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SCALING LAWS AND CONVERGENCE FOR THE ADVECTION–DIFFUSION EQUATION

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In this paper we study the convergence of stochastic processes related to a random partial differential equation (PDE with random coefficients) of heat equation propagation type in a Kolmogorov's random velocity field. Then we are able to improve the results of Avellanda and Majda in the case of "shear-flow" advection–diffusion because we prove a convergence in law of the solution of the RPDE instead of just convergence of the moments.

1. Introduction. The work of Avellaneda and Majda [3] consists of the study of the following "shear-flow" advection–diffusion equation:

$$\begin{aligned} &\frac{\partial u}{\partial t} - v_\varepsilon(x,t)\frac{\partial u}{\partial y} = \frac{1}{2}\nu_0 \Delta u, \\ &u(x,y,t=0) = u_0(\varepsilon x, \varepsilon y), \end{aligned} \tag{1}$$

where $v_\varepsilon(x,t)$ is a Kolmogorov random velocity field (see [3]) depending on two parameters ε and δ (we will denote by $\langle\ \rangle$ as the expectation with respect to the statistics of v_ε) and ε is a scaling parameter.

The success and the originality of the work is of two types.

1. They get the correct scaling laws in this "shear-flow" advection–diffusion equation,
2. They succeed in explicitly calculating the renormalized Green function in the case $0 < \delta < 4$.

More precisely about the first point, they prove that there exists an exact renormalization depending on the parameters of v_ε, so that the nth order moments $\langle u^n(x/\varepsilon, y/\varepsilon, t/\rho^2(\varepsilon))\rangle$ have a nontrivial limit when ε tends to 0 provided that a good choice of $\rho(\varepsilon)$ is made. For that, they use a Fourier transform with respect to the variable y and apply the Feynman–Kac formula to the resulting equation. Then they can identify the "good" scaling law and the limiting function.

This result suggests that the random solution $u(x/\varepsilon, y/\varepsilon, t/\rho^2(\varepsilon))$ might converge in law. The aim of this paper is to prove that this is indeed the case. For the sake of this, we use a completely probabilistic approach, studying the convergence of the stochastic processes underlying the PDE (1).

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After defining the process studied and giving the main theorem (Section 2), we prove the convergence of finite-dimensional laws and the tightness (Section 3) so that we can give results about the convergence of the solution (1) (Section 4).

REMARK. In this paper we did not try to see if our method could reach the Green's function in the case $0 < \delta < 2$, but we can hope that our entirely stochastic method can give results in that direction.

2. Setting of the problem.

2.1. *A simple mathematical model for turbulent transport*. The advection–diffusion of a passive scalar u by an incompressible velocity field v is given by the general equation

$$\begin{aligned} \frac{\partial u}{\partial t} - (v.\nabla)u &= \frac{1}{2}\nu_0 \Delta u, \\ u(x, t = 0) &= u_0(x), \\ \operatorname{div}(v) &= 0. \end{aligned} \tag{2}$$

The preceding problem is important in many applications such as turbulence or diffusion of tracers in heterogeneous porous media (see [3] for precise references). It is difficult, especially when the velocity possesses a continuous range of excited scales, an energy cascade and a random description.

The statistical description that we use here is relevant in Kolmogorov's hypothesis for fluids with high Reynolds number (see [5]). It appears then that there are two distinct length scales L_0 and $L_d (L_d \to 0$ when the Reynolds number Re tends to ∞) so that the velocity energy spectrum has a universal form (in d-space dimension) for wave number k in the range $L_0^{-1} < |k| < L_d^{-1}$ given by

$$\langle |\hat{v}(k)|^2 \rangle = K|k|^{1-d-5/3}.$$

We note that the energy spectrum is assumed to vanish for large k; meanwhile the small k behavior is not universal.

As a simple model, we will work with a rescaled velocity field

$$v_\varepsilon(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ikx} |k|^{(1-\delta)/2} \Psi_0^{1/2}\left(\frac{|k|}{\varepsilon}\right) \Psi_\infty^{1/2}(|k|)\, dW_k, \tag{3}$$

where we have the following:

1. $\{W_k,\ k \in \mathbb{R}\}$ is a one-dimensional Brownian motion;
2. $\int |k|^\alpha \Psi_\infty(k)\, dk \le C_\alpha$ for all $\alpha > 0$, $\Psi_\infty \ge 0$ is continuous at 0, $\Psi_\infty(0) = 1$ and Ψ_∞ is bounded;
3. $\Psi_0(k) = 0$ for $|k| < k_0$, $\Psi_0(k) = 1$ for $|k| > k_1$, Ψ_0 is continuous.

The hypothesis of a shear layer helps us by providing an explicit stochastic process (underlying the random PDE) that we will study. It would be a bit more difficult in a nonsheared model.

REMARKS. In this model, $\varepsilon \to 0$ corresponds to $\mathrm{Re} \to \infty$ in the physical problem. Besides Ψ_0 is related to the comportment of the fluid for low wave numbers. Finally, we can notice that k_1 corresponds to L_0^{-1} and that L_d^{-1} marks the value where Ψ_∞ begins to vanish.

