The Annals of Applied Probability
2000, Vol. 10, No. 4, 1100–1120

FRACTIONAL BROWNIAN MOTIONS IN A LIMIT OF TURBULENT TRANSPORT[1]

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We show that the motion of a particle advected by a random Gaussian velocity field with long-range correlations converges to a fractional Brownian motion in the long time limit.

1. Introduction. The motion of a particle advected by a random velocity field is governed by

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{V}(t, \mathbf{x}(t)), \tag{1}$$

where $\mathbf{V}(t, \mathbf{x}) = (V_1(t, \mathbf{x}), \ldots, V_d(t, \mathbf{x}))$ is a random, mean-zero, time- stationary, space-homogeneous incompressible velocity field in dimension $d \geq 2$.

In certain situations, it is believed that the *convergence* of the Taylor–Kubo formula (see [8] and [14]) given by

$$\int_0^\infty \{\mathbf{E}[V_i(t, \mathbf{0})V_j(0, \mathbf{0})] + \mathbf{E}[V_j(t, \mathbf{0})V_i(0, \mathbf{0})]\}\, dt \tag{2}$$

is a criterion for convergence of turbulent motion to Brownian motion in the long time limit. Indeed, it has been shown that the solution of

$$\frac{d\mathbf{x}_\varepsilon(t)}{dt} = \frac{1}{\varepsilon}\mathbf{V}\left(\frac{t}{\varepsilon^2}, \mathbf{x}_\varepsilon(t)\right), \qquad \mathbf{x}_\varepsilon(0) = 0, \tag{3}$$

converges in law, as $\varepsilon \to 0$, to the Brownian motion with diffusion coefficients given by the Taylor–Kubo formula when the velocity field is sufficiently *mixing in time* (see [2], [6], [7] and [9]). Moreover, the solution of (3) converges to the same Brownian motion for a family of *nonmixing* Gaussian, Markovian flows with power-law spectra as long as the Taylor–Kubo formula converges (see [3]). In this paper, for the same family of power-law spectra, we show that, when the Taylor–Kubo formula *diverges*, the solution of the following equation

$$\frac{d\mathbf{x}_\varepsilon(t)}{dt} = \varepsilon^{1-2\delta}\mathbf{V}\left(\frac{t}{\varepsilon^{2\delta}}, \mathbf{x}_\varepsilon(t)\right), \qquad \mathbf{x}_\varepsilon(0) = 0, \tag{4}$$

Received June 1999; revised December 1999.

[1]The research of Fannjiang was supported in part by NSF Grant DMS-96-00119. This work was finished while Komorowski was visiting the Department of Statistics, University of California, Berkeley.

AMS 1991 *subject classifications*. Primary 60F05,76F05,76R05; secondary 58F25.

Key words and phrases. Turbulent diffusion, mixing, fractional Brownian motion.

with some $\delta \neq 1$ depending on the velocity spectrum, converges, as $\varepsilon \to 0$, to a fractional Brownian motion (FBM), as introduced in [10] (see also [13]).

We define the family of velocity fields with power-law spectra as follows. Let $(\Omega, \mathscr{V}, P)$ be a probability space of which each element is a velocity field $\mathbf{V}(t, \mathbf{x})$, $(t, \mathbf{x}) \in R \times R^d$, satisfying the following properties:

(H1) $\mathbf{V}(t, \mathbf{x})$ is time stationary, space homogeneous, centered, that is, $\mathbf{E}\{\mathbf{V}\} = \mathbf{0}$, and Gaussian. Here $\mathbf{E}$ stands for the expectation with respect to the probability measure P.

(H2) The two-point correlation tensor $\mathbf{R} = [R_{ij}]$ is given by

$$\begin{aligned} R_{ij}(t, \mathbf{x}) &= \mathbf{E}\big[V_i(t, \mathbf{x}) V_j(0, \mathbf{0})\big] \\ &= \int_{R^d} \cos(\mathbf{k} \cdot \mathbf{x}) e^{-|\mathbf{k}|^{2\beta} t} \widehat{\mathbf{R}}_{ij}(\mathbf{k})\, d\mathbf{k}, \end{aligned} \tag{5}$$

with the spatial spectral density

$$\widehat{\mathbf{R}}(\mathbf{k}) = \frac{a(|\mathbf{k}|)}{|\mathbf{k}|^{2\alpha+d-2}} \left(\mathbf{I} - \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2} \right), \tag{6}$$

where $a : [0, +\infty) \to R_+$ is a compactly supported, continuous, nonnegative function. The factor $\mathbf{I} - \mathbf{k} \otimes \mathbf{k}/|\mathbf{k}|^2$ in (6) is a result of incompressibility.

(H3) $\alpha < 1$, $\beta \geq 0$ and $\alpha + \beta > 1$.

It can be readily checked that the correlation function (5) is temporally integrable and, hence, the Taylor–Kubo formula is convergent if and only if $\alpha + \beta < 1$.

The function $\exp(-|\mathbf{k}|^{2\beta} t)$ in (5) is called the *time correlation function* of the flow $\mathbf{V}$. For $\beta > 0$, the velocity field lacks the spectral gap and, thus, is not mixing in time. As the time correlation function is exponential, the Gaussian velocity field is an Ornstein–Uhlenbeck process. Because the function a has a compact support we may assume, without loss of generality, that $\mathbf{V}$ is jointly continuous in both $(t, \mathbf{x})$ and is C^∞ in $\mathbf{x}$ almost surely. For $\alpha < 1$, the spectral density $\widehat{\mathbf{R}}(\mathbf{k})$ is integrable in $\mathbf{k}$ and, thus, (5)–(6) defines a random velocity field with a finite second moment. The exponent α is directly related to the decay exponent of $\mathbf{R}$. Namely, $|\mathbf{R}|(0, \mathbf{x}) \sim |\mathbf{x}|^{\alpha-1}$ for $|\mathbf{x}| \gg 1$. As α increases to 1, the decay exponent of $\mathbf{R}$ decreases to 0.

Our main result is summarized in the following theorem (see also Figure 1).

THEOREM 1. *Under assumptions* (H1)–(H3)*, the solution of* (4) *with the scaling exponent*

$$\delta := \frac{\beta}{\alpha + 2\beta - 1} \tag{7}$$

converges in law, as ε tends to 0, to a fractional Brownian motion $\mathbf{B}_H(t)$, that is, to a Gaussian process with stationary increments whose covariance is given by

$$\mathbf{E}\big[\mathbf{B}_H(t) \otimes \mathbf{B}_H(t)\big] = \mathbf{D} t^{2H}, \tag{8}$$

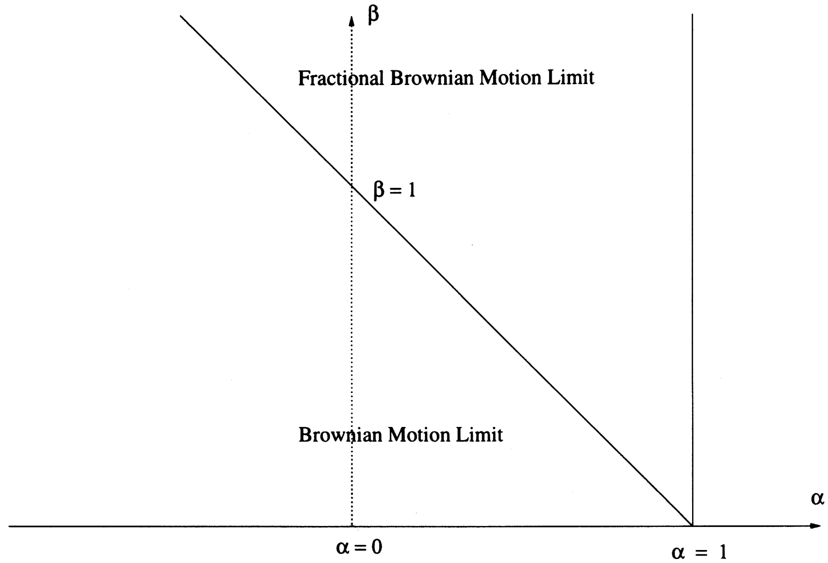

FIG. 1. *Phase diagram for scaling limit.*

with the coefficients $\mathbf{D}$

$$\mathbf{D} = \int_{R^d} \frac{e^{-|\mathbf{k}|^{2\beta}} - 1 + |\mathbf{k}|^{2\beta}}{|\mathbf{k}|^{2\alpha+4\beta-1}} \left(\mathbf{I} - \frac{\mathbf{k} \otimes \mathbf{k}}{|\mathbf{k}|^2} \right) \frac{a(0)}{|\mathbf{k}|^{d-1}} \, d\mathbf{k} \tag{9}$$

and the Hurst exponent H

$$\frac{1}{2} < H = \frac{1}{2} + \frac{\alpha + \beta - 1}{2\beta} < 1. \tag{10}$$

Moreover, we show that the process $\mathbf{x}_\varepsilon(t)$ is asymptotically, as $\varepsilon \to 0$, the same as the process

$$\mathbf{y}_\varepsilon(t) := \mathbf{x}_\varepsilon(0) + \varepsilon \int_0^{t/\varepsilon^{2\delta}} \mathbf{V}(s, \mathbf{x}_\varepsilon(0)) \, ds$$

(see Section 4). Namely, the spatial dependence of the Lagrangian velocity is frozen. As a result, the asymptotic motions of N particles starting at $\mathbf{x}^1_\varepsilon(0)$, $\mathbf{x}^2_\varepsilon(0), \ldots, \mathbf{x}^N_\varepsilon(0)$, can be easily deduced and they are in stark contrast to the case of diffusive scaling (cf. [2], [3]).

REMARK. Molecular diffusion can be added to the equation of motion so that instead of (1) we may consider an Itô stochastic differential equation

$$d\mathbf{x}(t) = \mathbf{V}\big(t, \mathbf{x}(t)\big) \, dt + \sqrt{2\kappa} \, d\mathbf{B}(t),$$

with $\mathbf{B}(t)$, $t \geq 0$, the standard Brownian motion, independent of $\mathbf{V}$ and $\kappa \geq 0$. This, however, would not influence our results.

