ON CONVERGENCE RATES OF GIBBS SAMPLERS FOR UNIFORM DISTRIBUTIONS

By Gareth O. Roberts[1] and Jeffrey S. Rosenthal[2]

University of Cambridge and University of Toronto

We consider a Gibbs sampler applied to the uniform distribution on a bounded region $R \subseteq \mathbf{R}^d$. We show that the convergence properties of the Gibbs sampler depend greatly on the smoothness of the boundary of R. Indeed, for sufficiently smooth boundaries the sampler is uniformly ergodic, while for jagged boundaries the sampler could fail to even be geometrically ergodic.

1. Introduction. This paper considers the use of Gibbs samplers applied to the uniform distribution on a bounded open region $R \subseteq \mathbf{R}^d$. We shall show that, subject to C^2 smoothness of the boundary of R, such Gibbs samplers are always uniformly ergodic. We shall also show that, even with certain types of "pointy" boundaries, the Gibbs samplers are still geometrically ergodic.

By way of contrast, it has recently been shown by Bélisle (1997) that if the boundary of R is sufficiently *irregular*, then the Gibbs sampler can converge arbitrarily slowly. Our results thus complement those of Bélisle.

We note that our interest in Gibbs samplers arises partially from our interest in "slice sampler" or "auxiliary variable" algorithms, whereby sampling from a complicated $(d-1)$-dimensional density f is achieved by applying the Gibbs sampler to the uniform distribution on the d-dimensional region underneath the graph of f. Thus, Gibbs samplers for uniform distributions promise to be a very important subject in the future. For further details, see Higdon (1997), Damien, Wakefield and Walker, (1997), Mira and Tierney (1997) and Roberts and Rosenthal (1997b).

We begin with some definitions. Let $R \subseteq \mathbf{R}^d$ be a bounded open connected region in d-dimensional Euclidean space, and let $\pi(\cdot)$ be the uniform distribution on R [i.e., $\pi(A) = \lambda(A \cap R)/\lambda(R)$ for Borel sets $A \subseteq \mathbf{R}^d$, where λ is d-dimensional Lebesgue measure]. Let $\mathbf{X}^{(0)}$ be some random variable taking values in R. The *random-scan Gibbs sampler* proceeds as follows. Given a point $\mathbf{X}^{(n)} \in \mathbf{R}^d$, it chooses $I_{n+1} \in \{1, 2, \ldots, d\}$ uniformly at random. It then chooses $\mathbf{X}^{(n+1)}$ uniformly from the one-dimensional set

$$\{(X_1^{(n)}, \ldots, X_{I-1}^{(n)}, y, X_{I+1}^{(n)}, \ldots, X_d^{(n)}); \ y \in \mathbf{R}\} \cap R,$$

that is, from the intersection of R with a line through $\mathbf{X}^{(n)}$ parallel to the ith coordinate axis. This process is repeated for $n = 0, 1, 2, \ldots$.

REMARK. Other versions of this algorithm are available. For example, instead of choosing a single coordinate I_{n+1} to update, it is possible to update all d coordinates in sequence, one at a time; this is the *deterministic-scan Gibbs sampler*. Also the Gibbs sampler may be defined for nonuniform distributions by sampling from the full conditional distributions on the one-dimensional sets instead of sampling uniformly. For further details, see, for example, Gelfand and Smith (1990), Smith and Roberts (1993) and Tierney (1994).

The random-scan Gibbs sampler algorithm thus implicitly defines Markov chain transition probabilities $\mathscr{L}(\mathbf{X}^{(n+1)}\,|\,\mathbf{X}^{(n)})$. It is easily checked that the resulting Markov chain is reversible with respect to $\pi(\cdot)$. Furthermore, the Markov chain is easily seen to be π-irreducible and aperiodic. Thus, from the general theory of Markov chains on general state spaces [see, e.g., Nummelin (1984), Meyn and Tweedie (1993) and Tierney (1994), Section 3], we will have that

$$\left\|\mathscr{L}(\mathbf{X}^{(n)}) - \pi(\cdot)\right\| \equiv \sup_{A \subseteq \mathbf{R}^d}\left|\mathbf{P}(\mathbf{X}^{(n)} \in A) - \pi(A)\right| \to 0, \qquad n \to \infty.$$

(Here $\|\cdots\|$ is the *total variation distance* metric.)

