(X_{\tau_{i+1}} - X_{\tau_i})$ and positive quantities $\tau_{i+1} - \tau_i, |X_{\tau_{i+1}} - X_{\tau_i}|(\tau_{i+1} - \tau_i)$, and $(X_{\tau_{i+1}} -$

$X_{\tau_i})^2$ are controlled by conditions (T5), (T6) and (T7). These conditions reduce the bias of the price change over the rebalancing time interval, which could have contributed the replication error.

REMARK 3.2. The conditions above are in fact natural. (T1), (T2), (T3) and (T4) essentially exclude small probabilities of large gaps between rebalancings. Furthermore, if the quantity $X_{\tau_{i+1}} - X_{\tau_i}$ is replaced by its absolute value in (T5), (T6) and (T7), then one would expect terms that are O_i rather than o_i. These conditions are, therefore, nothing more than constraints on the symmetry of the rebalancing condition in the price variable.

Partial differential equation assumptions. It will be shown in Theorem 3.1 that the solution to PDE (5) with initial condition (6) together with (2) prescribes a replicating strategy for the contingent claim $u(X_T)$ in the limit $\delta \to 0$. The required regularity properties to the solution $h(t, x)$ are the following:

(P1) Here h_t, h_x and h_{xx} are Lipschitz continuous in both spatial and temporal arguments.

(P2) The singularities (i.e., nondifferentiability with respect to spatial argument) of h_t, h_x and h_{xx} are described by the graph $(t, F(t))$ where F: $[0,T] \to \mathbb{R}^m$ is a Lipschitz continuous function. The coordinates of F will be denoted by F_i, $i = 1, \dots, m$. Thus, $\{F_i(t), 1 \le i \le m\}$ is the set of singularities at time t.

(P3) For each $t, h_{xx}(t, \cdot)$ is a difference between two convex functions. In other words, the second derivative of $h_{xx}(t, \cdot)$ exists in the distributional sense and is a signed Borel measure.

(P4) The left derivative of $h_{xx}(t, \cdot)$, denoted by $h_{xxx}(t, \cdot)$, has a polynomial growth order.

A few remarks about the regularity assumptions (P1) through (P4) are in order. PDE (5) can be written parsimoniously as:

$$h_t + F(t, x, h_{xx}) = 0,$$

where F is convex and piecewise linear in its third argument. Were F in fact twice continuously differentiable in its third argument, then existence, uniqueness and regularity results are available. For example, Theorem 23 in Chapter IX of Dong (1991) implies existence of a solution to (5) which is C^4 in x and C^2 in t. This would be adequate for the results in this section to hold. This regularity result, however, does not apply in our situation due to the lack of smoothness of F. When the payoff $u(x)$ is strictly convex and when $\sigma^2(t, x)$ satisfies appropriate conditions [e.g., if $\sigma^2(t, x)$ is $C^{2+\alpha}$ for some $\alpha > 0$ and is such that there exist positive constants a and A such that $ax^2 \le \sigma^2(t, x) \le Ax^2$], then, since h remains convex for all t, (5) has the regularity above. In this case there are no point of inflection to cause difficulties. When the payoff is strictly concave, regularity of the solution of (5) is also immediate, provided that $\sigma^2(t, x)$, $\lambda(t, x)$ and $\theta(t, x)$ are all $C^{2+\alpha}$ and that $\frac{1}{2}\sigma^2(t, x) + x(\lambda/\theta)$ is uniformly bounded above and below by multiples of x^2. The results that we present in the following theorems are new even in these cases of restricted convexity. When the payoff function $u(x)$ is of mixed convexity, assumptions (P1) through (P4) are required for our results to hold and we are unaware of existing regularity results from which these conditions would immediately follow. Theorem 3.1 below shows that selecting PDE (5) in conjunction with the stopping times above was the right thing to do. Before proceeding to this result we must establish a few basic consequences of the stopping time assumptions, which will be used throughout the remainder of the paper. The first lemma controls the number of rebalancing times that occur before time T.

LEMMA 3.1. *Under assumptions* (T1) *and* (T2), $E(N) = O(\delta^{-1})$ *and* $E(N^2) = O(\delta^{-2})$.

PROOF. For any positive integer n, assumption (T1) implies that

$$\theta_0^2\delta^2 E\left[\left(\sum_{i=1}^{n} \mathbb{1}(N > i)\right)^2\right] \le E\left[\left(\sum_{i=1}^{n} E\left[(\tau_{i+1} - \tau_i)|\mathscr{F}_{\tau_i}\right]\mathbb{1}(N > i)\right)^2\right]. \tag{7}$$

We will bound (7) with two terms, the first of which is

$$E\left[\left(\sum_{i=1}^{n} (\tau_{i+1} - \tau_i)\mathbb{1}(N > i)\right)^2\right] \le T^2.$$

The second term is

$$\begin{aligned} &E\left[\left(\sum_{i=1}^{n} \left\{(\tau_{i+1} - \tau_i) - E\left[(\tau_{i+1} - \tau_i)|\mathscr{F}_{\tau_i}\right]\right\}\mathbb{1}(N > i)\right)^2\right] \\ &\quad = E\left[\sum_{i=1}^{n} \operatorname{Var}\left[\tau_{i+1} - \tau_i|\mathscr{F}_{\tau_i}\right]\mathbb{1}(N > i)\right] \\ &\quad \le \delta^2 E[Z^2]^{1/2} E\left[\left(\sum_{i=1}^{n} \mathbb{1}(N > i)\right)^2\right]^{1/2} \end{aligned}$$

for some $Z = Z(\delta) \in L^8(\Omega)$ with $\sup_\delta E[|Z(\delta)|^2] < \infty$. This follows from the fact that $\operatorname{Var}[\tau_{i+1} - \tau_i|\mathscr{F}_{\tau_i}] = O_i(\delta^2)$ and Hölder's inequality. Then (7) implies

$$\theta_0^2\delta^2 E\left[\left(\sum_{i=1}^{n} \mathbb{1}(N > 1)\right)^2\right] \le 2T^2 + 2\delta^2 E[Z^2]^{1/2} E\left[\left(\sum_{i=1}^{n} \mathbb{1}(N > i)\right)^2\right]^{1/2}$$

for all n, and consequently,

$$E\left[\left(\sum_{i=1}^{n} \mathbb{1}(N > i)\right)^2\right] \le 4\frac{E(Z^2)}{\theta_0^4} + 8\frac{T^2}{\theta_0^2\delta^2}.$$

Since $(\sum_{i=1}^{n}\mathbb{1}(N > i))^2 = \sum_{i=1}^{\infty}(n \wedge i)^2\mathbb{1}(N = i + 1)$, the result follows from Fatou's lemma. □

The next lemma utilizes this control of N to bound terms in the form of a martingale difference array.

LEMMA 3.2. *Let M_i be a martingale difference array with respect to $\mathscr{F}_{\tau_i}$ such that*

$$E\left[M_i^2 | \mathscr{F}_{\tau_i}\right] \le \delta^\alpha X,$$

where $X \in L^2(\Omega)$. Then

$$E\left[\left(\sum_{i=1}^{N} M_i\right)^2\right] \le C\delta^{\alpha-1}\sqrt{E(X^2)}. \tag{8}$$

PROOF. For each positive integer n, we have

$$\begin{aligned} E\left[\left(\sum_{i=1}^{n} M_i \mathbb{1}(N > i)\right)^2\right] &= E\left[\sum_{i=1}^{n} \mathbb{1}(N > i) E\left[M_i^2 | \mathscr{F}_{\tau_i}\right]\right] \\ &\le \delta^\alpha \sqrt{E(X^2)}\sqrt{E(N^2)}, \end{aligned}$$

where the last equality follows from Hölder's inequality. The result now follows from Lemma 3.1 and by noting that an L^2 bounded martingale converges in L^2. □

We now proceed to the first theorem, which establishes the modified "Black–Scholes" PDE.

THEOREM 3.1. *Suppose that portfolio rebalancing is given by* (2) *with stopping times which satisfy* (T1) *through* (T3) *and with $h(t, x)$ the solution to* (5) *with initial condition* (6) *satisfying* (P1). *Then with $\kappa = \delta^{1/2}$ the payoff of the replicating strategy converges to the desired payoff in probability*: $\hat{u}(X) \to_P u(X_T)$.

PROOF. Using Lemma 3.1, (T1) and (T2), one can easily check that $\sum_{i=1}^{N}(\tau_i - \tau_{i-1})^2$ converges to 0 in L^1 as $\delta \to 0$. As a result, $\sup_{1 \le i \le N}(\tau_i - \tau_{i-1}) \to_P 0$. Since h_x is continuous, we have

$$\sum_{\tau_i < T} h_x(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i}) \to_P \int_0^T h_x(t, X_t)\, dX_t.$$

We will show that

$$\delta^{1/2} \sum_{\tau_i < T} h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i})|X_{\tau_{i+1}} \to_P \int_0^T |h_{xx}(t, X_t)|X_t \frac{\lambda}{\theta}(t, X_t)\, dt.$$

The result then follows from Itô's formula using PDE (5). The first step is to replace $X_{\tau_{i+1}}$ in the summand above with X_{τ_i} by noting that

$$\delta^{1/2} \sum_{\tau_i < T} \left| h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) \right| \left(X_{\tau_{i+1}} - X_{\tau_i} \right) \to_P 0,$$

which follows from the fact that $\sum |h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i})|(X_{\tau_{i+1}} - X_{\tau_i})$ is stochastically bounded. Second,

$$\begin{aligned} \delta^{1/2} \sum_{\tau_i < T} \Big| h_x(\tau_{i+1}, X_{\tau_{i+1}}) &- h_x(\tau_i, X_{\tau_i}) \\ &- h_x(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i}) \Big| X_{\tau_i} \to_P 0. \end{aligned} \tag{9}$$

This follows from the fact that

$$\begin{aligned} &\left| h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i}) \right| \\ &\quad \le \left| h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_{i+1}}) \right| \\ &\qquad + \left| h_x(\tau_i, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i}) \right| \\ &\quad = \left| h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_{i+1}}) \right| \\ &\qquad + \left| \left\{ \int_0^1 h_{xx}\big(\tau_i, X_{\tau_i} + r(X_{\tau_{i+1}} - X_{\tau_i})\big) - h_{xx}(\tau_i, X_{\tau_i})\, dr \right\} (X_{\tau_{i+1}} - X_{\tau_i}) \right| \\ &\quad = O\left(|\tau_{i+1} - \tau_i| + |X_{\tau_{i+1}} - X_{\tau_i}|^2 \right), \end{aligned}$$

by (P1). Consequently, $\sum |h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})| X_{\tau_i}$ is stochastically bounded yielding (9). It remains to verify that

$$\begin{aligned} \delta^{1/2} \sum_{\tau_i < T} &\left| h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i}) \right| X_{\tau_i} \\ &\to_P \int_0^T |h_{xx}(t, X_t)| X_t \frac{\lambda}{\theta}(t, X_t)\, dt. \end{aligned} \tag{10}$$