Notation. $\langle\ \rangle$ will be used for the expectation with respect to the Brownian motion W.

We can remark that v_ε is a stationary Gaussian process. Indeed the stationarity follows from that of the first two moments. We note that we have $\langle v_\varepsilon(x)\rangle = 0$ and moreover,

$$\begin{aligned}\langle v_\varepsilon(x_1), \bar{v}_\varepsilon(x_2)\rangle &= \frac{1}{2\pi}\int_{\mathbb{R}} \exp(ik(x_1 - x_2))|k|^{1-\delta}\Psi_0\left(\frac{|k|}{\varepsilon}\right)\Psi_\infty(|k|)\,dk\\ &= \langle v_\varepsilon(x_1 - x_2), \bar{v}_\varepsilon(0)\rangle.\end{aligned}$$

2.2. *Renormalization.* We are looking for an exact renormalization so that we can get a nontrivial limit equation for the mean field $\langle u\rangle$. Let us define

$$u_\varepsilon(x, y, t) = u\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{t}{\rho^2(\varepsilon)}\right). \tag{4}$$

Then u_ε is a solution of the RPDE (random partial differential equation):

$$\begin{aligned}\frac{\partial u_\varepsilon}{\partial t} &= \frac{1}{2}\nu_0\frac{\varepsilon^2}{\rho^2(\varepsilon)}\,\Delta u_\varepsilon + \frac{\varepsilon}{\rho^2(\varepsilon)}v_\varepsilon\left(\frac{x}{\varepsilon}\right)\frac{\partial u_\varepsilon}{\partial y},\\ u_\varepsilon(x, y, t = 0) &= u_0(x, y).\end{aligned} \tag{5}$$

However, we know from [4] that if

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}} \frac{\langle|\hat{v}_\varepsilon(k)|^2\rangle}{k^2}\,dk < \infty,$$

then with the classical choice of $\rho(\varepsilon) = \varepsilon$, the limiting equation is of diffusion type. It is the good scaling law for homogenization results, the usual proof using essentially a central limit theorem (see, e.g., [12]). In fact, the preceding integral is finite when $\delta < 0$. That is why we study the case $\delta > 0$ because the usual proof does not fit here.

2.3. *Stochastic processes associated with the RPDE.* Let us introduce (B_t), a new two-dimensional Brownian motion, independent of W. We are going to study the following processes:

$$\begin{aligned}X_t^{1,\varepsilon} &= x + \sqrt{\nu_0}\,\frac{\varepsilon}{\rho(\varepsilon)}B_t^1,\\ X_t^{2,\varepsilon} &= y + \sqrt{\nu_0}\,\frac{\varepsilon}{\rho(\varepsilon)}B_t^2 + \frac{\varepsilon}{\rho^2(\varepsilon)}\int_0^t v_\varepsilon\left(\frac{x}{\varepsilon} + \sqrt{\nu_0}\,\frac{1}{\rho(\varepsilon)}B_s^1\right)ds.\end{aligned}$$

They are related to the RPDE (2) in the sense that we have

$$u_\varepsilon(x,y,t) = E\big(u_0\big(X_t^{1,\varepsilon}, X_t^{2,\varepsilon}\big)\big),$$

where $E(\ )$ is the expectation with respect to B.

We can remark that if $\varepsilon/\rho(\varepsilon) \to 0$ then $X^{1,\varepsilon} \to x$ and $\sqrt{\nu_0}(\varepsilon/\rho(\varepsilon))B_\cdot^2 \to 0$. Moreover, we know that v_ε is a stationary process. Therefore, we will only look in this particular case at the following process:

$$Y_t^\varepsilon = y + \frac{\varepsilon}{\rho^2(\varepsilon)} \int_0^t v_\varepsilon\left(\frac{\sqrt{\nu_0}}{\varphi(\varepsilon)} B_s^1\right) ds. \tag{6}$$

Studying the convergence of this stochastic process, we are able to give the following results:

2.4. *Theorem and Corollary.*

THEOREM. (a) "*Inviscid hyperscaling region*": *in the case* $2 < \delta < 4$, *we choose* $\rho(\varepsilon) = \varepsilon^{1-\delta/4}$. *In this case the process* Y^ε *converges in law with respect to B and W towards Y which is equal* (*in law*) *to*

$$y + t\sqrt{(1/2\pi)\int_{\mathbb{R}} \Psi_0(|k|)|k|^{1-\delta}\,dk}\,N$$

where $N \sim \mathcal{N}(0,1)$ *and is independent of* B.