2. Multiple stochastic integrals. By the spectral theorem (see, e.g., [1]) we assume without loss of any generality that there exist two independent, identically distributed, real vector-valued, Gaussian spectral measures $\widehat{\mathbf{V}}_l(t,\cdot)$, $l=0,1$, such that

$$\mathbf{V}(t,\mathbf{x})=\int \widehat{\mathbf{V}}_0(t,\mathbf{x},d\mathbf{k}), \tag{11}$$

where

$$\widehat{\mathbf{V}}_0(t,\mathbf{x},d\mathbf{k}):=c_0(\mathbf{k}\cdot\mathbf{x})\widehat{\mathbf{V}}_0(t,d\mathbf{k})+c_1(\mathbf{k}\cdot\mathbf{x})\widehat{\mathbf{V}}_1(t,d\mathbf{k}),$$

with $c_0(\phi)\equiv\cos(\phi)$, $c_1(\phi)\equiv\sin(\phi)$. Define also

$$\widehat{\mathbf{V}}_1(t,\mathbf{x},d\mathbf{k}):=-c_1(\mathbf{k}\cdot\mathbf{x})\widehat{\mathbf{V}}_0(t,d\mathbf{k})+c_0(\mathbf{k}\cdot\mathbf{x})\widehat{\mathbf{V}}_1(t,d\mathbf{k}).$$

We have the relations

$$\partial\widehat{\mathbf{V}}_0(t,\mathbf{x},d\mathbf{k})/\partial x_j=k_j\widehat{\mathbf{V}}_1(t,\mathbf{x},d\mathbf{k}), \tag{12}$$

$$\partial\widehat{\mathbf{V}}_1(t,\mathbf{x},d\mathbf{k})/\partial x_j=-k_j\widehat{\mathbf{V}}_0(t,\mathbf{x},d\mathbf{k}). \tag{13}$$

Clearly, $\int\widehat{\mathbf{V}}_1(t,\mathbf{x},d\mathbf{k})$ is a random field distributed identically to and independently of $\mathbf{V}$. We define the *multiple stochastic integral*

$$\int\cdots\int\psi(\mathbf{k}_1,\ldots,\mathbf{k}_N)\widehat{\mathbf{V}}_{l_1}(t_1,\mathbf{x}_1,d\mathbf{k}_1)\otimes\cdots\otimes\widehat{\mathbf{V}}_{l_N}(t_N,\mathbf{x}_N,d\mathbf{k}_N) \tag{14}$$

for any $l_1,\ldots,l_N\in\{0,1\}$ and a suitable family of functions ψ by using the Fubini theorem [see (15)]. For $\psi_1,\ldots,\psi_N\in\mathscr{S}(R^d)$, the Schwartz space, and $l_1,\ldots,l_N\in\{0,1\}$ we set

$$\begin{aligned}&\int\cdots\int\psi_1(\mathbf{k}_1)\cdots\psi_N(\mathbf{k}_N)\widehat{\mathbf{V}}_{l_1}(t_1,\mathbf{x}_1,d\mathbf{k}_1)\otimes\cdots\otimes\widehat{\mathbf{V}}_{l_N}(t_N,\mathbf{x}_N,d\mathbf{k}_N)\\&\quad:=\int\psi_1(\mathbf{k}_1)\widehat{\mathbf{V}}_{l_1}(t_1,\mathbf{x}_1,d\mathbf{k}_1)\otimes\cdots\otimes\int\psi_N(\mathbf{k}_N)\widehat{\mathbf{V}}_{l_N}(t_N,\mathbf{x}_N,d\mathbf{k}_N).\end{aligned} \tag{15}$$

We then extend the definition of multiple integration to the closure $\mathscr{H}$ of the Schwartz space $\mathscr{S}((R^d)^N,R)$ under the norm

$$\begin{aligned}\|\psi\|^2:=&\int\cdots\int\psi(\mathbf{k}_1,\ldots,\mathbf{k}_N)\psi(\mathbf{k}_1',\ldots,\mathbf{k}_N')\\&\times\mathbf{E}\big[\widehat{\mathbf{V}}_{l_1}(t_1,\mathbf{x}_1,d\mathbf{k}_1)\otimes\cdots\otimes\widehat{\mathbf{V}}_{l_N}(t_N,\mathbf{x}_N,d\mathbf{k}_N)\\&\quad\cdot\widehat{\mathbf{V}}_{l_1}(t_1,\mathbf{x}_1,d\mathbf{k}_1')\otimes\cdots\otimes\widehat{\mathbf{V}}_{l_N}(t_N,\mathbf{x}_N,d\mathbf{k}_N')\big].\end{aligned} \tag{16}$$

The expectation is to be calculated by the formal rule

$$\begin{aligned}&\mathbf{E}\big[\widehat{V}_{l,i}(t,\mathbf{x},d\mathbf{k})\widehat{V}_{l',i'}(t',\mathbf{x}',d\mathbf{k}')\big]\\&\quad=e^{-|\mathbf{k}|^{2\beta}|t-t'|}\delta_{l,l'}c_0\big(\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')\big)\widehat{R}_{i,i'}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}')\,d\mathbf{k}\,d\mathbf{k}'.\end{aligned}$$

This approach to spectral integration follows [12].

When $\mathbf{i}=(i_1,\ldots,i_d)$, $i_1,\ldots,i_d\in\{1,2,\ldots,d\}$, is fixed and $\mathbf{l}=(l_1,\ldots,l_N)$, $l_1,\ldots,l_N\in\{0,1\}$, we shall denote the corresponding component of the stochastic integral by $\Psi_{\mathbf{l},\mathbf{i}}$.

Note that $\Psi_{\mathbf{l},\mathbf{i}} \in H^N(\mathbf{V})$—the Hilbert space obtained as a completion of the space of Nth-degree polynomials in variables $\int \psi(\mathbf{k})\widehat{\mathbf{V}}(t,\mathbf{x},\mathbf{k})$ with respect to the standard L^2-norm.

PROPOSITION 1. *For any* $(t_1,\mathbf{x}_1),\dots,(t_N,\mathbf{x}_N) \in R \times R^d$ *and* $p > 0$, $\Psi_{\mathbf{l},\mathbf{i}}$ *belongs to* $L^p(\Omega)$ *and*

$$\big(\mathbf{E}|\Psi_{\mathbf{l},\mathbf{i}}|^p\big)^{1/p} \le C\big(\mathbf{E}|\Psi_{\mathbf{l},\mathbf{i}}|^2\big)^{1/2}, \tag{17}$$

with the constant C *depending only on* p, N *and the dimension* d. *Moreover,* $\Psi_{\mathbf{l},\mathbf{i}}$ *is differentiable in the mean square sense with*

$$\begin{aligned}
&\nabla_{\mathbf{x}_j}\Psi_{\mathbf{l},\mathbf{i}}(t_1,\dots,t_N,\mathbf{x}_1,\dots,\mathbf{x}_N)\\
&\quad=(-1)^{l_j}\int\cdots\int \mathbf{k}_j\psi(\mathbf{k}_1,\dots,\mathbf{k}_N)\widehat{V}_{l_1,i_1}(t_1,\mathbf{x}_1,d\mathbf{k}_1)\cdots\widehat{V}_{1-l_j,i_j}(t_j,\mathbf{x}_j,d\mathbf{k}_j)\\
&\qquad\cdots\widehat{V}_{l_N,i_N}(t_N,\mathbf{x}_N,d\mathbf{k}_N).
\end{aligned}\tag{18}$$

The proof of Proposition 1 is standard and follows directly from the well-known hypercontractivity property for Gaussian measures (see, e.g., [5], Theorem 5.1 and its corollaries), so we do not repeat it here.

The field $\mathbf{V}$ is Markovian, that is,

$$\begin{aligned}
&\mathbf{E}\bigg[\int \psi(\mathbf{k})\widehat{\mathbf{V}}_l(t,\mathbf{x},d\mathbf{k})\big|\mathscr{V}_{-\infty,s}\bigg]\\
&\qquad = \int e^{-|\mathbf{k}|^{2\beta}(t-s)}\psi(\mathbf{k})\widehat{\mathbf{V}}_l(s,\mathbf{x},d\mathbf{k}), \qquad l=0,1,
\end{aligned}\tag{19}$$

for all $\psi \in \mathscr{S}(R^d,R)$, where $\mathscr{V}_{a,b}$ denotes the σ-algebra generated by random variables $\mathbf{V}(t,\mathbf{x})$, for $t \in [a,b]$ and $\mathbf{x} \in R^d$.

To calculate a mathematical expectation of multiple products of Gaussian random variables, it is convenient to use a graphical representation, borrowed from quantum field theory. We refer to, for example, Glimm and Jaffe [4] and Janson [5]. A *Feynman diagram* $\mathscr{F}$ (of order $n \ge 0$ and rank $r \ge 0$) is a graph consisting of a set $B(\mathscr{F})$ of n vertices and a set $E(\mathscr{F})$ of r edges without common endpoints. So there are r pairs of vertices, each joined by an edge, and $n-2r$ unpaired vertices, called *free vertices*. Note that $B(\mathscr{F})$ is a set of positive integers. An edge whose endpoints are $m,n \in B$ is represented by $\widehat{mn}$ (unless otherwise specified, we always assume $m < n$); an edge includes its endpoints. A diagram $\mathscr{F}$ is said to be *based on* $B(\mathscr{F})$. Denote the set of free vertices by $A(\mathscr{F})$, so $A(\mathscr{F}) = \mathscr{F} \setminus E(\mathscr{F})$. The diagram is *complete* if $A(\mathscr{F})$ is empty and *incomplete*, otherwise. Denote by $\mathscr{G}(B)$ the set of all diagrams based on B, by $\mathscr{G}_c(B)$ the set of all complete diagrams based on B and by $\mathscr{G}_i(B)$ the set of all incomplete diagrams based on B. A diagram $\mathscr{F}' \in \mathscr{G}_c(B)$ is called a *completion* of $\mathscr{F} \in \mathscr{G}_i(B)$ if $E(\mathscr{F}) \subseteq E(\mathscr{F}')$.