We rewrite $\delta^{1/2} |h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})| X_{\tau_i}$ as follows:

$$\begin{aligned} &\delta^{1/2} X_{\tau_i} \left| h_{xx}(\tau_i, X_{\tau_i}) \right| \left\{ |X_{\tau_{i+1}} - X_{\tau_i}| E\left[|X_{\tau_{i+1}} - X_{\tau_i}| \middle| \mathscr{F}_{\tau_i} \right] \right\} \\ &\quad - \left| h_{xx}(\tau_i, X_{\tau_i}) \right| X_{\tau_i} \frac{\lambda}{\theta}(\tau_i, X_{\tau_i}) \left\{ (\tau_{i+1} - \tau_i) - E\left[\tau_{i+1} - \tau_i \middle| \mathscr{F}_{\tau_i} \right] \right\} \\ &\quad + \left| h_{xx}(\tau_i, X_{\tau_i}) \right| X_{\tau_i} \frac{\lambda}{\theta}(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i) \\ &\quad + \left| h_{xx}(\tau_i, X_{\tau_i}) \right| X_{\tau_i} \left\{ 1 - \frac{\lambda}{\theta}(\tau_i, X_{\tau_i}) \right\} o_i(\delta^{3/2}). \end{aligned}$$

Each of the first two terms constitute a martingale difference array and satisfies the condition of Lemma 3.2 with $\alpha = 2$. Thus the sum of these two terms over the partition converges to 0 in probability. It can be checked that the sum of the last term over the partition converges to 0 in probability as well. Therefore, we have (10). □

Next we compute the replication error in the L^2 sense. We define the leading order conditional variance matrix V as

$$
\begin{aligned}
\operatorname{Var}\Big[\big(X_{\tau_{i+1}} - X_{\tau_i}\big)^2 | \mathscr{F}_{\tau_i}\Big] &= V_1(\tau_i, X_{\tau_i})\delta^2 + o_i(\delta^2),\\
\operatorname{Var}\Big[\tau_{i+1} - \tau_i | \mathscr{F}_{\tau_i}\Big] &= V_2(\tau_i, X_{\tau_i})\delta^2 + o_i(\delta^2),\\
\operatorname{Var}\Big[\delta^{1/2}|X_{\tau_{i+1}} - X_{\tau_i}| | \mathscr{F}_{\tau_i}\Big] &= V_3(\tau_i, X_{\tau_i})\delta^2 + o_i(\delta^2),\\
\operatorname{Cov}\Big[\big(X_{\tau_{i+1}} - X_{\tau_i}\big)^2, (\tau_{i+1} - \tau_i) | \mathscr{F}_{\tau_i}\Big] &= V_{12}(\tau_i, X_{\tau_i})\delta^2 + o_i(\delta^2),\\
\operatorname{Cov}\Big[(\tau_{i+1} - \tau_i), \delta^{1/2}|X_{\tau_{i+1}} - X_{\tau_i}| | \mathscr{F}_{\tau_i}\Big] &= V_{23}(\tau_i, X_{\tau_i})\delta^2 + o_i(\delta^2),\\
\operatorname{Cov}\Big[\big(X_{\tau_{i+1}} - X_{\tau_i}\big)^2, \delta^{1/2}|X_{\tau_{i+1}} - X_{\tau_i}| | \mathscr{F}_{\tau_i}\Big] &= V_{13}(\tau_i, X_{\tau_i})\delta^2 + o_i(\delta^2).
\end{aligned}
\tag{11}
$$

Again, we assume the functions on the right-hand side of (11) are continuous and have polynomial growth.

THEOREM 3.2. *Suppose that the rebalancing times satisfy* (T1) *through* (T7), *and the solution to* (5) $h(t, x)$ *satisfies* (P1) *through* (P4). *With* $\kappa = \delta^{1/2}$ *the replication error is given by*

$$
E\Big[\big(\hat{u}(X) - u(X_T)\big)^2\Big] = \delta E\int_0^T K(t, X_t)\theta^{-1}(t, X_t)\,dt + o(\delta), \tag{12}
$$

where

$$
\begin{aligned}
K = {}& \tfrac{1}{4}h_{xx}^2 V_1 + h_t^2 V_2 + x^2 h_{xx}^2 V_3 + h_{xx}h_t V_{12}\\
&+ x h_{xx}|h_{xx}|V_{13} + 2x h_t |h_{xx}| V_{23}.
\end{aligned}
\tag{13}
$$

The proof of Theorem 3.2 is extremely lengthy and is quarantined in the next section. The set of conditions (T) which yield the asymptotics for the replication error, while numerous, are in fact relatively easy to check in many natural situations. To illustrate this point, we conclude this section by using Theorems 3.1 and 3.2 in the context of the stock level strategies considered in Henrotte (1992) where the analog of Theorem 3.1 was established and a form for the replication error for a specific configuration of price levels was conjectured. We will also present a similar treatment to the class of continuous stochastic time change strategies, which includes the deterministic time change strategies considered in Grannan and Swindle (1994). Therefore, not only will the following enterprise clarify the use of the results in this paper, but it will establish heretofore unknown expressions for the replication error for this class of strategies presented in Henrotte (1992).

EXAMPLE 3.1. We begin by defining exactly what is meant by stock level strategies. Consider a function $f(x)$ which is smooth, positive and strictly increasing. Let $\alpha = 1 + \delta^{1/2}$ and define a sequence of stock levels (a partition of $[0, \infty)$) via

$$\{x_n = f^{-1}[\alpha^n f(X_0)],\ n = 0, \pm 1, \pm 2, \dots\}, \tag{14}$$

where X_0 is the initial stock price. We will take $\mu(t, x) = \mu x$ and $\sigma(t, x) = \sigma x$, where μ and σ are constant, so that X_t is a geometric Brownian motion. These stock levels are the levels at which portfolio rebalancings will occur. If $f(x) = x$, then the levels are set as $\alpha^n X_0$. Such exponentially growing gaps are consistent with the scaling properties of the geometric Brownian motion.

We define the rebalancing times via

$$\tau_{i+1} = \inf\{t > \tau_i\colon X_t = x_{n+1} \text{ or } X_t = x_{n-1} | X_{\tau_i} = x_n\} \tag{15}$$

so that a rebalancing occurs when the price traverses one of the gaps between levels.

We will now use Theorems 3.1 and 3.2 to identify a limiting hedging strategy and its associated replication error for a layout of intervals prescribed by the function $f(x)$. It will be useful at times to work with $y = \log(x)$, and we will let $\Delta_n = x_{n+1} - x_{n-1}$ and $\hat{\Delta}_n = y_{n+1} - y_{n-1}$. From (14), we observe

$$f(x_{n+1}) - f(x_{n-1}) = (\alpha - \alpha^{-1}) f(x_n) = 2\delta^{1/2} f(x_n) + O(\delta), \tag{16}$$

while Taylor expansion yields $f(x_{n+1}) - f(x_{n-1}) = f'(x_n)(x_{n+1} - x_{n-1}) + O(\delta)$. Therefore (16) and the Taylor expansion of $\log(x)$ result in

$$\Delta_n = 2\delta^{1/2} \frac{f(x_n)}{f'(x_n)} + O(\delta) \quad \text{and} \quad \hat{\Delta}_n = 2\delta^{1/2} \frac{f(x_n)}{f'(x_n)} \frac{1}{x_n} + O(\delta). \tag{17}$$

We will begin by dealing with conditions on the moments of $\tau_{i+1} - \tau_i$, namely conditions (T1), (T2) and (T4). We claim that

$$\begin{aligned} E\big[(\tau_{i+1} - \tau_i) | X_{\tau_i} = x_n\big] &= \frac{\hat{\Delta}_n^2}{4\sigma^2} + o_i(\delta^{3/2}) \\ &= \delta \left[\frac{f(x_n)}{\sigma x_n f'(x_n)} \right]^2 + o_i(\delta^{3/2}), \end{aligned} \tag{18}$$

$$\begin{aligned} E\big[(\tau_{i+1} - \tau_i)^2 | X_{\tau_i} = x_n\big] &= \frac{5}{48\sigma^4} \hat{\Delta}_n^4 + o_i(\delta^2) \\ &= \delta^2 \frac{5}{3} \left[\frac{f(x_n)}{\sigma x_n f'(x_n)} \right]^4 + o_i(\delta^2), \end{aligned} \tag{19}$$

where the second equalities are an application of (17). To verify (18) and (19), Girsanov's theorem and expressions for hitting times of the standard Brownian motion yield

$$
\begin{aligned}
&E\Big[\exp\{-\nu(\tau_{i+1}-\tau_i)\}\mathbb{1}(X_{\tau_{i+1}}=x_{n-1})|X_{\tau_i}=x_n\Big] \\
&\quad=\exp\left[\frac{1}{2}\left(1-\frac{2\mu}{\sigma^2}\right)(y_n-y_{n-1})\right] \\
&\qquad\times\frac{\sinh\Big[(y_{n+1}-y_n)/\sigma\sqrt{2\nu+\big[\big(\mu-\frac{1}{2}\sigma^2\big)/\sigma\big]^2}\,\Big]}{\sinh\Big[(y_{n+1}-y_{n-1})/\sigma\sqrt{2\nu+\big[\big(\mu-\frac{1}{2}\sigma^2\big)/\sigma\big]^2}\,\Big]}
\end{aligned}
\tag{20}
$$

[see Karatzas and Shreve (1991)]. Differentiating with respect to ν and expanding yields

$$
E\Big[(\tau_{i+1}+\tau_i)\,\mathbb{1}\big(X_{\tau_{i+1}}=x_{n-1}\big)|X_{\tau_i}=x_n\Big]\sim\frac{\hat{\Delta}_n^2}{8\sigma^2}.
\tag{21}
$$

An identical calculation for $\mathbb{1}(X_{\tau_{i+1}}=x_{n+1})$ combines to yield (18) and, therefore, (T1). This also shows that (T5) and (T6) hold. Second derivatives of (20) with respect to ν yield (19) and (T2). A fourth derivative results in (T4).

Next we proceed to the moments of $X_{\tau_{i+1}}-X_{\tau_i}$. Note that

$$
P\big(X_{\tau_{i+1}}=x_{n+1}|X_{\tau_i}=x_n\big)=\frac{\phi(x_n)-\phi(x_{n-1})}{\phi(x_{n+1})-\phi(x_{n-1})},
\tag{22}
$$

where $\phi(x)=x^{1-(2\mu/\sigma^2)}$ is a scale function for X_t. Also note that $x_{n+1}-x_n=\Delta_n/2+O(\delta)$, as well as $x_n-x_{n-1}=\Delta_n/2+O(\delta)$. Then (17) and Taylor expansion of (22) yield $\lambda(t,x)=f(x)/f'(x)$ in (T3). Verification of (T7) is straightforward. We have now verified the conditions for Theorem 3.1 [aside from the conditions on the PDE (P) which we do not address, and take as conditions on the payoff $u(x)$]. So, the limiting replication hedging strategy for these stopping times is given by the solution to

$$
h_t+\frac{1}{2}\sigma^2x^2\left[1+\frac{xf'}{f}(x)\operatorname{sign}(h_{xx})\right]h_{xx}=0.
\tag{23}
$$

This result was obtained by Henrotte (1991) for the case where $u(x)$ is convex, in which case $\operatorname{sign}(h_{xx})=1$.

