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EXPLICIT SOLUTION TO THE MULTIVARIATE SUPER-REPLICATION PROBLEM UNDER TRANSACTION COSTS

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We consider a multivariate financial market with transaction costs as in Kabanov. We study the problem of finding the minimal initial capital needed to hedge, without risk, European-type contingent claims. We prove that the value of this stochastic control problem is given by the cost of the cheapest buy-and-hold strategy. This is an extension of the already known result in the one-dimensional case. An important feature of our analysis is that we do not make use of the dual formulation of the problem, as in the previous literature.

1. Introduction. In the context of the Black and Scholes one-dimensional financial market with proportional transaction costs, Davis and Clark (1994) conjectured that the minimal initial wealth needed to superreplicate a European call option is just the price of one share of the underlying asset. In other words, the cheapest buy-and-hold strategy solves the superreplication problem. The conjecture was proved by analytic methods by Soner, Shreve and Cvitanić (1995) and (independently, and for more general models and contingent claims) by Levental and Skorohod (1995) by probabilistic methods.

In a one-dimensional Markov diffusion model, a simple proof of this conjecture was provided by Cvitanić, Pham and Touzi (1999) for general contingent claims. Their approach relies on the dual formulation of the superreplication cost [Jouini and Kallal (1995) and Cvitanić and Karatzas (1996)] which reduces the problem to a singular stochastic control problem in standard form.

In a recent paper, Kabanov (1999) provided an extension of the dual formulation of the superreplication problem to the case of currency markets with proportional transaction costs. This framework is a natural multidimensional version of the models discussed above. The multivariate superreplication problem under transaction costs presents some important difficulties which are not apparent in the one-dimensional model. In particular, we were not able to extend Cvitanić, Pham and Touzi's (1999) proof to this context.

Instead, we relate the superreplication problem to some convenient auxiliary superreplication problems defined on fictitious financial markets without transaction costs. Definition of such fictitious financial markets is obtained by use of the solvency cone introduced by Kabanov (1999). We then use a dynamic programming principle stated directly on the auxiliary problems as in Soner and Touzi (1998, 1999). We prove that such a dynamic programming equation allows us to characterize the value of the auxiliary control problem as a viscos-

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ity supersolution of a suitable Hamilton–Jacobi–Bellman partial differential equation. The remaining arguments are similar to Cvitanić, Pham and Touzi (1999).

The paper is organized as follows. After setting some notations in Section 2, we describe the model and the superreplication problem in Section 3. The main results of the paper are stated in Section 4 with a partial argument; the proof is concluded in Section 9 after some preparation in the Sections in between. In Section 5, we introduce a parameterization of the polar associated with the solvency cone of Kabanov (1999). In Section 6, we define auxiliary stochastic control problems which are interpreted as superreplication problems on fictitious financial markets without transaction costs. Section 7 contains the dynamic programming principle suited with the auxiliary control problem. Viscosity properties of the value function of the auxiliary problem are then reported in Section 8. Finally, Section 10 contains some examples.

2. Notation. We denote by $\cdot$ the natural scalar product in $\mathbb{R}^n$ and $\|\cdot\|$ the associated norm. Given a vector $x \in \mathbb{R}^n$, its ith component is denoted by x^i. $\mathbb{M}^{n,p}$ denotes the set of all real-valued matrices with n rows and p columns. Given a matrix $M \in \mathbb{M}^{n,p}$, we denote by M^{ij} the component corresponding to the ith row and the jth column. $\mathbb{M}_+^{n,p}$ denotes the subset of $\mathbb{M}^{n,p}$ whose elements have non-negative entries. If $n = p$, we simply denote $\mathbb{M}^n$ and $\mathbb{M}_+^n$ for $\mathbb{M}^{n,n}$ and $\mathbb{M}_+^{n,n}$. Since $\mathbb{M}^{n,p}$ can be identified with $\mathbb{R}^{np}$, we define the norm on $\mathbb{M}^{n,p}$ as the norm of the associated element of $\mathbb{R}^{np}$. Transposition is denoted by *. Given a square matrix $M \in \mathbb{M}^n$, we denote by $\operatorname{Tr}[M] := \sum_{i=1}^n M^{ii}$ the associated trace.

Given n scalars $x_1, \dots, x_n$, we denote by $\operatorname{Vect}[x_i,\ i = 1, \dots, n]$ the vector of $\mathbb{R}^n$ defined by the components $x_1, \dots, x_n$. For all $x \in \mathbb{R}^n$, $\operatorname{diag}[x]$ denotes the diagonal matrix of $\mathbb{M}^n$ whose ith diagonal element is x^i. Given a matrix $M \in \mathbb{M}^{n,p}$, we denote by $\bar{M}$ the matrix in $\mathbb{M}^{n+1,p}$ obtained from M by adding the first row of one. The same notation prevails for vectors in $\mathbb{R}^n$.

We denote by $\mathbf{1}_i$ the vector of $\mathbb{R}^n$ defined by $\mathbf{1}_i^j = 1$ if $j = i$ and 0 otherwise.

Given a smooth function φ mapping $\mathbb{R}^n$ into $\mathbb{R}^p$, we denote by $D\varphi$ the Jacobian matrix of φ, that is, $(D\varphi)^{ij} = \partial\varphi^i/\partial x^j$. If $x = (y, z)$, $D_y\varphi$ denotes the (partial) Jacobian matrix of φ with respect to the y variable. In the case $p = 1$, we denote by $D^2\varphi$ the Hessian matrix of φ, that is, $(D^2\varphi)^{ij} = \partial^2\varphi/\partial x^i \partial x^j$. If $x = (y, z)$, we define the matrices $D^2_{yy}\varphi$, $D^2_{zz}\varphi$ and $D^2_{yz}\varphi$ accordingly.

Given a filtered probability space $(\Omega, \mathscr{F}, P, \{\mathscr{F}(t),\ 0 \le t \le T\})$ and a scalar $p \ge 0$, we denote by $L^p(t)$ the set of all $\mathscr{F}(t)$-measurable random variables with finite L^p norm. This notation is extended naturally to stopping times. For $p = 0$, $L^0(t)$ is the set of all $\mathscr{F}(t)$-measurable random variables. Finally, we will use the convention $\inf \varnothing = +\infty$.

3. The model. Let T be a finite time horizon and $(\Omega, \mathscr{F}, P)$ be a complete probability space supporting a d-dimensional Brownian motion $\{B(t), 0 \le t \le T\}$. We shall denote by $\mathbb{F} = \{\mathscr{F}(t),\ 0 \le t \le T\}$ the P-augmentation of the filtration generated by B.

3.1. *The financial market.* We consider a financial market which consists of one bank account, with constant price process S^0, normalized to unity, and d risky assets $S := \{S^1, \dots, S^d\}^*$. The price process $S = \{S(t), 0 \le t \le T\}$ is an $\mathbb{R}^d$-valued stochastic process defined by the following stochastic differential system:

$$dS(t) = \operatorname{diag}[S(t)]\sigma(t, S(t))\, dB(t), \qquad 0 < t \le T. \tag{3.1}$$

Here $\sigma(.,.)$ is an $\mathbb{M}^d$-valued function. We shall assume throughout that the function $\operatorname{diag}[s]\sigma(t,s)$ satisfies the usual Lipschitz and linear growth conditions in order for the process S to be well defined and that $\sigma(t,s)$ is invertible for all $(t,s) \in [0,T] \times (0,\infty)^d$. We also assume that

$$P[S(u) \in A | \mathscr{F}(t)] > 0, \quad P\text{-a.s., } 0 \le t < u \le T \tag{3.2}$$

for all Borel subsets A of $(0,\infty)^d$.

REMARK 3.1. A sufficient condition for (3.2) to be verified is that, for all $(t,s) \in [0,T] \times (0,\infty)^d$, matrix $\sigma(t,s)$ satisfies the uniform ellipticity condition

$$\exists\ \varepsilon > 0 : \ \|\sigma(t,s)\xi\| \ \ge \ \varepsilon \|\xi\|^2 \quad \text{for all } \xi \in \mathbb{R}^d .$$

Notice that, with our notation, $\bar{S} := (S^0, S^1, \dots, S^d)^*$.

REMARK 3.2. As usual, the assumption that the interest rate of the bank account is zero could be easily dispensed with by discounting. Also, there is no loss of generality in defining S as a local martingale since we can always reduce the model to this context by appropriate change of measure (under mild conditions on the initial coefficients).

A *trading strategy* is an $\mathbb{M}_+^{d+1}$-valued process L, with initial value $L(0-) = 0$, such that L^{ij} is $\mathbb{F}$-adapted right-continuous and nondecreasing for all $i, j = 0, \dots, d$. Here, L^{ij} describes the cumulative amount of funds transferred from asset i to asset j.

Proportional transaction costs in this financial market are described by matrix $\lambda \in \mathbb{M}_+^{d+1}$. This means that transfers from asset i to asset j are subject to proportional transaction costs λ^{ij} for all $i, j = 0, \dots, d$.