Let $B = \{1,2,3,\dots,n\}$. Denote by $\mathscr{F}_{|k}$ the subdiagram of $\mathscr{F}$, based on $\{1,\dots,k\}$. Define $A_k(\mathscr{F}) = A(\mathscr{F}_{|k})$. A special class of diagrams, denoted by $\mathscr{G}_s(B)$, plays an important role in the subsequent analysis: a diagram $\mathscr{F}$ of order n belongs to $\mathscr{G}_s(B)$ if $A_k(\mathscr{F})$ is not empty for all $k = 1, \dots, n$. We shall adopt the following multiindex notation. For any $P \in Z^+$, multiindex $\mathbf{n} = (n_1, \dots, n_P)$, $|\mathbf{n}|$ stands for $\sum n_p$. If $P' \le P$ we denote $\mathbf{n}_{|P'} := (n_1, \dots, n_{P'})$. In addition, if k is any number we set $\mathbf{n} \cdot k := (n_1, \dots, n_P, k)$. We work out the conditional expectation for multiple spectral integrals using the Markov property (19).

PROPOSITION 2. *For any function $\psi \in \mathscr{H}$ and $l_1, \dots, l_N \in \{0, 1\}$, $i_1, \dots, i_N \in \{1, \dots, d\}$,*

$$\mathbf{E}\left[\int \cdots \int \psi(\mathbf{k}_1, \dots, \mathbf{k}_N) \widehat{V}_{l_1, i_1}(t, \mathbf{x}_1, d\mathbf{k}_1) \cdots \widehat{V}_{l_N, i_N}(t, \mathbf{x}_N, d\mathbf{k}_N) \bigg| \mathscr{V}_{-\infty, s}\right]$$

$$= \sum_{\mathscr{F} \in \mathscr{G}(\{1,\dots,N\})} \int \cdots \int \exp\left\{ - \sum_{m \in A(\mathscr{F})} |\mathbf{k}_m|^{2\beta}(t-s)\right\} \tag{20}$$

$$\times\ \psi(\mathbf{k}_1, \dots, \mathbf{k}_N) \widehat{V}_{s, \mathbf{x}_1, \dots, \mathbf{x}_N}(d\mathbf{k}_1, \dots, d\mathbf{k}_N; \mathscr{F}),$$

with

$$\widehat{V}_{s, \mathbf{x}_1, \dots, \mathbf{x}_N}(d\mathbf{k}_1, \dots, d\mathbf{k}_N; \mathscr{F})$$

$$:= \prod_{m \in A(\mathscr{F})} \widehat{V}_{l_m, i_m}(s, \mathbf{x}_m, d\mathbf{k}_m) \prod_{\widehat{mn} \in E(\mathscr{F})} \left[1 - e^{-\left(|\mathbf{k}_m|^{2\beta} + |\mathbf{k}_n|^{2\beta}\right)(t-s)}\right] \tag{21}$$

$$\times \mathbf{E}[\widehat{V}_{l_m, i_m}(s, \mathbf{x}_m, d\mathbf{k}_m) \widehat{V}_{l_n, i_n}(s, \mathbf{x}_n, d\mathbf{k}_n)].$$

PROOF. Without loss of generality we consider $\psi(\mathbf{k}_1, \dots, \mathbf{k}_N) = \mathbf{1}_{A_1}(\mathbf{k}_1) \cdots \mathbf{1}_{A_N}(\mathbf{k}_N)$ for some Borel sets $A_1, \dots, A_N$. Note that $\widehat{\mathbf{V}}_l(t, A_i) = \widehat{\mathbf{V}}_l^0(t, A_i) + \widehat{\mathbf{V}}_l^1(t, A_i)$, where $\widehat{\mathbf{V}}_l^0(t, \cdot)$ is the orthogonal projection of $\widehat{\mathbf{V}}_l(t, \cdot)$ on $L^2_{-\infty, t}$ and $\widehat{\mathbf{V}}_l^1(t, \cdot)$ its complement. Here $L^2_{a, b}$ denotes L^2 closure of the linear span over $\mathbf{V}(s, \mathbf{x})$, $a \le s \le b$, $\mathbf{x} \in R^d$. The conditional expectation in (20) equals

$$\sum_{\mathscr{F} \in \mathscr{G}(\{1,\dots,N\})} \prod_{\widehat{mn} \in E(\mathscr{F})} \mathbf{E}\left[\widehat{V}^1_{i_m, l_m}(t, A_m) \widehat{V}^1_{i_n, l_n}(t, A_n)\right] \prod_{m \in A(\mathscr{F})} \widehat{V}^0_{i_m, l_m}(t, A_m).$$

The statement follows upon the application of the relations

$$\widehat{\mathbf{V}}_l^0(t, A) = \int_A e^{-|\mathbf{k}|^{2\beta}(t-s)} \widehat{\mathbf{V}}_l(s, d\mathbf{k})$$

and

$$\mathbf{E}\left[\widehat{\mathbf{V}}_l^1(t, A) \otimes \widehat{\mathbf{V}}_{l'}^1(t, B)\right]$$

$$= \int_A \int_B \delta_{l, l'} \left\{ \mathbf{E}\left[\widehat{\mathbf{V}}_l(t, d\mathbf{k}) \otimes \widehat{\mathbf{V}}_{l'}(t, d\mathbf{k}')\right] - \mathbf{E}\left[\widehat{\mathbf{V}}_l^0(t, d\mathbf{k}) \otimes \widehat{\mathbf{V}}_{l'}^0(t, d\mathbf{k}')\right]\right\}. \qquad \square$$

3. Proof of tightness. Let us start with the following result, which establishes, among other things, that the family of continuous trajectory processes $\mathbf{x}_\varepsilon(t)$, $t \geq 0$, is tight.

LEMMA 1. *For the family of trajectories given by* (4) *we have*

$$\lim_{\varepsilon\downarrow 0} \mathbf{E}\big[(\mathbf{x}_\varepsilon(t)-\mathbf{x}_\varepsilon(u))\otimes(\mathbf{x}_\varepsilon(t)-\mathbf{x}_\varepsilon(u))\big]=\mathbf{D}(t-u)^{2H} \quad \textit{if } t>u,$$

where H and $\mathbf{D}$ are given by (9) *and* (10)*, respectively.*

PROOF. First, let us observe that since $\mathbf{x}_\varepsilon(t)$ has stationary increments it is enough to prove the lemma for $u=0$. By the stationarity of $\mathbf{V}(s,\varepsilon\mathbf{x}(s))$ (see [11]), we can write that

$$\lim_{\varepsilon\downarrow 0} \mathbf{E}\big[\mathbf{x}_\varepsilon(t)\otimes\mathbf{x}_\varepsilon(t)\big]=\lim_{\varepsilon\downarrow 0}\varepsilon^2\int_0^{t/\varepsilon^{2\delta}} ds\int_0^s \mathbf{E}\big[\mathbf{V}(s',\varepsilon\mathbf{x}(s'))\otimes\mathbf{V}(0,\mathbf{0})\big]\,ds'. \tag{22}$$

Thus (22) equals

$$2\sum_{n=1}^{N}\mathscr{I}_n+\mathscr{R}_N, \tag{23}$$

where

$$\mathscr{I}_n=\varepsilon^{n+1}\int_0^{t/\varepsilon^{2\delta}} ds\int_0^s ds_1\cdots\int_0^{s_{n-1}}\mathbf{E}\big[\mathbf{W}_{n-1}(s_1,\dots,s_n,\mathbf{0})\otimes\mathbf{V}(0,\mathbf{0})\big]\,ds_n$$

and

$$\mathbf{W}_0(s_1,\mathbf{x})=\mathbf{V}(s_1,\mathbf{x}),$$

$$\mathbf{W}_n(s_1,\dots,s_{n+1},\mathbf{x})=\mathbf{V}(s_{n+1},\mathbf{x})\cdot\nabla\mathbf{W}_{n-1}(s_1,\cdots,s_n,\mathbf{x}) \quad \text{for } n=1,2,\dots,$$

with the remainder term

$$\mathscr{R}_N=2\varepsilon^{N+2}\int_0^{t/\varepsilon^{2\delta}} ds\int_0^s ds_1\cdots\int_0^{s_N}\mathbf{E}\big[\mathbf{W}_N(s_1,\dots,s_{N+1},\varepsilon\mathbf{x}(s_{N+1}))\otimes\mathbf{V}(0,\mathbf{0})\big]\,ds_{N+1}. \tag{24}$$

Estimates of $\mathscr{I}_n$. Elementary calculations show that

$$\lim_{\varepsilon\downarrow 0}\mathscr{I}_1=\mathbf{D}t^{2H}.$$

Since $\mathbf{V}$ is Gaussian we deduce that

$$\mathbf{E}\mathscr{I}_n=\mathbf{0},$$

when $n\geq 2$ is even. We now show that

$$\lim_{\varepsilon\downarrow 0}\mathbf{E}\mathscr{I}_n=\mathbf{0} \tag{25}$$

for $n\geq 3$ odd. The i, jth entry of the matrix $\mathscr{I}_n$ is given by

$$\mathscr{I}^n_{i,j}=2\varepsilon^{n+1}\int_0^{t/\varepsilon^{2\delta}} ds\int_0^s ds_1\cdots\int_0^{s_{n-1}}\mathbf{E}\big[\mathbf{E}_0 W_{n-1,\,i}(s_1,\dots,s_n,\mathbf{0})\times V_j(0,\mathbf{0})\big]\,ds_n. \tag{26}$$

We would like to express the conditional expectation appearing in (26) in terms of spectral measures associated with the velocity field. To do so, we introduce first the so-called *proper* functions of order $n, \sigma : \{1, \dots, n\} \to \{0, 1\}$ that appear in the statement of the next lemma. The proper function of order 1 is unique and is given by $\sigma(1) = 0$. Any proper function, σ', of order $n+1$ is generated from a proper function σ of order n as follows. For some $p \le n$,

$$\begin{aligned} &\sigma'(n+1) := 0, \\ &\sigma'(k) := \sigma(k) \quad \text{for } k \le n \text{ and } k \ne p, \\ &\sigma'(p) := 1 - \sigma(p). \end{aligned} \tag{27}$$

In other words, each proper function σ of order n generates n different proper functions of order $n+1$. Thus, the total number of proper functions of order n is $(n-1)!$. In the remainder of the paper, we sometimes write σ_k instead of $\sigma(k)$.