A natural question is the rate at which this convergence takes place. It is shown by Bélisle (1997) that, without further restrictions on R, this convergence can be arbitrarily slow: for any sequence $\{b_n\}$ converging to 0, Bélisle shows that R and $\mathbf{X}^{(0)}$ can be chosen so that $\|\mathscr{L}(\mathbf{X}^{(n)}) - \pi(\cdot)\| \geq b_n$ for all sufficiently large n. However, it is reasonable to expect that if regularity conditions are imposed on R, then convergence will be faster.

Recall [cf. Meyn and Tweedie (1993) and Tierney (1994)] that a Markov chain with state space $\mathscr{X}$ and stationary distribution $\pi(\cdot)$ is *geometrically ergodic* if there is $\rho < 1$, a subset $\mathscr{X}_0 \subseteq \mathscr{X}$ with $\pi(\mathscr{X}_0) = 1$ and $M: \mathscr{X}_0 \to \mathbf{R}$ such that

$$\left\|\mathscr{L}(\mathbf{X}^{(n)}\,|\,\mathbf{X}^{(0)} = x_0) - \pi(\cdot)\right\| \leq M(x_0)\rho^n, \qquad n \in \mathbf{N},\ x_0 \in \mathscr{X}_0.$$

The chain is *uniformly ergodic* if it is geometrically ergodic with M constant (or, equivalently, with M bounded above). We note that geometric or uniform ergodicity ensures that the chain does *not* converge arbitrarily slowly in the sense of Bélisle.

In this paper, we shall show that for certain regions R (e.g., if the boundary of R is C^2), the corresponding Gibbs sampler is uniformly ergodic (Section 2). For slightly less regular regions R, the Gibbs sampler is still geometrically ergodic (Section 3).

2. Uniform ergodicity. In this section we shall derive conditions on R which ensure uniform ergodicity of the corresponding random-scan Gibbs sampler for the uniform distribution on R.

We recall [see, e.g., Nummelin (1984) and Meyn and Tweedie (1993)] that, given a Markov chain on a state space $\mathscr{X}$, a subset $C \subseteq \mathscr{X}$ is *small* [or (n_0, a, ν)-*small*] if for some $n_0 \in \mathbf{N}$, $a > 0$, and probability distribution $\nu(\cdot)$

on $\mathscr{X}$, we have

$$P^{n_0}(x, \cdot) \geq a\,\nu(\cdot), \qquad x \in C.$$

We note that if $B \subseteq C$ and C is small, then B is also small (with the same n_0, a and ν). We further recall [cf. Meyn and Tweedie (1993), Theorem 16.0.2] that a Markov chain is uniformly ergodic *if and only if* the entire state space $\mathscr{X}$ is small, that is, if and only if the above condition is satisfied with $C = \mathscr{X}$.

We begin with a simple lemma.

LEMMA 1. *Let R be a bounded region in $\mathbf{R}^d$, and let C be a d-dimensional rectangle which lies entirely inside R. Then C is small for the Gibbs sampler on the uniform distribution on R (with either random- or deterministic-scan).*

PROOF. If C has widths $a_1, a_2, \ldots, a_d$ and if R is bounded by a rectangle with widths $A_1, A_2, \ldots, A_d$, then the deterministic-scan Gibbs sampler starting inside C is clearly at least $\prod_i (a_i/A_i)$ times the uniform measure on C. For random-scan, we just need an extra factor of $d!/d^d$, which is the probability that the first d directions chosen include each direction precisely once. We thus obtain that

$$P_{DS}(x, \cdot) \geq \left(\prod_{i=1}^{d} \frac{a_i}{A_i}\right)\mathscr{U}_C(\cdot) \quad \text{and} \quad P_{RS}(x, \cdot) \geq (d!/d^d)\left(\prod_{i=1}^{d} \frac{a_i}{A_i}\right)\mathscr{U}_C(\cdot),$$

where P_{DS} and P_{RS} are the deterministic-scan and random-scan Gibbs samplers, respectively, and where $\mathscr{U}_C$ is the uniform distribution on C. □

To make use of this lemma, we require a general result about small sets. [A similar result is presented in Meyn and Tweedie (1993), Proposition 5.5.5(ii).]

PROPOSITION 2. *For an irreducible aperiodic Markov chain, the finite union of small sets (each of positive stationary measure) is small.*

PROOF. By induction, it suffices to consider just two small sets. Suppose that C_1 is $(n_1, \varepsilon_1, \nu_1)$-small, and that C_2 is $(n_2, \varepsilon_2, \nu_2)$-small.