(b) "*Random nonlocal diffusivity*": *if* $0 < \delta < 2$, *we choose* $\rho(\varepsilon) = \varepsilon^{2/(2+\delta)}$. *In this case, the process* Y^ε *converges in law with respect to B and W to Y which is equal* (*in law*) *to*

$$y + \big(1/\sqrt{2\pi}\big)\int_0^t ds \int_{\mathbb{R}} \exp\big(i\sqrt{\nu_0}\,kB_s\big)|k|^{(1-\delta)/2}\,dW_k.$$

COROLLARY. *Let us define* $\bar{f}_u(x,y,t) = \lim_{\varepsilon\to 0}\langle f(u(x/\varepsilon, y/\varepsilon, t/\rho^2(\varepsilon)))\rangle$ *and*

$$a(\delta) = \sqrt{\frac{1}{2\pi}\int_{\mathbb{R}} \Psi_0(|k|)|k|^{1-\delta}\,dk}\,;$$

then in the peculiar case $2 < \delta < 4$ *we find that* $\bar{f}_u$ *is solution of the PDE*:

$$\frac{\partial \bar{f}_u}{\partial t} = ta^2(\delta)\frac{\partial^2 \bar{f}_u}{\partial^2 y}$$

$$\bar{f}_u(x,y,t=0) = f(u_0)(x,y).$$

3. Proof of the theorem. We want to prove the convergence of the process Y^ε. We know that

$$Y_\cdot^\varepsilon \to Y_\cdot \text{ in law} \Leftrightarrow \{(Y_\cdot^\varepsilon),\ \varepsilon > 0\} \text{ is tight and the finite-dimensional laws are convergent.} \tag{7}$$

3.1. *Convergence of finite-dimensional laws.*

LEMMA 3.1. (i) *In the case* $2 < \delta < 4$, *finite-dimensional laws are convergent.*
(ii) *In the case* $0 < \delta < 2$, *finite-dimensional laws are convergent.*

PROOF. We are here in the particular case where we have an explicit expression of the studied process. We have

$$Y_t^\varepsilon = y + \frac{\varepsilon}{\rho^2(\varepsilon)\sqrt{2\pi}} \int_0^t ds \int_{\mathbb{R}} \exp\left(i \frac{k}{\rho(\varepsilon)} \sqrt{\nu_0} B_s\right) \times |k|^{(1-\delta)/2} \Psi_0^{1/2}\left(\frac{|k|}{\varepsilon}\right) \Psi_\infty^{1/2}(|k|)\, dW_k. \tag{8}$$

In case (i), using a change of variable $k' = k/\varepsilon$, the good choice of $\rho(\varepsilon)$ and the fact that $\sqrt{\varepsilon}\, W_k = W_{\varepsilon k}$ (in law), we have that a.s. in B and $\tilde{W}$,

$$\begin{aligned}
\tilde{Y}_t^\varepsilon &= y + \frac{1}{\sqrt{2\pi}} \int_0^t ds \int_{\mathbb{R}} \exp\left(i \frac{\varepsilon}{\rho(\varepsilon)} k \sqrt{\nu_0} B_s\right) |k|^{(1-\delta)/2} \Psi_0^{1/2}(|k|) \Psi_\infty^{1/2}(\varepsilon |k|)\, d\tilde{W}_k \\
&\xrightarrow[\varepsilon \to 0]{} y + \frac{1}{\sqrt{2\pi}} \int_0^t ds \int_{\mathbb{R}} |k|^{(1-\delta)/2} \Psi_0^{1/2}(|k|)\, d\tilde{W}_k \\
&= y + \frac{t}{\sqrt{2\pi}} \int_{\mathbb{R}} |k|^{(1-\delta)/2} \Psi_0^{1/2}(|k|)\, d\tilde{W}_k \\
&= \tilde{Y}_t,
\end{aligned}$$

by a simple argument of dominated convergence theorem and moreover,

$$Y_t^\varepsilon = \tilde{Y}_t^\varepsilon \text{ (in law with respect to } W).$$

In case (ii), with the same method using the change of variable $k' = k/\rho(\varepsilon)$, the good choice of $\rho(\varepsilon)$ and the fact that $\sqrt{\rho(\varepsilon)}\, W_k = W_{\rho(\varepsilon)k}$ (in law), we have that a.s. in B and $\tilde{W}$,

$$\begin{aligned}
\tilde{Y}_t^\varepsilon &\xrightarrow[\varepsilon \to 0]{} y + \frac{1}{\sqrt{2\pi}} \int_0^t ds \int_{\mathbb{R}} \exp\left(ik\sqrt{\nu_0} B_s\right) |k|^{(1-\delta)/2}\, d\tilde{W}_k \\
&= \tilde{Y}_t
\end{aligned}$$

and moreover $Y_t^\varepsilon = \tilde{Y}_t^\varepsilon$ (in law).

Therefore, we have the convergence of the finite-dimensional laws because we can proceed the same way for $(Y_{t_1}, \dots, Y_{t_n})_{t_1 < t_2 < \cdots < t_n < \infty}$. Moreover, we get an exact expression of the limit. □

3.2. *Tightness.* We know that Y^{ε} is a continuous process, so we will use the following Kolmogrov criterion (see, e.g., [6]) in order to prove the tightness:

$$\{(Y^{\varepsilon}_{\cdot}),\ \varepsilon > 0\} \text{ is tight if}$$

$$\begin{cases} \{(Y^{\varepsilon}_0),\ \varepsilon > 0\} \text{ is tight} \\ \exists\, \alpha, \beta, C > 0 \text{ so that } \forall\,(s,t),\ E\langle |Y^{\varepsilon}_t - Y^{\varepsilon}_s|^{\alpha}\rangle \le C.|t-s|^{1+\beta}. \end{cases} \tag{9}$$

Since $Y^{\varepsilon}_0 = y$, the tightness of $\{Y^{\varepsilon}_0\}$ is obvious, and the tightness of $\{(Y^{\varepsilon}_{\cdot}),\ \varepsilon > 0\}$ follows in case $2 < \delta < 4$ from the following lemma.