LEMMA 2. *Let $n \ge 1$ and $s_1 \ge s_2 \ge \cdots \ge s_n \ge s_{n+1}$, $i \in \{1, \dots, d\}$, $\mathbf{x} \in R^d$. We have then that*

$$\begin{aligned} &\mathbf{E}_{s_{n+1}} W_{n-1,\,i}(s_1, \dots, s_n, \mathbf{x}) \\ &\quad := \mathbf{E}\big[W_{n-1,\,i}(s_1, \dots, s_n, \mathbf{x}) \big| \mathscr{V}_{-\infty,\, s_{n+1}}\big] \\ &\quad = \sum \int \cdots \int \varphi^{(n)}_{\mathbf{i},\,\sigma}(\mathbf{k}_1, \dots, \mathbf{k}_n) \exp\Big\{ - \sum_{m \in A_n(\mathscr{F})} |\mathbf{k}_m|^{2\beta} (s_n - s_{n+1}) \Big\} \\ &\qquad\qquad \times P_{n-1}(\mathscr{F}) Q(\mathscr{F}) \prod_{m \in A_n(\mathscr{F})} \widehat{V}_{i_m,\,\sigma_m}(s_{n+1}, \mathbf{x}, d\mathbf{k}_m), \end{aligned} \tag{28}$$

where $\varphi^{(n)}_{\mathbf{i},\,\sigma}$ are some functions, with $\sup|\varphi^{(n)}_{\mathbf{i},\,\sigma}| \le 1$,

$$P_{n-1}(\mathscr{F}) = \prod_{j=1}^{n-1} \Big(\sum_{m \in A_j(\mathscr{F})} |\mathbf{k}_m| \Big) \exp\Big\{ - \sum_{m \in A_j(\mathscr{F})} |\mathbf{k}_m|^{2\beta} (s_j - s_{j+1}) \Big\}, \tag{29}$$

$$Q(\mathscr{F}) = \prod_{\widehat{mm'} \in E(\mathscr{F})} \mathbf{E}\big[\widehat{V}_{i_m,\,\sigma_m}(0, d\mathbf{k}_m) \widehat{V}_{i_{m'},\,\sigma_{m'}}(0, d\mathbf{k}_{m'})\big]. \tag{30}$$

The summation is over all multiindices $\mathbf{i}$ *of length n, whose first component equals i, all $\mathscr{F} \in \mathscr{G}_s$ and all proper functions σ of order n.*

Before proving Lemma 2, we apply it to show (25). Notice that according to (28),

$$\begin{aligned} &\int_0^{t/\varepsilon^{2\delta}} ds \int_0^s ds_1 \cdots \int_0^{s_{n-1}} \mathbf{E}_0 W_{n-1,i}(s_1, \dots, s_n, \mathbf{0})\, ds_n \\ &\quad = \sum \int_0^{t/\varepsilon^{2\delta}} ds \int \cdots \int \tilde{\varphi}^{(n)}_{\mathbf{i},\,\sigma}(\mathbf{k}_1, \dots, \mathbf{k}_n) \exp\Big\{ - \sum_{m \in A_n(\mathscr{F})} |\mathbf{k}_m|^{2\beta} s \Big\} \\ &\qquad\qquad \times P'_{n-1}(\mathscr{F}) Q(\mathscr{F}) \prod_{m \in A_n(\mathscr{F})} \widehat{V}_{i_m,\,\sigma_m}(0, d\mathbf{k}_m) \end{aligned}$$

for $i = 1, \dots, d$. Here, adopting the convention $s_{n+1} := 0$, we set

$$\tilde{\varphi}^{(n)}_{\mathbf{i},\sigma}(\mathbf{k}_1, \dots, \mathbf{k}_n) := \frac{\int_0^s ds_1 \cdots \int_0^{s_{n-1}} \varphi^{(n)}_{\mathbf{i},\sigma}(\mathbf{k}_1, \dots, \mathbf{k}_n) \times \prod_{j=1}^n \exp\{-\sum_{m \in A_j(\mathscr{F})} |\mathbf{k}_m|^{2\beta}(s_j - s_{j+1})\}\, ds_1 \cdots ds_n}{\int_0^s ds_1 \cdots \int_0^s \prod_{j=1}^n \exp\{-\sum_{m \in A_j(\mathscr{F})} |\mathbf{k}_m|^{2\beta} s_j\}\, ds_1 \cdots ds_n}$$

and

$$P'_{n-1}(\mathscr{F}) = \prod_{j=1}^{n-1} \left\{ \left(\sum_{m \in A_j(\mathscr{F})} |\mathbf{k}_m| \right) \times \frac{1 - \exp\{-\sum_{m \in A_j(\mathscr{F})} |\mathbf{k}_m|^{2\beta} s\}}{\sum_{m \in A_j(\mathscr{F})} |\mathbf{k}_m|^{2\beta}} \right\}.$$

It is elementary to check that, due to $|\varphi^{(n)}_{\mathbf{i},\sigma}| \le 1$,

$$|\tilde{\varphi}^{(n)}_{\mathbf{i},\sigma}(\mathbf{k}_1, \dots, \mathbf{k}_n)| \le 1. \tag{31}$$

Using Lemma 2, we infer that the left-hand side of (26) equals

$$\begin{aligned} &2\varepsilon^{n+1} \sum \int_0^{t/\varepsilon^{2\delta}} ds \int \cdots \int \frac{\tilde{\varphi}^{(n)}_{\mathbf{i},\sigma}(\mathbf{k}_1, \dots, \mathbf{k}_n)}{\sum_{m \in A_n(\mathscr{F})} |\mathbf{k}_m|^{2\beta}} P'_{n-1}(\mathscr{F}) Q(\mathscr{F}) \\ &\qquad \times \mathbf{E}\left[\prod_{m \in A_n(\mathscr{F}) \cup \{n+1\}} \widehat{V}_{i_m, \sigma_m}(0, d\mathbf{k}_m) \right]. \end{aligned} \tag{32}$$

Here the summation extends over all multiindices $\mathbf{i} = (i_1, \dots, i_{n+1})$ such that $i_1 = i$, $i_{n+1} = j$, all Feynman diagrams $\mathscr{F} \in \mathscr{G}_s$ and all proper functions σ of order n. Using an elementary inequality stating that

$$\frac{1 - e^{-x/\varepsilon^{2\delta}}}{x} \le \frac{C}{\varepsilon^{2\delta} + x}$$

for a certain constant C independent of ε, x, we conclude that the absolute value of (32) is less than or equal to

$$2t\varepsilon^{n+1-2\delta} \sum \int_0^K \cdots \int_0^K \frac{p_{n-1,\varepsilon}(\mathscr{F})}{\varepsilon^{2\delta} + \sum_{m \in A_n(\mathscr{F})} k_m^{2\beta}} \prod_{\widehat{mm'}} \frac{\delta(k_m - k_{m'}) dk_m dk_{m'}}{k_m^{2\alpha-1}}, \tag{33}$$

with

$$p_{n-1,\varepsilon}(\mathscr{F}) := \prod_{j=1}^{n-1} \frac{\sum_{m \in A_j(\mathscr{F})} k_m}{\varepsilon^{2\delta} + \sum_{m \in A_j(\mathscr{F})} k_m^{2\beta}}.$$

Suppose first that $2\beta > 1$. There exists then a constant C, depending only on n, β and K such that, for any $m_j \in A_j(\mathscr{F})$,

$$\frac{\sum_{m \in A_j(\mathscr{F})} k_m}{\varepsilon^{2\delta} + \sum_{m \in A_j(\mathscr{F})} k_m^{2\beta}} \le C \frac{\varepsilon^{\delta/\beta} + k_{m_j}}{\varepsilon^{2\delta} + k_{m_j}^{2\beta}}. \tag{34}$$

So,

$$\frac{p_{n-1,\varepsilon}(\mathscr{F})}{\varepsilon^{2\delta}+\sum_{m\in A_n(\mathscr{F})}k_m^{2\beta}} \le C\prod_{j=1}^{n-1}\frac{\varepsilon^{\delta/\beta}+k_{m_j}}{\varepsilon^{2\delta}+k_{m_j}^{2\beta}}\times\frac{1}{\varepsilon^{2\delta}+k_{m_n}^{2\beta}} \tag{35}$$

for any choice of $m_j \in A_j(\mathscr{F})$. Let $m_j := j$ if j is *not* the right endpoint of an edge of the diagram $\mathscr{F}$. Otherwise, let m_j be the closest free vertex to the left of j.