Then, given an initial holdings vector $x \in \mathbb{R}^{d+1}$ and a trading strategy L, the portfolio holdings $X_x^L = (X_x^{i,L})_{i=0,\dots,d}$ are defined by the dynamics,

$$X_x^{i,L}(0-) = x^i,$$

$$dX_x^{i,L}(t) = X_x^{i,L}(t)\, \frac{d\bar{S}^i(t)}{\bar{S}^i(t)} + \sum_{j=0}^{d} \left[dL^{ji}(t) - (1+\lambda^{ij})\, dL^{ij}(t) \right], \quad 0 \le t \le T$$

for all $i = 0, \dots, d$. The dynamics of the portfolio holdings process can be written alternatively in terms of the number of shares transferred from one asset to another. Set $l_k^{ij}(t) := \int_0^t dL^{ij}(r)/\bar{S}^k(r)$ for all $i, j = 0, \dots, d$ and $k \in \{i, j\}$. Here, l_i^{ij} (resp. l_j^{ij}) is the cumulated transfer from asset i to asset j

in terms of number of shares of asset i (resp. j). Then, from the previous dynamics, we get

$$X_x^{i,L}(t) = x^i + \bar{S}^i(t) \sum_{j=0}^{d} \left(l_i^{ji}(t) - (1 + \lambda^{ij}) l_i^{ij}(t) \right), \qquad 0 \le t \le T .$$

3.2. *The superreplication problem.* Following Kabanov (1999), we define the *solvency region*,

$$K := \left\{ x \in \mathbb{R}^{d+1} \colon \exists\, a \in \mathbb{M}_+^{d+1},\ x^i + \sum_{j=0}^{d} (a^{ji} - (1 + \lambda^{ij}) a^{ij}) \ge 0;\ i = 0, \dots, d \right\}.$$

The elements of K can be interpreted as the vectors of portfolio holdings such that the no-bankruptcy condition is satisfied: the liquidation value of the portfolio holdings x, through some convenient transfers, is nonnegative. Another economic interpretation is that the portfolio holdings $-x$ can be reached from zero initial portfolio holdings through some convenient transfers.

Clearly, the set K is a closed convex cone containing the origin. Following Kabanov (1999), we introduce the partial ordering $\succeq$ induced by K, defined by

$$\text{for all } x_1, x_2 \in \mathbb{R}^{d+1},\ x_1 \succeq x_2 \text{ if and only if } x_1 - x_2 \in K .$$

A trading strategy L is said to be *admissible* for the initial holdings $x \in \mathbb{R}^{d+1}$ if the no-bankruptcy condition

$$X_x^L(t) \succeq 0, \quad P\text{-a.s., } 0 \le t \le T \tag{3.3}$$

holds. We shall denote by $\mathscr{A}(x)$ the set of all admissible trading strategies.

A *contingent claim* is a $(d+1)$-dimensional $\mathscr{F}(T)$-measurable random variable $g(S(T))$. Here, g maps $(0, \infty)^d$ into $\mathbb{R}^{d+1}$ and satisfies

$$g \text{ is lower semicontinuous and } g(s) \succeq 0 \text{ for all } s \text{ in } (0, \infty)^d. \tag{3.4}$$

In the rest of the paper, we shall identify a contingent claim with its pay-off function g. For all $i = 0, \dots, d$, the random variable $g^i(S(T))$ represents a target position in asset i.

The *superreplication problem* of contingent claim g is then defined by

$$v(0, S(0)) := \inf \left\{ w \in \mathbb{R} :\ \exists\, L \in \mathscr{A}(w\mathbf{1}_0),\ X_{w\mathbf{1}_0}^L(T) \succeq g(S(T))\ P\text{-a.s.} \right\}, \tag{3.5}$$

that is, $v(0, S(0))$ is the minimal initial capital which allows hedging the contingent claim g through some admissible trading strategy.

REMARK 3.3. Given an initial portfolio holding $x \in \mathbb{R}^{d+1}$, we can generalize the definition of v by setting

$$v(0, S(0), x) := \inf \left\{ w \in \mathbb{R} \colon \exists\, L \in \mathscr{A}(x + w\mathbf{1}_0),\ X_{x+w\mathbf{1}_0}^L(T) \succeq g(S(T))\ P\text{-a.s.} \right\}.$$

That is, $v(0, S(0), x)$ is the minimal initial cash increment needed to hedge the contingent claim g when starting with the initial holding x. This generalized problem can also be solved by the techniques developed in this paper [see the survey paper of Touzi (1999)].

4. The main result. Following Kabanov (1999), we introduce the positive polar of K,

$$K' := \{\xi \in \mathbb{R}^{d+1}:\ \xi \cdot x \geq 0,\ \forall\ x \in K\}\ ,$$

and we denote by Λ the subset of $\mathbb{R}^d$,

$$\Lambda := \{r \in \mathbb{R}^d:\ \bar{r} \in K'\},$$

that is, $\{1\} \times \Lambda$ is the section of the positive polar cone K' with the hyperplane $\{\xi \in \mathbb{R}^{d+1}:\ \xi^0 = 1\}$. The partial ordering $\succeq$ can be characterized in terms of Λ by

$$x_1\ \succeq\ x_2 \text{ if and only if } \bar{r} \cdot (x_1 - x_2) \geq 0 \quad \text{for all } r \in \Lambda\ .$$

Notice that Λ is not empty since $\mathbf{1} = \sum_{i=1}^{d} \mathbf{1}_i \in \Lambda$. This is easily checked from the definition of K (by summing up the $d+1$ inequalities and using the fact that the transfer matrix a as well as the transaction costs matrix λ have nonnegative entries).

REMARK 4.1. An important property of the polar cone K' is that $K' \setminus \{0\} \subset (0, \infty)^{d+1}$. This claim will be proved in Lemma 5.1. It follows that $\Lambda \subset (0, \infty)^d$.

REMARK 4.2. The set Λ is a compact subset of $\mathbb{R}^d$. This claim will be justified in Remark 5.2.

Next, we introduce the functions

$$G(z) := \sup_{r \in \Lambda} \bar{r} \cdot g\left(\operatorname{diag}[r]^{-1} z\right) \quad \text{for all } z \text{ in } (0, \infty)^d$$

and

$$\hat{g}(s) := \sup_{r \in \Lambda} G^{\text{conc}}(\operatorname{diag}[r] s) \quad \text{for all } s \text{ in } (0, \infty)^d,$$

where G^{conc} is the concave envelope of G.

The main result of this paper requires the following condition.

ASSUMPTION 4.1. $\lambda^{ij} + \lambda^{ji} > 0$ for all $i, j = 0, \ldots, d,\ i \neq j$.

REMARK 4.3. Assumption 4.1 is necessary and sufficient for K' to have a nonempty interior.

THEOREM 4.1. *Under Assumption* 4.1, *the solution of the superreplication problem is given by*

$$v(0, S(0)) = \hat{g}(S(0))\ .$$

The proof of the last result will be provided in subsequent sections of the paper. We now give an economic interpretation of the result.

Let $w \geq 0$ be some initial capital. A buy-and-hold strategy is an admissible strategy L in $\mathscr{A}(w\mathbf{1}_0)$ such that the number of shares of the ith asset $X^{i,L}_{w\mathbf{1}_0}(t)/\bar{S}^i(t)$ is constant over the time interval $[0,T]$, for all $i = 0,\ldots,d$. In other words, the number of shares induced by the strategy L is unchanged during the time interval $[0,T]$, so that no transfers are operated after time 0. The cost of the cheapest buy-and-hold strategy is clearly given by

$$h(S(0)) := \inf\Big\{w \in \mathbb{R}:\ \exists\ \Delta \in \mathbb{R}^{d+1},\ w\mathbf{1}_0 \succeq \operatorname{diag}[\bar{S}(0)]\Delta \\ \text{and } \operatorname{diag}[\bar{z}]\Delta \succeq g(z),\ \text{for all } z \text{ in}(0,\infty)^d\Big\}.$$

THEOREM 4.2. *For all s in $(0,\infty)^d$, we have $\hat{g}(s) = h(s)$.*

Finally, as a direct consequence of the last two theorems, we have a corollary.

COROLLARY 4.1. *Under Assumption* 4.1, *the value of the superreplication problem is the cost of the cheapest buy-and-hold strategy*. *Moreover*, *existence holds for the optimization problem* (3.5).

PROOF OF THEOREM 4.2. (i) We first prove that $\hat{g}(s) \leq h(s)$. Consider some arbitrary scalar $w > h(s)$. By definition of $h(s)$, there exists some Δ in $\mathbb{R}^{d+1}$ such that, for all z in $(0,\infty)^d$,

$$\bar{r}'\cdot(w\mathbf{1}_0 - \operatorname{diag}[\bar{s}]\Delta) \geq 0 \quad \text{and} \quad \bar{r}\ \cdot(\operatorname{diag}[\bar{z}]\Delta - g(z)) \geq 0 \quad \text{for all}\ \ r,\ r' \in \Lambda\ .$$

Using the fact that $\bar{r}^0 = 1$, this provides

$$w + \Delta\cdot(\bar{z} - \operatorname{diag}[\bar{r}']\bar{s})\ \geq \bar{r}\cdot g\left(\operatorname{diag}[r]^{-1}z\right) \quad \text{for all } r,\ r'\ \in \Lambda,$$

and taking supremum over r, we get

$$w + \Delta\cdot(\bar{z} - \operatorname{diag}[\bar{r}']\bar{s}) \geq G(z) \quad \text{for all } z\ \in\ (0,\infty)^d \text{ and } \bar{r}'\ \in \Lambda.$$

This proves that $w \geq G^{\mathrm{conc}}(\operatorname{diag}[r']s)$ for all r' in Λ, and therefore $w \geq \hat{g}(s)$. The required inequality follows from the arbitrariness of $w > h(s)$.

