ON THE MAXIMUM WORKLOAD OF A QUEUE FED BY FRACTIONAL BROWNIAN MOTION

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Consider a queue with a stochastic fluid input process modeled as fractional Brownian motion (fBM). When the queue is stable, we prove that the maximum of the workload process observed over an interval of length t grows like $\gamma(\log t)^{1/(2-2H)}$, where $H > 1/2$ is the self-similarity index (also known as the Hurst parameter) that characterizes the fBM and can be explicitly computed. Consequently, we also have that the typical time required to reach a level b grows like $\exp\{b^{2(1-H)}\}$. We also discuss the implication of these results for statistical estimation of the tail probabilities associated with the steady-state workload distribution.

1. Introduction. Triggered by measurements and statistical analysis of traffic in high-speed networks, recent research has focused on stochastic models of network traffic that have the properties of *long-range dependence* and *self-similarity*; see, for example, Leland, Taqqu, Willinger and Willson (1993), Beran, Sherman, Taqqu and Willinger (1995) and Erramilli, Narayan and Willinger (1997) for a discussion of the statistical evidence that favors such models. Perhaps the most theoretically important traffic model that exhibits these properties is fractional Brownian motion (fBM). In fact, just as Brownian motion is supported as a model of short-range dependency on the basis of Donsker's theorem and its generalizations, fBM can be viewed as a natural limiting approximation to a broad class of more physically plausible models that describe how traffic in a network is generated from its individual sources; see Heath, Resnick and Samorodnitsky (1997, 1998), Konstantopoulos and Lin (1996) and Whitt (1998). Because of both theoretical and statistical evidence that supports fBM as a possible traffic model, there is significant interest in trying to reach an understanding of the implications of such long-range dependent traffic for the performance of queues (because queueing effects occur at the buffers to the switches in a network).

Because fBM is highly non-Markovian (i.e., there is no finite-dimensional supplementary variable representation of fBM that makes it Markov), it is a challenging process to analyze as an input to a queue. At this point in time, we are aware of only two sets of results that describe the performance of a queue

fed by fBM. The first such result is an asymptotic, owing to Norros (1994) and Duffield and O'Connell (1995), for the tail probabilities of the steady-state workload in such a queue. For a description of the result, see Section 2 of this paper. These results were refined recently by Massoulie and Simonian (1999) and Hüsler and Piterbarg (1999) to give the exact tail asymptotic [see also Narayan (1998)]. The second set of results, owing to Krishnan (1996), takes advantage of the self-similarity of fBM to obtain a family of parameter scaling relationships that hold for the steady-state distribution of a queue fed by fBM.

Our objective in this paper is to establish several additional results that serve to enhance our understanding of queues fed by fBM. Our main focus here is on studying the maximum of the workload process over an interval of length t. The analysis of such maximum r.v.'s has a long history within the queueing literature; see, for example, Cohen (1968) and Iglehart (1972), and the recent survey by Asmussen (1998). The principal results in this paper are as follows:

1. The derivation of the asymptotic behavior of the maximum of the workload process over an interval of length t as $t \to \infty$; see Theorem 1 and Proposition 2.
2. The development of an asymptotic approximation for the time required by the workload process to first hit level b when $b \to \infty$; see Theorem 2.
3. Some remarks on estimating buffer loss probabilities from observed traffic and buffer dynamics when the input is fBM; see Proposition 3 (the convergence rate of the associated estimators is very slow, however).

In Section 2 of this paper, we describe the precise model considered here and discuss the main results. Section 3 contains proofs.

2. Main results. A real-valued process $B_H = (B_H(t): t \geq 0)$ is said to be a fBM with *self-similarity* index $H \in [1/2, 1)$ if $B_H(0) = 0$, B_H has continuous sample paths and B_H is a zero-mean stationary increments Gaussian process with

$$\mathrm{Cov}(B_H(t), B_H(s)) = 1/2 \Big\{ t^{2H} + s^{2H} - |t - s|^{2H} \Big\}$$

for $s, t \geq 0$. The case $H = 1/2$ corresponds to standard Brownian motion. When $H > 1/2$, the correlation between two unit-length intervals separated by time t decays as t^{2H-2}, so that the autocorrelation sequence for $(B_H(k) - B_H(k-1): k \geq 1)$ is nonsummable. Thus, B_H describes a long-range dependent process. Additional properties and constructions of fBM are described in Samorodnitsky and Taqqu [(1994), Section 7.2], and Mandelbrot and Van Ness (1968).