Consequently, the expression on the right-hand side of (35) is less than or equal to

$$C\prod_{\widehat{mm'}\in E(\mathscr{F})}\frac{\varepsilon^{\delta/\beta}+k_m}{\varepsilon^{2\delta}+k_m^{2\beta}}\prod_{m\in A_n(\mathscr{F})\cup\{n+1\}}\frac{(\varepsilon^{\delta/\beta}+k_m)^{q_m+1-\delta_{m,m_n}}}{(\varepsilon^{2\delta}+k_m^{2\beta})^{q_m+1}}, \tag{36}$$

where $E(\mathscr{F})$ denotes the set of the edges of the diagram $\mathscr{F}$ and $q_m \ge 0$ is the number of *left* endpoints between a free vertex $m \in A_n(\mathscr{F})$ and the next free vertex $m' \in A_n(\mathscr{F})$, also by a convention $q_{n+1} := -1$. Note that

$$c_n + 2\sum_{m\in A_n(\mathscr{F})\cup\{n+1\}} q_m = n-2, \tag{37}$$

with c_n denoting the cardinality of the set $A_n(\mathscr{F})$. Applying (36) to (33), we deduce that

$$\begin{aligned}|\mathscr{I}_n| \le Ct\varepsilon^{n+1-2\delta}\sum\left(\int_0^K\frac{(\varepsilon^{\delta/\beta}+k)dk}{(\varepsilon^{2\delta}+k^{2\beta})k^{2\alpha-1}}\right)^{(n-c_n)/2}\\ \times\prod_{\widehat{mm'}\in\mathscr{F}'}\int_0^K\frac{(k+\varepsilon^{\delta/\beta})^{2+q_m+q_{m'}-r_{m,m'}}}{(k^{2\beta}+\varepsilon^{2\delta})^{2+q_m+q_{m'}}}\times\frac{dk}{k^{2\alpha-1}}.\end{aligned} \tag{38}$$

Here the summation extends over all Feynman diagrams $\mathscr{F}$ from $\mathscr{G}_s(\{1,\dots,n\})$ and all complete diagrams $\mathscr{F}'$ made of the vertices of $A_n(\mathscr{F})\cup\{n+1\}$ and $r_{m,m'} := \delta_{m,m_n}+\delta_{m',m_n}$. Using the definition of δ [see (7)], it is elementary to observe that

$$\int_0^K\frac{(\varepsilon^{\delta/\beta}+k)\,dk}{(\varepsilon^{2\delta}+k^{2\beta})k^{2\alpha-1}} \le C(1+\varepsilon^{\gamma})$$

and

$$\int_0^K\frac{(k+\varepsilon^{\delta/\beta})^{2+q_m+q_{m'}-r_{m,m'}}}{(k^{2\beta}+\varepsilon^{2\delta})^{2+q_m+q_{m'}}}\times\frac{dk}{k^{2\alpha-1}} \le C\Big(1+\varepsilon^{\gamma(\widehat{mm'})}\Big),$$

with

$$\gamma := \frac{3-2\alpha-2\beta}{\alpha+2\beta-1}, \tag{39}$$

$$\gamma(\widehat{mm'}) := \frac{4-2\alpha-4\beta+(q_m+q_{m'})(1-2\beta)-r_{m,m'}}{\alpha+2\beta-1}. \tag{40}$$

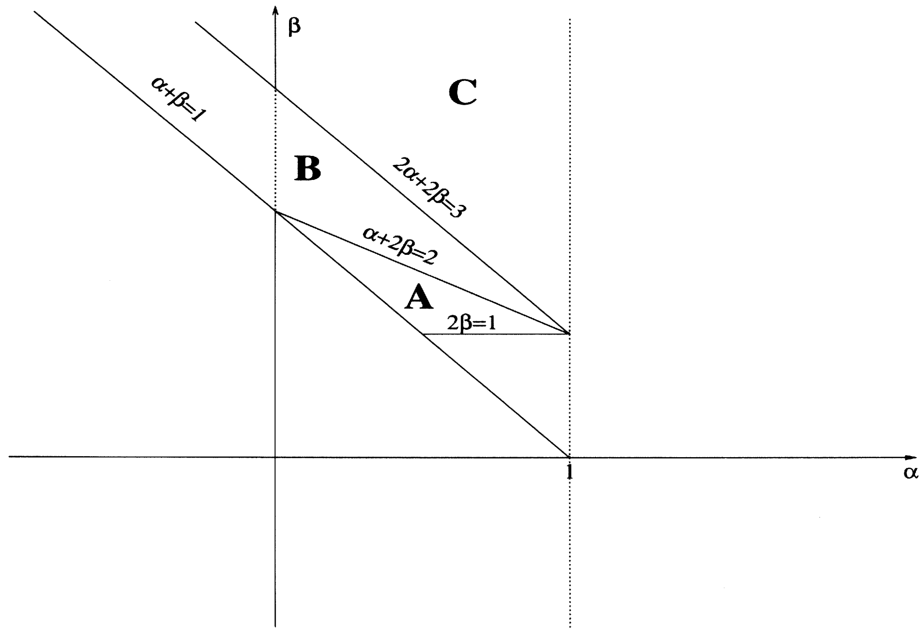

FIG. 2. *Regions A, B and C for estimating $\mathscr{I}_n$.*

Hence we obtain

$$|\mathscr{I}_n| \leq C\varepsilon^{n+1-2\delta}\big(1+\varepsilon^{(n-c_n)\gamma/2}\big)\varepsilon^{\kappa}, \tag{41}$$

with

$$\kappa := \frac{2r(2-\alpha-2\beta)+\sum'\big[(1-2\beta)(q_m+q_{m'})-r_{m,m'}\big]}{\alpha+2\beta-1}.$$

Here the summation $\sum'$ extends over the edges $\widehat{mm'}$ of the diagram $\mathscr{F}'$ for which $\gamma(\widehat{mm'})<0$ and

$$r \leq (c_n+1)/2 \tag{42}$$

denotes the number of such edges. In case there are no such edges we set $\kappa := 0$. The estimate of the right-hand side of (41) consists of considering all possible situations depending on signs of the expressions $3-2\alpha-2\beta$, $2-\alpha-2\beta$ appearing in (39) and (40). As shown in Figure 2, there are three cases corresponding to three regions (A, B and C) in the (α,β) plane. In each region we can deduce that

$$|\mathscr{I}_n| \leq C\varepsilon^{(n-1)\mu}. \tag{43}$$

Indeed, in region A we have

$$\kappa \geq \frac{4(1-\alpha-\beta)+(1-2\beta)(n-c_n)}{2(\alpha+2\beta-1)}$$

and $\mu := (2(\alpha+\beta)-1)/2(\alpha+2\beta-1)$; in region B we have

$$\kappa \geq \frac{-2\alpha + c_n(3-2\alpha-2\beta)+(1-2\beta)n}{2(\alpha+2\beta-1)}$$

and $\mu := (2(\alpha+\beta)-1)/2(\alpha+2\beta-1)$; in region C we have κ as in the previous case and $\mu := 1/(\alpha+2\beta-1)$.

When, on the other hand, $2\beta < 1$, we conclude that $p_{n-1,\varepsilon}(\mathscr{F}) \leq C$ for a certain constant $C > 0$. From (33) we obtain that

$$\begin{aligned}|\mathscr{I}_n| &\leq Ct\varepsilon^{n+1-2\delta} \sum \int_0^K \cdots \int_0^K \frac{1}{\varepsilon^{2\delta} + \sum_{m\in A_n(\mathscr{F})} k_m^{2\beta}} \prod_{\widehat{mm'}} \frac{\delta(k_m - k_{m'})dk_m dk_{m'}}{k_m^{2\alpha-1}} \\ &\leq Ct\varepsilon^{n+1-2\delta}\varepsilon^{2(1-\alpha-\beta)\delta/\beta} \int_0^{K/\varepsilon^{-\delta/\beta}} \frac{dk}{(k^{2\beta}+1)k^{2\alpha-1}} \leq Ct\varepsilon^{n-1}.\end{aligned} \tag{44}$$

In conclusion, we deduce that all terms $\mathscr{I}_n$ vanish as $\varepsilon \downarrow 0$ when $n \geq 2$.

Estimates of $\mathscr{R}_N$. Note that according to (24)

$$\mathscr{R}_N = 2\varepsilon^{N+2} \int_0^{t/\varepsilon^{2\delta}} ds \int_0^s ds_1 \cdots \int_0^{s_N} \mathbf{E}\big[\mathbf{E}_{s_{N+1}} \mathbf{W}_N\big(s_1, \dots, s_{N+1}, \varepsilon\mathbf{x}(s_{N+1})\big) \otimes \mathbf{V}(0,\mathbf{0})\big]\, ds_{N+1}.$$

By the Cauchy–Schwarz inequality we get that

$$\begin{aligned}|\mathscr{R}_N|^2 &\leq 4t^2\varepsilon^{4(1-\delta)+2N}\mathbf{E}|\mathbf{V}(0,\mathbf{0})|^2 \\ &\quad\times \max_{0\leq s\leq t/\varepsilon^{2\delta}} \mathbf{E}\bigg| \int \cdots \int_{s\geq s_1 \geq \cdots \geq s_{N+1}\geq 0} \mathbf{E}_{s_{N+1}} \\ &\qquad\times \mathbf{W}_N\big(s_1, \dots, s_N, s_{N+1}, \varepsilon\mathbf{x}(s_{N+1})\big)\, ds_1 \cdots ds_{N+1}\bigg|^2.\end{aligned} \tag{45}$$

The stationarity of the Lagrangian velocity field implies that the maximum in (45) is equal to

$$\begin{aligned}&\max_{0\leq s\leq t/\varepsilon^{2\delta}} \mathbf{E}\bigg|\int_0^s ds' \int \cdots \int_{s'\geq s_1\geq\cdots\geq s_N\geq 0} \mathbf{E}_0 \mathbf{W}_N(s_1,\dots,s_N,0,\mathbf{0}) ds_1 \cdots ds_N\bigg|^2 \\ &\quad\leq C \max_{0\leq s\leq t/\varepsilon^{2\delta}} \mathbf{E}\bigg|\int_0^s ds' \int\cdots\int_{s'\geq s_1\geq\cdots\geq s_N\geq 0} \mathbf{E}_0 \nabla \mathbf{W}_{N-1}(s_1,\dots,s_{N-1},s_N,\mathbf{0}) \\ &\qquad\times ds_1\cdots ds_N\bigg|^2 \mathbf{E}|\mathbf{V}(0,\mathbf{0})|^2.\end{aligned} \tag{46}$$

In the last line we used that $\mathbf{W}_N \in H^N(\mathbf{V})$ and the resulting hypercontractive property of L^p-norms with respect to a Gaussian measure on the space $H^N(\mathbf{V})$ (cf. Proposition 1). Using subsequently Lemma 2 to represent the conditional expectations on the right-hand side of (46), we deduce that the left-hand side of the preceding formula is less than or equal to

$$
\begin{aligned}
(47)\qquad & C\frac{t^2}{\varepsilon^{4\delta}}\times \mathbf{E}\Bigg|\sum\int\cdots\int \psi_{\mathbf{i},\sigma}(\mathbf{k}_1,\dots,\mathbf{k}_N)P_N(\mathscr{F})Q(\mathscr{F})\\
&\qquad\times\prod_{m\in A_N(\mathscr{F})}\widehat{V}_{i_m,\sigma_m}(0,d\mathbf{k}_m)\Bigg|^2,
\end{aligned}
$$

with some $|\psi_{\mathbf{i},\sigma}|\le 1$. The summation in (47) above extends over all Feynman diagrams $\mathscr{F}\in\mathscr{G}_s$, the relevant proper functions σ and multiindices $\mathbf{i}$.