LEMMA 3.2. *In case $2 < \delta < 4$ there exists C such that*

$$E\big(\langle |Y^{\varepsilon}_t - Y^{\varepsilon}_s|^2\rangle\big) \le C(t-s)^2.$$

PROOF. Let us estimate

$$\begin{aligned} E\big(\langle |Y^{\varepsilon}_t - Y^{\varepsilon}_s|^2\rangle\big) &= \frac{\varepsilon^2}{2\pi\rho^4(\varepsilon)} \int_s^t \int_s^t \int_{\mathbb{R}} E\left(\exp\left(i\sqrt{\nu_0}\,\frac{k}{\rho(\varepsilon)}(B_u - B_v)\right)\right) \\ &\qquad \times \psi_0\left(\frac{|k|}{\varepsilon}\right)\psi_\infty(|k|)|k|^{1-\delta}\,dk\,du\,dv \\ &= \frac{\varepsilon^2}{2\pi\rho^4(\varepsilon)} \int_s^t \int_s^t \int_{\mathbb{R}} \exp\left(-\frac{\nu_0 k^2}{2\rho^2(\varepsilon)}|u-v|\right) \\ &\qquad \times \psi_0\left(\frac{|k|}{\varepsilon}\right)\psi_\infty(|k|)|k|^{1-\delta}\,dk\,du\,dv \\ &= \frac{\varepsilon^2}{\pi\rho^4(\varepsilon)} \int_s^t \int_s^u \int_{\mathbb{R}} \exp\left(-\frac{\nu_0 k^2}{2\rho^2(\varepsilon)}(u-v)\right) \\ &\qquad \times \psi_0\left(\frac{|k|}{\varepsilon}\right)\psi_\infty(|k|)|k|^{1-\delta}\,dk\,du\,dv. \end{aligned}$$

Defining

$$\begin{aligned} f_{s,t}(k) &= \int_s^t \int_s^u \exp\left(-\frac{\nu_0 k^2}{2\rho^2(\varepsilon)}(u-v)\right) du\,dv \\ &= \frac{2\rho^2(\varepsilon)}{\nu_0 k^2}(t-s) \\ &\quad + \frac{4\rho^4(\varepsilon)}{\nu_0^2 k^4}\left(\exp\left(-\frac{\nu_0 k^2}{2\rho^2(\varepsilon)}(t-s)\right) - 1\right), \end{aligned} \tag{10}$$

we have that

$$E\big(\langle |Y_t^\varepsilon - Y_s^\varepsilon|^2\rangle\big) = \frac{\varepsilon^2}{\pi\rho^4(\varepsilon)}\int_{\mathbb{R}} f_{s,t}(k)\,\psi_0\bigg(\frac{|k|}{\varepsilon}\bigg)\psi_\infty(|k|)|k|^{1-\delta}\,dk.$$

Now from the inequality $e^{-at} \le 1 - at + (a^2t^2/2)$, we deduce that

$$E\big(\langle |Y_t^\varepsilon - Y_s^\varepsilon|^2\rangle\big) \le (t-s)^2\frac{\varepsilon^2}{2\pi\rho^4(\varepsilon)}\int_{\mathbb{R}} \psi_0\bigg(\frac{|k|}{\varepsilon}\bigg)\psi_\infty(|k|)|k|^{1-\delta}\,dk.$$

With the choice of $\rho(\varepsilon) = \varepsilon^{1-\delta/4}$ and after a change of variables $k' = k/\varepsilon$ in the integral, we can get

$$E\big(\langle |Y_t^\varepsilon - Y_s^\varepsilon|^2\rangle\big) \le \frac{(t-s)^2}{2\pi}\int_R \psi_0(|k|)\,\psi_\infty(\varepsilon|k|)|k|^{1-\delta}\,dk.$$

Besides, from the dominated convergence theorem we have

$$\int_{\mathbb{R}} \psi_0(|k|)\,\psi_\infty(\varepsilon|k|)|k|^{1-\delta}\,dk \xrightarrow[\varepsilon\to 0]{} \int_{\mathbb{R}} \psi_0(|k|)|k|^{1-\delta}\,dk. \tag{11}$$

Since this integral is finite as soon as $\delta > 2$, there exists C so that

$$E\big(\langle |Y_t^\varepsilon - Y_s^\varepsilon|^2\rangle\big) \le C(t-s)^2. \qquad \square$$

REMARK 1. Here, the fact that $\delta > 0$ ensures that $\varepsilon/\rho(\varepsilon) \to 0$.