By irreducibility, since $\pi(C_2) > 0$, there is $m \in \mathbf{N}$ and $\delta > 0$, such that $\nu_1 P^m(C_2) \equiv \int_R P^m(x, C_2)\nu_1(dx) \geq \delta$. It follows that $P^{n_1+m+n_2}(x, \cdot) \geq \varepsilon_1\delta\varepsilon_2\nu_2(\cdot)$ for $x \in C_1$. Also $P^{n_2}(x, \cdot) \geq \varepsilon_2\nu_2(\cdot) \geq \varepsilon_1\delta\varepsilon_2\nu_2(\cdot)$ for $x \in C_2$. Thus, $\sum_{n=1}^{\infty} P^n(x, \cdot) \geq \varepsilon_1\delta\varepsilon_2\nu_2(\cdot)$ for $x \in C_1 \cup C_2$. Hence, $C_1 \cup C_2$ is "petite" in the sense of Meyn and Tweedie [(1993), page 121].

Then by irreducibility and aperiodicity, it follows [cf. Meyn and Tweedie (1993), Theorem 5.5.7] that $C_1 \cup C_2$ must be small. □

We now put these results together. For $\mathbf{x} \in \mathbf{R}^d$, we shall write $B(x, \varepsilon)$ for the open L^∞ cube centered at $\mathbf{x}$ of radius ε, that is,

$$B(\mathbf{x}, \varepsilon) = \left\{\mathbf{y} \in \mathbf{R}^d;\; x_i - \varepsilon < y_i < x_i + \varepsilon,\; i = 1, 2, \ldots, d\right\}.$$

THEOREM 3. *Let R be a bounded open connected region in $\mathbf{R}^d$. Let R_ε be the set of all $\mathbf{x} \in R$ such that $B(\mathbf{x}, \varepsilon)$ lies entirely inside R; that is,*

$$R_\varepsilon = \{\mathbf{x} \in \mathbf{R}^d; \ B(\mathbf{x}, \varepsilon) \subseteq R\}.$$

Then R_ε is small for the random-scan Gibbs sampler for the uniform distribution on R.

PROOF. Let K be the closure of R_ε. Then K is compact, and $\mathrm{dist}(K, R^C) \geq \varepsilon/2 > 0$. Put an open L^∞ cube of radius $\varepsilon/2$ around each point of K. By compactness, K is contained in a finite union of these open L^∞ cubes, that is, $K \subseteq B(\mathbf{x}_1, \varepsilon/2) \cup \cdots \cup B(\mathbf{x}_r, \varepsilon/2)$ for some $\mathbf{x}_1, \ldots, \mathbf{x}_r \in K$. By the lemma, each $B(\mathbf{x}_i, \varepsilon/2)$ is small. By the proposition, their finite union is also small (since the connectedness of R implies the irreducibility of the Gibbs sampler). Since K is contained in this finite union, and since $R_\varepsilon \subseteq K$, therefore K and R_ε are small, too. $\square$

This immediately implies the following theorem.

THEOREM 4. *Let R be a bounded open connected region of $\mathbf{R}^d$. Suppose there is $m \in \mathbf{N}$, $\varepsilon > 0$ and $\delta > 0$ such that*

$$P^m(\mathbf{x}, R_\varepsilon) \geq \delta, \qquad \mathbf{x} \in R$$

(where P is the corresponding random-scan Gibbs sampler). Then P is uniformly ergodic.

PROOF. From the previous theorem, we have $P^{n_0}(\mathbf{x}, \cdot) \geq a\nu(\cdot)$ for all $\mathbf{x} \in R_\varepsilon$, for some $n_0, a > 0$ and $\nu(\cdot)$. Then $P^{n_0+m}(\mathbf{x}, \cdot) \geq \delta a\nu(\cdot)$ for all $\mathbf{x} \in R$. The result follows. $\square$

We conclude this section by studying a particular case in which we can verify the conditions of the above theorem, namely for regions R whose boundaries are sufficiently smooth. We begin by showing that all such regions have the property that there is a fixed $a > 0$ such that every point in their closure is contained in some ball of radius a lying entirely inside the region. Intuitively, by rolling a radius-a soccer ball around the interior of R, we could touch every point in the closure of R.