(ii) We now prove the converse inequality. Since Λ is a compact subset of $\mathbb{R}^d$ and G^{conc} is continuous, there exists some $\hat{r} \in \Lambda$ such that

$$\hat{g}(s) = G^{\mathrm{conc}}(\operatorname{diag}[\hat{r}]s)\ . \tag{4.1}$$

Recall that the concave envelope is characterized by

$$G^{\mathrm{conc}}(z) = \min\left\{c \in \mathbb{R}:\ \exists\ \zeta \in \mathbb{R}^d,\ c + \zeta\cdot(z' - z) \geq G(z') \text{ for all } z' \in (0,\infty)^d\right\}.$$

Also, it is well known that the solution of the above optimization problem is given by any element of the subgradient $\partial G^{\mathrm{conc}}(z)$ of the concave function G^{conc} at z. Hence

$$\forall\ \zeta \in \partial G^{\mathrm{conc}}(\operatorname{diag}[\hat{r}]s)\,, \quad \hat{g}(s) + \zeta\cdot(z - \operatorname{diag}[\hat{r}]s) \geq G(z); \qquad z \in (0,\infty)^d\ . \tag{4.2}$$

We claim that

$$\exists\ \hat{\zeta} \in \partial G^{\mathrm{conc}}(\mathrm{diag}[\hat{r}]s):\quad \mathrm{diag}[s]\hat{\zeta}\cdot(r-\hat{r}) \le 0 \quad \text{for all } r \in \Lambda\,. \tag{4.3}$$

We leave the proof of the previous claim to part (iii). Now, let $\hat{\Delta}$ be the vector of $\mathbb{R}^{d+1}$ defined by $\hat{\Delta}^i = \hat{\zeta}^i$ for $i = 1, \ldots, d$ and $\hat{\Delta}^0 = \hat{g}(s) - \hat{\zeta}\cdot\mathrm{diag}[\hat{r}]s$. Then from (4.2), $\hat{\Delta}\cdot\bar{z} \ge G(z)$ for all $z \in (0,\infty)^d$, and by definition of G, we see that $\hat{\Delta}\cdot\bar{z} \ge \bar{r}\cdot g(\mathrm{diag}[r]^{-1}z)$ for all $(r,z) \in \Lambda\times(0,\infty)^d$. By a trivial change of variables, this provides

$$\bar{r}\cdot\Big(\mathrm{diag}[\bar{z}]\hat{\Delta} - g(z)\Big) \ge 0 \quad \text{for all } r \in \Lambda \text{ and } z \in (0,\infty)^d\,.$$

By definition of the normalized polar Λ, this proves that

$$\mathrm{diag}[\bar{z}]\hat{\Delta} \succeq g(z) \quad \text{for all } z \in (0,\infty)^d\,.$$

Now, rewriting (4.2) in terms of $\hat{\Delta}$ and using (4.3) yields

$$\hat{g}(s)\mathbf{1}_0 \ \succeq\ \mathrm{diag}[\bar{s}]\hat{\Delta}\,,$$

which together with the previous inequality implies that $\hat{g}(s) \ge h(s)$.

(iii) In order to conclude the proof, it remains to verify (4.3). Let ε be an arbitrary parameter in $(0,1)$ and $r \in \Lambda$. Since Λ is convex, we have

$$\begin{aligned}\hat{g}(s) &\ge G^{\mathrm{conc}}(\mathrm{diag}[(1-\varepsilon)\hat{r}+\varepsilon r]s)\\ &= \hat{g}(s) + \varepsilon(r-\hat{r})\cdot\mathrm{diag}[s]\zeta_\varepsilon\,,\end{aligned}$$

where ζ_ε is an element of $\partial G^{\mathrm{conc}}(z_\varepsilon)$ for some z_ε lying in the interval defined by the bounds $\mathrm{diag}[\hat{r}]s$ and $\mathrm{diag}[\hat{r}+\varepsilon(r-\hat{r})]s$. Since G^{conc} is concave and z_ε converges to the interior point $\mathrm{diag}[\hat{r}]s$, the sequence (ζ_ε) converges to some $\hat{\zeta} \in \partial G^{\mathrm{conc}}(\mathrm{diag}[\hat{r}]s)$. Then claim (4.3) is obtained by passing to the limit in the last inequality. □

5. Parameterization. By direct computation, it is easily checked that

$$K' = \bigcap_{i,j=0}^{d} H^{ij} \quad \text{where } H^{ij} = \Big\{\xi \in \mathbb{R}_+^{d+1}\colon\ \xi^j - (1+\lambda^{ij})\xi^i \le 0\Big\}$$

[see Kabanov (1999)]. In the sequel, we shall denote $\partial H^{ij} := \{\xi \in \mathbb{R}_+^{d+1}\colon\ \xi^j - (1+\lambda^{ij})\xi^i = 0\}$.

LEMMA 5.1. (i) *K' is a closed convex polyhedral cone.*
(ii) *Let ξ be a nonzero element of K'. Then $\xi^i > 0$ for all $i = 0, \ldots, d$.*

PROOF. Part (i) follows from Rockafellar [(1970), page 171, Theorem 19.1] and the fact that K' is a finite intersection of half-hyperplanes. To see that part (ii) holds, consider some $\xi \in K'$ with $r^i > 0$ for some $i = 0, \ldots d$. Then since $\xi \in H^{ji}$ for all $j = 0, \ldots, d$ we have $\xi^i - (1+\lambda^{ji})\xi^j \le 0$. □

Since K' is a closed convex polyhedral cone, it is finitely generated [see Rockafellar 1970, page 171, Theorem 19.1]. Then, we can define a generating family $\{\tilde{e}_1, \dots, \tilde{e}_n\}$, that is,

$$K' = \left\{ \xi \in \mathbb{R}^{d+1}\colon\ \xi = \sum_{i=1}^{n} y^i \tilde{e}_i \text{ for some } y \in \mathbb{R}^n_+ \right\} .$$

By part (ii) of Lemma 5.1, the generating vectors can be normalized by

$$\tilde{e}_i^0 = 1 \quad \text{for all } i = 1, \dots, n . \tag{5.1}$$

Therefore, denoting by e_i the $\mathbb{R}^d$ vector defined by the d last components of $\tilde{e}_i$, we can rewrite the generating family as $\{\bar{e}_1, \dots, \bar{e}_n\}$.

EXAMPLE 5.1. In the one-dimensional case $d = 1$, the generating vectors of K' are given by

$$\bar{e}_1 = (1, 1 + \lambda^{01}) \quad \text{and} \quad \bar{e}_2 = (1, (1 + \lambda^{10})^{-1}).$$

This is the case studied by Cvitanić, Pham and Touzi (1999).

For the two-dimensional case $d = 2$, we also have explicitly the vectors of the generating family under some condition on the transaction costs matrix; see Section 10. In the general case, we do not have an explicit form of the family of generators. However, the main result of this paper does not require this information.

REMARK 5.1. Since K' has nonempty interior by Remark 4.3, the range of the family $\{\bar{e}_1, \dots, \bar{e}_n\}$ is $d + 1$. In particular, $n \geq d + 1$.

In order to have a parameterization of Λ, we define the following function f mapping $(0, \infty)^n$ into $\mathbb{R}^d$ by

$$f^i(y) = \left(\sum_{j=1}^{n} y^j \bar{e}_j^0 \right)^{-1} \sum_{j=1}^{n} y^j \bar{e}_j^i = \left(\sum_{j=1}^{n} y^j \right)^{-1} \sum_{j=1}^{n} y^j e_j^i ; \qquad i = 1, \dots, d,$$

so that the set Λ can be written in terms of the function f,

$$\Lambda = \{ f(y)\colon\ y \in (0, \infty)^n \} .$$

The following result is the keystone of our analysis.

LEMMA 5.2. *Let Assumption* 4.1 *hold. Then, for all y in $(0, \infty)^n$, the range of the Jacobian matrix $Df(y)$ is d.*

PROOF. For all $k = 1, \dots, n$, we introduce the matrices $N_k := [e_1, \dots, e_k]$. Recall that $\bar{N}_k$ is obtained from N_k by adding the first row of one, so that $\bar{N}_k := [\bar{e}_1, \dots, \bar{e}_k]$. Then, direct computation shows that

$$\left(\sum_{i=1}^{n} y^i \right) Df(y) = N_n - N_n \tilde{y} \quad \text{where } \tilde{y} = \left(\sum_{i=1}^{n} y^i \right)^{-1} y.$$

It follows that $Df(y)$ and $N_n - N_n\tilde{y}$ have the same range.

(i) We first show that range$[N_n - N_n\tilde{y}]$ =range$[N_n]$. To see this, observe that the ith column of $N_n - N_n\tilde{y}$ is given by $\tilde{e}_i := e_i - \sum_{j=1}^n \tilde{y}^j e_j = (1-\tilde{y}^i)e_i - \sum_{i\neq j,\, j=1}^n \tilde{y}^j e_j$. Since $\sum_{i=1}^n \tilde{y}^i = 1$ and $\tilde{y}^i > 0$ for all i, we have $(1-\tilde{y}^i) > 0$. Then the families $\{\tilde{e}_i,\ i = 1,\dots,n\}$ and $\{e_i,\ i = 1,\dots,n\}$ have same the range.