REMARK 2. In case $0 < \delta < 2$, the preceding estimation is not enough. Indeed, the "good" choice of $\rho(\varepsilon) = \varepsilon^{2/(2+\delta)}$ leads to a divergent integral as $\varepsilon \to 0$. Precisely, we have (after a change of variables $k' = k/\rho(\varepsilon)$ in the integral),

$$E\big(\langle |Y_t^\varepsilon - Y_s^\varepsilon|^2\rangle\big) \le \frac{(t-s)^2}{2\pi}\int_R \psi_0\bigg(\frac{\rho(\varepsilon)}{\varepsilon}|k|\bigg)\psi_\infty(\rho(\varepsilon)|k|)|k|^{1-\delta}\,dk$$

and

$$\int_{\mathbb{R}} \psi_0\bigg(\frac{\rho(\varepsilon)}{\varepsilon}|k|\bigg)\psi_\infty(\rho(\varepsilon)|k|)|k|^{1-\delta}\,dk \xrightarrow[\varepsilon\to 0]{} \infty.$$

This result is not surprising. In fact, $f_{s,t}(k) \underset{k\to\infty}{\sim} (2\rho^2(\varepsilon)/\nu_0 k^2)(t-s)$, which corresponds to a choice of $\beta = 0$ in (9). So we need to consider a moment of $Y_t^\varepsilon - Y_s^\varepsilon$ of order strictly bigger than 2.

LEMMA 3.3. *In case* $0 < \delta < 2$ *there exists* C'_M *such that*

$$\forall (t,s) \in [0,M]^2, \qquad E\big(\langle |Y_t^\varepsilon - Y_s^\varepsilon|^4\rangle\big) \le C'_M(t-s)^2.$$

PROOF. First, let us define

$$\begin{aligned} q_{s,t}(k) = \int_s^t\int_s^t\int_s^t\int_s^t du\,dv\,dw\,dx \\ \times E\Big(\exp i\sqrt{\nu_0}\,k(B_u - B_v) + i\sqrt{\nu_0}\,k(B_w - B_x)\Big) \end{aligned} \tag{12}$$

Now let us compute $E(\langle |Y_t^\varepsilon - Y_s^\varepsilon|^4\rangle)$. We know that

$$Y_t^\varepsilon - Y_s^\varepsilon = \frac{\varepsilon}{\sqrt{2\pi}\,\rho^2(\varepsilon)}\int_{\mathbb{R}}\left(\int_s^t \exp\left(i\frac{\sqrt{\nu_o}\,k}{\rho(\varepsilon)}B_u\right)du\right) \times |k|^{(1-\delta)/2}\Psi_0^{1/2}\left(\frac{|k|}{\varepsilon}\right)\psi_\infty^{1/2}(|k|)\,dW_k$$

so we can deduce that a.s. with respect to B, $(Y_t^\varepsilon - Y_s^\varepsilon)$ is a centered Gaussian random variable $M_{t,s}$ and then, because of $\langle |M|^4\rangle = 3\langle |M|^2\rangle^2$, we have

$$\begin{aligned}\langle |Y_t^\varepsilon - Y_s^\varepsilon|^4\rangle &= \frac{3\varepsilon^4}{4\pi^2\rho^8(\varepsilon)}\left[\int_{\mathbb{R}}\left|\int_s^t \exp\left(i\frac{\sqrt{\nu_0}\,k}{\rho(\varepsilon)}B_u\right)du\right|^2 p^\varepsilon(k)\,dk\right]^2\\ &= \frac{3\varepsilon^4}{\pi^2\rho^8(\varepsilon)}\left[\int_{\mathbb{R}}\left|\int_s^t \exp\left(i\frac{\sqrt{\nu_0}\,k}{\rho(\varepsilon)}B_u\right)du\right|^2 p^\varepsilon(k)\,dk\right]^2,\end{aligned}$$

where

$$p^\varepsilon(k) = \psi_0\left(\frac{|k|}{\varepsilon}\right)\psi_\infty(|k|)|k|^{1-\delta}.$$

Then, after the change of scale $\rho(\varepsilon) = \varepsilon^{2/(2+\delta)}$ and the change of variable $k' = k/\rho(\varepsilon)$ in the preceding integral, we get

$$\begin{aligned}E\langle |Y_t^\varepsilon - Y_s^\varepsilon|^4\rangle &= \frac{3}{\pi^2}\int_{\mathbb{R}^{+2}} E\left[\left|\int_s^t \exp\left(i\sqrt{\nu_0}\,kB_u\right)du\right|^2 \times\left|\int_s^t \exp\left(i\sqrt{\nu_0}\,k'B_u\right)du\right|2\right]\\ &\qquad \times p^\varepsilon(\rho(\varepsilon)k)\,p^\varepsilon(\rho(\varepsilon)k')\,dk\,dk'\\ &\le \frac{3}{\pi^2}\int_{\mathbb{R}^{+2}}\sqrt{E\left[\left|\int_s^t \exp\left(i\sqrt{\nu_0}\,B_u\right)du\right|^4\right]}\\ &\qquad \times\sqrt{E\left[\left|\int_s^t \exp\left(i\sqrt{\nu_0}\,k'B_u\right)du\right|^4\right]}\\ &\qquad \times p^\varepsilon(\rho(\varepsilon)k)\,p^\varepsilon(\rho(\varepsilon)k')\,dk\,dk'\\ &= \frac{3}{\pi^2}\left[\int_{\mathbb{R}^+}\sqrt{q_{s,t}(k)}\,p^\varepsilon(\rho(\varepsilon)k)\,dk\right]^2\\ &\xrightarrow[\varepsilon\to 0]{} \frac{3}{\pi^2}\left[\int_{\mathbb{R}^+}\sqrt{q_{s,t}(k)}\,k^{1-\delta}\,dk\right]^2.\end{aligned} \tag{13}$$

REMARK. At this step of the proof, we see that in this case the cut-offs do not appear any more.