LEMMA 5. *Let R be a bounded open region in $\mathbf{R}^d$ whose boundary is a $(d-1)$-dimensional C^2 manifold. Write $\overline{R}$ for the closure of R. For $\mathbf{x} \in \overline{R}$, let*

$$\eta(\mathbf{x}) = \sup\{r > 0; \ \mathbf{x} \in \overline{D(\mathbf{y}, r)} \text{ for some } \mathbf{y} \in R \text{ such that } D(\mathbf{y}, r) \subseteq R\},$$

where $D(\mathbf{y}, r) = \{\mathbf{z} \in \mathbf{R}^d; \ \sum_{i=1}^d (z_i - y_i)^2 < r^2\}$ is an L^2-ball centered at $\mathbf{y}$. Then there is $a > 0$ such that $\eta(\mathbf{x}) \geq a$ for all $\mathbf{x} \in \overline{R}$.

PROOF. It clearly suffices to consider only points $\mathbf{x} \in \partial R$, where ∂R is the boundary of R: indeed we have $\inf_{\mathbf{x} \in \overline{R}} \eta(\mathbf{x}) = \inf_{\mathbf{x} \in \partial R} \eta(\mathbf{x})$, since any $\mathbf{x} \in R$ which is not in one of the radius-a circles touching the boundary is at least a distance a away from all boundary points.

Since ∂R is C^2, for each $\mathbf{x} \in \partial R$ the curvatures of all geodesics in ∂R passing through $\mathbf{x}$ have a finite supremum $K(\mathbf{x})$. Furthermore, by compactness of ∂R, there is $K < \infty$ such that $K(\mathbf{x}) \leq K$ for all $\mathbf{x} \in \partial R$. This means that, given an L^2 ball of radius $\leq 1/K$ which is tangent to ∂R at $\mathbf{x}$, the boundary ∂R does not curve enough to intersect this ball at any point of ∂R (aside from $\mathbf{x}$ itself) whose geodesic distance to $\mathbf{x}$ along ∂R is less than or equal to $1/K$. That is, such L^2 balls can only intersect ∂R at $\mathbf{x}$ and at points whose geodesic distance to $\mathbf{x}$ along ∂R is at least $1/K$. Now, by compactness, there is a positive smallest distance from $\mathbf{x}$ to the other points of intersection. We conclude that $\eta(\mathbf{x}) > 0$, for each $\mathbf{x} \in \partial R$.

To continue, we write $S(\mathbf{x}, r)$ for the L^2 ball (i.e., sphere) of radius r which is tangent to ∂R at the point $\mathbf{x} \in \partial R$. Now, since ∂R is a manifold, therefore each $\mathbf{x} \in \partial R$ has a neighborhood $\mathcal{N}(\mathbf{x})$ on which ∂R is diffeomorphic to $\mathbf{R}^{d-1}$. If $y \in \partial R$ is sufficiently close to $\mathbf{x}$, then $S(y, \eta(\mathbf{x}) - \delta)$ is entirely contained in the union of $\mathcal{N}(\mathbf{x})$ and $S(\mathbf{x}, \eta(\mathbf{x}))$ (for appropriate small δ). But from this it follows that $\liminf_{y \to x} \eta(y) \geq \min(\eta(\mathbf{x}), 1/K)$.

Hence, we see that the function $\min(\eta(\mathbf{x}), 1/K)$ is both positive and lower semi-continuous on $\mathbf{x} \in \partial R$. Hence, again by compactness, it has a positive minimum on ∂R. The result follows. $\square$

To make use of this lemma, we need a second lemma.

LEMMA 6. *Consider the random-scan Gibbs sampler for the uniform distribution on $R \subseteq \mathbf{R}^d$. Suppose there is $a > 0$ such that $\eta(\mathbf{x}) \geq a$ for all $\mathbf{x} \in \overline{R}$, with $\eta(\mathbf{x})$ as in Lemma 5. Then for any fixed sufficiently small $\varepsilon > 0$, there is $\delta > 0$ such that $P(x, R_\varepsilon) \geq \delta$ for all $x \in R$.*

PROOF. Given $\mathbf{x} \in R$, let $\mathbf{y} \in R$ be such that $\mathbf{x} \in D(\mathbf{y}, a) \subseteq R$. [Such $\mathbf{y}$ exists since $\eta(\mathbf{x}) \geq a$.] Let $\mathbf{u} = (\mathbf{y} - \mathbf{x})/|\mathbf{y} - \mathbf{x}|$ be the unit vector from $\mathbf{x}$ towards $\mathbf{y}$.