Next we will extract the leading order behavior of the replication error for the stock level strategies, which constitutes a previously unknown result. The only additional ingredient which is required is a calculation of the covariance matrix given in (11). In the case of these stock level strategies, this is particularly straightforward. Using (22) and expanding in δ yields that $V_1, V_3, V_{12}, V_{13}, V_{23}$ are zero. The only term that contributes to the replication error in the leading order term is

$$
V_2=\eta-\theta^2=\frac{2}{3}\left[\frac{f}{\sigma xf'}\right]^4.
$$

Therefore, the replication error is

$$E\Big[\big(\hat{u}(X) - u(X_T)\big)^2\Big] = \delta\frac{2}{3}E\int_0^T\left[\frac{f}{\sigma xf'}\right]^2 h_t^2\,dt + o(\delta).$$

EXAMPLE 3.2. Consider a continuous time change $Y_t = X_{A_t}$ where

$$A_t = \inf\left[r > 0\colon \int_0^r \theta^{-1}(s, X_s)\,ds > t\right]. \tag{24}$$

As in (T1), $\theta(t, x)$ is continuous and bounded below by $\theta_0 > 0$. Furthermore, it has a polynomial growth order in x. It can be shown [see, e.g., Ethier and Kurtz (1985)] that (Y, A) is an $\mathbb{R}^2$ diffusion

$$\begin{aligned} dY_t &= \sigma\cdot\theta^{1/2}(A_t, Y_t)\,dB_t + \mu\cdot\theta(A_t, Y_t)\,dt,\\ dA_t &= \theta(A_t, Y_t)\,dt, \end{aligned}$$

where B is a standard Brownian motion defined by

$$B_t = \int_0^t \theta^{-1}(A_s, Y_s)\,dW_{A_s}.$$

Rebalancing times are uniformly spaced in the changed time domain, that is, the ith rebalancing time will be $A_{i\delta}$. Then

$$\lambda(t,) = \sqrt{\frac{2}{\pi}}\,\sigma\cdot\theta^{1/2}(t, x),$$

and by Theorem 3.1 PDE (5) becomes

$$h_t(t, x) + \frac{1}{2}\sigma^2(t, x)h_{xx}(t, x) + \sqrt{\frac{2}{\pi}}\,x\sigma\cdot\theta^{-1/2}(t, x)|h_{xx}|(t, x) = 0.$$

Next, to obtain the expression for the leading order replication error, we find

$$\begin{aligned} V_1(t, x) &= 2\sigma^4\theta^2(t, x),\\ V_3(t, x) &= \left(1 - \frac{2}{\pi}\right)\sigma^2\theta(t, x),\\ V_{13}(t, x) &= \sqrt{\frac{2}{\pi}}\,\sigma^3\theta^{3/2}(t, x), \end{aligned}$$

while the other components of V vanish. Therefore, by Theorem 3.2 we have

$$E\Big[\big(\hat{u}(X) - u(X_T)\big)^2\Big] = \delta E\int_0^T K(t, X_t)\theta^{-1}(t, X_t)\,dt + o(\delta),$$

where

$$K\theta^{-1} = \sigma^2\left(\frac{\sigma^2\theta}{2} + \left(1 - \frac{2}{\pi}\right)x^2\right)h_{xx}^2 + \sqrt{\frac{2}{\pi}}\,\sigma^3\theta^{1/2}xh_{xx}|h_{xx}|.$$

For other situations, for example, a hybrid of the temporal strategies and the stock level strategies where rebalancing occurs upon hitting a rectangle consisting of two stock levels of order $\delta^{1/2}$ apart, and a time boundary of order δ ahead, all the terms of the covariance matrix are nontrivial. More complex strategies involve more work in evaluating moments. This is the only significant limitation in applications of Theorems 3.1 and 3.2.

4. Proof of Theorem 3.2. We prove Theorem 3.2 in several steps. Using the telescoping sum, we rewrite $u(X_T) - \hat{u}(X_T)$ as

$$\sum_{\tau_i < T} \left\{ h(\tau_{i+1}, X_{\tau_{i+1}}) - h(\tau_i, X_{\tau_i}) - h_x(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i}) \right\} \\ + \sum_{\tau_i < T} \delta^{1/2} \left| h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) \right| X_{\tau_{i+1}}.$$

Our goal is to rearrange this into

$$\sum_{\tau_i < T} \tfrac{1}{2} h_{xx}(\tau_i, X_{\tau_i}) \left\{ (X_{\tau_{i+1}} - X_{\tau_i})^2 - E\left[(X_{\tau_{i+1}} - X_{\tau_i})^2 | \mathscr{F}_{\tau_i} \right] \right\} \\ + \sum_{\tau_i < T} h_t(\tau_i, X_{\tau_i}) \left\{ \tau_{i+1} - \tau_i - E\left[\tau_{i+1} - \tau_i | \mathscr{F}_{\tau_i} \right] \right\} \\ + \sum_{\tau_i < T} X_{\tau_i} \left| h_{xx}(\tau_i, X_{\tau_i}) \right| \left\{ \delta^{1/2} |X_{\tau_{i+1}} - X_{\tau_i}| - E\left[\delta^{1/2} |X_{\tau_{i+1}} - X_{\tau_i}| | \mathscr{F}_{\tau_i} \right] \right\}$$

by eliminating negligible terms. This is a sum of martingale difference arrays, and hence (12) will follow from continuity of the conditional variance matrix (11).

DEFINITION 4.1. We will say that $\{J_i, 0 \le i < N\}$ is *δ-negligible*, or simply J_i is *δ-negligible*, if

$$E \left| \sum_{i=0}^{N-1} J_i \right|^2 = o(\delta).$$

Terms that are δ-negligible do not contribute to the leading order of the replication error. The following four results will be useful in establishing when terms are in fact δ-negligible.

LEMMA 4.1. *Consider a sequence of $\mathscr{F}_{\tau_i}$-measurable random variables $X_i(\delta)$, such that $\sup_i E[|X_i(\delta)|^4] \to 0$ and satisfy $\sup_{i<N} |X_i(\delta)| \in L^8(\Omega)$ uniformly in δ. Then $\delta^{3/2} X_i(\delta)$ is δ-negligible. So terms that are $o_i(\delta^{3/2})$ are in fact δ-negligible.*

PROOF. The martingale difference array

$$M_i = X_i(\delta)\frac{\delta^{3/2}}{E\left[\tau_{i+1}-\tau_i|\mathscr{F}_{\tau_i}\right]}\left\{\tau_{i+1}-\tau_i - E\left[\tau_{i+1}-\tau_i|\mathscr{F}_{\tau_i}\right]\right\}$$

satisfies

$$E\left[M_i^2|\mathscr{F}_{\tau_i}\right] \le \delta^3 \sup_i |X_i(\delta)|^2 \frac{Z}{\theta_0^2},$$

where $Z \in L^6$. This follows from $\mathrm{Var}[\tau_{i+1}-\tau_i|\mathscr{F}_{\tau_i}] = O_i(\delta^2)$. Then Lemma 3.2 implies that M_i is δ-negligible. It remains to control

$$\sum_{i=0}^{N-1} X_i(\delta)\frac{\delta^{3/2}}{E\left[\tau_{i+1}-\tau_i|\mathscr{F}_{\tau_i}\right]}(\tau_{i+1}-\tau_i). \tag{25}$$

Since τ_i's consist of a partition of $[0,T]$ and $\theta \ge \theta_0 > 0$, the square of (25) is bounded by $\delta\sum_{i=0}^{N-1}|X_i(\delta)|^2(\tau_{i+1}-\tau_i)$. Then it suffices to show that $\sum_{i=0}^{N-1}|X_i(\delta)|^2E[\tau_{i+1}-\tau_i|\mathscr{F}_{\tau_i}]$ converges to 0 in L^1. Put $Y_i(\delta)=E[\tau_{i+1}-\tau_i|\mathscr{F}_{\tau_i}]/\delta$ and pick a nonnegative function $q(\delta)$ such that $q(\delta)\to\infty$ as $\delta\to 0$ and $q(\delta)\sup_i E[|X_i(\delta)|^4]^{1/2}\to 0$. Then

$$\begin{aligned}&\sum_{i=0}^{N-1}|X_i(\delta)|^2E\left[\tau_{i+1}-\tau_i|\mathscr{F}_{\tau_i}\right]\\ &\qquad \le N\delta\sup_{i<N}|X_i(\delta)|^2Y_i(\delta)\mathbb{1}(N>n(\delta)) + \delta\sum_{i=0}^{n(\delta)}|X_i(\delta)|^2Y_i(\delta),\end{aligned}$$

where $n(\delta)$ is the smallest integer greater than $E(N)+q(\delta)\sqrt{\mathrm{Var}(N(\delta))}$. The expectation of the first term is bounded by

$$\delta E[N^2]^{1/2}E\left[\sup_{i<N}|X_i(\delta)|^8\right]^{1/4}E\left[\sup_{i<N}|Y_i(\delta)|^8\right]^{1/8}P(N>n(\delta))^{1/8},$$

which tends to 0 as $\delta\to 0$. Next, we observe that

$$\delta E\left[\sum_{i=0}^{n(\delta)}|X_i(\delta)|^2Y_i(\delta)\right] \le \delta n(\delta)\sup_i E\left[|X_i(\delta)|^4\right]^{1/2}\sup_i E\left[|Y_i(\delta)|^2\right]^{1/2}$$

tends to 0 as well. □

The next lemma combines the results of the previous lemma and Lemma 3.2.

LEMMA 4.2. *Consider a sequence of* $\mathscr{F}_{\tau_{i+1}}$*-measurable random variables* $X_i(\delta)$ *such that* $E[X_i(\delta)|\mathscr{F}_{\tau_i}] = o_i(\delta^{3/2})$ *and* $E[X_i^2(\delta)|\mathscr{F}_{\tau_i}] = O_i(\delta^{2+\varepsilon})$ *for some* $\varepsilon>0$. *Then* $X_i(\delta)$ *is* δ*-negligible.*

PROOF. Here $E[X_i(\delta)|\mathscr{F}_{\tau_i}]$ being δ-negligible follows from the previous lemma. The martingale difference array

$$X_i(\delta) - E\left[X_i(\delta)|\mathscr{F}_{\tau_i}\right]$$

is also δ-negligible due to Lemma 3.2. □

By virtue of Lemma 4.2, we will check conditional moments to identify δ-negligible terms. The next lemma facilitates estimating conditional moments.