(ii) We now prove that range$[N_n] = d$ which concludes the proof. By Remark 5.1, the matrix $\bar{N}_{d+1}$ is invertible, after possibly changing the order of the $\bar{e}_i$'s. Then clearly,

$$\operatorname{range}[N_n] = \operatorname{range}[N_{d+1}] = \operatorname{range}[J\bar{N}_{d+1}] \quad \text{with } J = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ & I_d & \\ 0 & & \end{pmatrix}$$

and I_d is the identity matrix of $\mathbb{R}^d$. The required result follows from the fact that $\bar{N}_{d+1}$ is invertible and range$[J] = d$. $\square$

REMARK 5.2. From Lemma 5.1(ii), $f^i(y) > 0$ for all $y \in (0,+\infty)^n$. Moreover, since $\bar{f} \in H^{0i}$ for all $i = 1,\dots,d$, it follows that $f^i(y) = \bar{f}^i(y) \le (1+\lambda^{0i})\bar{f}^0(y) = (1+\lambda^{0i})$. Hence f is bounded and

$$0 < f^i(y) \le (1+\lambda^{0i}) \quad \text{for all } i = 1,\dots,d \text{ and } y \in (0,\infty)^n.$$

Also, from the expression of the Jacobian matrix given in the above proof, it is easily checked that $Df(y)\operatorname{diag}[y]$ is bounded.

6. An auxiliary control problem. In order to alleviate notations, we introduce the functions

$$F(y) := \operatorname{diag}[f(y)]$$

Fix some arbitrary parameter $\mu > 0$. For all $y_0 > 0$, we define the continuous function α^{y_0} on $[0,T]\times(0,\infty)^{d+n}\times\mathbb{M}^{n,d}\times\mathbb{R}^n$ as

$$\alpha^{y_0}(t,s,y,a,b) \tag{6.1}$$
$$:= \begin{cases} A(t,s,y,a,b), & \text{if } \displaystyle\sum_{i=1}^d\sum_{j=1}^n \left(\left|\ln\frac{s^i}{S^i(0)}\right| + \left|\ln\frac{y^j}{y_0^j}\right|\right) < \mu, \\ \text{constant}, & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} A(t,s,y,a,b) = \sigma(t,s)^{-1}F(y)^{-1}\{&Df(y)\operatorname{diag}[y]b \\ &+\tfrac{1}{2}\operatorname{Vect}\left[\operatorname{Tr}\left(D^2f^i(y)\operatorname{diag}[y]aa^*\operatorname{diag}[y]\right),\ i=1,\dots,d\right] \\ &+\operatorname{Vect}\left[(Df(y)\operatorname{diag}[y]a\sigma^*(t,s))_{ii},\ i=1,\dots,d\right]\}. \end{aligned}$$

Let $\mathscr{D}$ be the set of all bounded progressively measurable processes $(a,b) = \{(a(t),b(t)),\ 0\le t\le T\}$ where a and b are valued, respectively, in $\mathbb{M}^{n,d}$ and

$\mathbb{R}^n$. For all y in $(0,\infty)^n$ and (a, b) in $\mathscr{D}$, we introduce the controlled process $Y_y^{(a,b)}$ defined on $[0,T]$ defined as the solution of the stochastic differential equation

$$\begin{aligned} dY(t) &= \mathrm{diag}[Y(t)][(b(t)+a(t)\alpha^y(t, S(t), Y(t), a(t), b(t)))\,dt \\ &\qquad\qquad + a(t)\,dB(t)], \\ Y(0) &= y. \end{aligned} \tag{6.2}$$

We do not write the dependence of $Y_y^{(a,b)}$ with respect to μ to alleviate the notations. Notice that, since $\alpha^{y_0}(t, s, y, a, b)$ is a random Lipschitz function of y, the process $Y_y^{(a,b)}$ is well defined on $[0,T]$.

For each (a,b) in $\mathscr{D}$, we define the process $Z_y^{(a,b)}$ by

$$Z_y^{(a,b)}(t) = F(Y_y^{(a,b)}(t))S(t) \quad \text{for all } 0 \le t \le T. \tag{6.3}$$

Let ϕ be a progressively measurable process valued in $\mathbb{R}^{d+1}$ and satisfying

$$\sum_{i=1}^{d}\int_0^T |\phi^i(t)|^2 d\langle Z_y^{i,(a,b)}(t)\rangle < \infty, \qquad P\text{-a.s.} \tag{6.4}$$

Then, given $w \ge 0$, we introduce the process $W_{w,y}^{(a,b)^\phi}$ defined by

$$W_{w,y}^{(a,b)^\phi}(t) = w + \int_0^t \phi(r)\cdot d\bar{Z}_y^{(a,b)}(r), \qquad 0 \le t \le T, \tag{6.5}$$

and we denote by $\mathscr{B}^{(a,b)}(w, y)$ the set of all such processes ϕ satisfying the additional condition

$$W_{w,y}^{(a,b)^\phi}(t) \ge 0, \qquad P\text{-a.s., } 0 \le t \le T. \tag{6.6}$$

We finally define the auxiliary stochastic control problems

$$\begin{aligned} &u^{(a,b)}(0, y, F(y)S(0)) \\ &\quad := \inf\Big\{w \in \mathbb{R}\colon \exists\, \phi \in \mathscr{B}^{(a,b)}(w,\ y)\ , \\ &\qquad\qquad W_{w,y}^{(a,b)^\phi}(T) \ge \bar{f}\Big(Y_y^{(a,b)}(T)\Big)\cdot g(S(T))P\text{-a.s.}\Big\} \end{aligned} \tag{6.7}$$

and

$$u(0, y, F(y)S(0)) := \sup_{(a,b)\in\mathscr{D}} u^{(a,b)}(0, y, F(y)S(0))\ . \tag{6.8}$$

Before stating the main result of this section, we provide an economic interpretation of the auxiliary control problems $u^{(a,b)}$. For all $(a, b) \in \mathscr{D}$, the process $Z_y^{(a,b)}$ describes the price process of d risky assets in a fictitious financial market without transaction costs. The process ϕ is a portfolio strategy on the fictitious financial market: ϕ^i is the number of shares of risky asset i held at each time, for $i = 1, \dots, d$. The process $W_{w,y}^{(a,b)^\phi}$ describes the wealth induced by portfolio strategy ϕ and initial wealth w, under the self-financing condition. Hence, $u^{(a,b)}$ is the superreplication problem for a conveniently modified contingent claim on the auxiliary market. The main feature of the price process $Z_y^{(a,b)}$ is the following. Let x, $x' \in \mathbb{R}^{d+1}$ be two vectors of portfolio holdings such that $x'^i = x^i + \sum_{i,j}(a^{ji} - (1+\lambda^{ij})a^{ij})$, $i = 0, \dots, d$ for some transfer matrix $a \in \mathbb{M}_+^{d+1}$. Then

$$\mathrm{diag}[\bar{S}(t)]^{-1}(x'-x) \cdot \bar{Z}_y^{(a,b)}(t) \le 0 ,$$

that is, portfolio rebalancement on the fictitious financial market without transaction costs is cheaper than on the initial market with transaction costs.

The above formal discussion provides an intuitive justification of the following connection between the control problems u and v.

PROPOSITION 6.1. *For all y in $(0,\infty)^n$, we have*

$$v(0, S(0)) \ge u(0, y, F(y)S(0)) .$$

PROOF. Let $(a,b) \in \mathscr{D}$, $w > v(0, S(0))$, $y \in (0,\infty)^n$, set $x := w\mathbf{1}_0$ and consider some portfolio strategy $L \in \mathscr{A}(x)$ such that $X_x^L(T) \succeq g(S(T))$, P-a.s., that is,

$$X_x^L(T) - g(S(T)) \in K, \qquad P\text{-a.s.} \tag{6.9}$$

Set $l_k^{ij}(t) := \int_0^t dL^{ij}(r)/\bar{S}^k(r)$ for all $i, j = 0, \dots, d$, $k \in \{i, j\}$ and $0 \le t \le T$. Here $l_i^{ij}(t)$ (resp. $l_j^{ij}(t)$) is the cumulated transfer from asset i to asset j in terms of number of shares of asset i (resp. j). In terms of l, the wealth process can be written

$$X_x^{i,L}(t) = x^i + \bar{S}^i(t) \sum_{j=0}^d \left(l_i^{ji}(t) - l_i^{ij}(t)(1+\lambda^{ij}) \right), \qquad i = 0, \dots, d,\ 0 \le t \le T.$$

For all $i = 0, \dots, d$, define

$$\phi^i(t) := \sum_{j=0}^d \left(l_i^{ji}(t) - l_i^{ij}(t)(1+\lambda^{ij}) \right) \quad \text{for } 0 \le t \le T,$$

so that

$$X_x^{i,L}(t) = x^i + \bar{S}^i(t)\phi^i(t), \qquad i = 0, \dots, d,\ 0 \le t \le T.$$