Now, we have $\sum_{i=1}^d (\mathbf{e}_i \cdot \mathbf{u})^2 = 1$, where $\{\mathbf{e}_i\}$ are the standard unit basis. Hence there is some coordinate i with $\mathbf{e}_i \cdot \mathbf{u} \geq 1/\sqrt{d}$. It follows that there are $\delta' > 0$ and $a' < a$ which depend on d but not on $\mathbf{x}$, such that if the chain starts at $\mathbf{x}$ and updates the ith coordinate, it has probability at least δ' of ending up within a' of $\mathbf{y}$. Hence, since we had probability $1/d$ of choosing to update the ith coordinate, it follows that $P(\mathbf{x}, D(\mathbf{y}, a')) \geq \delta'/d$, for all $\mathbf{x} \in R$. Also, note that $D(\mathbf{y}, a') \subseteq R_\varepsilon$ with $\varepsilon = a - a'$. Hence, setting $\delta = \delta'/d$ and $\varepsilon = a - a'$, the result follows. $\square$

REMARK. This lemma makes use of the fact that we are using the random-scan Gibbs sampler. For the deterministic-scan Gibbs sampler, the situation is in fact more complicated. For example, suppose $R \subseteq \mathbf{R}^2$ contains the unit

ball in $\mathbf{R}^2$ and also contains an open neighborhood of the point $(10, -1)$. Suppose the deterministic-scan Gibbs sampler begins at the point $(0, -0.99)$ and first updates the x coordinate. This could bring it to near $(10, -1)$ or some other faraway point, and this is very difficult to control. (By contrast, for the random-scan Gibbs sampler, we would have probability $1/2$ of updating the y coordinate first, in which case we might happily move to near the origin, and thus definitely be inside R_ε.) On the other hand, if R is assumed to be *convex*, then no such difficulties arise, and our proof goes through with minor changes to the case of the deterministic-scan Gibbs sampler.

Combining Lemma 6 with Theorem 4, we immediately obtain the following theorem.

THEOREM 7. *Let R be a bounded open connected region in $\mathbf{R}^d$, whose boundary is a $(d-1)$-dimensional C^2 manifold (or, such that $\eta(\mathbf{x}) \geq a > 0$ for all $\mathbf{x} \in \overline{R}$ as in Lemma 6). Then the random-scan Gibbs sampler for the uniform distribution on R is uniformly ergodic.*

REMARKS.

1. Based on Theorem 7, it is not surprising that the slowly mixing examples studied by Bélisle (1997) involve regions which do not have C^2 boundaries (see, e.g., his Figure 3).
2. Theorem 7 is somewhat analogous to results about the spectral gap of the Laplacian for Brownian motion in a region; see, for example, Bañuelos and Carroll (1994) and references therein.

Finally, we note the following. Even if we do not have $\eta(\mathbf{x}) \geq a > 0$ for all $\mathbf{x} \in \overline{R}$, we may still have uniform ergodicity. For example, if the boundary of R has some nondifferentiable "pointy" regions (e.g., the vertices, if R is an irregular polygon), but if these pointy regions are angled such that their apexes each contain some coordinate direction (i.e., some line segment parallel to some coordinate axis), then the conditions of Theorem 4 are still satisfied since there is probability bounded away from 0 of leaving the pointy region in a single step. (Similarly, if the apex boundary is *exactly parallel* to a coordinate direction, then there is probability bounded away from 0 of leaving in two steps.) However, if the pointy regions are "tilted" so that their apex does *not* contain a coordinate direction, not even on its boundary, then the chain is clearly *not* uniformly ergodic; indeed, the closer the chain is to the vertex point, the longer it will take the chain to move away from this point. Nevertheless, we shall see in the next section that such chains are still geometrically ergodic.

3. Geometric ergodicity. In this section we consider regions $R \subseteq \mathbf{R}^d$ which do not have a C^2 boundary. For such regions, the results of the previous section do not apply, and, indeed, the Gibbs samplers for such regions may not be uniformly ergodic in general. However, we are able to show that they

are still geometrically ergodic, in certain cases. For simplicity we concentrate primarily on the two-dimensional case $d = 2$, though we also provide one three-dimensional result.

We begin with the case where R is a triangle. We recall from the previous section that, if the triangle is such that all vertices have apex which contains a coordinate direction, then the associated Gibbs sampler is uniformly ergodic. Thus, we instead consider the case where one of the vertices is "tilted" and does not contain a coordinate direction.