Let $\Gamma(t)$ be the cumulative amount of work input to the system over $[0, t]$. We assume that $\Gamma(t) = \lambda t + \sigma B_H(t)$ for $t \geq 0$ and $\sigma, \lambda > 0$, so that the input to the system is fBM. Given that $\Gamma = (\Gamma(t): t \geq 0)$ has continuous sample paths, we view Γ as a fluid inflow to the queue. If the service mechanism deterministically serves work at rate $\mu > 0$, then the workload present in the

system at time t is given by

$$(1) \qquad\qquad W(t) = \Gamma(t) - \mu t - \min_{0 \le s \le t}[\Gamma(s) - \mu s];$$

see Harrison (1985) for additional details on this representation of the work-load. Put $X(t) = \Gamma(t) - \mu t$. Because $X = (X(t) : t \ge 0)$ evolves freely of any boundary behavior, we call X the *free process*. On the other hand, $W = (W(t) : t \ge 0)$ is nonnegative. The mapping that carries the free process X into the nonnegative process W is called the *regulator mapping*. We therefore call the workload process W that appears here a *regulated* fBM process. We prefer the term "regulator mapping" over "reflection mapping" to differentiate this map from the Skorohod mapping that appears in the theory of reflected diffusions; see Lions and Sznitman (1984).

Let $\rho := \lambda / \mu$ and suppose that the traffic intensity $\rho < 1$. Then

$$(2) \qquad\qquad W(t) \Rightarrow W(\infty)$$

as $t \to \infty$, where $W(\infty) = \max\{X(s) : s \ge 0\}$. The tail asymptotics of the steady-state workload are known.

PROPOSITION 1.   *Suppose $\rho < 1$. Then*:

(i) $\mathbb{P}(W(\infty) > b) \ge \mathbb{P}(X(t^*) > b)$, *where* $t^* := Hb/((1 - H)\mu(1 - \rho))$;
(ii) $b^{2H-2} \log \mathbb{P}(W(\infty) > b) \to -\theta^*$ *as* $b \to \infty$, *where*

$$\theta^* := \frac{(\mu(1 - \rho))^{2H}}{2\sigma^2} \frac{1}{H^{2H}(1 - H)^{2(1-H)}}.$$

The lower bound (i) is owing to Norros (1994), whereas the asymptotic (ii) can be found in Duffield and O'Connell (1995). Related results appear in Chang, Yao and Zajic (1996) and O'Connell and Procissi (1999). Because $W(\infty)$ is expressed easily in terms of X, these types of tail asymptotics can be attacked directly in terms of the free process alone. It is also known [see Konstantopoulos, Zazanis and De Veciana (1996)] that if $\rho < 1$, one can construct a probability space supporting both the process X and a stationary process $W^* = (W^*(t) : t \ge 0)$ such that:

(i) $W^*(t) \stackrel{\mathscr{D}}{=} W(\infty)$ for $t \ge 0$, where $\stackrel{\mathscr{D}}{=}$ denotes equality in distribution,
(ii) $W^*(t) = \Gamma(t) - \mu t + \{W^*(0) \vee L^*(t)\}$ for $t \ge 0$,

where $L^*(t) = -\min\{\Gamma(s) - \mu s : s \in [0, t]\}$ is the nondecreasing process "regulating" the fBM. Thus, W^* is a stationary version of the workload process for our system, in which the input process is fBM.

Our main focus in this paper is to study the two maximum r.v.'s:

$$M(t) = \max_{0 \le s \le t} W(s),$$

$$M^*(t) = \max_{0 \le s \le t} W^*(s).$$

It is natural to expect that these two r.v.'s behave in an asymptotically identical fashion for large t. Furthermore, it is known, in substantial generality, that the asymptotic behavior of $M^*(t)$ is closely related to the tail behavior of $W^*(t)$. In particular, under suitable mixing conditions on W^*, $M^*(t)$ should behave asymptotically like the maximum of $\lfloor at \rfloor$ i.i.d. copies of $W(\infty)$, for some $a \in (0, 1)$; see Leadbetter, Lindgren and Rootzén (1983) for such results. In principle, this then yields the asymptotic behavior of $M^*(t)$ (because the maximum behavior for i.i.d. sequences is well known).

The difficulty with this approach is the verification of the requisite mixing properties in our present setting. Such a methodology is particularly effective when W^* is regenerative; see Asmussen (1998) for many examples of queueing-related maximum processes that can be readily studied by taking advantage of the regenerative cycle structure of W^*. Such regenerations are easily identified for many commonly used short-range dependent input processes (e.g., Markov-modulated arrivals). However, because of the non-Markov nature of fBM, it is unclear that any regenerative structure is present in regulated fBM. In addition, the more general mixing conditions that permit us to view $M^*(t)$ as the maximum of i.i.d. r.v.'s seem difficult to verify directly, given that W^* is non-Markov and has long-range dependent input. However, our main result (based on a different style of argument) proves that the asymptotics suggested in the preceding text are indeed correct in a suitable asymptotic sense.

THEOREM 1. *If $\rho < 1$, then*

$$\frac{M^*(t)}{(\log t)^{1/(2(1-H))}} \Rightarrow \left(\frac{1}{\theta^*}\right)^{1/(2-2H)} \quad as \ t \to \infty,$$

$$\frac{M(t)}{(\log t)^{1/(2(1-H))}} \Rightarrow \left(\frac{1}{\theta^*}\right)^{1/(2-2H)} \quad as \ t \to \infty.$$

In fact, the foregoing convergence is actually in L_p for all $p \in [1, \infty)$.