For the exact computation of $q_{s,t}(k)$, see the Appendix. The idea consists in cutting the integral in parts that we can easily estimate, using the properties of the Brownian motion and the symmetric roles of the integrants. At the end, we estimate that

$$
\begin{aligned}
k^8 q_{s,t}(k) &= 87 - 30\nu_0 k^2(t-s) + 4\nu_0^2 k^4 (t-s)^2 \\
&\quad - \frac{784}{9}\exp\left(-\frac{\nu_0 k^2}{2}(t-s)\right) \\
&\quad - \frac{40}{3}\nu_0 k^2 (t-s)\exp\left(-\frac{\nu_0 k^2}{2}(t-s)\right) \\
&\quad + \frac{1}{9}\exp\left(-2\nu_0 k^2(t-s)\right)
\end{aligned}
\tag{14}
$$

so that the preceding integral is finite. Indeed, the large k behavior of the integrand is

$$\sqrt{q_{s,t}(k)}\, k^{1-\delta} \underset{k\to\infty}{\sim} \nu_0 k^{-1-\delta}(t-s) \quad \text{and} \quad \int_1^{+\infty} k^{-1-\delta}\, dk < \infty \quad \text{if } \delta > 0$$

and the small k behavior of the integrand is

$$\sqrt{q_{s,t}(k)}\, k^{1-\delta} \underset{k\to 0}{\sim} \tau\nu_0^2 (t-s)^2 k^{1-\delta} \quad \text{and} \quad \int_0^1 |k|^{1-\delta}\, dk < \infty \quad \text{if } \delta < 2.$$

Moreover, if $s, t \in [0, M]$, we have $(t-s)^4 \le M^2(t-s)^2$. So we have finally

$$E\left(\langle |Y_t^\varepsilon - Y_s^\varepsilon|^4\rangle\right) \le C'_M (t-s)^2$$

and the theorem is proved. □

4. From stochastic process to PDE. Because of the convergence in law of the studied stochastic process, we are able to give results about the limiting mean field.

COROLLARY 4.1.

$$\langle u_\varepsilon(x,y,t)\rangle \xrightarrow[\varepsilon\to 0]{} \bar{u}(x,y,t) \textit{ where } \bar{u}(x,y,t) = \langle E(u_0(x, Y_t))\rangle.$$

PROOF. Because of the theorem in Section 2.4, we have that the process

$$\left\{\left(X_t^{1,\varepsilon}, X_t^{2,\varepsilon}\right), t \in \mathbb{R}\right\} \to \left\{(x, Y_t), t \in \mathbb{R}\right\}$$

in law with respect to B and W. However, we know that

$$u_\varepsilon(x,y,t) = E\left(u_0\left(X_t^{1,\varepsilon}, X_t^{2,\varepsilon}\right)\right)$$

and then

$$\langle u_\varepsilon(x,y,t)\rangle = \langle E(u_0(X_t^{1,\varepsilon}, X_t^{2,\varepsilon}))\rangle \tag{15}$$

$$\to \langle E(u_0(x, Y_t))\rangle \tag{16}$$

□

In fact, in the proof of the convergence of finite-dimensional laws, we get an a.s. convergence for a similar process (equal in law to the one that we study), so that we have the following corollary.

COROLLARY 4.2. (i) *Let us define* $\bar{f}_u(x,y,t) = \lim_{\varepsilon\to 0}\langle f(u(x/\varepsilon, y/\varepsilon, t/\rho^2(\varepsilon)))\rangle$. *Then* $\bar{f}_u(x,y,t) = \langle f(E(u_0(x,Y_t)))\rangle$;

(ii) *In the peculiar case* $2 < \delta < 4$ *we find that* $\bar{f}_u$ *is the solution of the PDE*

$$\frac{\partial \bar{f}_u}{\partial t} = ta^2(\delta)\frac{\partial^2 \bar{f}_u}{\partial^2 y},$$

$$\bar{f}_u(x,y,t=0) = f(u_0)(x,y),$$

where $a(\delta) = \sqrt{(1/2\pi)\int_{\mathbb{R}}\Psi_0(|k|)|k|^{1-\delta}\,dk}$.

PROOF. (i) Let us fix $t \geq 0$. We have seen that

$$f(u_\varepsilon)(x,y,t) = f(E(u_0(X_t^{1,\varepsilon}, X_t^{2,\varepsilon})))$$

and that it has the same limits as $f(E(u_0(x,Y_t^\varepsilon)))$ as ε tends to 0.