LEMMA 4.3. *Let ξ_t be an $\mathscr{F}_t$-measurable process dominated by $C(1+|X_t|^p)$ for some $p>0$ and all $t\in[0,T]$ almost surely. Then*

$$E\left[\left(\int_{\tau_i}^{\tau_{i+1}}\xi_s\,ds\right)^2\middle|\mathscr{F}_{\tau_i}\right]=O_i(\delta^2). \tag{26}$$

PROOF. First, note that

$$\begin{aligned} P\Big(\sup_{\tau_i\le t\le\tau_{i+1}}|X_t-X_{\tau_i}|>1|\mathscr{F}_{\tau_i}\Big) &\le 2^4E\left[\sup_{\tau_i\le t\le t_{i+1}}\left|\int_{\tau_i}^t\sigma(s,X_s)\,dW_s\right|^4\middle|\mathscr{F}_{\tau_i}\right]\\ &\quad+2^4E\left[\sup_{\tau_i\le t\le\tau_{i+1}}\left|\int_{\tau_i}^t\mu(s,X_s)\,ds\right|^4\middle|\mathscr{F}_{\tau_i}\right]\\ &\le O_i(\delta^{3/2}). \end{aligned}$$

The first inequality follows from Chebyshev's inequality, and the last line follows from (T4), the Burkholder–Davis–Gundy inequality and Hölder's inequality. Then

$$P\Big(\sup_{\tau_i\le t\le t_{i+1}}|\xi_t-\xi_{\tau_i}|>C_{\tau_i}|\mathscr{F}_{\tau_i}\Big)\le O_i(\delta^{3/2}), \tag{27}$$

where $C_{\tau_i}=2C(1+(1+|X_{\tau_i}|)^p$, since $|\xi_t-\xi_{\tau_i}|\le C(1+|X_t|^p)+C(1+|X_{\tau_i}|^p)$. On the event $\{\sup_{\tau_i\le t\le\tau_{i+1}}|\xi_t-\xi_{\tau_i}|\le C_{\tau_i}\}$, we have $\sup_{\tau_i\le t\le\tau_{i+1}}|\xi_t|\le|\xi_{\tau_i}|+C_{\tau_i}$, and the result follows by (T2). On the complementary event, Hölder's inequality shows that (26) is dominated by

$$\begin{aligned} &E\big[(\tau_{i+1}-\tau_i)^4|\mathscr{F}_{\tau_i}\big]^{1/2}E\Big[\sup_{\tau_i\le t\le\tau_{i+1}}|\xi_t|^6|\mathscr{F}_{\tau_i}\Big]^{1/6}P\Big[\sup_{\tau_i\le t\le\tau_{i+1}}|\xi_t-\xi_{\tau_i}|>C_{\tau_i}|\mathscr{F}_{\tau_i}\Big]^{1/3}\\ &\qquad=O_i(\delta^2) \end{aligned}$$

by (T4) and (27). Therefore we have (26). □

As an application of Lemma 4.3, $E[\sup_{\tau_i\le t\le\tau_{i+1}}|X_t-X_{\tau_i}|^4|\mathscr{F}_{\tau_i}]$ is of $O_i(\delta^2)$, since it is bounded by

$$c\cdot 2^4E\left[\left(\int_{\tau_i}^{\tau_{i+1}}\sigma^2(t,X_t)\,dt\right)^2\middle|\mathscr{F}_{\tau_i}\right]+T^22^4E\left[\left(\int_{\tau_i}^{\tau_{i+1}}\mu^2(t,X_t)\,dt\right)^2\middle|\mathscr{F}_{\tau_i}\right],$$

where c is the constant in the Burkholder–Davis–Gundy inequality.

LEMMA 4.4. *Let H_t be an $\mathscr{F}_t$-measurable process bounded by $C(1+|X_t|^p)$ for some $p>0$ and all $t\in[0,T]$ almost surely. In addition, assume that*

$$E\Big[\sup_{\tau_i\le t\le\tau_{i+1}}|H_t-H_{\tau_i}|^4|\mathscr{F}_{\tau_i}\Big]=o_i(1). \tag{28}$$

Then $\int_{\tau_i}^{\tau_{i+1}}(X_t-X_{\tau_i})H_t\,dt$ is δ-negligible.

PROOF. First, we replace H_t by H_{τ_i} by showing that the following is δ-negligible:

$$\int_{\tau_i}^{\tau_{i+1}}(X_t-X_{\tau_i})(H_t+H_{\tau_i})\,dt. \tag{29}$$

Both

$$E\bigg[\sup_{\tau_i\le t\le\tau_{i+1}}\bigg|\int_{\tau_i}^{t}\sigma(s,X_s)\,dW_s\bigg|^4\bigg|\mathscr{F}_{\tau_i}\bigg]\quad\text{and}\quad E\bigg[\bigg|\int_{\tau_i}^{\tau_{i+1}}|\mu(t,X_t)|\,dt\bigg|^4\bigg|\mathscr{F}_{\tau_i}\bigg]$$

are at most $O_i(\delta^2)$. The estimate of the first term follows from the Burkholder–Davis–Gundy inequality and Lemma 4.3. Thus

$$E\Big[\sup_{\tau_i\le t\le\tau_{i+1}}|X_t-X_{\tau_i}|^4|\mathscr{F}_{\tau_i}\Big]=O_i(\delta^2). \tag{30}$$

Using Hölder's inequality, the conditional expectation of (29) with respect to $\mathscr{F}_{\tau_i}$ is seen to be bounded above by

$$\begin{aligned}&E\Big[\sup_{\tau_i\le t\le\tau_{i+1}}|H_t-H_{\tau_i}|^4|\mathscr{F}_{\tau_i}\Big]^{1/4}E\Big[\sup_{\tau_i\le t\le\tau_{i+1}}|X_t-X_{\tau_i}|^4|\mathscr{F}_{\tau_i}\Big]^{1/4}\\&\qquad\times E\big[(\tau_{i+1}-\tau_i)^2|\mathscr{F}_{\tau_i}\big]^{1/2},\end{aligned}$$

which is of $o_i(\delta^{3/2})$ by (T4), (30) and (28). Similarly, we find that the second conditional moment is $O_i(\delta^{9/4})$. Then by Lemma 4.2, (29) is δ-negligible. The remaining task is to control

$$\begin{aligned}H_{\tau_i}\int_{\tau_i}^{\tau_{i+1}}(X_t-X_{\tau_i})\,dt&=H_{\tau_i}(X_{\tau_{i+1}}-X_{\tau_i})(\tau_{i+1}-\tau_i)\\&\qquad-\int_{\tau_i}^{\tau_{i+1}}(t-\tau_i)H_{\tau_i}\,dX_t.\end{aligned} \tag{31}$$

The first term of (31) is δ-negligible due to (T6) and Lemma 4.2. The mean of the second term is of $o_i(\delta^{3/2})$ since the martingale part vanishes. The second moment being $O_i(\delta^{2+\varepsilon})$ for some $\varepsilon>0$ can be easily checked. Therefore (31) is δ-negligible. □

Now we start eliminating δ-negligible terms from the replication error.

PROPOSITION 4.1. *Under the hypotheses of Theorem* 3.2,

$$E\Big[\big(X_{\tau_{i+1}} - X_{\tau_i}\big)^2 - \sigma^2(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i)\Big|\mathscr{F}_{\tau_i}\Big]$$

and

$$\begin{aligned}(32)\qquad &\int_{\tau_i}^{\tau_{i+1}} h_x(t, X_t) - h_x(\tau_i, X_{\tau_i})\,dX_t \\ &\quad - \tfrac{1}{2} h_{xx}(\tau_i, X_{\tau_i}) \times \Big[\big(X_{\tau_{i+1}} - X_{\tau_i}\big)^2 - \sigma^2(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i)\Big]\end{aligned}$$

are δ-negligible.

PROOF. By Itô's formula, we may decompose $\int_{\tau_i}^{\tau_{i+1}} \sigma^2(t, X_t) - \sigma^2(\tau_i, X_{\tau_i})\,dt$ into

$$\begin{aligned}&\int_{\tau_i}^{\tau_{i+1}} \int_{\tau_i}^{t} (\sigma^2) x(s, X_s) - (\sigma^2)_x(\tau_i, X_{\tau_i})\,dX_s\,dt \\ &\quad + \int_{\tau_i}^{\tau_{i+1}} (\sigma^2)_x(\tau_i, X_{\tau_i})(X_t - X_{\tau_i})\,dt \\ &\quad + \int_{\tau_i}^{\tau_{i+1}} \int_{\tau_i}^{t} \tfrac{1}{2}\sigma^2(\sigma^2)_{xx}(s, X_s) + (\sigma^2)_t(s, X_s)\,ds\,dt.\end{aligned}$$

The second term is δ-negligible due to Lemma 4.4, and the others being δ-negligible follows from Lemma 4.3. Therefore, to show that $E[(X_{\tau_{i+1}} - X_{\tau_i})^2 - \sigma^2(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i)|\mathscr{F}_{\tau_i}]$ is δ-negligible, it suffices to show the same for

$$\begin{aligned}&\tfrac{1}{2} E\Big[\big(X_{\tau_{i+1}} - X_{\tau_i}\big)^2 - \int_{\tau_i}^{\tau_{i+1}} \sigma^2(t, X_t)\,dt\Big|\mathscr{F}_{\tau_i}\Big] \\ &\qquad = E\Big[\int_{\tau_i}^{\tau_{i+1}} (X_t - X_{\tau_i})\mu(t, X_t)\,dt\Big|\mathscr{F}_{\tau_i}\Big].\end{aligned}$$

First note that $\int_{\tau_i}^{\tau_{i+1}} (X_t - X_{\tau_i})\mu(t, X_t)\,dt$ is δ-negligible as can be seen by setting $H_t = \mu(t, X_t)$ and applying Lemma 4.4. In addition,

$$(33)\qquad E\Big[\int_{\tau_i}^{\tau_{i+1}} (X_t - X_{\tau_i})\mu(t, X_t)\,dt\Big|\mathscr{F}_{\tau_i}\Big] - \int_{\tau_i}^{\tau_{i+1}} (X_t - X_{\tau_i})\mu(t, X_t)\,dt$$

is a martingale difference array. To apply Lemma 3.2 with M_i being (33), we must verify that $E[M_i^2|\mathscr{F}_{\tau_i}]$ is bounded above by $\delta^{2+\varepsilon}X$ for some $\varepsilon > 0$, which follows from Hölder's inequality and the polynomial growth condition on $\mu(t, x)$.