The cases of heavy traffic and unstable queues are the subject of the following proposition.

PROPOSITION 2.

(i) *If $\rho > 1$, then*

$$t^{-H}(M(t) - \mu(\rho - 1)t) \Rightarrow \sigma \mathcal{N}(0, 1) \quad as \ t \to \infty.$$

(ii) *If $\rho = 1$, then*

$$t^{-H}M(t) \Rightarrow \sigma \xi \quad as \ t \to \infty,$$

where $\xi := \max_{0 \leq r \leq 1} \max_{0 \leq v \leq r}[B_H(r) - B_H(v)]$.

Let $T(b) := \inf\{t \geq 0\colon W(t) \geq b\}$ and note that $\{T(b) \leq t\} = \{M(t) \geq b\}$. Because of this relationship between $T(b)$ and $M(t)$, Theorem 1 and Proposition 2 together yield asymptotic approximations for $T(b)$ that are valid for large b.

THEOREM 2.

(i) *If $\rho < 1$, then*

$$\frac{\log T(b)}{b^{2(1-H)}} \Rightarrow \theta^* \quad \text{as } b \to \infty.$$

(ii) *If $\rho = 1$, then*

$$\frac{T(b)}{b^{1/H}} \Rightarrow (\sigma \xi)^{-1/H} \quad \text{as } b \to \infty.$$

(iii) *If $\rho > 1$, then*

$$b^{-H}\left(T(b) - \frac{b}{\mu(\rho - 1)}\right) \Rightarrow \left(\frac{1}{\mu(\rho - 1)}\right)^{1+H} \sigma \cdot \mathcal{N}(0, 1) \quad \text{as } b \to \infty.$$

If W were regenerative, (i) could be obtained by appealing, for example, to the regenerative approach described in Glasserman and Kou (1995). However, as discussed earlier, it is unclear how to implement this idea in the current fBM setting.

We conclude this section with some discussion of the implications of the results for estimation of loss probabilities in finite buffer queues based on real-time measurement of traffic. Such loss-probability estimators could potentially be useful in admission control for high-speed networks. To connect the infinite capacity model considered so far in this paper to a finite buffered system, we view the exceedence probability $\mathbb{P}(W(\infty) > b)$ as a surrogate for the loss probability in a buffer of size b fed by fBM. Recall that if b is large, Proposition 1 asserts that $\mathbb{P}(W(\infty) > b)$ is essentially determined by θ^* and H. Thus, $\mathbb{P}(W(\infty) > b)$ can be roughly (i.e., in a logarithmic scale) estimated once estimators for θ^* and H are determined. In the short-range dependent context, several authors have proposed estimating θ^* from observed traffic using the maximum workload r.v. $M(t)$; see Berger and Whitt (1995) and Hsu and Walrand (1996). Theorem 1 states that if H is known, then θ^* can also be successfully estimated from long-range dependent traffic using $M(t)$.

Of course, in general, H itself would also need to be estimated from incoming traffic. Assume that the process Γ is observable, so that we can form the r.v. $H(t)$ from the traffic observed over $[0, t]$, where

$$H(t) := \left(\log \max_{0 \leq s \leq t}\left[X(s) - s\frac{X(t)}{t}\right]\right) \Big/ \log t.$$

PROPOSITION 3.

(i) *For $0 \leq t \leq 1$, let $B_H^0(t) = B_H(t) - tB_H(1)$ be the fractional Brownian bridge process. Then*

$$(\log t)(H(t) - H) \Rightarrow \zeta$$

as $t \to \infty$, where $\zeta := \log \max\{\sigma B_H^0(s) : 0 \leq s \leq 1\}$.

(ii) *If $\rho < 1$, then*

$$\frac{(M(t))^{2(1-H(t))}}{\log t} \Rightarrow \frac{1}{\theta^*}$$

as $t \to \infty$.

Proposition 3 asserts that the parameters H and θ^* can be consistently estimated, using the maximum workload r.v. $M(t)$, when the input process is fractional Brownian motion.

REMARK 1.   Another natural estimator to consider in this context is the so-called moving average estimator, which is constructed as follows. Fix $a(t)$ and $m(t) := \lfloor t/a(t) \rfloor$. Chop up the observation window $[0, t]$ into $m(t)$ subwindows of length $a(t)$ each, and a remainder that is a fraction of $a(t)$ in length. Let $M_i(t) := \sup\{W(s) : s \in [(i-1)a(t), ia(t)]\}$ for $i = 1, 2, \ldots, m(t)$ and let $H(t)$ be as in Proposition 3. Then the moving average estimator is defined to be

$$\gamma(t) := \frac{1}{m(t)} \sum_{i=1}^{m(t)} \frac{(M_i(t))^{2(1-H(t))}}{(\log a(t))}.$$

Using the results of Theorem 1 and Proposition 3, we can establish that $\gamma(t) \Rightarrow (\theta^*)^{-1}$.