Thus, we have

$$
(48)\qquad \mathscr{R}_N^2\le Ct^4\varepsilon^{2N+4(1-2\delta)}\sum\int_0^K\cdots\int_0^K p^2_{N,\varepsilon}(\mathscr{F})\prod_{\widehat{mm'}}\frac{\delta(k_m-k_{m'})\,dk_m\,dk_{m'}}{k_m^{2\alpha-1}}.
$$

Here the summation extends over all possible diagrams $\mathscr{F}\in\mathscr{G}_s(\{1,\dots,N\})$, $\mathscr{F}'\in\mathscr{G}_c(A_N(\mathscr{F})\cup N+A_N(\mathscr{F}))$. The product is taken over all edges of $\mathscr{F}'$ with $A_N(\mathscr{F})$ denoting the set of free edges of $\mathscr{F}$. Let c_N be the cardinality of $A_N(\mathscr{F})$. Arguing as in (38), when $2\beta>1$, we obtain that, for $q_m\ge 0$, $m\in A_N(\mathscr{F})\cup N+A_N(\mathscr{F})$ as in (36) satisfying

$$
2c_N+2\sum q_m=2N
$$

we have

$$
\begin{aligned}
(49)\qquad |\mathscr{R}_N|^2 &\le Ct^4\varepsilon^{2N+4(1-2\delta)}\sum\left(\int_0^K\frac{(\varepsilon^{\delta/\beta}+k)\,dk}{(\varepsilon^{2\delta}+k^{2\beta})k^{2\alpha-1}}\right)^{N-c_N}\\
&\qquad\times\prod_{\widehat{mm'}}\int_0^K\left(\frac{k+\varepsilon^{\delta/\beta}}{k^{2\beta}+\varepsilon^{2\delta}}\right)^{2+q_m+q_{m'}}\times\frac{dk}{k^{2\alpha-1}}.
\end{aligned}
$$

The ranges of the sum and the product in (49) remain the same as in (48). Repeating the argument made after (42), we deduce that there exists $\mu_1>0$ such that

$$
(50)\qquad |\mathscr{R}_N|^2\le Ct^4\varepsilon^{N\mu_1-8\delta}.
$$

The same inequality, with $\mu_1=1$, holds also when $2\beta\le 1$. This can be deduced repeating the corresponding argument used to obtain (44). We infer therefore that $\mathscr{R}_N$ vanishes as $\varepsilon\downarrow 0$ for a sufficiently large N. In conclusion, we proved that the left-hand side of (22) tends to $\mathbf{D}t^{2H}$ as $\varepsilon\downarrow 0$, provided that $\alpha+\beta>1$. □

REMARK. The foregoing argument can be used to infer, via an application of the hypercontractivity property of the L^p-norms over Gaussian measures on $H^N(\mathbf{V})$, that for an arbitrary $p\ge 1$ and $T>0$ there exists a constant $C>0$ such that, for any $T\ge t\ge s\ge 0$, $\varepsilon>0$,

$$
(51)\qquad \mathbf{E}|\mathbf{x}_\varepsilon(t)-\mathbf{x}_\varepsilon(s)|^p\le C(t-s)^{2Hp}.
$$

PROOF OF LEMMA 2. We prove the lemma by induction. The case $n = 1$ is obvious by choosing $\varphi_i^{(0)} \equiv 1$. Suppose that the result holds for n. For the sake of convenience we assume without loss of generality that $s_{n+2} = 0$. Then

$$\begin{aligned}&\mathbf{E}_0 W_{n+1,\,i}(s_1,\dots,s_{n+1},\mathbf{x})\\&\quad=\mathbf{E}_0\big\{\mathbf{V}(s_{n+1},\mathbf{x})\cdot\nabla\mathbf{E}_{s_{n+1}}W_{n,\,i}(s_1,\dots,s_n,\mathbf{x})\big\}.\end{aligned}\tag{52}$$

By virtue of the inductive assumption we can represent $\mathbf{E}_{s_{n+1}} W_{n,\,i}$ using (28) and as a result (52) becomes

$$\begin{aligned}&\sum\mathbf{E}_0\bigg[\int\cdots\int\varphi^{(n)}_{\mathbf{i},\,\sigma}(\mathbf{k}_1,\dots,\mathbf{k}_n)\\&\qquad\times\exp\bigg\{-\sum_{m\in A_n(\mathscr{F})}|\mathbf{k}_m|^{2\beta}(s_n-s_{n+1})\bigg\}P_{n-1}(\mathscr{F})Q(\mathscr{F})\\&\qquad\times\widehat{\mathbf{V}}_0(s_{n+1},\,\mathbf{x},\,d\mathbf{k}_{n+1})\cdot\nabla\bigg\{\prod_{m\in A_n(\mathscr{F})}\widehat{V}_{i_m,\,\sigma_m}(s_{n+1},\,\mathbf{x},\,d\mathbf{k}_m)\bigg\}\bigg].\end{aligned}\tag{53}$$

To calculate (53), we decompose each $\widehat{V}_{\sigma,\,i}(s,\mathbf{x},d\mathbf{k})$ as

$$\widehat{V}_{\sigma,\,i}(s,\mathbf{x},d\mathbf{k})=\widehat{V}^0_{\sigma,\,i}(s,\mathbf{x},d\mathbf{k})+\widehat{V}^1_{\sigma,\,i}(s,\mathbf{x},d\mathbf{k})\tag{54}$$

and use (13)–(12) where

$$\widehat{V}^0_{\sigma,\,i}(s,\mathbf{x},d\mathbf{k})=e^{-|\mathbf{k}|^{2\beta}(s-t)}\widehat{V}_{\sigma,\,i}(t,\mathbf{x},d\mathbf{k})\tag{55}$$

is the orthogonal projection of $\widehat{V}_{\sigma,\,i}$ on $\mathscr{V}_{-\infty,\,t}$. Expression (53) becomes

$$\begin{aligned}&\sum\mathbf{E}_0\bigg[\int\cdots\int\varphi^{(n)}_{\mathbf{i},\,\sigma}(\mathbf{k}_1,\dots,\mathbf{k}_n)\\&\qquad\times\exp\bigg\{-\sum_{m\in A_n(\mathscr{F})}|\mathbf{k}_m|^{2\beta}(s_n-s_{n+1})\bigg\}P_{n-1}(\mathscr{F})Q(\mathscr{F})\mathscr{K}(\mathscr{F})\bigg],\end{aligned}\tag{56}$$

with

$$\begin{aligned}\mathscr{K}(\mathscr{F}):=&\sum_{\substack{\varrho=\{\varrho_j\}\\ j\in A_n(\mathscr{F})\cup\{n+1\}}}\widehat{\mathbf{V}}_0^{\varrho_{n+1}}(s,\mathbf{x},d\mathbf{k}_{n+1})\\&\qquad\times\nabla\bigg\{\prod_{m\in A_n(\mathscr{F})}\widehat{V}^{\varrho_m}_{\sigma_m,\,i_m}(s,\mathbf{x},d\mathbf{k}_m)\bigg\}.\end{aligned}$$

The term corresponding to $\varrho_j \equiv 1$ vanishes, as is clear from the following calculation:

$$
\begin{aligned}
&\mathbf{E}\bigg\{\int\cdots\int \varphi^{(n)}_{\mathbf{i},\sigma}(\mathbf{k}_1,\ldots,\mathbf{k}_n)P_{n-1}(\mathscr{F})Q(\mathscr{F})\\
&\qquad \times \widehat{\mathbf{V}}^1_0(s,\mathbf{x},d\mathbf{k}_{n+1})\cdot\nabla\bigg(\prod_{m\in A_n(\mathscr{F})}\widehat{V}^1_{\sigma_m,i_m}(s,\mathbf{x},d\mathbf{k}_m)\bigg)\bigg\}\\
&\quad = \nabla\cdot\mathbf{E}\bigg\{\int\cdots\int \varphi^{(n)}_{\mathbf{i},\sigma}(\mathbf{k}_1,\ldots,\mathbf{k}_n)P_{n-1}(\mathscr{F})Q(\mathscr{F})\\
&\qquad\qquad \times \widehat{\mathbf{V}}^1_0(s,\mathbf{x},d\mathbf{k}_{n+1})\prod_{m\in A_n(\mathscr{F})}\widehat{V}^1_{\sigma_m,i_m}(s,\mathbf{x},d\mathbf{k}_m)\bigg\} = 0
\end{aligned}
\tag{57}
$$

by homogeneity of the velocity field. By (13)–(12),

$$
\begin{aligned}
&\widehat{\mathbf{V}}_0(s,\mathbf{x},d\mathbf{k}_{n+1})\cdot\nabla\bigg\{\prod_{m\in A_n(\mathscr{F})}\widehat{V}_{\sigma_m,i_m}(s,\mathbf{x},d\mathbf{k}_m)\bigg\}\\
&\quad = \sum_{m'\in A_n(\mathscr{F})}\mathbf{k}_{m'}\cdot\widehat{\mathbf{V}}_0(s,\mathbf{x},d\mathbf{k}_{n+1})\times\prod_{m\in A_n(\mathscr{F})}\widehat{V}_{\sigma^{m'}_m,i_m}(s,\mathbf{x},d\mathbf{k}_m),
\end{aligned}
\tag{58}
$$

where

$$
\sigma^{m'}_m := \begin{cases} 1-\sigma_{m'}, & \text{if } m'=m,\\ \sigma_m, & \text{otherwise.}\end{cases}
\tag{59}
$$