Also observe that $\phi^i(0-) = 0$ since $L(0-) = 0$. Notice that

$$\begin{aligned}
\bar{Z}_y^{(a,b)}(t) \cdot d\phi(t) &= \sum_{i,j=0}^d \left(dL^{ji}(t)\bar{f}^i(Y_y^{(a,b)}(t)) - dL^{ji}(t)(1+\lambda^{ji})\bar{f}^j(Y_y^{(a,b)}(t)) \right) \\
&= \sum_{i,j=0}^d dL^{ji}(t) \left(\bar{f}^i(Y_y^{(a,b)}(t)) - (1+\lambda^{ji})\bar{f}^j(Y_y^{(a,b)}(t)) \right) \\
&\le 0 ,
\end{aligned}$$

where we used the fact that $f(Y_y^{(a,b)}(\cdot)) \in \Lambda$, L^{ji} is nondecreasing and the expression of $\{1\} \times \Lambda$ as intersection of the half-hyperplanes H^{ij}. Since ϕ is a bounded variation process and $\phi(0-) = 0$, this proves that

$$\bar{Z}_y^{(a,b)}(t) \cdot \phi(t) \leq \int_0^t \phi(r) \cdot d\bar{Z}_y^{(a,b)}(r) \; = \; W_{w,y}^{(a,b)^\phi}(t) - w.$$

Then

$$\begin{aligned} W_{w,y}^{(a,b)^\phi}(t) &\geq w + \phi(t) \cdot \mathrm{diag}[\bar{f}(Y_y^{(a,b)}(t))]\bar{S}(t) \\ &= w + \sum_{i=0}^{d} \phi^i(t)\bar{f}^i(Y_y^{(a,b)}(t))\bar{S}^i(t) \\ &= X_x^L(t) \cdot \bar{f}(Y_y^{(a,b)}(t)), \qquad P\text{-a.s. for } 0 \leq t \leq T\,. \end{aligned} \tag{6.10}$$

Since $f(Y_y^{(a,b)}(\cdot)) \in \Lambda$, it follows from (6.9) that $\bar{f}(Y_y^{(a,b)}(T)) \cdot (X_x^L(T) - g(S(T))) \geq 0$ P-a.s., and then

$$W_{w,y}^{(a,b)^\phi}(T) \geq \bar{f}(Y_y^{(a,b)}(T)) \cdot g(S(T)), \qquad P\text{-a.s.}$$

Now, we claim that $\phi \in \mathscr{B}^{(a,b)}(w, y)$. Then the last inequality proves that $w \geq u^{a,b}(0, y, F(y)S(0))$ and therefore $v(0, S(0)) \geq u(0, y, F(y)S(0))$ from the arbitrariness of $w > v(0, S(0))$, $y \in (0, \infty)^n$ and $(a, b) \in \mathscr{D}$.

Hence, in order to conclude the proof, it remains to show that $\phi \in \mathscr{B}^{(a,b)}(w, y)$. Using (6.10), the admissibility condition (3.3) and the fact that $f(Y_y^{(a,b)}(\cdot)) \in \Lambda$, we see that

$$W_{w,y}^{(a,b)^\phi}(t) \geq 0, \qquad P\text{-a.s.}, \quad 0 \leq t \leq T\,.$$

Finally, for all $i = 1, \ldots, d$,

$$\begin{aligned} &\int_0^T |\phi^i(t)|^2 d\left\langle Z_y^{i,(a,b)}(t)\right\rangle \\ &\quad = \int_0^T |\phi^i(t)S^i(t)|^2 \sum_{j=1}^{d} \Bigg(\Big\{F(Y_y^{(a,b)}(t))\sigma(t, S(t))\Big\}^{ij} \\ &\qquad\qquad + \Big\{Df(Y_y^{(a,b)}(t))\mathrm{diag}[Y_y^{(a,b)}(t)]a(t)\Big\}^{jj}\Bigg)^2 dt \\ &\quad < +\infty, \quad P\text{-a.s.}, \end{aligned}$$

since ϕ is a bounded variation process and σ, a, $f(y)$ and $Df(y)\mathrm{diag}[y]$ are bounded; see Remark 5.2. □

We conclude this section by the following result which explains the reason for adopting the parameterization of the control process $Y_y^{(a,b)}$ in (6.2).

Consider the stopping time

$$\theta_y^{(a,b)} := \inf\left\{ t > 0 : \sum_{j=1}^{n}\sum_{i=1}^{d}\left(\left|\ln(S^i(t)/S^i(0))\right| + \left|\ln(Y_y^{j,(a,b)}(t)/y^j)\right|\right) \geq \mu \right\}$$

and the exponential

$$M_y^{(a,b)}(t) := \mathscr{E}\left(-\int_0^t \alpha^y(r, S(r), Y_y^{(a,b)}(r), a(r), b(r)) \cdot dB(r)\right), \qquad t \geq 0\,.$$

As defined, function α^y is bounded. Then, the process $\left\{M_y^{(a,b)}(t),\ t \geq 0\right\}$ is well defined and is a martingale. We then introduce the probability measure $Q_y^{(a,b)}$ equivalent to P by

$$\frac{dQ_y^{(a,b)}}{dP} = M_y^{(a,b)}(T)\,.$$

LEMMA 6.1. *The stopped process* $\left\{Z_y^{(a,b)}(t \wedge \theta_y^{(a,b)}),\ t \geq 0\right\}$ *is a* $Q_y^{(a,b)}$-*martingale*.

PROOF. Define the process

$$B_y^{(a,b)}(t) := B(t) + \int_0^t \alpha^y(r, S(r), Y_y^{(a,b)}(r), a(r), b(r))\, dr, \qquad 0 \leq t \leq T.$$

Then by definition of the stopping time $\theta_y^{(a,b)}$ and Girsanov's theorem, we see that the process $\left\{B_y^{(a,b)}(t \wedge \theta_y^{(a,b)}),\ t \geq 0\right\}$ is a Brownian motion under $Q_y^{(a,b)}$. Applying Itô's lemma on the stochastic interval $[0, T \wedge \theta_y^{(a,b)}]$, we get by direct computation,

$$\begin{aligned} dZ_y^{(a,b)}(t) = \operatorname{diag}[Z_y^{(a,b)}(t)]\Big(&\sigma(t, F(Y_y^{(a,b)}(t))^{-1} Z_y^{(a,b)}(t)) \\ &+ F^{-1}(Y_y^{(a,b)}(t)) Df(Y_y^{(a,b)}(t)) \operatorname{diag}[Y_y^{(a,b)}(t)] a(t)\Big)\, dB_y^{(a,b)}(t), \end{aligned}$$

The required result follows from the fact that the diffusion term in the above stochastic differential equation is bounded on the stochastic interval $[0, \theta_y^{(a,b)}]$. □

REMARK 6.1. In the one-dimensional case, Cvitanić, Pham and Touzi (1999) solved the superreplication problem under transaction costs by means of the dual formulation of the problem. The keystone of their analysis was to define processes $Z_y^{(a,b)}$ which are martingales under $Q_y^{(a,b)}$. In our multidimensional framework, this property holds only up to the stopping time $\theta_y^{(a,b)}$. Therefore, we cannot use the dual formulation of the problem [see Kabanov (1999)] as in the one-dimensional case.

REMARK 6.2. Notice that process $Z_y^{(0,0)}$ defined by $Z_y^{(0,0)}(t) = F(y)S(t)$ for all $0 \le t \le T$ is a P-martingale.

7. Dynamic programming. We first extend the definition of the processes $\left(W^{(a,b)^\phi}, Y^{(a,b)}, Z^{(a,b)}\right)$ to the case where the time origin is defined by some t in $[0, T]$. Let $(w, y, z) \in \mathbb{R}_+ \times (0, \infty)^n \times (0, \infty)^d$. We define the process $\left(W_{t,w,y,z}^{(a,b)^\phi}, Y_{t,y,z}^{(a,b)}, Z_{t,y,z}^{(a,b)}\right)$ by the dynamics (6.5), (6.2), (6.3) and the initial condition $\left(W_{t,w,y,z}^{(a,b)^\phi}(t), Y_{t,y,z}^{(a,b)}(t), Z_{t,y,z}^{(a,b)}(t)\right) = (w, y, z)$. We define accordingly the set of admissible controls $\mathscr{B}^{(a,b)}(t, w, y, z)$, the stopping time $\theta_{t,y,z}^{(a,b)}$ and the probability measure $Q_{t,y,z}^{(a,b)}$.

The dynamic stochastic control problems associated with (6.7) and (6.8) are then given by

$$
\begin{aligned}
u^{(a,b)}&(t, y, z)\\
&:= \inf\Big\{w \in \mathbb{R}\colon\ \exists\ \phi \in \mathscr{B}^{(a,b)}(t, w, y, z),\\
&\qquad W_{t,w,y,z}^{(a,b)^\phi}(T) \ge \bar{f}(Y_{t,y,z}^{(a,b)}(T)) \cdot g\Big(F(Y_{t,y,z}^{(a,b)}(T))^{-1} Z_{t,y,z}^{(a,b)}(T)\Big), P\text{-a.s.}\Big\},\\
u(t, y, z) &:= \sup_{(a,b)\in\mathscr{D}} u^{(a,b)}(t, y, z).
\end{aligned}
$$

The following result is adapted from Soner and Touzi (1998, 1999).