Next we show that (32) is δ-negligible. By Itô's formula and Lemma 4.4, it can be shown that

$$h_{xx}(\tau_i, X_{\tau_i}) \int_{\tau_i}^{\tau_{i+1}} \sigma^2(t, X_t)\,dt - h_{xx}(\tau_i, X_{\tau_i})\sigma^2(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i)$$

is also δ-negligible. Then we only need to control

$$(34)\qquad \int_{\tau_i}^{\tau_{i+1}} h_x(t, X_t) - h_x(\tau_i, X_{\tau_i})\,dX_t - \int_{\tau_i}^{\tau_{i+1}} h_{xx}(\tau_i, X_{\tau_i})(X_t - X_{\tau_i})\,dX_t.$$

Rewrite $h_x(t, X_t) - h_x(\tau_i, X_{\tau_i})$ as follows:

$$h_x(t, X_t) - h_x(\tau_i, X_t) + h_x(\tau_i, X_t) - h_x(\tau_i, X_{\tau_i})$$
$$= h_x(t, X_t) - h_x(\tau_i, X_t) + \int_0^1 h_{xx}\big(\tau_i, X_{\tau_i} + r(X_t - X_{\tau_i})\big)\, dr\big(X_t - X_{\tau_i}\big).$$

Noting that $|h_x(t, X_t) - h_x(\tau_i, X_t)|$ is of $O(|t - \tau_i|)$, we are left with

$$\int_{\tau_i}^{\tau_{i+1}} \int_0^1 h_{xx}\big(\tau_i, X_{\tau_i} + r(X_t - X_{\tau_i})\big) - h_{xx}(\tau_i, X_{\tau_i})\, dr\big(X_t - X_{\tau_i}\big)\, dX_t.$$

By the Lipschitz continuity of h_{xx}, the integrand is bounded by $K|X_t - X_{\tau_i}|^2$, for some constant $K > 0$, and therefore (34) is δ-negligible. □

The next difficulty to be surmounted is to deal with the region where h_{xx} is small. It is at the points of inflection, identified by the graphs $(t, F(t))$ [recall condition (P2) on the solution of the PDE for $h(t, x)$], where higher order derivatives may not exist and more work is required. The procedure will be to isolate the regions in the space–time plane near the points of inflection where the nondifferentiability issue arises and to control the replication error separately in a different fashion than in the rest of the plane where the solution is smooth. The following technical lemma will be essential in this endeavor.

LEMMA 4.5. *Let Y be a continuous semimartingale such that we have the following*:

(i) $E(\sup_{\tau_i \le t \le \tau_{i+1}} |Y_t - Y_{\tau_i}| | \mathscr{F}_{\tau_i}) = O_i(\delta^{1/2})$;

(ii) $[Y, Y]_t = \int_0^t \xi_s\, ds$ *where* ξ_t *is dominated by* $C(1 + |X_t|^p)$ *for some* $p > 0$ *and all* $t \in [0, T]$ *almost surely*;

(iii) $y \to E|L_T^y(Y)|^2$ *is bounded in a compact interval, where* $L_T^y(Y)$ *is the local time of* Y *up to time* T *at level* y.

Then $\delta^{2/5} \mathbb{1}(|Y_{\tau_i}| \le \delta^c) \int_{\tau_i}^{\tau_{i+1}} d[Y, Y]_t$ *is* δ*-negligible for each* $c > 1/4$.

PROOF. Note that

$$\begin{aligned}
\mathbb{1}\big(|Y_{\tau_i}| \le \delta^c\big) \int_{\tau_i}^{\tau_{i+1}} d[Y, Y]_t &= \int_{\tau_i}^{\tau_{i+1}} \mathbb{1}\big(|Y_{\tau_i}| \le \delta^c, |Y_t| \le \delta^{1/4}\big)\, d[Y, Y]_t \\
&\quad + \int_{\tau_i}^{\tau_{i+1}} \mathbb{1}\big(|Y_{\tau_i}| \le \delta^c, |Y_t| > \delta^{1/4}\big)\, d[Y, Y]_t \\
&\le \int_{\tau_i}^{\tau_{i+1}} \mathbb{1}\big(|Y_t| \le \delta^{1/4}\big)\, d[Y, Y]_t \\
&\quad + \int_{\tau_i}^{\tau_{i+1}} \mathbb{1}\big(|Y_t - Y_{\tau_i}| \ge \delta^{1/4} - \delta^c\big)\, d[Y, Y]_t.
\end{aligned}$$

Using the space–time change formula [see Corollary 1 on page 168 of Protter (1990)], we have

$$\sum_{\tau_i < T} \int_{\tau_i}^{\tau_{i+1}} \mathbb{1}\left(|Y_t| \le \delta^{1/4}\right) d[Y,Y]_t = \int_{\mathbb{R}} \mathbb{1}(|y| \le \delta^{1/4}) L_T^y(Y)\, dy$$

and therefore

$$\begin{aligned} E&\left(\delta^{2/5} \sum_{\tau_i < T} \int_{\tau_i}^{\tau_{i+1}} \mathbb{1}\left(|Y_t| \le \delta^{1/4}\right) d[Y,Y]_t\right)^2 \\ &\le 2\delta^{21/20} \int \mathbb{1}\left(|y| \le \delta^{1/4}\right) E|L_T^y(Y)|^2\, dy, \end{aligned}$$

which is of $o(\delta)$ by the assumptions above. It remains to show the following is δ-negligible:

$$\begin{aligned} &\delta^{2/5} \int_{\tau_i}^{\tau_{i+1}} \mathbb{1}\left(|Y_t - Y_{\tau_i}| \ge \delta^{1/4} - \delta^c\right) d[Y,Y]_t \\ &\quad \le \delta^{2/5}(\tau_{i+1} - \tau_i)\, \mathbb{1}\left(\sup_{\tau_i \le t \le \tau_{i+1}} |Y_t - Y_{\tau_i}| \ge \delta^{1/4} - \delta^c\right) \sup_{\tau_i \le t \le \tau_{i+1}} \xi_t. \end{aligned} \tag{35}$$

Markov's inequality with the assumptions above implies

$$P\left(\sup_{\tau_i \le t \le t_{i+1}} |Y_t - Y_{\tau_i}| \ge \delta^{1/4} - \delta^c | \mathscr{F}_{\tau_i}\right) = O_i(\delta^{1/4}),$$

and Hölder's inequality yields

$$\begin{aligned} E&\left[(\tau_{i+1} - \tau_i)\, \mathbb{1}\left(\sup_{\tau_i \le t \le \tau_{i+1}} |Y_t - Y_{\tau_i}| \ge \delta^{1/4} - \delta^c\right) \sup_{\tau_i \le t \le \tau_{i+1}} \xi_t | \mathscr{F}_{\tau_i}\right] \\ &\le E\left[(\tau_{i+1} - \tau_i)^2 | \mathscr{F}_{\tau_i}\right]^{1/2} P\left(\sup_{\tau_i \le t \le \tau_{i+1}} |Y_t - Y_{\tau_i}| \ge \delta^{1/4} - \delta^c | \mathscr{F}_{\tau_i}\right)^{4/9} \\ &\quad \times E\left[\sup_{\tau_i \le t \le \tau_{i+1}} \xi_t^{18} | \mathscr{F}_{\tau_i}\right]^{1/18} \\ &= o_i(\delta^{11/10}). \end{aligned}$$

The second moment of (35) can be shown to be $O_i(\delta^{23/20})$ in a similar fashion. □

We now proceed to isolate regions around the points of inflection. Let $Y_t^j = X_t - F_j(t)$, $j = 1, \dots, m$. Then each Y^j satisfies the conditions of Lemma 4.5, and hence

$$\delta^{8/20}\, \mathbb{1}\left(|Y_{\tau_i}^j| \le \delta^{7/20}\right) \int_{\tau_i}^{\tau_{i+1}} \sigma^2(t, X_t)\, dt$$

is δ-negligible. Define

$$D(t,\delta) = \bigcup_{j=1}^{m} \left[F_j(t) - \delta^{7/20}, F_j(t) + \delta^{7/20}\right]. \tag{36}$$

Then we have

$$\delta^{8/20}\mathbb{1}\big(X_{\tau_i}\in D(\tau_i,\delta)\big)\int_{\tau_i}^{\tau_{i+1}}\sigma^2(t,X_t)\,dt$$
$$\le \delta^{8/20}\sum_{j=1}^{m}\mathbb{1}\big(|Y^j_{\tau_i}|\le\delta^{7/20}\big)\int_{\tau_i}^{\tau_{i+1}}\sigma^2(t,X_t)\,dt,$$

which is δ-negligible.

Proposition 4.2. *Under the hypothesis of Theorem* 3.2,

$$\int_{\tau_i}^{\tau_{i+1}}X_t\frac{\lambda}{\theta}(t,X_t)|h_{xx}(t,X_t)|\,dt - X_{\tau_i}\frac{\lambda}{\theta}(\tau_i,X_{\tau_i})\big|h_{xx}(\tau_i,X_{\tau_i})\big|(\tau_{i+1}-\tau_i)$$

is δ-negligible.

Proof. The function $g(t,x)=x(\lambda/\theta)(t,x)|h_{xx}(t,x)|$ is Lipschitz continuous, so that we can remove the t dependence by noting that

$$\int_{\tau_i}^{\tau_{i+1}}g(t,X_t)-g(\tau_i,X_t)\,dt$$

is δ-negligible. We next claim that

$$\left|\int_{\tau_i}^{\tau_{i+1}}g(\tau_i,X_t)\,dt-g(\tau_i,X_{\tau_i})(\tau_{i+1}-\tau_i)\right|\mathbb{1}\big(X_{\tau_i}\in D(\tau_i,\delta)\big)\tag{37}$$

is δ-negligible. Due to Lipschitz continuity of g, this is dominated by

$$K\sup_{\tau_i\le t\le\tau_{i+1}}|X_t-X_{\tau_i}|(\tau_{i+1}-\tau_i)\,\mathbb{1}\big(X_{\tau_i}\in D(\tau_i,\delta)\big)$$

for some positive constant K. Note that

$$P\big(\sigma^2(\tau_i,X_{\tau_i})\le\delta^{1/20}\big)+P\Big(\sup_{\tau_i\le t\le\tau_{i+1}}|X_t-X_{\tau_i}|>\delta^{9/20}|\mathscr{F}_{\tau_i}\Big)=o_i(1).$$

Then,

$$\sup_{\tau_i\le t\le\tau_{i+1}}|X_t-X_{\tau_i}|(\tau_{i+1}-\tau_i)\,\mathbb{1}\Big(\sigma^2(\tau_i,X_{\tau_i})\le\delta^{1/20}\quad\text{or}\quad\sup_{\tau_i\le t\le\tau_{i+1}}|X_t-X_{\tau_i}|>\delta^{9/20}\Big)$$

is δ-negligible. Since

$$\sigma^{-2}(\tau_i,X_{\tau_i})\sup_{\tau_i\le t\le\tau_{i+1}}|X_t-X_{\tau_i}|\mathbb{1}\Big(\sigma^2(\tau_i,X_{\tau_i})>\delta^{1/20},\ \sup_{\tau_i\le t\le\tau_{i+1}}|X_t-X_{\tau_i}|\le\delta^{9/20}\Big)\le\delta^{8/20},$$

it is enough to show that

$$\delta^{8/20}I\big(X_{\tau_i}\in D(\tau_i,\delta)\big)\sigma^2(\tau_i,X_{\tau_i})(\tau_{i+1}-\tau_i)$$

is δ-negligible. As in the proof of Proposition 4.1, $\sigma^2(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i)$ can be replaced by $\int_{\tau_i}^{\tau_{i+1}} \sigma^2(t, X_t)\, dt$. Lemma 4.5 implies that

$$\delta^{8/20} \mathbb{1}\left(X_{\tau_i} \in D(\tau_i, \delta)\right) \int_{\tau_i}^{\tau_{i+1}} \sigma^2(t, X_t)\, dt$$

is δ-negligible. Therefore (37) is δ-negligible.