3. Proofs.   For brevity, set

$$\beta := \frac{1}{2(1-H)}.$$

Otherwise, notation and definition follow Section 2.

3.1. *Proof of Theorem* 1.   The proof of the theorem involves establishing the usual upper and lower bounds; that is, our goal is to prove that for arbitrary positive constant $\delta > 0$, we have

(3)
$$\mathbb{P}\left(\frac{M^*(t)}{(\log t)^\beta} \geq \left(\frac{1-\delta}{\theta^*}\right)^\beta\right) \xrightarrow{t \to \infty} 1,$$

(4)
$$\mathbb{P}\left(\frac{M^*(t)}{(\log t)^\beta} \geq \left(\frac{1+\delta}{\theta^*}\right)^\beta\right) \xrightarrow{t \to \infty} 0,$$

from which the convergence in probability follows. We then argue that essentially the same proof yields convergence for $M(t)$ as well. Finally, we prove

the uniform integrability of the family $\{M^*(t)/(\log t)^\beta\}_{t \geq 2}$, which establishes the L^p convergence.

PROOF OF THE LOWER BOUND (3). Again, we break up the proof into several steps.

Step 1. Fix $\Delta \in (0, t)$, so that

(5)
$$W^*(t) \geq X(t) - \inf_{0 \leq s \leq t} X(s)$$
$$\geq X(t) - X(t - \Delta).$$

This gives

$$M^*(t) = \max_{0 \leq s \leq t} W^*(s)$$
$$\geq \max_{k=1,2,\ldots,\lfloor t/\Delta \rfloor} W^*(k\Delta)$$
$$\geq \max_{k=1,2,\ldots,\lfloor t/\Delta \rfloor} [X(k\Delta) - X((k-1)\Delta)]$$
$$:= \max_{1 \leq k \leq \lfloor t/\Delta \rfloor} Y_k^{(\Delta)}$$

and note that

$$Y_k^{(\Delta)} := X(k\Delta) - X((k-1)\Delta)$$
$$\overset{\mathscr{D}}{=} \sigma B_H(\Delta) - \mu(1 - \rho)\Delta$$
$$\overset{\mathscr{D}}{=} \sigma \Delta^H B_H(1) - \mu(1 - \rho)\Delta$$

using the properties of stationary increments and self-similarity of fBM. Set

$$Z_i = \frac{Y_i^{(\Delta)} + \mu(1 - \rho)\Delta}{\Delta^H \sigma}$$

with $\sigma^2 = \text{Var } B_1$. Then, because fBM is a Gaussian process, we have that the sequence $\{Z_i\}_{i=1}^{\lfloor t/\Delta \rfloor}$ is a stationary sequence of standardized Gaussian r.v.'s, the so-called *fractional Gaussian noise*. In addition,

$$\rho_Z(\ell) := \frac{\mathbb{E}[(Y_1^{(\Delta)} + \mu(1 - \rho)\Delta)(Y_{1+\ell}^{(\Delta)} + \mu(1 - \rho)\Delta)]}{\sigma^2 \Delta^{2H}}$$
$$= \frac{1}{\sigma^2 \Delta^{2H}} \mathbb{E}[\sigma B_H(\Delta)][\sigma B_H((1 + \ell)\Delta) - \sigma B_H(\ell\Delta)]$$
$$= \frac{1}{2}[(\ell + 1)^{2H} - (2\ell)^{2H} + (\ell - 1)^{2H}],$$

where the last step follows from the covariance structure of the fractional Gaussian noise sequence [cf. Proposition 7.2.9 in Samorodnitsky and

Taqqu (1994)]. Thus

$$\rho_Z(\ell) \sim H(2H - 1)\ell^{2H-2}$$

as $\ell \to \infty$.

Step 2. Note that $\{Z_i\}_{i=1}^{\lfloor t/\Delta \rfloor}$ is a stationary sequence of standard Gaussian r.v.'s, with $\rho_Z(\ell) \log \ell \to 0$. Thus, we can appeal to Theorem 4.3.3 in Leadbetter et al. (1983), which states that for a sequence of real numbers u_m and a standardized Gaussian sequence, say $\{Z_i\}_{i=1}^m$, with the preceding properties, we have

$$\mathbb{P}\left(\bigvee_{i=1}^m Z_i \geq u_m\right) \to 1 + e^{-\tau}$$

if and only if

$$m\mathbb{P}(Z > u_m) \to \tau$$

as $m \to \infty$, with $\tau \in [0, \infty]$. To apply the theorem, we let $\Delta = \Delta(t)$ depend on t and choose $\Delta(t)$ carefully. Fix $\delta \in (0, 1)$ and set