By (55), (54), (58) and definition (29), (56) further reduces to

$$
\begin{aligned}
&\sum\int\cdots\int\sum_{i_{n+1}=1}^{d}\sum_{m'\in A_n(\mathscr{F})}\sum_{\mathscr{F}'}\varphi^{(n)}_{\mathbf{i},\sigma}\frac{k_{m',i_{n+1}}}{\sum_{m\in A_n(\mathscr{F})}|\mathbf{k}_m|}\\
&\qquad\times\exp\bigg\{-\sum_{m\in A(\mathscr{F}')}|\mathbf{k}_m|^{2\beta}s_{n+1}\bigg\}P_n(\mathscr{F})Q(\mathscr{F})\\
&\qquad\times\prod_{m\in A(\mathscr{F}')}\widehat{V}_{\sigma^{m'}_m,i_m}(t,\mathbf{x},d\mathbf{k}_m)\\
&\qquad\times\prod_{\widehat{pq}\in E(\mathscr{F}')}\Big[1-e^{-(|\mathbf{k}_p|^{2\beta}+|\mathbf{k}_q|^{2\beta})(s-t)}\Big]\\
&\qquad\times\mathbf{E}\big[\widehat{V}_{\tilde{\sigma}_{p,m'},i_p}(0,\mathbf{0},d\mathbf{k}_p)\widehat{V}_{\tilde{\sigma}_{q,m'},i_q}(0,\mathbf{0},d\mathbf{k}_q)\big],
\end{aligned}
\tag{60}
$$

with $\tilde{\sigma}_{1,\,m'} = 0$ and $\tilde{\sigma}_{j+1,m'} = \sigma_j^{m'}$ and all incomplete Feynman diagrams $\mathscr{F}'$ based on the set $A_n(\mathscr{F}) \cup \{n+1\}$. Lemma 2 follows with

$$\varphi^{(n+1)}_{\mathbf{i},\,\sigma'_{m'}}(\mathbf{k}_1,\dots,\mathbf{k}_{n+1}) := \varphi^{(n)}_{\mathbf{i},\,\sigma}(\mathbf{k}_1,\dots,\mathbf{k}_n)\frac{k_{m',i_{n+1}}}{\sum_{m'\in A_n(\mathscr{F})}|\mathbf{k}_{m'}|} \times \prod_{\widehat{pq}\in E(\mathscr{F}')}\big[1-e^{-(|\mathbf{k}_p|^{2\beta}+|\mathbf{k}_q|^{2\beta})s_{n+1}}\big]. \qquad \square$$

4. Proof of weak convergence. It is a straightforward matter to verify that the Gaussian processes

$$\mathbf{y}_\varepsilon(t) := \varepsilon\int_0^{t/\varepsilon^{2\delta}}\mathbf{V}(s,\mathbf{0})\,ds, \qquad t\ge 0, \tag{61}$$

converge weakly to the fractional Brownian motion $\mathbf{B}_H(t)$, $t \ge 0$, given by (8). In addition, we have

$$\limsup_{\varepsilon\downarrow 0}\mathbf{E}|\mathbf{y}_\varepsilon(t)|^p < +\infty$$

for any $p \ge 1, t \ge 0$.

We now prove that

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\mathbf{E}\Big\{\big[x_{\varepsilon,\,i_1}(t_1)-x_{\varepsilon,\,i_1}(t_2)\big]^{p_1}\cdots\big[x_{\varepsilon,\,i_M}(t_M)-x_{\varepsilon,\,i_M}(t_{M+1})\big]^{p_M}\Big\}\\ &\quad=\mathbf{E}\Big\{\big[B_{H,\,i_1}(t_1)-B_{H,\,i_1}(t_2)\big]^{p_1}\cdots\big[B_{H,\,i_M}(t_M)-B_{H,\,i_M}(t_{M+1})\big]^{p_M}\Big\}.\end{aligned} \tag{62}$$

Equation (62) implies that the limiting law of the family of processes $\mathbf{x}_\varepsilon(t)$, $t \ge 0$, whose tightness, as $\varepsilon \downarrow 0$, has been established in the previous section, is that of the fractional Brownian motion $B_H(t)$, $t \ge 0$. Equation (62) is a consequence of

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\Big|\mathbf{E}\Big\{\big[x_{\varepsilon,\,i_1}(t_1)-x_{\varepsilon,\,i_1}(t_2)\big]^{p_1}\cdots\big[x_{\varepsilon,\,i_M}(t_M)-x_{\varepsilon,\,i_M}(t_{M+1})\big]^{p_M}\\ &\qquad-\big[y_{\varepsilon,\,i_1}(t_1)-y_{\varepsilon,\,i_1}(t_2)\big]^{p_1}\cdots\big[y_{\varepsilon,\,i_M}(t_M)-y_{\varepsilon,\,i_M}(t_{M+1})\big]^{p_M}\Big\}\Big|=0,\end{aligned} \tag{63}$$

with $\mathbf{y}_\varepsilon(t) = (y_{\varepsilon,\,1}(t),\dots,y_{\varepsilon,\,d}(t))$. Equation (63) follows from the next lemma.

LEMMA 3. *For any positive integers $M, p_1,\dots,p_M$, multiindices $\mathbf{i}_j \in \{1,\dots,d\}^{p_j}$ for $j = 1,\dots,M$ and $t_1 \ge \dots \ge t_M \ge t_{M+1} = 0$, we have*

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\big|\mathbf{E}\big[Z^{(p_1)}_{\varepsilon,\,\mathbf{i}_1}(t_2,t_1)\cdots Z^{(p_M)}_{\varepsilon,\,\mathbf{i}_M}(t_{M+1},t_M)\\ &\qquad-W^{(p_1)}_{\varepsilon,\mathbf{i}_1}(t_2,t_1)\cdots W^{(p_M)}_{\varepsilon,\,\mathbf{i}_M}(t_{M+1},t_M)\big]\big|=0.\end{aligned} \tag{64}$$

Here for any integer $N \geq 1$, *multiindex* $\mathbf{i} = (i_1, \ldots, i_N) \in \{1, \cdots, d\}^N$ *and* $t \geq s$, *we define*

$$Z_{\varepsilon,\mathbf{i}}^{(N)}(s,t) := \varepsilon^N \int \cdots \int_{\triangle_N(s,t)} \prod_{p=1}^{N} V_{i_p}(s_p, \varepsilon\mathbf{x}(s_p)) ds_1 \cdots ds_N$$

and

$$W_{\varepsilon,\mathbf{i}}^{(N)}(s,t) := \varepsilon^N \int \cdots \int_{\triangle_N(s,t)} \prod_{p=1}^{N} V_{i_p}(s_p, \mathbf{0})\, ds_1 \cdots ds_N,$$

with $\triangle_N(s,t) := \{(s_1, \ldots, s_N) \;:\; t/\varepsilon^{2\delta} \geq s_1 \geq \cdots \geq s_N \geq s/\varepsilon^{2\delta}\}$.

PROOF. To avoid cumbersome expressions that may obscure the essence of the proof, we consider only the special case of $M = 1$ and $t_1 = t$, $t_2 = 0$. The general case follows from exactly the same argument. We shall proceed with the induction argument on $p_1 = P$. The case when $P = 1$ is trivial because the stationarity of the relevant processes implies that the expression under the limit in (64) vanishes. In fact, as a consequence of the remark made after the proof of Lemma 2, we can conclude that, for any $q \geq 1$,

$$\limsup_{\varepsilon \downarrow 0} \mathbf{E}\left| Z_{\varepsilon,\mathbf{i}}^{(1)}(0,t)\right|^q < +\infty.$$

Assume now that (64) has been proven for a certain $P - 1 \geq 1$ and that for any $q \geq 1$ we have

$$\limsup_{\varepsilon \downarrow 0} \mathbf{E}\left| Z_{\varepsilon,\mathbf{i}}^{(P-1)}(0,t)\right|^q < +\infty. \tag{65}$$

In analogy with (23) we write that

$$\mathbf{E} Z_{\varepsilon,\mathbf{i}}^{(P)}(0,t) = \sum_{n=0}^{N-1} \mathscr{I}_n(0,t) + \mathscr{R}_N(0,t), \tag{66}$$

with

$$\begin{aligned} \mathscr{I}_n(0,t) := \varepsilon^{P+n+1} \int \cdots \int_{\Delta_P^{(n)}(0,t)} \mathbf{E}\Big\{ & \mathbf{E}_{s_2} W_{i_1}^{n}\big(\mathbf{s}_1^{(n)},\ \varepsilon\mathbf{x}(s_2)\big) \\ & \times \prod_{p=2}^{P} V_{i_p}\big(s_p, \varepsilon\mathbf{x}(s_p)\big)\Big\}\, d\mathbf{s}_1^{(n)}\, ds_2 \cdots ds_P, \end{aligned} \tag{67}$$

$$\begin{aligned} \mathscr{R}_N(0,t) := \varepsilon^{P+N+1} \int \cdots \int_{\Delta_P^{(N)}(0,t)} \mathbf{E}\Big\{ & \mathbf{E}_{s_{1,N+1}} W_{i_1}^{N}\Big(\mathbf{s}_1^{(N)}, \varepsilon\mathbf{x}(s_{1,N+1})\Big) \\ & \times \prod_{p=2}^{P} V_{i_p}\big(s_p, \varepsilon\mathbf{x}(s_p)\big)\Big\} d\mathbf{s}_1^{(N)} ds_2 \cdots ds_P. \end{aligned} \tag{68}$$

Here

$$\Delta_P^{(n)}(s,t) := \{(\mathbf{s}_1^{(n)}, s_2, \dots, s_P) \;:\; t/\varepsilon^{2\delta} \ge \mathbf{s}_1^{(n)} \ge s_2 \ge \cdots \ge s_P \ge s/\varepsilon^{2\delta}\},$$

with $\mathbf{s}_1^{(n)} := (s_{1,1}, \dots, s_{1,n+1})$. We say that $t \ge \mathbf{s}_1 \ge s$, where $\mathbf{s} = (s_1, \dots, s_n)$ is an ordered n-tuple, that $s_1 \ge \cdots \ge s_n$, when $t \ge s_1 \ge s_n \ge s$.