PROPOSITION 8.   *Let $R \subseteq \mathbf{R}^2$ be the width-1 triangle with lower angle θ and upper angle ϕ, that is,*

$$R = \big\{(x, y) \in \mathbf{R}^2;\ 0 < x < 1,\ x \tan(\theta) < y < x \tan(\phi)\big\},$$

where $0 < \theta < \phi < \pi/2$. Then the Gibbs sampler (with either random- or deterministic-scan) for the uniform distribution on R is geometrically ergodic.

PROOF.   We recall from the previous section that the subset $C = \{(x, y) \in R;\ x > \tan(\theta)/\tan(\phi)\}$, say, is small for the Gibbs sampler. Thus, by standard Markov chain theory [see, e.g., Nummelin (1984) and Meyn and Tweedie (1993), Theorem 15.0.1], we will be done if we can find a *drift function* $V: R \to [1, \infty)$ and $\lambda < 1$ such that

$$PV(x, y) \equiv \int_R V(\mathbf{z})P((x, y), d\mathbf{z}) \le \lambda\, V(x, y), \qquad (x, y) \in R\backslash C.$$

To continue, we consider the drift function $V(x, y) = 1/x$. To compute $PV(x, y)$, for ease of computation we shall focus on the deterministic-scan Gibbs sampler on R which updates first the y coordinate and then the x coordinate, rather than on the random-scan Gibbs sampler. This is not a restriction since it is known [see, e.g., Roberts and Rosenthal (1997a), Proposition 5] that if the deterministic-scan Gibbs sampler is geometrically ergodic, then so is the random-scan Gibbs sampler.

We compute that, for the deterministic-scan Gibbs sampler, if $(x, y) \notin C$,

$$
\begin{aligned}
PV(x, y) &= \frac{1}{x \tan(\phi) - x \tan(\theta)} \\
&\quad \times \int_{x\tan(\theta)}^{x\tan(\phi)} \frac{1}{w \cot(\theta) - w \cot(\phi)} \int_{w\cot(\phi)}^{w\cot(\theta)} V(z, w)\, dz\, dw \\
&= \lambda\, V(x, y),
\end{aligned}
$$

where

$$\lambda = \lambda(\theta, \phi) = [\log(\cot(\theta)/\cot(\phi))]^2/[(\tan(\phi) - \tan(\theta))(\cot(\theta) - \cot(\phi))].$$

(Note that we actually have equality here, even though we only require an inequality.) Now, we have $\lambda(\theta, \phi) < 1$ whenever $0 < \theta < \phi < \pi/2$; indeed, if

we set $f(\varepsilon) = \lambda(\theta, \theta + \varepsilon)$, then to second order in ε, as $\varepsilon \to 0^+$, we have

$$f(\varepsilon) \approx 1 - \varepsilon^2/(3\sin^2(2\theta)) < 1.$$

The geometric ergodicity follows. □

It is possible to combine Proposition 8 with the results of Section 2. For example, we have the following theorem.

THEOREM 9. *Suppose R is a region in $\mathbf{R}^2$ whose boundary is a one-dimensional C^2 manifold except at a finite number of points. Suppose further that in a neighborhood of each of these exceptional points, R coincides with a triangle (as in Proposition 8). Then the random-scan Gibbs sampler for the uniform distribution on R is geometrically ergodic.*

OUTLINE OF PROOF. As noted at the end of Section 2, the Gibbs sampler is uniformly ergodic except near those exceptional points whose vertices are "tilted," that is, have apexes which do not contain any coordinate direction. For such tilted vertices, it is possible to choose $\varepsilon > 0$ small enough that $R \setminus R_\varepsilon$ breaks up into a finite number of connected components, one near each exceptional point, such that it is impossible to get from one of these components to another in a single step. Once we have done that, then we define a drift function V to be equal to 1 on R_ε, and equal to the appropriate drift function (as in the proof of Proposition 8) on each of the different connected components of $R \setminus R_\varepsilon$. Then, separately from each connected component, the Gibbs sampler has geometric drift back to the small set R_ε. Hence, as in Proposition 8, the result follows. □

Similar results are available for higher-dimensional regions R having "vertices" on the boundary. We illustrate this with a particular example, a "tilted cone" with base at the origin, tilted so that it does not contain any coordinate direction.