$$\alpha(t) := \left(\frac{1-\delta}{\theta^*} \log t\right)^\beta,$$

$$u(t) := \frac{\alpha(t) + \mu(1 - \rho)\Delta(t)}{\sigma \Delta^H(t)},$$

$$\tau(t) := m(t)\mathbb{P}(Z_1 > u(t))$$

with $m(t) := \lfloor t/\Delta(t) \rfloor$. Noting that

$$\mathbb{P}\left(\bigvee_{i=1}^{m(t)} Y_i^{(\Delta)} \geq \alpha(t)\right) = \mathbb{P}\left(\bigvee_{i=1}^{m(t)} Z_i \geq u(t)\right),$$

we are left with the task of showing that $\tau(t) \to \infty$, which will establish the lower bound. The crucial step is to choose $\Delta(t)$ carefully, so that for $\alpha(t)$ as before, $\tau(t) \to \infty$. The only feasible choice turns out to be

$$\Delta(t) = \left(\frac{2(1 - \epsilon)\sigma^2 H^2}{(\mu(1 - \rho))^2} \log t\right)^{1/(2-2H)}$$

for any $\epsilon \in (0, \delta]$. Plugging this value into $u(t)$, straightforward algebra verifies that indeed $\tau(t) \to \infty$ as required. In verifying this, we use the standard bound on the Gaussian tail probability, namely, $\mathbb{P}(Z_1 > u(t)) \geq c(u(t))^{-1} \exp\{-u^2(t)/2\}$. Unfolding the arguments, we have that

$$\mathbb{P}\left(\frac{M^*(t)}{(\log t)^\beta} \geq \left(\frac{1-\delta}{\theta^*}\right)^\beta\right) \xrightarrow{t \to \infty} 1$$

as required.

PROOF OF THE UPPER BOUND (4).   The proof will be broken up into several steps.

Step 1. Consider the following discretization of the continuous time problem. Recall that $M^*(t) = \sup\{W^*(s) : s \in [0, t]\}$. Set

$$Y_i = \sup_{s \in [i-1, i)} W^*(s).$$

Then, trivially, $M^*(t) \leq \bigvee_{i=1}^{[t]} Y_i$. Observe that by stationarity of W^* it follows that (Y_i) form a sequence of identically distributed random variables. Fix $\delta > 0$. Then, using the union bound, we have

$$\mathbb{P}\left(M^*(t) \geq \left(\frac{1+\delta}{\theta^*}\right)^\beta (\log t)^\beta\right) \leq \lceil t \rceil \mathbb{P}\left(Y_1 > \left(\frac{1+\delta}{\theta^*}\right)^\beta (\log t)^\beta\right).$$

Step 2. The goal here is to estimate the tail behavior of Y_1. Start by observing that

(6)
$$Y_1 \leq W^*(0) + \max_{0 \leq s \leq 1}\left(X(s) - \min_{0 \leq \tau \leq s} X(\tau)\right)$$
$$\leq W^*(0) + \max_{0 \leq s \leq 1} X(s) - \min_{0 \leq s \leq 1} X(s).$$

Thus,

$$P_1 := \mathbb{P}\left(Y_1 > \left(\frac{1+\delta}{\theta^*}\right)^\beta (\log t)^\beta\right)$$

$$\leq \mathbb{P}\left(W^*(0) + \max_{0 \leq s \leq 1} X(s) - \min_{0 \leq s \leq 1} X(s) \geq \left(\frac{1+\delta}{\theta^*}\right)^\beta (\log t)^\beta\right)$$

$$\leq \underbrace{\mathbb{P}\left(W^*(0) \geq \left(\frac{1+\delta/2}{\theta^*}\right)^\beta (\log t)^\beta\right)}_{Q_1}$$

$$+ \underbrace{\mathbb{P}\left(\max_{0 \leq s \leq 1} X(s) \geq \frac{1}{2}\left(\frac{\delta/2}{\theta^*}\right)^\beta (\log t)^\beta\right)}_{Q_2}$$

$$+ \underbrace{\mathbb{P}\left(-\min_{0 \leq s \leq 1} X(s) \geq \frac{1}{2}\left(\frac{\delta/2}{\theta^*}\right)^\beta (\log t)^\beta\right)}_{Q_3},$$

where we used the fact that $\beta > 1$ implies $(1+\delta)^\beta > (1+\delta/2)^\beta + (\delta/2)^\beta$.