PROPOSITION 7.1 (Dynamic programming). *Fix some $(t, y, z) \in [0, T) \times (0, \infty)^{n+d}$ and $(a, b) \in \mathscr{D}$. Then for all scalar $w > u(t, y, z)$, there exists some control ϕ in $\mathscr{B}^{(a,b)}(t, w, y, z)$ such that*

$$
W_{t,w,y,z}^{(a,b)^\phi}(\theta) \ge u\left(\theta, Y_{t,y,z}^{(a,b)}(\theta), Z_{t,y,z}^{(a,b)}(\theta)\right), \quad P\text{-a.s.}
$$

for all $[t, T]$-valued stopping time θ.

PROOF. Fix $w > u(t, y, z)$ and $(a, b) \in \mathscr{D}$. By definition of the control problem $u^{(a,b)}$, there exists $\phi \in \mathscr{B}^{(a,b)}(t, w, y, z)$ such that

$$
W_{t,w,y,z}^{(a,b)^\phi}(T) \ge \bar{f}(Y_{t,y,z}^{(a,b)}(T)) \cdot g\left(F(Y_{t,y,z}^{(a,b)}(T))^{-1} Z_{t,y,z}^{(a,b)}(T)\right), \quad P\text{-a.s.} \tag{7.1}
$$

Fix some $[t, T]$-valued stopping time θ. Since for each $(a, b) \in \mathscr{D}$ the coefficients of (6.2) are (random) Lipschitz, there exists a unique solution to (6.2). Then clearly,

$$
Y_{t,y,z}^{(a,b)}(T) = Y_{\theta, Y_{t,y,z}^{(a,b)}(\theta), Z_{t,y,z}^{(a,b)}(\theta)}^{(a,b)}(T), \quad P\text{-as.}
$$

From the definition of $W^{(a,b)^\phi}_{t,w,y,z}$ and $Z^{(a,b)}_{t,y,z}$, the same property holds for these processes. Then, by direct substitution, we see that ϕ is an admissible super-replicating strategy for the contingent claim, when starting with the initial conditions $\left(\theta, W^{(a,b)^\phi}_{t,w,y,z}(\theta), Y^{(a,b)}_{t,y,z}(\theta), Z^{(a,b)}_{t,y,z}(\theta)\right)$ for P-almost every $\omega \in \Omega$. By definition of the dynamic stochastic control problem u, this proves that

$$W^{(a,b)^\phi}_{t,w,y,z}(\theta) \geq u^{(a,b)}(\theta, Y^{(a,b)}_{t,y,z}(\theta), Z^{(a,b)}_{t,y,z}(\theta)), \quad P\text{-a.s.}$$

Since $\left(W^{(a,b)^\phi}_{t,w,y,z}(\theta), Y^{(a,b)}_{t,y,z}(\theta), Z^{(a,b)}_{t,y,z}(\theta)\right)$ depends on (a, b) only through the stochastic interval $[t, \theta]$, we may take supremum on the right-hand side, and we get the required result from the arbitrariness of w, (a, b) and θ. □

COROLLARY 7.1. *Fix some $(t, y, z) \in [0, T) \times (0, \infty)^{n+d}$ and consider some scalar $w > u(t, y, z)$. Then for all (a, b) in $\mathscr{D}$ and $t \leq r \leq T$, we have*

$$w \geq E^{Q^{(a,b)}_{t,y,z}}\Big[u\Big(r \wedge (t + \theta^{(a,b)}_{t,y,z}), Y^{(a,b)}_{t,y,z}(r \wedge (t + \theta^{(a,b)}_{t,y,z})), Z^{(a,b)}_{t,y,z}(r \wedge (t + \theta^{(a,b)}_{t,y,z}))\Big)\Big].$$

PROOF. By the dynamic programming equation of the previous proposition, for all (a, b) in $\mathscr{D}$, there exists some control ϕ in $\mathscr{B}^{(a,b)}(t, w, y, z)$ such that

$$\begin{aligned} &W^{(a,b)^\phi}_{t,w,y,z}(r \wedge (t + \theta^{(a,b)}_{t,y,z})) \\ &\quad \geq u^{(a,b)}\Big(r \wedge (t + \theta^{(a,b)}_{t,y,z}), Y^{(a,b)}_{t,y,z}(r \wedge (t + \theta^{(a,b)}_{t,y,z})), Z^{(a,b)}_{t,y,z}(r \wedge (t + \theta^{(a,b)}_{t,y,z}))\Big), \end{aligned}$$

P-a.s. for all $t \leq r \leq T$. Then the required result is obtained by taking expectations under $Q^{(a,b)}_{t,y,z}$ and using Lemma 6.1 together with the admissibility conditions (6.4)–(6.6) and Fatou's lemma. □

8. Viscosity property of the auxiliary control problem. We denote by u_* the *lower semicontinuous envelope* of u,

$$u_*(t, y, z) = \liminf_{(t', y', z') \to (t, y, z)} u(t', y', z').$$

We shall use the notation

$$\Gamma^a(t, y, z) := \operatorname{diag}[z]\left(\sigma(t, F(y)^{-1}z) + F(y)^{-1}Df(y)\operatorname{diag}[y]a\right).$$

Then, for all control $(a, b) \in \mathscr{D}$ and $(t, y, z) \in [0, T) \times (0, \infty)^{n+d}$, the dynamics of the process $(Y^{(a,b)}_{t,y,z}, Z^{(a,b)}_{t,y,z})$ on the stochastic interval $[t, t + \theta^{(a,b)}_{t,y,z}]$ is given by

$$dY^{(a,b)}_{t,y,z}(r) = \operatorname{diag}[Y^{(a,b)}_{t,y,z}(r)]\left(b(r)dr + a(r)\,dB^{(a,b)}_{t,y,z}(r)\right), \tag{8.1}$$

$$dZ^{(a,b)}_{t,y,z}(r) = \Gamma^{a(t)}\left(r, Y^{(a,b)}_{t,y,z}(r), Z^{(a,b)}_{t,y,z}(r)\right) dB^{(a,b)}_{t,y,z}(r), \tag{8.2}$$

where $B^{(a,b)}_{t,y,z}$ is a Brownian motion under the equivalent probability measure $Q^{(a,b)}_{t,y,z}$; see the proof of Lemma 6.1.

PROPOSITION 8.1. *Function $u_*(t, y, z)$ is a lower semicontinuous viscosity supersolution of the Hamilton–Jacobi–Bellman equation*,

$$\inf_{(a,b)\in\mathbb{M}^{n,d}\times\mathbb{R}^n} -\mathscr{L}^a\varphi \; - \mathscr{G}^{a,b}\varphi = 0 \quad \text{on } [0,T)\times(0,\infty)^{n+d}, \tag{8.3}$$

where

$$\mathscr{L}^a\varphi(t,y,z) = D_t\varphi(t,y,z) + \tfrac{1}{2}\mathrm{Tr}\left[\Gamma^{a*}D^2_{zz}\varphi\Gamma^a\right](t,y,z)$$

$$\mathscr{G}^{a,b}\varphi(t,y,z) = \mathrm{diag}[y]b\cdot D_y\varphi(t,y,z) + \tfrac{1}{2}\mathrm{Tr}\left[D^2_{yy}\varphi(t,y,z)\mathrm{diag}[y]aa^*\mathrm{diag}[y]\right]$$
$$+\mathrm{Tr}\left[\Gamma^a(t,y,z)a^*\mathrm{diag}[y]D^2_{yz}\varphi(t,y,z)\right] .$$

Moreover, for all (y,z) in $(0,\infty)^{n+d}$, we have

$$u_*(T,y,z) \ge \bar{f}(y)\cdot g\left(F(y)^{-1}z\right). \tag{8.4}$$

PROOF. We first prove (8.4). Let ε be an arbitrary positive scalar and $(t,y,z)\in[0,T)\times(0,\infty)^{n+d}$. Set $w := u(t,y,z)$. By definition of the control problem $u^{(0,0)}$, there exists some control ϕ in $\mathscr{B}^{(0,0)}(t, w+\varepsilon, y, z)$ such that

$$W^{(0,0)^\phi}_{t,w+\varepsilon,y,z}(T) \ge \bar{f}(y)\cdot g\left(F(y)^{-1}Z^{(0,0)}_{t,y,z}(T)\right), \quad P\text{-a.s.}$$

Since $W^{(0,0)^\phi}_{t,w+\varepsilon,y,z}$ is a nonnegative P-local martingale, it is a P-supermartingale (see Remark 6.2), and we get by taking expectations under P and sending ε to zero,

$$u(t,y,z) \ge E\left[\bar{f}(y)\cdot g\left(F(y)^{-1}Z^{(0,0)}_{t,y,z}(T)\right)\right]. \tag{8.5}$$

The required result is obtained by sending t to T and using Fatou's lemma as well as the lower semicontinuity of g.