PROPOSITION 10. *Suppose $R \subseteq \mathbf{R}^3$ is the tilted cone*

$$R = \left\{ (x, y, z) \in \mathbf{R}^3; \ 0 < x < 1, \ z^2 + \frac{(\alpha x - y)^2}{1 + \alpha^2} < c\frac{(x + \alpha y)^2}{1 + \alpha^2} \right\},$$

for some $\alpha > 0$ and $0 < c < 1$. Then the Gibbs sampler (with either random- or deterministic-scan) for the uniform distribution on R is geometrically ergodic.

PROOF. We use the same drift function $V(x, y, z) = 1/x$ as before. We consider the deterministic-scan Gibbs sampler which updates first z, then y, and then x. [The corresponding result for the random-scan Gibbs sampler then follows once again from Roberts and Rosenthal (1997a), Proposition 5.] Clearly updating z does not change the value of V, so it suffices to consider the effect of updating x and y conditional on a fixed value of z.

Now, conditional on $z = 0$, the point (x, y) is restricted to the *triangle*

$$R \cap \{z = 0\} = \{(x, y, 0) \in \mathbf{R}^3; \ x \tan(\theta) < y < x \tan(\phi)\},$$

for some $0 < \theta < \phi < \pi/2$. Furthermore, conditional on a particular value of $z \neq 0$, the point (x, y) is restricted to a *hyperbola* lying inside (and asymptotic to) the triangle $R \cap \{z = 0\}$, whose proximity to this triangle depends on z.

To proceed, let P_{z_0} be the two-dimensional random-scan Gibbs sampler for the uniform distribution on $R \cap \{z = z_0\}$, that is, which acts on the coordinates x and y while leaving the value of z fixed at $z = z_0$. Then P_0 is the usual two-dimensional random-scan Gibbs sampler on the triangle $R \cap \{z = 0\}$, and hence by Proposition 8, P_0 is geometrically ergodic with $P_0 V(x, y, z) \leq \lambda V(x, y, z)$ for some $\lambda < 1$.

Now, we claim that for any choice of $z_0 \in \mathbf{R}$ such that $R \cap \{z = z_0\}$ is nonempty, we have $P_{z_0} V(x, y, z_0) \leq P_0 V(x, y, z_0)$. Indeed, for fixed z_0 we have

$$(\dagger) \quad P_{z_0} V(x, y, z) = \frac{1}{y_2(x) - y_1(x)} \int_{y_1(x)}^{y_2(x)} \frac{1}{x_2(w) - x_1(w)} \int_{x_1(w)}^{x_2(w)} (1/z) \, dz \, dw,$$

where $y_1(x)$, $y_2(x)$, $x_1(w)$, and $x_2(w)$ are defined by

$$R \cap \{(x, t); \ t \in \mathbf{R}\} = \{(x, t); \ y_1(x) < t < y_2(x)\}$$

and

$$R \cap \{(t, w); \ t \in \mathbf{R}\} = \{(t, w); \ x_1(x) < w < x_2(x)\}.$$

It is furthermore verified that there are functions $d(x)$ and $D(y)$ (which also depend on θ, ϕ, and z_0) such that

$$y_1(x) = x \cot(\theta) + d(x); \qquad y_2(x) = x \cot(\phi) - d(x);$$

$$x_1(y) = y \tan(\phi) + D(y); \qquad x_2(y) = y \tan(\theta) - D(y);$$

that is, the interval $(x_1(y), x_2(y))$ is symmetrically embedded in the interval $(y \cot(\phi), y \cot(\theta))$ [and similarly for $(y_1(x), y_2(x))$].

To show that $P_{z_0} V \leq P_0 V$, we observe that, for fixed $0 < a < b$ and $0 \leq k < (b - a)/2$, the quantity $1/(b - a - 2k) \int_{a+k}^{b-k} (1/z) \, dz$ as a function of k is maximized at $k = 0$. Applying this observation twice to $(\dagger)$ shows that $P_{z_0} V(x, y, z) \leq P_0 V(x, y, z)$ as desired.

It follows that the deterministic-scan Gibbs sampler on R is again geometrically ergodic, with at least as small a value of λ as the corresponding value from Proposition 8. $\square$

Finally, we turn our attention to showing that certain Gibbs samplers are *not* geometrically ergodic. We begin with a result, following Lawler and Sokal

(1988), which may be viewed as a generalization of Roberts and Tweedie (1996), Theorem 5.1.