The argument used in the proof of Lemma 2 together with (65) shows that

$$\lim_{\varepsilon \downarrow 0} \mathscr{I}_n(0,t) = 0$$

for $n \ge 1$ and

$$\lim_{\varepsilon \downarrow 0} \mathscr{R}_N(0,t) = 0,$$

provided that N is chosen sufficiently large. Asymptotically then, as $\varepsilon \downarrow 0$ the behavior of $\mathbf{E}Z_{\varepsilon,\mathbf{i}}^{(P)}(0,t)$ is the same as that of the term

$$\begin{aligned} \mathscr{I}_0(0,t) := \varepsilon^{P+1} \int \cdots \int_{\Delta_P^{(0)}(0,t)} \mathbf{E}\Big\{ & V_{i_1}(s_1, \varepsilon\mathbf{x}(s_2)) \\ & \times \prod_{p=2}^{P} V_{i_p}(s_p, \varepsilon\mathbf{x}(s_p))\Big\} ds_1 \cdots ds_P. \end{aligned} \tag{69}$$

In order to deal with (69), we need a generalization of the argument used in the proof of Lemma 2. Let us introduce some additional notation. For any multiindex $\mathbf{i} = (i_1, \dots, i_p)$ and $p \ge 1$ we define $W_{\mathbf{i}}^{p,n}$ by induction as follows. We set

$$\begin{aligned} & W_{i_1,\dots,i_p}^{p,0}(s_1, \dots, s_p, \mathbf{x}) \\ & \quad := V_{i_1}(s_1, \mathbf{x}) \cdots V_{i_p}(s_p, \mathbf{x}) - \mathbf{E}\{V_{i_1}(s_1, \mathbf{x}) \cdots V_{i_p}(s_p, \mathbf{x})\} \end{aligned}$$

and assuming that $W_{i_1,\dots,i_p}^{p,n}(s_1, \dots, s_{p-1}, \mathbf{s}_p^{(n)}, \mathbf{x})$ has been defined for any ordered $n+1$-tuple $\mathbf{s}_p^{(n)} = (s_{p,1}, \dots, s_{p,n+1}) \le s_{p-1}$ we set

$$\begin{aligned} & W_{i_1,\dots,i_p}^{p,n+1}(s_1, \dots, s_{p-1}, \mathbf{s}_p^{(n+1)}, \mathbf{x}) \\ & \quad := \nabla W_{i_1,\dots,i_p}^{p,n}(s_1, \dots, \mathbf{s}_p^{(n)}, \mathbf{x}) \cdot \mathbf{V}(s_{p,n+2}, \mathbf{x}) \end{aligned}$$

for any ordered $n+2$-tuple $\mathbf{s}_p^{(n+1)} = (s_{p,1}, \dots, s_{p,n+1}, s_{p,n+2})$. Expanding the left-hand side of (69) in analogy with (23), we obtain

$$\begin{aligned} \mathscr{I}_0(0,t) = \varepsilon^{P+1} \int \cdots \int_{\Delta_P^{(0)}(0,t)} & \mathbf{E}\{V_{i_1}(s_1, \varepsilon\mathbf{x}(s_2)) V_{i_2}(s_2, \varepsilon\mathbf{x}(s_2))\} \\ & \times \mathbf{E}\Big\{\prod_{p=3}^{P} V_{i_p}(s_p, \varepsilon\mathbf{x}(s_p))\Big\}\, ds_1\, ds_2 \cdots ds_P \\ & + \sum_{n=0}^{N-1} \mathscr{I}_{1,n}(0,t) + \mathscr{R}_{1,N}(0,t), \end{aligned}$$

where

$$\mathscr{I}_{1,n}(0,t) := \varepsilon^{P+n+1} \int \cdots \int_{\Delta_P^{(1,n)}(0,t)} \mathbf{E}\Big\{ \mathbf{E}_{s_3} W^{2,n}_{i_1,i_2}\big(s_1, \mathbf{s}_2^{(n)}, \varepsilon \mathbf{x}(s_3)\big) \times \prod_{p=3}^{P} V_{i_p}\big(s_p, \varepsilon \mathbf{x}(s_p)\big)\Big\}\, ds_1\, d\mathbf{s}_2^{(n)}\, ds_2 \cdots ds_P, \tag{70}$$

$$\mathscr{R}_{1,N}(0,t) := \varepsilon^{P+N+1} \int \cdots \int_{\Delta_P^{(1,N)}(0,t)} \mathbf{E}\Big\{ \mathbf{E}_{s_{2,N+1}} W^{2,N}_{i_1,i_2}\Big(s_1, \mathbf{s}^{(N)}, \varepsilon \mathbf{x}(s_{2,N+1})\Big) \times \prod_{p=3}^{P} V_{i_p}\big(s_p, \varepsilon \mathbf{x}(s_p)\big)\Big\}\, ds_1\, d\mathbf{s}_2^{(N)}\, ds_3 \cdots ds_P, \tag{71}$$

$$\Delta_P^{(1,n)}(0,\ t) := \Big\{\Big(s_1, \mathbf{s}_2^{(n)}, s_3, \ldots, s_P\Big)\ :\ t/\varepsilon^{2\delta} \ge s_1 \ge \mathbf{s}_2^{(n)} \ge \cdots \ge s_P \ge 0\Big\}.$$

We represent the conditional expectations appearing in (70) and (71) using a generalization (Lemma 4) of Lemma 2.

To formulate it, we need a generalized notion of a proper function, which we call a p-proper function. Let p be a positive integer. The p-proper function of order 1 is unique and is given by $\sigma(i) = 0, i = 1, \ldots, p$. Any p-proper function, σ', of order $n+1$ is generated from a p-proper function σ of order n as follows. For some $q \le p+n$,

$$\begin{aligned} &\sigma'(p+n+1) := 0,\\ &\sigma'(k) := \sigma(k) \quad \text{for } k \le n+p \text{ and } k \ne q,\\ &\sigma'(q) := 1 - \sigma(q). \end{aligned} \tag{72}$$

We also distinguish a special class of Feynman diagram $p\mathscr{G}_s(B)$: a diagram $\mathscr{F}$ of order $n+p$ belongs to $p\mathscr{G}_s(B)$ if $A_k(\mathscr{F})$ is not empty for all $k = p, \ldots, n+p$.

LEMMA 4. *For any positive integer* $p, s_1 \ge \cdots \ge s_{p-1} \ge \mathbf{s}_p^{(n-1)} \ge s$, *a multiindex* $\mathbf{i} = (i_1, \ldots, i_p) \in \{1, \ldots, d\}^p$, *we have*

$$\begin{aligned} \mathbf{E}_s W^{p,n-1}_{\mathbf{i}}\big(s_1, \ldots, s_{p-1}, \mathbf{s}_p^{(n-1)}, \mathbf{x}\big) &= \sum \int \cdots \int \varphi^{(p,n)}_{\mathbf{j},\sigma}(\mathbf{k}_1, \ldots, \mathbf{k}_{p+n})\\ &\quad \times \exp\Big\{ - \sum_{m \in A_{n+p}(\mathscr{F})} |\mathbf{k}_m|^{2\beta}(s_{p,n} - s)\Big\} P_{p,n-1}(\mathscr{F}) Q(\mathscr{F})\\ &\quad \times \prod_{m \in A_{n+p}(\mathscr{F})} \widehat{V}_{i_m,\sigma_m}(s, \mathbf{x}, d\mathbf{k}_m), \end{aligned} \tag{73}$$

where $\varphi_{\mathbf{j},\sigma}^{(p,n)}$ *are functions satisfying* $|\varphi_{\mathbf{j},\sigma}^{(p,n)}| \le 1$ *and*

$$
\begin{aligned}
P_{p,n}(\mathscr{F}) = {} & \prod_{j=p}^{n+p-1} \Bigg(\sum_{m \in A_j(\mathscr{F})} |\mathbf{k}_m| \Bigg) \\
& \times \exp\Bigg\{ - \sum_{m \in A_j(\mathscr{F})} |\mathbf{k}_m|^{2\beta} (s_{p,\,j-p} - s_{p,\,j-p+1}) \Bigg\}, \qquad (74)\\
Q(\mathscr{F}) = {} & \prod_{\widehat{mm'} \in E(\mathscr{F})} \mathbf{E}\big[\widehat{V}_{i_m,\sigma_m}(0, d\mathbf{k}_m) \widehat{V}_{i_{m'},\sigma_{m'}}(0, d\mathbf{k}_{m'})\big].
\end{aligned}
$$

The summation is over all multiindices $\mathbf{j} = (j_1, \dots, j_{n+p})$*, such that* $\mathbf{j}_{|p} = \mathbf{i}$*, all* $\mathscr{F} \in p\mathscr{G}_s$ *and all* p*-proper functions* σ *of order* n*. Here by a convention* $s_{p,0} := s_{p-1}$.

The proof of Lemma 4 is exactly the same as that of Lemma 2 and is omitted.

Using Lemma 4 and the argument presented in the foregoing to demonstrate that $\mathscr{I}_0(0, t)$ is asymptotically equal to $\mathbf{E} Z_{\varepsilon,\mathbf{i}}^{(P)}(0, t)$, as $\varepsilon \downarrow 0$, we can show that

$$
\begin{aligned}
\varepsilon^{P+1} \int \cdots \int_{\Delta_P(0,t)} \mathbf{E}\Bigg\{ & V_{i_1}(s_1, \varepsilon \mathbf{x}(s_3)) V_{i_2}(s_2, \varepsilon \mathbf{x}(s_3)) \\
& \times \prod_{p=3}^{P} V_{i_p}(s_p, \varepsilon \mathbf{x}(s_p)) \Bigg\} ds_1 ds_2 \cdots ds_P
\end{aligned}
$$

is asymptotically equal to $\mathbf{E} Z_{\varepsilon,\mathbf{i}}^{(P)}(0, t)$, as $\varepsilon \downarrow 0$. Repeating the preceding argument p times, we obtain (64). Finally, we notice that the hypercontractivity properties of the L^p norms over Gaussian measure space allow us to conclude that (65) holds with $P - 1$ replaced by P. $\square$

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