We now prove (8.3). Fix $(t,y,z)\in[0,T)\times(0,\infty)^{n+d}$ and some control $(a,b)\in\mathscr{D}$ such that the process (a,b) is constant on a neighborhood of t. Let φ be an arbitrary $C^2([0,T)\times(0,\infty)^{n+d})$ function such that

$$0 = (u_* - \varphi)(t,y,z) = \min(u_* - \varphi) .$$

Let $(t_k,\ y_k,\ z_k)_{k\ge1}$ be a sequence in $[0,T)\times(0,\infty)^{n+d}$ satisfying

$$(t_k,y_k,z_k)\to(t,y,z) \quad \text{and} \quad u(t_k,y_k,z_k)\to u_*(t,y,z) \quad \text{as } k\to+\infty.$$

Set $w_k := u(t_k,y_k,z_k) + 1/k$ and $\beta_k := w_k - \varphi(t_k,y_k,z_k)$ and observe that

$$\beta_k \to 0 \quad \text{as } k\to+\infty .$$

For ease of notation, we set $\theta_k := \theta^{(a,b)}_{t_k,y_k,z_k}$ and $Q_k := Q^{(a,b)}_{t_k,y_k,z_k}$. We introduce the stopping time

$$h_k := \theta_k \wedge \left(\sqrt{\beta_k} + h1_{\{\beta_k\neq0\}}\right) \quad \text{for some } h>0 .$$

Observe that $\theta^{(a,b)}_{t,y,z} > 0$ P-a.s. and 0,

$$\liminf_{k\to+\infty} \theta_k \geq \tfrac{1}{2}\theta^{(a,b)}_{t,y,z} > 0 \tag{8.6}$$

This follows from the fact that $(Y^{(a,b)}_{t_k,y_k,z_k}, Z^{(a,b)}_{t_k,y_k,z_k}) \to (Y^{(a,b)}_{t,y,z}, Z^{(a,b)}_{t,y,z})$ for P-a.e. $\omega \in \Omega$, uniformly on compact subsets [see Protter (1990), Theorem 37, page 246]. From Corollary 7.1, it follows that

$$w_k \geq E^{Q_k}\left[u\left(t_k+h_k, Y^{(a,b)}_{t_k,y_k,z_k}(t_k+h_k), Z^{(a,b)}_{t_k,y_k,z_k}(t_k+h_k)\right)\right] .$$

Since $u \geq u_* \geq \varphi$, we may replace u by φ in the previous inequality and we get by Itô's lemma,

$$\beta_k - E^{Q_k}\left[\int_{t_k}^{t_k+h_k}\left(\mathscr{L}^a\varphi + \mathscr{G}^{a,b}\varphi\right)\left(r, Y^{(a,b)}_{t_k,y_k,z_k}(r), Z^{(a,b)}_{t_k,y_k,z_k}(r)\right)\, dr\right] \geq 0. \tag{8.7}$$

We now consider two cases.

First case. Suppose that the set $\{k \geq 1: \ \beta_k = 0\}$ is finite. Then there exists a subsequence renamed $(\beta_k)_{k\geq 1}$ such that $\beta_k \neq 0$ for all $k \geq 1$. Dividing by $\sqrt{\beta_k}$ and sending k to infinity, we get by dominated convergence and the right continuity of the filtration,

$$\liminf_{k\to+\infty} -\frac{1}{\sqrt{\beta_k}}\int_{t_k}^{t_k+h_k}(\mathscr{L}^a\varphi + \mathscr{G}^{a,b})\varphi(r, Y^{(a,b)}_{t_k,s_k,y_k}(r), Z^{(a,b)}_{t_k,y_k,z_k}(r))\, dr \geq 0 .$$

The required result is a direct consequence of (8.6) and the following lemma whose proof will be carried out later.

LEMMA 8.1. *Let* $\psi\colon [0,T)\times(0,\infty)^{n+d} \to \mathbb{R}$ *be locally Lipschitz in* (t,y,z) *then*

$$\frac{1}{\sqrt{\beta_k}}\int_{t_k}^{t_k+h_k}\left[\psi(r, Y^{(a,b)}_{t_k,y_k,z_k}(r), Z^{(a,b)}_{t_k,y_k,z_k}(r)) - \psi(t,y,z)\right]\, dr \to 0$$

$$\text{as } k \to +\infty, \ P\text{-a.s.}$$

along some subsequence.

Second case. If the set $\{k \geq 1: \ \beta_k = 0\}$ is not finite, then there exists a subsequence renamed $(\beta_k)_{k\geq 1}$ such that $\beta_k = 0$ for all $k \geq 1$. Then, we follow the same line of arguments as in the first case, by dividing (8.7) by h and sending h to zero.

PROOF OF LEMMA 8.1. Since $\psi(t, y, z)$ is locally Lipschitz in (t, y, z), we have

$$\begin{aligned}
&\left|\frac{1}{\sqrt{\beta_k}}\int_{t_k}^{t_k+h_k}\Big[\psi(r, Y^{(a,b)}_{t_k, y_k, z_k}(r), Z^{(a,b)}_{t_k, y_k, z_k}(r)) - \psi(t, y, z)\Big]\, dr\right| \\
&\quad \le C\,\frac{1}{\sqrt{\beta_k}}\int_{t_k}^{t_k+h_k}\Big(|r-t| + \|Z^{(a,b)}_{t_k, y_k, z_k}(r) - z\| + \|Y^{(a,b)}_{t_k, y_k, z_k}(r) - y\|\Big)\, dr \\
&\quad \le C\,\frac{h_k}{\sqrt{\beta_k}}\Bigg(h_k + |t_k - t| + \sup_{t_k\le r\le t_k+h_k}||Z^{(a,b)}_{t_k, y_k, z_k}(r) - z|| \\
&\qquad\qquad + \sup_{t_k\le r\le t_k+h_k}||Y^{(a,b)}_{t_k, y_k, z_k}(r) - y||\Bigg),
\end{aligned}$$

for some constant C. In order to obtain the required result, we shall prove that

$$\sup_{t_k\le r\le t_k+h_k}\|Z^{(a,b)}_{t_k, y_k, z_k}(r) - z\| \longrightarrow 0$$

and

$$\sup_{t_k\le r\le t_k+h_k}\|Y^{(a,b)}_{t_k, y_k, z_k}(r) - y\| \longrightarrow 0, \quad P\text{-a.s.}$$

as $k \to \infty$. We only report the proof of the second convergence result. The first one is obtained by the same line of argument. Since b, α^{y_k}, a and $Y^{(a,b)}_{t_k, y_k, z_k}$ are bounded on $[t_k, t_k + h_k]$,

$$\|Y^{(a,b)}_{t_k, y_k, z_k}(r) - y\| \le \|y_k - y\| + h_k C' + \left\|\int_{t_k}^{s}\tilde{a}(\tau, Y^{(a,b)}_{t_k, y_k, z_k}(\tau))\, dB(\tau)\right\|$$

for some constant C', where we denoted $\tilde{a}(t, y) = \mathrm{diag}[y]a(t)$. Therefore,

$$\begin{aligned}
\sup_{t_k\le r\le t_k+h_k}\|Y^{(a,b)}_{t_k, y_k, z_k}(r) - y\| &\le \|y_k - y\| + h_k C' \\
&\quad + \sup_{t_k\le r\le t_k+h_k}\left\|\int_{t_k}^{r}\tilde{a}(\tau, Y^{(a,b)}_{t_k, y_k, z_k}(\tau))\, dB(\tau)\right\|.
\end{aligned}$$

The first two terms on the right-hand side converge to zero. As for the third term, it follows from Doob's maximal inequality for submartingales that

$$\begin{aligned}
&E\left[\left(\sup_{t_k\le r\le t_k+h_k}\left\|\int_{t_k}^{r}\tilde{a}(\tau, Y^{(a,b)}_{t_k,y_k,z_k}(\tau))\, dB(\tau)\right\|\right)^2\right] \\
&\qquad \le 4E\left[\int_{t_k}^{t_k+h_k}\left\|\tilde{a}\tilde{a}^*(\tau, Y^{(a,b)}_{t_k, y_k, z_k}(\tau))\right\|\, d\tau\right].
\end{aligned}$$

Since a is bounded and $Y^{(a,b)}_{t_k, y_k, z_k}$ is bounded on $[t_k, t_k + h_k]$, uniformly in k, this proves that

$$\sup_{t_k\le r\le t_k+h_k}\left\|Y^{(a,b)}_{t_k, y_k, z_k}(r) - y\right\| \longrightarrow 0 \quad \text{as } k \to \infty \text{ in } L^2(P),$$

and therefore P-a.s. along some subsequence. $\square$

REMARK 8.1. In the previous proof, we established inequality (8.5). Since $f(\cdot)$ is valued in Λ, (3.4) implies in particular that

$$u(t, y, z) \geq 0 \quad \text{for all} \ \ (t, y, z) \in [0, T) \times (0, \infty)^{n+d} .$$

LEMMA 8.2. (i) Function $u_*(t, y, z)$ is independent of y.
(ii) Under Assumption 4.1, function $u_*(t, y, z)$ is nonincreasing in t and concave in z.