Step 3. Our goal is to prove that $\lceil t \rceil Q_i \to 0$ for $i = 1, 2, 3$. First consider Q_2 and note

$$\max_{0 \leq s \leq 1} X(s) = \max_{0 \leq s \leq 1}(\sigma B_H(s) - \mu(1-\rho)s) \leq \max_{0 \leq s \leq 1} \sigma B_H(s).$$

To clinch the result, we need an estimate on the tail behavior of the maximum of standard fBM on a fixed interval. We appeal to Theorem 5.5 of Adler (1990), applied as in his Corollary 5.6, which, when specialized to the case of fBM, yields

$$
(7) \qquad \mathbb{P}\Big(\max_{0 \le s \le 1} B_H(s) > x\Big) \sim \mathbb{P}(Z > x),
$$

where $Z \sim \mathcal{N}(0, 1)$. This implies that

$$
\lceil t \rceil Q_2 \to 0,
$$

because, for $\beta > 1$, standard estimates on the Gaussian tail give $\lceil t \rceil \mathbb{P}(Z > c(\log t)^\beta) \to 0$, and $Q_2 \sim \mathbb{P}(Z > c(\log t)^\beta)$ using (7), where $c > 0$ is a generic constant that premultiplies the logarithmic term. We next consider Q_3. The key is to note that

$$
-\min_{0 \le s \le 1} X(s) \stackrel{\mathscr{D}}{=} \max_{0 \le s \le 1}(\mu(1 - \rho)s + \sigma B_H(s))
$$

$$
\le \mu(1 - \rho) + \sigma \max_{0 \le s \le 1} B_H(s)
$$

because $(B_H(s) : s \in [0, 1]) \stackrel{\mathscr{D}}{=} (-B_H(s) : s \in [0, 1])$. Therefore, upper bounding Q_3 by the sum of the probabilities involving the maximum and the minimum, respectively, the previously established result for Q_2 can be applied [with different constants premultiplying $(\log t)^\beta$] to give that $\lceil t \rceil Q_3 \to 0$. It remains to show that $\lceil t \rceil Q_1 \to 0$ or, equivalently, $a(t) := \log \lceil t \rceil + \log Q_1 \to -\infty$. The result in Proposition 1 implies that

$$
\lim_{t \to \infty} \frac{\log P\Big\{W^*(0) > \big((1 + \delta/2)/\theta^*\big)^\beta (\log t)^\beta\Big\}}{\log t} = -(1 + \delta/2).
$$

Consequently we have that

$$
a(t) = \log(t) + \log Q_1
$$

$$
= \log(t)\left(1 + \frac{\log P\Big\{W^*(0) > \big((1 + \delta/2)/\theta^*\big)^\beta (\log t)^\beta\Big\}}{\log t}\right)
$$

$$
\to -\infty.
$$

Thus, $\lceil t \rceil Q_1 \to 0$. Going back to Step 1 we see that $\lceil t \rceil \mathbb{P}(Y_1 > x(t)) \to 0$. Thus

$$
\mathbb{P}\left(\frac{M^*(t)}{(\log t)^\beta} \ge \left(\frac{1 + \delta}{\theta^*}\right)^\beta\right) \to 0,
$$

which establishes the upper bound.

PROOF SKETCH FOR $M(t)$. Note that for the process $M(t)$, $W(0) = 0$ because the free process $X(t)$ starts at 0. The proof of the lower bound then holds, with equality replacing the first inequality in (5). The upper bound

on the tail probability of Y_1 in (6) holds with $W(t)$ replacing $W^*(t)$, and the bounds on P_1 still hold because $W^*(0) \geq 0$ a.s. The rest of the arguments deal with estimates on the tails of the free process and carry through without change.

PROOF OF L_p CONVERGENCE. Fix $p \in [1, \infty)$. It suffices to show that the sequence $(M^*(t)/(\log t)^\beta)^p$ is uniformly integrable. A sufficient condition for this is

$$(8) \qquad \sup_{t \geq 2} \mathbb{E}\left[\frac{M^*(t)}{(\log t)^\beta}\right]^{p+1} < \infty,$$

where the estimator can be arbitrarily defined as 0 for $t \leq 2$. To this extent, define

$$K_1 = \inf\left\{y > 0 : \text{such that } \frac{\log P(W^*(0) > x)}{x^{1/\beta}} \leq -\frac{\theta^*}{2}, \ \forall x \geq y\right\}$$

and note that $K_1 < \infty$ follows from Proposition 1, that is,

$$\limsup_{x \to \infty} \frac{\log P\{W^*(0) > x\}}{x^{1/\beta}} \leq -\frac{\theta^*}{2}.$$

Let

$$K_2 = \inf\left\{x \geq 0 : \mathbb{P}\left(\max_{0 \leq s \leq 1} B_H(s) \geq y\right) \leq 2\mathbb{P}(Z \geq y), \ \forall y > x\right\},$$

where $Z \sim \mathcal{N}(0, 1)$. The finiteness of K_2 follows from Step 3 of the proof of the upper bound (3.4). Then, setting $K = \max\{K_1, K_2, 4/\theta^*\}$, we have

$$\mathbb{E}\left[\frac{M^*(t)}{(\log t)^\beta}\right]^{p+1} = \int_0^\infty (p+1)y^p \mathbb{P}(M^*(t) > y(\log t)^\beta)\,dy$$