We now move to the space–time region away from the points of inflection where we must control

$$\left| \int_{\tau_i}^{\tau_{i+1}} g(\tau_i, X_t)\, dt - g(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i) \right| \mathbb{1}\left(X_{\tau_i} \in D(\tau_i, \delta)^C\right),$$

where $D(\tau_i, \delta)^C$ is the complement of $D(\tau_i, \delta)$. Note that

$$\int_{\tau_i}^{\tau_{i+1}} \left| g(\tau_i, X_t) - g(\tau_i, X_{\tau_i}) \right| dt\, \mathbb{1}\left(\sup_{\tau_i \le t \le \tau_{i+1}} |X_t - X_{\tau_i}| > \delta^{9/20} \right)$$

is δ-negligible since

$$P\left(\sup_{\tau_i \le t \le \tau_{i+1}} |X_t - X_{\tau_i}| > \delta^{9/20} | \mathscr{F}_{\tau_i} \right) = o_i(1).$$

Then it suffices to show that

$$\int_{\tau_i}^{\tau_{i+1}} \left(g(\tau_i, X_t) - g(\tau_i, X_{\tau_i}) \right) \mathbb{1}\left(|X_t - X_{\tau_i}| \le \delta^{9/20},\, X_{\tau_i} \in D(\tau_i, \delta)^C \right) dt$$

is δ-negligible. If $X_{\tau_i} \in D(\tau_i, \delta)^C$, $g(\tau_i, \cdot\,)$ is continuously differentiable in the neighborhood of X_{τ_i} with radius $\delta^{7/20}$. Thus we have

$$\begin{aligned}
&\int_{\tau_i}^{\tau_{i+1}} \left(g(\tau_i, X_t) - g(\tau_i, X_{\tau_i}) \right) \mathbb{1}\left(|X_t - X_{\tau_i}| \le \delta^{9/20},\, X_{\tau_i} \in D(\tau_i, \delta)^C \right) dt \\
&\quad = \int_{\tau_i}^{\tau_{i+1}} \int_0^1 g_x\left(\tau_i, X_{\tau_i} + r(X_t - X_{\tau_i})\right) dr \left(X_t - X_{\tau_i}\right) \\
&\qquad\qquad \times \mathbb{1}\left(|X_t - X_{\tau_i}| \le \delta^{9/20},\, X_{\tau_i} \in D(\tau_i, \delta)^C \right) dt.
\end{aligned}$$

We decompose X into a martingale and a finite variation process. Applying Lemma 3.5 with

$$H_t = \int_0^1 g_x\left(\tau_i, X_{\tau_i} + r(X_t - X_{\tau_i})\right) dr\, \mathbb{1}\left(|X_t - X_{\tau_i}| \le \delta^{9/20},\, X_{\tau_i} \in D(\tau_i, \delta)^C \right)$$

for $t \in [\tau_i, \tau_{i+1})$ yields that the term with the martingale is δ-negligible. The remaining term is clearly δ-negligible. □

Before proceeding, we will summarize the consequences of Proposition 4.1 and Proposition 4.2. Applying Itô's formula with PDE (5), we have

$$h(\tau_{i+1}, X_{\tau_{i+1}}) - h(\tau_i, X_{\tau_i}) = \int_{\tau_i}^{\tau_{i+1}} h_x(t, X_t)\, dX_t - \int_{\tau_i}^{\tau_{i+1}} X_t \frac{\lambda}{\theta}(t, X_t) \left| h_{xx}(t, X_t) \right| dt.$$

Now, Propositions 4.1 and 4.2 imply that $h(\tau_{i+1}, X_{\tau_{i+1}}) - h(\tau_i, X_{\tau_i}) - h_x(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})$ is equivalent to

$$\tfrac{1}{2} h_{xx}(\tau_i, X_{\tau_i}) \left[(X_{\tau_{i+1}} - X_{\tau_i})^2 - \sigma^2(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i) \right] \tag{38}$$

$$-X_{\tau_i} \frac{\lambda}{\theta}(\tau_i, X_{\tau_i}) \left| h_{xx}(\tau_i, X_{\tau_i}) \right| (\tau_{i+1} - \tau_i) \tag{39}$$

up to δ-negligible terms. Then (38) is equivalent to

$$\tfrac{1}{2} h_{xx}(\tau_i, X_{\tau_i}) \left\{ (X_{\tau_{i+1}} - X_{\tau_i})^2 - E\left[(X_{\tau_{i+1}} - X_{\tau_i})^2 | \mathscr{F}_{\tau_i} \right] \right\} - \tfrac{1}{2} h_{xx}(\tau_i, X_{\tau_i}) \sigma^2(\tau_i, X_{\tau_i}) \left\{ \tau_{i+1} - \tau_i - E\left[\tau_{i+1} - \tau_i | \mathscr{F}_{\tau_i} \right] \right\},$$

since $E[(X_{\tau_{i+1}} - X_{\tau_i}) - \sigma^2(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i) | \mathscr{F}_{\tau_i}]$ is δ-negligible by Proposition 4.1. An application of assumptions (T1) and (T3) yields that (39) is equivalent to

$$\begin{aligned} &-X_{\tau_i} \frac{\lambda}{\theta}(\tau_i, X_{\tau_i}) \left| h_{xx}(\tau_i, X_{\tau_i}) \right| \left\{ \tau_{i+1} - \tau_i - E\left[\tau_{i+1} - \tau_i | \mathscr{F}_{\tau_i} \right] \right\} \\ &\quad + X_{\tau_i} \left| h_{xx}(\tau_i, X_{\tau_i}) \right| \left\{ \delta^{1/2} |X_{\tau_{i+1}} - X_{\tau_i}| - E\left[\delta^{1/2} |X_{\tau_{i+1}} - X_{\tau_i}| | \mathscr{F}_{\tau_i} \right] \right\} \\ &\quad - \delta^{1/2} \left| h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i}) \right| X_{\tau_i}. \end{aligned}$$

up to δ-negligible terms. The fact that $h(t, x)$ satisfies PDE (5) yields

$$h_t(\tau_i, X_{\tau_i}) = -\frac{1}{2} \sigma^2(\tau_i, X_{\tau_i}) h_{xx}(\tau_i, X_{\tau_i}) - X_{\tau_i} \frac{\lambda}{\theta}(\tau_i, X_{\tau_i}) \left| h_{xx}(\tau_i, X_{\tau_i}) \right|,$$

which allows us to combine (38) and (39) as

$$\begin{aligned} &\tfrac{1}{2} h_{xx}(\tau_i, X_{\tau_i}) \left\{ (X_{\tau_{i+1}} - X_{\tau_i})^2 - E\left[(X_{\tau_{i+1}} - X_{\tau_i})^2 | \mathscr{F}_{\tau_i} \right] \right\} \\ &\quad + h_t(\tau_i, X_{\tau_i}) \left\{ \tau_{i+1} - \tau_i - E\left[\tau_{i+1} - \tau_i | \mathscr{F}_{\tau_i} \right] \right\} \\ &\quad + X_{\tau_i} \left| h_{xx}(\tau_i, X_{\tau_i}) \right| \left\{ \delta^{1/2} |X_{\tau_{i+1}} - X_{\tau_i}| - E\left[\delta^{1/2} |X_{\tau_{i+1}} - X_{\tau_i}| | \mathscr{F}_{\tau_i} \right] \right\} \\ &\quad - \delta^{1/2} \left| h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i}) \right| X_{\tau_i}. \end{aligned}$$

Recalling the bookkeeping formula (2), once we show that

$$\begin{aligned} &\delta^{1/2} \left| h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) \right| X_{\tau_{i+1}} \\ &\quad - \delta^{1/2} \left| h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i}) \right| X_{\tau_i} \end{aligned} \tag{40}$$

is δ-negligible, the replication error is to leading order in δ given by

$$E\Biggl\{\Biggl[\sum_{\tau_i<T}\tfrac{1}{2}h_{xx}(\tau_i,X_{\tau_i})\Bigl\{\bigl(X_{\tau_{i+1}}-X_{\tau_i}\bigr)^2-E\Bigl[\bigl(X_{\tau_{i+1}}-X_{\tau_i}\bigr)^2|\mathscr{F}_{\tau_i}\Bigr]\Bigr\}$$
$$+h_t(\tau_i,X_{\tau_i})\bigl\{\tau_{i+1}-\tau_i-E\bigl[\tau_{i+1}-\tau_i|\mathscr{F}_{\tau_i}\bigr]\bigr\}$$
$$+X_{\tau_i}\bigl|Vh_{xx}(\tau_i,X_{\tau_i})\bigr|\Bigl\{\delta^{1/2}|X_{\tau_{i+1}}-X_{\tau_i}|-E\bigl[\delta^{1/2}|X_{\tau_{i+1}}-X_{\tau_i}||\mathscr{F}_{\tau_i}\bigr]\Bigr\}\Biggr]^2\Biggr\}.$$

The cross terms (in i) above vanish, and (11) implies that the diagonal terms converge to (12). Therefore, the last remaining task is to very that (40) is δ-negligible. The next proposition shows that $X_{\tau_{i+1}}$ attached at the end of the first term of (40) can be replaced by X_{τ_i}.

PROPOSITION 4.3. *Under the hypotheses of Theorem* 3.2,

$$\delta^{1/2}\bigl|h_x(\tau_{i+1},X_{\tau_{i+1}})-h_x(\tau_i,X_{\tau_i})\bigr|\bigl(X_{\tau_{i+1}}-X_{\tau_i}\bigr)$$

is δ-negligible.

PROOF. Since $h_x(\tau_{i+1},X_{\tau_{i+1}})-h_x(\tau_i,X_{\tau_{i+1}})=O(|\tau_{i+1}-\tau_i|)$, it suffices to show that

$$\delta^{1/2}\bigl|h_x(\tau_i,X_{\tau_{i+1}})-h_x(\tau_i,X_{\tau_i})\bigr|\bigl(X_{\tau_{i+1}}-X_{\tau_i}\bigr)$$

is δ-negligible. By virtue of (T7), the following is δ-negligible:

$$\delta^{1/2}\bigl|h_{xx}(\tau_i,X_{\tau_i})\bigl(X_{\tau_{i+1}}-X_{\tau_i}\bigr)\bigr|\bigl(X_{\tau_{i+1}}-X_{\tau_i}\bigr).$$

The proof will be complete if we show that

$$\delta^{1/2}\bigl|h_x(\tau_i,X_{\tau_{i+1}})-h_x(\tau_i,X_{\tau_i})-h_{xx}(\tau_i,X_{\tau_i})\bigl(X_{\tau_{i+1}}-X_{\tau_i}\bigr)\bigr|\bigl(X_{\tau_{i+1}}-X_{\tau_i}\bigr)$$

is also δ-negligible. This follows from

$$\begin{aligned}&\bigl|h_x(\tau_i,X_{\tau_{i+1}})-h_x(\tau_i,X_{\tau_i})-h_{xx}(\tau_i,X_{\tau_i})\bigl(X_{\tau_{i+1}}-X_{\tau_i}\bigr)\bigr|\\&\quad=\left|\int_0^1 h_{xx}\bigl(\tau_i,X_{\tau_i}+r\bigl(X_{\tau_{i+1}}-X_{\tau_i}\bigr)\bigr)-h_{xx}(\tau_i,X_{\tau_i})\,dr\bigl(X_{\tau_{i+1}}-X_{\tau_i}\bigr)\right|\\&\quad=O\bigl(|X_{\tau_{i+1}}-X_{\tau_i}|^2\bigr).\end{aligned}$$

We must now show that

$$\delta^{1/2}X_{\tau_i}\Bigl\{\bigl|h_x(\tau_{i+1},X_{\tau_{i+1}})-h_x(\tau_i,X_{\tau_i})\bigr|-\bigl|h_{xx}(\tau_i,X_{\tau_i})\bigl(X_{\tau_{i+1}}-X_{\tau_i}\bigr)\bigr|\Bigr\}\tag{41}$$

is δ-negligible. When $h_{xx}(\tau_i,X_{\tau_i})$ is 0, this is of $O_i(\delta^{1/2})(|\tau_{i+1}-\tau_i|+|X_{\tau_{i+1}}-X_{\tau_i})|^2)$, and hence (40) is still too big. This is a problem only near the points of inflection, and the next proposition says this does not happen very often.