LEMMA 11. *Let $P(x, \cdot)$ be the transition probabilities for a Markov chain on a state space $\mathscr{X}$, having stationary distribution $\pi(\cdot)$. Suppose that, for any $\delta > 0$, there is a subset $A \subseteq \mathscr{X}$ with $0 < \pi(A) < 1$ such that*

$$\frac{\int_A P(\mathbf{x}, A^C)\pi(d\mathbf{x})}{\pi(A)\pi(A^C)} < \delta.$$

Then the Markov chain is not geometrically ergodic.

PROOF. We use the notion of *conductance* or *Cheeger's constant*, as in Lawler and Sokal (1988). Recall that this is defined by

$$\mathscr{K} = \inf_{A \subseteq R} \frac{\int_A P(\mathbf{x}, A^C)\pi(d\mathbf{x})}{\pi(A)\pi(A^C)},$$

where the infimum is taken over all measurable subsets of R, and the integral is taken with respect to the stationary distribution $\pi(\cdot)$. It follows from Lawler and Sokal (1988) that for a reversible Markov chain (such as the random-scan Gibbs sampler), we have $\mathscr{K} > 0$ if and only if the Markov chain is geometrically ergodic. But the hypothesis of the lemma implies that $\mathscr{K} = 0$. Hence the chain is not geometrically ergodic. □

Now, Proposition 8 considers the case where $R \subseteq \mathbf{R}^2$ has a pointed vertex which subtends a positive angle. One can still ask about the case where R has a "sharpened" vertex, that is, a vertex whose two adjoining boundary curves are asymptotically tangent. For such a case, it turns out that the Gibbs sampler is *not* geometrically ergodic, as the following result shows.

PROPOSITION 12. *Let $R \subseteq \mathbf{R}^2$ be the width-1 "sharpened" triangle with lower angle θ and power α, that is,*

$$R = \{(x, y) \in \mathbf{R}^2;\ 0 < x < 1,\ x\tan(\theta) < y < (x + x^\alpha)\tan(\theta)\},$$

where $0 < \theta < \pi/2$ and $1 < \alpha < \infty$. Then the Gibbs sampler (with either random- or deterministic-scan) for the uniform distribution on R is not geometrically ergodic.

PROOF. We shall show the result for the random-scan Gibbs sampler; the result for the deterministic-scan Gibbs sampler then follows from, for example, Roberts and Rosenthal (1997a), Proposition 5.

We shall apply Lemma 11. To that end, let $A \subseteq R$ be defined by

$$A = \{(x, y) \in R;\ y < (\varepsilon + \varepsilon^\alpha)\tan(\theta)\},$$

where $\varepsilon > 0$. Then we note that for $(x, y) \in A$, we have $P((x, y), A^C) = 0$ unless $x > \varepsilon$. Now, it is seen by inspection that

$$\pi\{(x, y) \in A; \ x > \varepsilon\} = \varepsilon^{2\alpha} \tan(\theta)/|R|.$$

Hence,

$$\int_A P(\mathbf{x}, A^C)\pi(d\mathbf{x}) \le \int_A \mathbf{1}_{\{x>\varepsilon\}}(\mathbf{z})\pi(d\mathbf{z}) = \varepsilon^{2\alpha} \tan(\theta)/|R|.$$

On the other hand, we have

$$\pi(A) > \int_0^\varepsilon \tan(\theta)\, t^\alpha\, dt/|R| = \tan(\theta)\, \varepsilon^{\alpha+1}/(\alpha+1)|R|.$$

It follows that, if we choose ε small enough so that $\pi(A^C) \le 1/2$, then

$$\frac{\int_A P(\mathbf{x}, A^C)\pi(d\mathbf{x})}{\pi(A)\pi(A^C)} \le 2\, \varepsilon^{\alpha-1}/(1+\alpha).$$

Since $\alpha > 1$, this converges to 0 as $\varepsilon \to 0^+$. Hence it can be made arbitrarily small, and the result follows from Lemma 11. □

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STATISTICAL LABORATORY
UNIVERSITY OF CAMBRIDGE
CAMBRIDGE CB2 1SB
UNITED KINGDOM
E-MAIL: g.o.roberts@statslab.cam.ac.uk

DEPARTMENT OF STATISTICS
UNIVERSITY OF TORONTO
TORONTO, ONTARIO
CANADA M5S 3G3
E-MAIL: jeff@utstat.toronto.edu