$$= \int_0^K (p+1)y^p \mathbb{P}(M^*(t) > y(\log t)^\beta)\,dy$$

$$(9) \qquad\qquad + \int_K^\infty (p+1)y^p \mathbb{P}(M^*(t) > y(\log t)^\beta)\,dy$$

$$\leq K^{p+1} + \underbrace{\int_K^\infty (p+1)y^p \lfloor t \rfloor \mathbb{P}(W^*(0) > y(\log t)^\beta/2)\,dy}_{I_t}$$

$$+ \underbrace{\int_K^\infty (p+1)y^p \lfloor t \rfloor \mathbb{P}\left(\max_{0 \leq s \leq 1} X(s) - \min_{0 \leq s \leq 1} X(s) > y(\log t)^\beta/2\right)dy}_{R_t},$$

where the inequality follows from Step 1 of the proof of the upper bound (4). Now,

$$
\begin{aligned}
I_t &= \int_K^\infty (p+1)y^p \lfloor t \rfloor \mathbb{P}(W^*(0) > y(\log t)^\beta/2)\, dy \\
&\leq \int_K^\infty (p+1)y^p \exp\left\{ \frac{(\log\lfloor t\rfloor)^\beta}{2}\left(1 + y\frac{\log \mathbb{P}(W^*(0) > (y(\log t)^\beta)/2)}{y(\log\lfloor t\rfloor)^\beta/2} \right) \right\} dy \\
&\overset{(a)}{\leq} \int_K^\infty (p+1)y^p \exp\left\{ ((\log\lfloor t\rfloor)^\beta/2)\left(1 - y\frac{\theta^*}{2} \right) \right\} dy \\
&\overset{(b)}{\leq} \int_K^\infty (p+1)y^p e^{(-\theta^*/8)y(\log\lfloor t\rfloor)^\beta}\, dy \\
&\leq \left(\frac{8}{\theta^*} \right)^{p+1} \frac{(p+1)!}{(\log\lfloor t\rfloor)^{(p+1)\beta}},
\end{aligned}
$$

where (a) and (b) follow from the definition of K. Using the bound on Q_3 in Step 4 of the proof of the upper bound and by definition of K, it is clear that there exists a $C < \infty$ such that $R_t \leq C/(\log t)^{(p+1)\beta}$ [in fact, it is clear that $R_t = o(I_t)$]. Combining (9), the upper bound on I_t from the preceding equation and the bound on R_t, we have proved (8). Thus, the sequence $(M^*(t)/(\log t)^\beta)^p$ is uniformly integrable. Putting all four parts of the proof together completes the proof of Theorem 1. $\square$

3.2. *Proof of Theorem* 2 *and Propositions* 2 *and* 3.

PROOF OF PROPOSITION 2.   For $\rho = 1$,

$$
\begin{aligned}
M(t) &\overset{\mathscr{D}}{=} \max_{0\leq s\leq t} \max_{0\leq u\leq s} [\sigma B_H(s) - \sigma B_H(u)] \\
&= \sigma \max_{0\leq r\leq 1} \max_{0\leq v\leq r} [B_H(rt) - B_H(vt)] \\
&\overset{\mathscr{D}}{=} \sigma t^H \max_{0\leq r\leq 1} \max_{0\leq v\leq r} [B_H(r) - B_H(v)] \\
&= \sigma t^H \xi.
\end{aligned}
$$

For $\rho > 1$, consider the sequence of unit increments of fBM (sometimes referred to as fractional Gaussian noise), which is a stationary Gaussian sequence. From the properties of its spectral density [cf. Propositions 7.2.9 and 7.2.10 in Samorodnitsky and Taqqu (1994)], we easily see that its spectral measure does not have point masses, which in turn, for stationary Gaussian processes, is a necessary and sufficient condition for ergodicity. Details of this argument can be found, for example, in Rozanov [(1967), page 163]. Consequently, the pointwise ergodic theorem gives

$$
\frac{B_H(t)}{t} \to 0 \quad \text{a.s.,}
$$

so $X(t) = \mu(\rho - 1)t + \sigma B_H(t) \to +\infty$ a.s. Set $\Psi := \inf\{X(t): t \geq 0\}$. Hence, there exists $T_0 < \infty$ s.t. for $t \geq T_0$,

$$W(t) = X(t) - \Psi,$$

so for $t \geq T_0$,

$$\max_{0 \leq s \leq t} X(s) - \Psi \leq \max_{0 \leq s \leq t} W(s) \leq \max_{0 \leq s \leq T_0} W(s) + \max_{0 \leq s \leq t} X(s) - \Psi.$$

whereas $\mu(\rho - 1) > 0$,

$$\max_{0 \leq s \leq t} X(s) - X(t) = \max_{0 \leq s \leq t}[\sigma B_H(s) + \mu(\rho - 1)s - \sigma B_H(t) - \mu(\rho - 1)t]$$