Since $Y_t^\varepsilon = \tilde{Y}_t^\varepsilon$ (in law with respect to W) and $\tilde{Y}_t^\varepsilon \to \tilde{Y}_t$ a.s. in B and W (see the definition of $\tilde{Y}$ in Lemma 3.1), we have that

$$f(E(u_0(x,\tilde{Y}_t^\varepsilon))) \xrightarrow[\varepsilon\to 0]{} f(E(u_0(x,\tilde{Y}_t))) \quad \text{a.s. in } W$$

and then

$$\langle f(E(u_0(x,\tilde{Y}_t^\varepsilon)))\rangle \xrightarrow[\varepsilon\to 0]{} \langle f(E(u_0(x,\tilde{Y}_t)))\rangle$$

$$= \langle f(E(u_0(x,Y_t)))\rangle.$$

(ii) The idea is to use the fact that $Y_.$ is deterministic with respect to $B_.$. Then

$$\bar{f}_u(x,y,t) = \langle f(u_0(x,Y_t))\rangle.$$

Moreover,

$$\tilde{Y}_t = \frac{t}{\sqrt{2\pi}}\int \Psi_0^{1/2}(|k|)|k|^{(1-\delta)/2}\,dW_k = ta(\delta)N \quad \text{(in law)},$$

where $N \sim \mathscr{N}(0,1)$ and $a(\delta) = \sqrt{(1/2\pi)\int_{\mathbb{R}}\Psi_0(|k|)|k|^{1-\delta}\,dk}$.

Then, we get

$$\bar{f}_u(x,y,t) = \frac{1}{a\sqrt{2\pi}}\int_{\mathbb{R}} f(u_0)(x, y+tu)\exp(-u^2/2a^2)\,du$$

so that

$$\begin{aligned}\frac{\partial \bar{f}_u(x,y,t)}{\partial t} &= \frac{1}{a\sqrt{2\pi}}\int_{\mathbb{R}} u\frac{\partial f(u_0)}{\partial y}(x, y+tu)\exp(-u^2/2a^2)\,du\\ &= \frac{ta^2}{a\sqrt{2\pi}}\int_{\mathbb{R}} \frac{\partial^2 f(u_0)}{\partial y^2}(x, y+tu)\exp(-u^2/2a^2)\,du\\ &= ta^2\frac{\partial^2 \bar{f}_u}{\partial y^2}(x,y,t). \qquad \square\end{aligned}$$

REMARK. For the difficult case $0 < \delta < 2$, we exhibit an exact expression of the limiting function $\bar{f}_u$. Maybe the interpretation of this form can provide results for the Green function.

5. Conclusion. With the tool of stochastic processes, we have been able to give additional information about this simple model of advection–diffusion RPDE. The study of such stochastic processes can provide general results for diffusion in more general random velocity fields [12], or for semi-linear RPDE (see [13], [18]). Therefore, we suggest that the presented method may succeed in the following areas:

1. The study of the time-dependent case (i.e., $v_\varepsilon(x,t)$; see [3], [9] and [10] for discussions and examples);
2. The study of some problems involving nonsheared fluid flows (see [2] for presentation of the Manhattan model);
3. The study of some problems of higher dimensions presenting nonstandard renormalizations as have been studied in [11].

REMARK. Indeed, a recent work of Carmona and Xu [7] provides a general model of turbulence involving two-dimensional nonsheared fluid flows, exhibits anomalous scaling and identifies the limit of the related stochastic processes.

APPENDIX

We compute

$$A = \int_s^t du \int_s^t dv \int_s^t dw \int_s^t dx\, E\big(\exp(ia(B_u + B_w - B_v - B_x))\big). \tag{17}$$

First, we can notice that u and w and v and x play a symmetric role, so that

$$A = 4\int_s^t du \int_s^u dw \int_s^t dv \int_s^v dx\, E\big(\exp(ia(B_u + B_w - B_v - B_x))\big).$$

Then, we are going to split this last integral in parts, ordering u, v, w and x. So, using properties of the Brownian motion, we will be able to reduce the whole job to the computation of two different integrals:

$$u > v > x > w \to (A1),$$
$$u > v > w > x \to (A2),$$
$$u > w > v > x \to (A3),$$
$$v > u > x > w \to (A4),$$
$$v > u > w > x \to (A5),$$
$$v > x > u > w \to (A6).$$

Noticing that A is an even function of a, we see that the pairs (u, w) and (v, x) play a symmetric role. Then $A1 = A5$, $A2 = A4$ and $A3 = A6$.

We will use the following formula:

$$\int_s^\alpha b(u-s)e^{b(u-s)}\,du = \left(\alpha - s - \frac{1}{b}\right)e^{b(\alpha-s)} + \frac{1}{b} \tag{18}$$

and we will denote $b = a^2/2$ in order to simplify the notation.