PROPOSITION 4.4. *Under the hypotheses of Theorem* 3.2,

$$\delta^{1/2} X_{\tau_i}\Big\{\big|h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i})\big| \\ - \big|h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\big|\Big\} \mathbb{1}\big(X_{\tau_i} \in D(\tau_i, \delta)\big)$$

is δ*-negligible.*

PROOF. The expressions $h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i})$ can be rewritten as

$$h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i}) + (X_{\tau_{i+1}} - X_{\tau_i}) \\ \times \int_0^1 \big[h_{xx}\big(\tau_i, X_{\tau_i} + r(X_{\tau_{i+1}} - X_{\tau_i})\big) - h_{xx}(\tau_i, X_{\tau_i})\big]\, dr \tag{42}$$

$$+ h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_{i+1}}). \tag{43}$$

Since h_x is Lipschitz, (43) is bounded by $K(\tau_{i+1} - \tau_i)$ where K is a positive constant. Thus

$$\delta^{1/2} X_{\tau_i}\big|h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_{i+1}})\big| \le \delta^{1/2} X_{\tau_i} K(\tau_{i+1} - \tau_i). \tag{44}$$

Note that $X_{\tau_i}\mathbb{1}(X_{\tau_i} \in D(\tau_i, \delta))$ is bounded by a constant due to the compactness of $\bigcup_{0 \le t \le T} D(t, \delta)$. Then, as in the proof of Proposition 4.2, (44) is δ-negligible. Thus

$$\delta^{1/2} X_{\tau_i}\big|h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_{i+1}})\big| \mathbb{1}\big(X_{\tau_i} \in D(\tau_i, \delta)\big)$$

is also δ-negligible. Then (42) is bounded above by

$$\int_0^1 \big|h_{xx}\big(\tau_i, X_{\tau_i} + r(X_{\tau_{i+1}} - X_{\tau_i})\big) - h_{xx}(\tau_i, X_{\tau_i})\big|\, dr |X_{\tau_{i+1}} - X_{\tau_i}| \\ \le K|X_{\tau_{i+1}} - X_{\tau_i}|^2$$

by Lipschitz continuity. Observing that

$$|X_{\tau_{i+1}} - X_{\tau_i}|^2 = 2\int_{\tau_i}^{\tau_{i+1}} (X_t - X_{\tau_i})\, dX_t + \int_{\tau_i}^{\tau_{i+1}} \sigma^2(t, X_t)\, dt$$

and that both

$$\delta^{1/2} \int_{\tau_i}^{\tau_{i+1}} (X_t - X_{\tau_i})\, dX_t \quad \text{and} \quad \delta^{1/2} \int_{\tau_i}^{\tau_{i+1}} \sigma^2(t, X_t)\, dt\, \mathbb{1}\big(X_{\tau_i} \in D(\tau_i, \delta)\big)$$

are δ-negligible results in the fact that

$$\delta^{1/2}\bigg|\int_0^1 h_{xx}\big(\tau_i, X_{\tau_i} + r(X_{\tau_{i+1}} - X_{\tau_i})\big) \\ - h_{xx}(\tau_i, X_{\tau_i})\, dr (X_{\tau_{i+1}} - X_{\tau_i})\bigg| \mathbb{1}\big(X_{\tau_i} \in D(\tau_i, \delta)\big)$$

is also δ-negligible. □

At this point the main concern is to control

$$\left|h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i})\right| - \left|h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\right|$$

away from the inflection points; that is, for $X_{\tau_i} \in D(\tau_i, \delta)^C$.

LEMMA 4.6. *Suppose that $A \neq 0$. Then*

$$|A + B| - |A| = \operatorname{sign}(A)B + 2|A + B|\mathbb{1}(-B \wedge 0 < A < -B \vee 0).$$

PROOF. It is true for $A + B = 0$, and hence suppose not. Then

$$\begin{aligned}|A + B| &= (A + B)\mathbb{1}(A > 0, A + B > 0) - (A + B)\mathbb{1}(A > 0, A + B < 0)\\ &\quad - (A + B)\mathbb{1}(A < 0, A + B < 0) + (A + B)\mathbb{1}(A < 0, A + B > 0)\\ &= (A + B)\mathbb{1}(A > 0) - 2(A + B)\mathbb{1}(A > 0, A + B < 0)\\ &\quad - (A + B)\mathbb{1}(A < 0) + 2(A + B)\mathbb{1}(A < 0, A + B > 0)\\ &= \operatorname{sign}(A)(A + B) + 2|A + B|\mathbb{1}(0 < A < -B \text{ or } -B < A < 0)\end{aligned}$$

and the result follows. □

Thus if $X_{\tau_i} \in D(\tau_i, \delta)^C$, then (41) can be rewritten as

$$2\delta^{1/2} X_{\tau_i}\left|h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i})\right| I(\tau_i) \tag{45}$$

$$\begin{aligned}&+ \delta^{1/2} X_{\tau_i} \operatorname{sign}\left[h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\right]\\ &\times\left\{h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\right\},\end{aligned} \tag{46}$$

where $I(\tau_i)$ is an indicator of the event that $|h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})|$ falls between

$$-\left[h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\right] \wedge 0$$

and

$$-\left[h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\right] \vee 0.$$

We begin by controlling (46).

PROPOSITION 4.5. *Under the hypothesis of Theorem* 3.2,

$$\begin{aligned}&\delta^{1/2} X_{\tau_i} \operatorname{sign}\left[h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\right]\\ &\quad\times\left\{h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\right\}\\ &\quad\times \mathbb{1}\left(X_{\tau_i} \in D(\tau_i, \delta)^C\right)\end{aligned}$$

is δ-negligible.

PROOF. First we show that the above is equivalent to

$$\begin{aligned}&\delta^{1/2} X_{\tau_i} \operatorname{sign}\left[h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\right]\\ &\quad\times\left\{h_{tx}(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i) + h_{xxx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})^2\right\}\\ &\quad\times \mathbb{1}\left(X_{\tau_i} \in D(\tau_i, \delta)^C\right).\end{aligned} \tag{47}$$

That is, we must show that the following is δ-negligible:

$$\begin{aligned}(48)\qquad &\delta^{1/2}X_{\tau_i}\Big|h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\\ &\qquad - h_{tx}(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i) - h_{xxx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})^2\Big|\\ &\quad \times \mathbb{1}\Big(X_{\tau_i} \in D(\tau_i, \delta)^C\Big).\end{aligned}$$

Since

$$P\bigg(\sup_{\tau_i \le t \le \tau_{i+1}} |X_t - X_{\tau_i}| > \delta^{9/20} | \mathscr{F}_{\tau_i}\bigg) = o_i(1),$$

$\mathbb{1}(X_{\tau_i} \in D(\tau_i, \delta)^C)$ can be replaced by $\mathbb{1}(\sup_{\tau_i \le t \le \tau_{i+1}} |X_t - X_{\tau_i}| \le \delta^{9/20},\ X_{\tau_i} \in D(\tau_i, \delta)^C)$. When this indicator turns on, h_x is smooth in the neighborhood of X_{τ_i} with radius $|X_{\tau_{i+1}} - X_{\tau_i}|$, and hence the terms inside absolute value can be

$$\begin{aligned}&\int_0^1 h_{tx}\big(\tau_i + r(\tau_{i+1} - \tau_i), X_{\tau_i} + r(X_{\tau_{i+1}} - X_{\tau_i})\big) - h_{tx}(\tau_i, X_{\tau_i})\, dr(\tau_{i+1} - \tau_i)\\ &\quad + \int_0^1 h_{xx}\big(\tau_i + r(\tau_{i+1} - \tau_i), X_{\tau_i} + r(X_{\tau_{i+1}} - X_{\tau_i})\big)\\ &\quad - h_{xx}\big(\tau_i, X_{\tau_i} + r(X_{\tau_{i+1}} - X_{\tau_i})\big)\, dr(X_{\tau_{i+1}} - X_{\tau_i})\\ &\quad + \int_0^1\int_0^r h_{xxx}\big(\tau_i, X_{\tau_i} + q(X_{\tau_{i+1}} - X_{\tau_i})\big) - h_{xxx}(\tau_i, X_{\tau_i})\, dq\, dr(X_{\tau_{i+1}} - X_{\tau_i})^2.\end{aligned}$$

Together with $\mathbb{1}(\sup_{\tau_i \le t \le \tau_{i+1}} |X_t - X_{\tau_i}| > \delta^{9/20})$, this is of $o_i(1)(|\tau_{i+1} - \tau_i| + |X_{\tau_{i+1}} - X_{\tau_i}|^2)$, and therefore (48) is δ-negligible.

It remains to show that (47) is also δ-negligible:

$$\begin{aligned}&\delta^{1/2}X_{\tau_i}\operatorname{sign}\big[h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\big]\\ &\quad \times h_{tx}(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i)\,\mathbb{1}\Big(X_{\tau_i} \in D(\tau_i, \delta)^C\Big)\end{aligned}$$

is δ-negligible due to (T5) and Lemma 4.2. Similarly,

$$\begin{aligned}&\delta^{1/2}X_{\tau_i}\operatorname{sign}\big[h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\big]\\ &\quad \times \Big\{h_{xxx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})^2\Big\}\,\mathbb{1}\Big(X_{\tau_i} \in D(\tau_i, \delta)^C\Big)\end{aligned}$$

is δ-negligible due to (T7) and Lemma 4.2. □

Now we complete the proof of Theorem 3.2 with the following result, which dispenses of (45).