$$\overset{\mathscr{D}}{=} \max_{0 \leq s \leq t}[\sigma B_H(t) - \sigma B_H(s) - \mu(\rho - 1)(t - s)]$$

$$\overset{\mathscr{D}}{=} \max_{0 \leq s \leq t}[\sigma B_H(s) - \mu(\rho - 1)s]$$

$$\Rightarrow \widetilde{W}(\infty)$$

with $\widetilde{W}(\infty) < \infty$ almost surely. So,

$$t^{-H}(M(t) - X(t)) \Rightarrow 0,$$

which concludes the proof. $\square$

PROOF OF THEOREM 2.   Claim (i) follows from the relationship $\{M^*(t) \geq b\} = \{T(b) \leq t\}$ and taking a sequence

$$(10) \qquad\qquad t_b := \exp\left\{b^{1/\beta} \frac{\theta^*}{1 + \delta}\right\}$$

so that $t_b \to \infty$ as $b \uparrow \infty$. Then, by Theorem 1,

$$(11) \qquad\qquad \mathbb{P}\left(M^*(t) > \left(\frac{1 + \delta}{\theta^*} \log t\right)^\beta\right) \to 0$$

and the convergence holds also along the subsequence t_b. In particular, substituting (10) into (11), we have

$$\mathbb{P}\left(\frac{T(b)}{\theta^* b^{1/\beta}} \leq 1 - \delta'\right) \to 0$$

with $\delta' := \delta/(1 + \delta)$. The upper bound follows similarly.

(ii) Follows from

$$\mathbb{P}(b^{-1/H} T(b) \leq t) = \mathbb{P}(M(tb^{1/H}) \geq b)$$

$$= \mathbb{P}\left((tb^{1/H})^{-H} M(tb^{1/H}) \geq t^{-H}\right)$$

$$\to \mathbb{P}(\sigma \xi \geq t^{-H})$$

$$= \mathbb{P}((\sigma \xi)^{-1/H} \leq t).$$

(iii) The argument follows along the lines of Theorem 6 of Glynn and Whitt (1988). Fix $x \in \mathbb{R}$. Then

$$\mathbb{P}\left(b^{-H}\left(T(b) - \frac{b}{\mu(\rho - 1)}\right) \leq x\right)$$

$$= \mathbb{P}\left(T(b) \leq \frac{b}{\mu(\rho - 1)} + xb^H\right)$$

$$= \mathbb{P}(M(t_b) \geq b); t_b := b/(\mu(\rho - 1)) + xb^H$$

$$= \mathbb{P}\left(t_b^{-H}(M(t_b) - \mu(\rho - 1)t_b) \geq (b - \mu(\rho - 1)t_b)t_b^{-H}\right)$$

$$\to \mathbb{P}\left(Z \geq -x(\mu(\rho - 1))^{1+H}\right) \qquad (\text{as } b \to \infty)$$

$$= \mathbb{P}\left((\mu(\rho - 1))^{-(1+H)}Z \leq x\right),$$

where $Z \sim \sigma \mathcal{N}(0, 1)$ by Theorem 1. Whereas x is arbitrary, the convergence holds for all continuity points of the limiting (Gaussian) cdf, and the proof is complete. $\square$

PROOF OF PROPOSITION 3.   Claim (i) follows from

$$\tilde{M}(t) := \sup_{0 \leq s \leq t}\left(X(s) - s\frac{X(t)}{t}\right)$$

$$= \sup_{0 \leq s \leq 1}\left(\sigma B_H(st) - st\frac{\sigma B_H(t)}{t}\right)$$

$$\overset{\mathscr{D}}{=} \sigma t^H \sup_{0 \leq s \leq 1}(B_H(s) - sB_H(1)).$$

Taking logs from both sides and rearranging gives the result.

(ii) The trick is to reduce the problem to that studied in Theorem 1, that is, replace $H(t)$ with H. Write

$$\frac{(M(t))^{2(1-H(t))}}{\log t} = \frac{(M(t))^{2(1-H)}}{\log t}\frac{(M(t))^{2(1-H(t))}}{(M(t))^{2(1-H)}}$$

$$= \underbrace{\left(\frac{M(t)}{(\log t)^{1/(2(1-H))}}\right)^{2(1-H)}}_{I_t}\underbrace{(M(t))^{2(H-H(t))}}_{II_t}$$

and observe that

$$I_t \Rightarrow \frac{1}{\theta^*}$$

by Theorem 1 and the continuous mapping theorem. As for II_t, observe that

$$\log II_t = 2(H - H(t))\log M(t).$$

Now, by part (i) we have

$$|H - H(t)| = O_p\left(\frac{1}{\log t}\right),$$

and by taking logs in Theorem 1, we have also

$$\frac{\log M(t)}{\log \log t} \Rightarrow \frac{1}{2(1 - H)}.$$

Thus, an application of the continuous mapping theorem gives

$$\log II_t \Rightarrow 0,$$

which concludes the proof. □

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