$(A1)$ $u > v > x > w$:

$$\begin{aligned} E\big(\exp(ia(B_u + B_w - B_v - B_x))\big) &= E\big(\exp(ia[(B_u - B_v) - (B_x - B_w)])\big) \\ &= \exp\left(-\frac{a^2}{2}(u - v + x - w)\right) \\ &= \exp\left(\frac{a^2}{2}(-u + v - x + w)\right). \end{aligned}$$

So

$$\begin{aligned} (A1) &= \int_s^t du \int_s^u dv \int_s^v dx \int_s^x dw \exp(b(-u + v - x + w)) \\ &= \int_s^t du \int_s^u dv \int_s^v dx \exp(b(-u + v - x))\left(\frac{\exp(bx) - \exp(bs)}{b}\right) \\ &= \frac{1}{b}\int_s^t du \int_s^u dv \exp(b(-u + v))\left[(v - s) + \frac{\exp(-b(v-s)) - 1}{b}\right] \\ &= \frac{1}{b^2}\int_s^t du \exp(-bu)\left[(u - s)\exp(bu) - \frac{\exp(bu)}{b} + \frac{\exp(bs)}{b}\right. \\ &\qquad\qquad \left. - \frac{\exp(bu) - \exp(bs)}{b} + (u - s)\exp(bs)\right] \\ &= \frac{1}{b^3}\left[b\frac{(t-s)^2}{2} - 2(t - s) - \frac{2(\exp(-b(t-s)) - 1)}{b}\right. \\ &\qquad \left. -(t - s)\exp(-b(t - s)) - \frac{\exp(-b(t-s))}{b} + \frac{1}{b}\right]. \end{aligned}$$

Then finally

$$(A1) = \frac{1}{a^8}\left(48 - 16a^2(t-s) + 2a^4(t-s)^2 - 48\exp\left(-\frac{a^2}{2}(t-s)\right) - 8a^2(t-s)\exp\left(\frac{a^2}{2}(t-s)\right)\right).$$

$(A2)$ $u > v > w > x$:

$$E\big(\exp(ia(B_u + B_w - B_v - B_x))\big) = E\big(\exp(ia[(B_u - B_v) + (B_w - B_x)])\big) = \exp\left(-\frac{a^2}{2}(u - v + w - x)\right) = \exp\left(\frac{a^2}{2}(-u + v - w + x)\right).$$

So

$$(A2) = \int_s^t du \int_s^u dv \int_s^v dw \int_s^w dx \exp(b(-u + v - w + x))$$

and finally $(A2) = (A1)$.

$(A3)$ $u > w > v > x$:

$$E\big(\exp(ia(B_u + B_w - B_v - B_x))\big) = E\big(\exp(ia[(B_u - B_w) + 2(B_w - B_v) + (B_v - B_x)])\big) = \exp\left(-\frac{a^2}{2}(u - w + 4w - 4v + v - x)\right) = \exp\left(\frac{a^2}{2}(-u - 3w + 3v + x)\right).$$

So

$$(A3) = \int_s^t du \int_s^u dw \int^w dv \int_s^v dx \exp(b(-u - 3w + 3v + x))$$
$$= \int_s^t du \int_s^u dw \int_s^w dv \exp(b(-u - 3w + 3v))\left(\frac{\exp(bv) - \exp(bs)}{b}\right)$$
$$= \frac{1}{b}\int_s^t du \int_s^u dw \exp(b(-u - 3w)) \times \left(\frac{\exp(4bw) - \exp(4bs)}{4b} - \exp(bs)\left(\frac{\exp(3bw) - \exp(3bs)}{3b}\right)\right)$$

$$= \frac{1}{12b^2}\int_s^t du \int_s^u dw \exp(-bu)$$
$$\times(3\exp(bw) - 4\exp(bs) + \exp(-b(3w - 4s)))$$
$$= \frac{1}{12b^2}\int_s^t du \exp(-bu)$$
$$\times\left(\frac{3(\exp(bu) - \exp(bs))}{b}\right.$$
$$\left. -4(u - s)\exp(4bs)\left(\frac{\exp(-3bs) - \exp(-3bu)}{3b}\right)\right)$$
$$= \frac{1}{36b^3}\int_s^t du(9 - 8\exp(-b(u - s))$$
$$-12b(u - s)\exp(-b(u - s)) - \exp(-4b(u - s)))$$
$$= \frac{1}{36b^3}\left[9(t - s) + 8\frac{\exp(-b(t - s)) - 1}{b}\right.$$
$$+12(t - s)\exp(-b(t - s)) + 12\frac{\exp(-b(t - s))}{b}$$
$$\left. -\frac{12}{b} + \frac{\exp(-4b(t - s)) - 1}{4b}\right].$$

Then finally

$$(A3) = \frac{1}{a^8}\left[-9 + 2a^2(t - s) + \frac{80}{9}\exp\left(-\frac{a^2}{2}(t - s)\right)\right.$$
$$\left. + \frac{16}{3}a^2(t - s)\exp\left(-\frac{a^2}{2}(t - s)\right) + \frac{1}{9}\exp(-2a^2(t - s))\right].$$

Finally we can get A with $A = 8[(A1) + (A2) + (A3)]$:

$$A = \frac{8}{a^8}\left[87 - 30a^2(t - s) + 4a^4(t - s)^2 - \frac{784}{9}\exp\left(-\frac{a^2}{2}(t - s)\right)\right.$$
$$\left. - \frac{40}{3}a^2(t - s)\exp\left(-\frac{a^2}{2}(t - s)\right) + \frac{1}{9}\exp(-2a^2(t - s))\right].$$

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