PROPOSITION 4.6. *Under the hypotheses of Theorem* 3.2, *the following is δ-negligible*:

$$\delta^{1/2}X_{\tau_i}\Big|h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i})\Big| I(\tau_i)\,\mathbb{1}\Big(X_{\tau_i} \in D(\tau_i, \delta)^C\Big).$$

Recall that $I(\tau_i)$ *indicates the event that* $|h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})|$ *falls between*

$$-\left[h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\right] \wedge 0$$

and

$$-\left[h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\right] \vee 0.$$

PROOF. By the definition of $I(\tau_i)$,

$$\begin{aligned}&\left|h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i})\right| I(\tau_i)\\ &\quad \le 2\left|h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\right|(\tau_i).\end{aligned}$$

Thus, as in the proof of Proposition 3.5, we will show that

$$\begin{aligned}&\delta^{1/2} X_{\tau_i}\left|h_{tx}(\tau_i, X_{\tau_i})(\tau_{i+1} - \tau_i) + \tfrac{1}{2} h_{xxx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})^2\right|\\ &\qquad \times I(\tau_i)\,\mathbb{1}\left(\sup_{\tau_i \le t \le \tau_{i+1}} |X_t - X_{\tau_i}| \le \delta^{9/20},\, X_{\tau_i} \in D(\tau_i, \delta)^C\right)\end{aligned}$$

is δ-negligible. We break $I(\tau_i)$ into three pieces:

$$\begin{aligned}I_1(\tau_i) &= I(\tau_i)\,\mathbb{1}\left(\left|h_{xx}(\tau_i, X_{\tau_i})\right| > \delta^{9/20}\right),\\ I_2(\tau_i) &= I(\tau_i)\,\mathbb{1}\left(\left|h_{xx}(\tau_i, X_{\tau_i})\right| \le \delta^{9/20}, \left|h_{xxx}(\tau_i, X_{\tau_i})\right| \le \delta^{1/20}\right),\\ I_3(\tau_i) &= I(\tau_i)\,\mathbb{1}\left(\left|h_{xx}(\tau_i, X_{\tau_i})\right| \le \delta^{9/20}, \left|h_{xxx}(\tau_i, X_{\tau_i})\right| > \delta^{1/20}\right).\end{aligned}$$

First, note that $E[I_1(\tau_i)|\mathscr{F}_{\tau_i}]$ is dominated by

$$\begin{aligned}P\Big(\delta^{9/20}|X_{\tau_{i+1}} - X_{\tau_i}| < \Big|h_x(\tau_{i+1}, X_{\tau_{i+1}}) &- h_x(\tau_i, X_{\tau_i})\\ &- h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})\Big|\Big|\mathscr{F}_{\tau_i}\Big).\end{aligned}$$

This is of $o(1)$ since $\delta^{-1}|h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})|$ is tight. Thus, $\delta^{1/2}X_{\tau_i}|h_x(\tau_{i+1}, X_{\tau_{i+1}}) - h_x(\tau_i, X_{\tau_i}) - h_{xx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})|$ $I_1(\tau_i)$ is δ-negligible. Next, we consider the second term. By the definition of I_2,

$$\begin{aligned}&\delta^{1/2}X_{\tau_i}\left|h_{xxx}(\tau_i, X_{\tau_i})(X_{\tau_{i+1}} - X_{\tau_i})^2\right| I_2(\tau_i)\\ &\qquad \times \mathbb{1}\left(\sup_{\tau_i \le t \le \tau_{i+1}} |X_t - X_{\tau_i}| \le \delta^{9/20},\, X_{\tau_i} \in D(\tau_i, \delta)^C\right)\end{aligned}$$

is dominated by $\delta^{11/20}X_{\tau_i}(X_{\tau_{i+1}} - X_{\tau_i})^2$, and hence it is δ-negligible. Differentiating PDE (5) with respect to x, we have

$$h_{tx}(t, x) = \alpha(t, x) h_{xxx}(t, x) + \beta(t, x) h_{xx}(t, x),$$

where

$$\alpha(t,x) = \frac{1}{2}\sigma^2(t,x) + x\frac{\lambda}{\theta}(t,x)\operatorname{sign}[h_{xx}(t,x)],$$

$$\beta(t,x) = \frac{1}{2}\frac{\partial\sigma^2}{\partial x}(t,x) + \frac{\partial}{\partial x}\left\{x\frac{\lambda}{\theta}(t,x)\right\}\operatorname{sign}[h_{xx}(t,x)].$$

Therefore, by definition of I_2,

$$\delta^{1/2}X_{\tau_i}\big|h_{tx}(\tau_i, X_{\tau_i})(\tau_{i+1}-\tau_i)\big|I_2(\tau_i)$$
$$\times\,\mathbb{1}\Big(\sup_{\tau_i\le t\le\tau_{i+1}}|X_t - X_{\tau_i}| \le \delta^{9/20},\ X_{\tau_i}\in D(\tau_i,\delta)^C\Big)$$

is also dominated by

$$\delta^{11/20}\big[\alpha(\tau_i, X_{\tau_i})(\tau_{i+1}-\tau_i) + \delta^{8/20}\beta(\tau_i, X_{\tau_i})\big]X_{\tau_i}(\tau_{i+1}-\tau_i),$$

and hence δ-negligible. Finally, we show that the third term is δ-negligible:

$$\delta^{1/2}X_{\tau_i}\Big|h_{tx}(\tau_i, X_{\tau_i})(\tau_{i+1}-\tau_i) + h_{xxx}(\tau_i, X_{\tau_i})\big(X_{\tau_{i+1}} - X_{\tau_i}\big)^2\Big|$$
$$\times I_3(\tau_i)\,\mathbb{1}\Big(\sup_{\tau_i\le t\le\tau_{i+1}}|X_t - X_{\tau_i}| \le \delta^{9/20},\ X_{\tau_i}\in D(\tau_i,\delta)^C\Big).$$

This is dominated by

$$\delta^{19/20}\big|\beta(\tau_i, X_{\tau_i})\big|X_{\tau_i}(\tau_{i+1}-\tau_i) \tag{49}$$

$$\begin{aligned}&+\delta^{1/2}X_{\tau_i}h_{xxx}(\tau_i, X_{\tau_i})\Big[\big(X_{\tau_{i+1}} - X_{\tau_i}\big)^2 + \big|\alpha(\tau_i, X_{\tau_i})\big|(\tau_{i-1}-\tau_i)\Big]\\ &\times I_3(\tau_i)\,\mathbb{1}\Big(\sup_{\tau_i\le t\le\tau_{i+1}}|X_t - X_{\tau_i}| \le \delta^{9/20},\ X_{\tau_i}\in D(\tau_i,\delta)^C\Big).\end{aligned} \tag{50}$$

The fact that (49) is δ-negligible is straightforward. Under (P3), $h_{xx}(\cdot, X)$ is a continuous semimartingale with a quadratic variation

$$\int_0^{\cdot} h_{xxx}^2(t, X_t)\sigma^2(t, X_t)\,dt,$$

where h_{xxx} is a left derivative of h_{xx}. By virtue of the set of conditions (P), $h_{xx}(\cdot, X)$ satisfies the conditions of Lemma 4.5, and hence

$$\delta^{2/5}\,\mathbb{1}\Big(\big|h_{xx}(\tau_i, X_{\tau_i})\big| \le \delta^{9/20}\Big)\int_{\tau_i}^{\tau_{i+1}} h_{xxx}^2(t, X_t)\sigma^2(t, X_t)\,dt \tag{51}$$

is δ-negligible. We may replace $(X_{\tau_{i+1}} - X_{\tau_i})^2$ by $\int_{\tau_i}^{\tau_{i+1}}\sigma^2(t, X_t)\,dt$. Also note that

$$\big\{(t,x) : |h_{xx}(t,x)| \le \delta^{9/20},\ t\in[0,T]\big\}$$

is compact. This and the definition of I_3 imply

$$\delta^{1/2}X_{\tau_i}h_{xxx}(\tau_i, X_{\tau_i})\big(X_{\tau_{i+1}} - X_{\tau_i}\big)^2 I_3(\tau_i)$$
$$\times\,\mathbb{1}\Big(\sup_{\tau_i\le t\le\tau_{i+1}}|X_t - X_{\tau_i}| \le \delta^{9/20},\ X_{\tau_i}\in D(\tau_i,\delta)^C\Big)$$

is δ-negligible. To complete the proof, it remains to show that

$$\delta^{2/5}\mathbb{1}\Big(\big|h_{xx}(\tau_i, X_{\tau_i})\big| \le \delta^{9/20}\Big)h_{xxx}^2(\tau_i, X_{\tau_i})\sigma^2(\tau_i, X_{\tau_i})(\tau_{i+1}-\tau_i)I_3(\tau_i)$$
$$\times \mathbb{1}\Big(\sup_{\tau_i \le t \le \tau_{i+1}} |X_t - X_{\tau_i}| \le \delta^{9/20},\ X_{\tau_i} \in D(\tau_i, \delta)^C\Big)$$

is also δ-negligible. This follows from the fact that

$$\int_{\tau_i}^{\tau_{i+1}} h_{xxx}^2(t, X_t)\sigma^2(t, X_t) - h_{xxx}^2(\tau_i, X_{\tau_i})\sigma^2(\tau_i, X_{\tau_i})\,dt$$
$$\times \mathbb{1}\Big(\sup_{\tau_i \le t \le \tau_{i+1}} |X_t - X_{\tau_i}| \le \delta^{9/20},\ X_{\tau_i} \in D(\tau_i, \delta)^C\Big)$$

is of $o_i(1)(\tau_{i+1} - \tau_i)$. □

5. Conclusion. The practical applications of the results of this paper are derived largely from the possibility of optimizing hedging methods over much larger classes of strategies than those previously considered. In the small transaction cost limit, many optimization questions revolve around the trade-off between initial portfolio value and replication error, and the primary limitation in optimization has to date been the lack of a unified treatment of the replication error. Here we have reduced this problem to the calculation of a few moments.

Consequently, for any class of strategies for which these moments can be calculated, one can in theory obtain the strategy, which, for example, minimizes replication error given an initial portfolio value. The only limitations are the numerical tasks involved.

A class of strategies which we are currently considering consists of time interval strategies with a weak stock price dependence. Specifically, these are strategies in which $\tau_{i+1} = \tau_i + \delta\theta(\tau_i, X_{\tau_i})$, so that the next rebalancing time is fixed given the previous rebalancing time and price. These strategies fall under the umbrella of the results in this paper, and in fact are rather straightforward extensions of the fixed time interval strategies. Nevertheless, although the replication error is immediately obtainable from Theorem 3.2, the numerical tasks associated with minimizing, for example, a weighted sum of initial portfolio value and replication error are nontrivial.

We expect that the additional work associated with optimization over this class of temporal strategies with weak price dependence will be worth the effort. The major limitation of strategies with fixed time intervals between rebalancing is that no consideration is made regarding the local Γ (curvature) of the payoff. These weakly price dependent strategies are perhaps the simplest class which allows for rapid rebalancing in regions of high Γ and infrequent rebalancing when Γ is small. It is the possibility of distinguishing between regions where hedging is important and where it is not that suggest that significant improvements over fixed time interval strategies are possible, even within the context of these simple strategies.

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