PROOF. The y-independence of function $u_*(t, y, z)$ is proved by sending b_i to $\pm\infty$, $1 \leq i \leq n$, in (8.3) and using Lemmas 5.3 and 5.4 of Cvitanić, Pham and Touzi (1999). We now prove that u_* is nonincreasing in t. Let $\hat{a}$ be any solution of

$$Df(y)\mathrm{diag}[y]\hat{a} = F(y)\sigma(t, F(y)^{-1}z) .$$

Notice that $\hat{a}$ is well defined by Lemma 5.2. Then $\Gamma^{\hat{a}}(t, y, z) = 0$. Since u_* is independent of its y variable, (8.3) shows that u_* is a viscosity supersolution of the equation $-\varphi_t = 0$. Then it follows from Lemma 5.3 in Cvitanić, Pham and Touzi (1999) that u_* is nonincreasing in t.

It remains to prove the concavity of u_* in the z variable. Let (n, ξ) be an arbitrary element of $\mathbb{N} \times \mathbb{R}^d$ and define $\check{a}$ as a solution of

$$\frac{1}{n} Df(y)\mathrm{diag}[y]\check{a} = F(y) \left(\mathrm{diag}[z]^{-1}\xi \mid \mathbf{0} \right) - F(y)\sigma(t, F(y)^{-1}z) ,$$

where $\mathbf{0}$ is the zero matrix of $\mathbb{M}^{d,\,d-1}$. Notice that $\check{a}$ is well defined by Lemma 5.2. Then it is easily checked that

$$\mathrm{Tr}\left[\Gamma^{\check{a}}(t, y, z)^* D^2_{zz}\varphi(t, y, z)\Gamma^{\check{a}}(t, y, z)\right] = n^2\xi^* D^2_{zz}\varphi(t, y, z)\xi ,$$

and therefore u_* is a viscosity supersolution of the equation $-\xi^* D^2_{zz}\varphi\xi = 0$ for all $\xi \in \mathbb{R}^d$. Now, let z_1 and z_2 be two arbitrary elements of $(0, \infty)^d$. Then the function

$$\psi(r) := u_*(t, r(z_1 - z_2) + z_2), \qquad r \in [0, 1],$$

is a viscosity supersolution of $-\varphi'' = 0$. This is easily seen by an appropriate change of basis of $\mathbb{R}^d$ and Lemma 5.3 of Cvitanić, Pham and Touzi (1999). Moreover, ψ is bounded from below by Remark 8.1. Then, by the same argument as in Proposition 5.2 of Cvitanić, Pham and Touzi (1999), it follows that ψ is concave and therefore

$$\begin{aligned} u_*(t, rz_1 + (1-r)z_2) &= \psi(r) \\ &\geq r\psi(1) + (1-r)\psi(0) \\ &= ru_*(t, z_1) + (1-r)u_*(t, z_2) \end{aligned}$$

for all $r \in [0, 1]$ and $t \in [0, T)$. The required result follows from the arbitrariness of z_1 and z_2 in $(0, \infty)^d$. $\square$

9. Proof of the main theorem. We first prove that $v(0, S(0)) \geq \hat{g}(S(0))$. By Lemma 8.2, u_* is nonincreasing on $[0, T)$ and independent of y. Then

$$u_*(0, y, z) = u_*(0, y', z) \geq u_*(t, y', z) \quad \text{for all } (t, y, y', z) \in [0, T) \times (0, \infty)^{2n+d}.$$

Since u_* is lower semicontinuous, we get from the terminal condition (8.4) of Proposition 8.1,

$$u_*(0, y, z) \geq \bar{f}(y') \cdot g(F(y')^{-1} z) \quad \text{for all } (y, y', z) \in (0, \infty)^{2n+d}.$$

Taking supremum over y', this provides $u_*(0, y, z) \geq G(z)$. Again by Lemma 8.2, u_* is concave in z, and therefore,

$$u_*(0, y, z) \geq G^{\text{conc}}(z) \quad \text{for all } (y, z) \in (0, \infty)^{n+d}.$$

Now from Proposition 6.1 and the fact that $u \geq u_*$, we see that

$$\begin{aligned} v(0, S(0)) &\geq \sup_{y \in (0,\infty)^n} u(0, y, F(y)S(0)) \\ &\geq \sup_{y \in (0,\infty)^n} u_*(0, y, F(y)S(0)) \\ &\geq \sup_{y \in (0,\infty)^n} G^{\text{conc}}(F(y)S(0)) \;=\; \hat{g}(S(0)) \,. \end{aligned}$$

It remains to prove the converse inequality. If $\hat{g}(s) = +\infty$, then the result follows from the previous inequality. Next suppose that $\hat{g}(s) < +\infty$. From Theorem 4.2, there exists some $\Delta \in \mathbb{R}^{d+1}$ such that

$$\hat{g}(S(0))\mathbf{1}_0 \;\succeq\; \text{diag}[\bar{S}(0)]\Delta \text{ and } \text{diag}[\bar{S}(T)]\Delta \;\succeq\; g(S(T)), \quad P\text{-a.s.} \tag{9.1}$$

From the left-hand side inequality, we see that there exists a matrix $a \in M_+^{d+1}$ such that for all $i = 0, \ldots, d$,

$$\left(\hat{g}(S(0))\mathbf{1}_0 - \text{diag}[\bar{S}(0)]\Delta\right)^i + \sum_{j=0}^{d} \left(a^{ji} - (1 + \lambda^{ij})a^{ij}\right) \geq 0.$$

Now define the trading strategy,

$$L(t) \;=\; L(0) := a, \qquad 0 \leq t \leq T \,.$$

Then it is easily checked that $L \in \mathscr{A}(\hat{g}(S(0))\mathbf{1}_0)$. From the right-hand side inequality of (9.1), it follows that $X^L_{\hat{g}(S(0))\mathbf{1}_0}(T) \succeq g(S(T))$. The required result then follows from the definition of the superreplication problem $v(0, S(0))$. $\square$

10. Examples.

10.1. *On the generating family of K'.* The solution of the superreplication problem is given in terms of a variational problem involving the normalized polar cone Λ. Hence, in order to compute explicitly the value function v, we need to characterize explicitly the generating family of the polyhedral cone K'.

We first provide a subfamily of the generating family $\{e_1, \dots, e_n\}$ of the polar cone K'. Consider the $\mathbb{R}^{d+1}$ vectors,

$$z_i := (1+\lambda^{0i})\left((1+\lambda^{0i})^{-1}, \dots, (1+\lambda^{di})^{-1}\right), \qquad i = 0, \dots, d,$$

$$z'_i := (1+\lambda^{i0})^{-1}\left((1+\lambda^{i0}), \dots, (1+\lambda^{id})\right), \qquad i = 0, \dots, d.$$

Then it is easily checked that $z_i,\ z'_i \in K'$ for all $i = 0, \dots, d$. Also, we have

$$z_i,\ z'_i\ \in \partial(K') \quad \text{for all } i = 0, \dots, d.$$

To see this, suppose that $z_i \in \operatorname{Int}(K')$ for some i. Then, the vector $\hat{z}_i := z_i + \varepsilon \mathbf{1}_i \in K'$ for some $\varepsilon > 0$. We end up with a contradiction by writing that $\hat{z}_i \in H^{0i}$. By the same argument, we get the result for the vectors z'_i.

Now observe that $z_i \in \partial H^{0i} \cap \left(\cap_{j\neq i} \partial H^{ji}\right)$ and $z'_i \in \partial H^{i0} \cap \left(\cap_{j\neq i} \partial H^{ij}\right)$, for all $i = 0, \dots, d$. Then, from the above discussion, it follows that the generating family $\{e_1, \dots, e_n\}$ can be constructed by completing the family $\{z_i, z'_i,\ i = 0, \dots, d\}$.

Consider the following (natural) conditions on the transaction costs matrix λ:

$$\lambda^{ii} = 0 \quad \text{for all } i = 0, \dots, d, \qquad \lambda^{ij} > 0 \quad \text{for all } i \neq j = 0, \dots, d \tag{10.1}$$

and

$$(1+\lambda^{ik})(1+\lambda^{kj}) > (1+\lambda^{ij}) \quad \text{for all } i, j, k = 0, \dots, d \text{ with } i, j \neq k\,. \tag{10.2}$$

Condition 10.2 is used to provide a complete characterization of the set of generators in the two-dimensional case; see Example 10.2.

EXAMPLE 10.1. In the one-dimensional case $d = 1$, $z_o = z'_1 = (1, (1+\lambda^{10})^{-1})$ and $z'_o = z_1 = (1, (1+\lambda^{01}))$. These are exactly the generating vectors of Example 5.1.

EXAMPLE 10.2. We consider here the two-dimensional case $d = 2$. In order to obtain a generating family of the polar cone K' by completing the family $\{z_i, z'_i, i = 0, \dots, 2\}$, we proceed as follows. Consider all vectors, with unit first component, defined by the intersection of hyperplanes ∂H^{ij} and ∂H^{kl}. This provides all candidates for the required generating vectors. Such a candidate is effectively a generating vector if and only if it lies in K'. By tedious calculation, it is easily checked that condition 10.2 rules out all such candidates except the vectors z_i and z'_i for $i = 0, \dots, 2$. Hence $\{z_i, z'_i,\ i = 0, \dots, 2\}$ is a generating family of K'.

In the general case, one can proceed as in Example 10.2: define the candidate generating vectors as intersections of d hyperplanes $\partial H^{i,j}$, then check whether such vectors lie in the polar cone K'. In contrast with the two-dimensional case, condition 10.2 does not allow characterizing those candidates which are effectively in K', and we are unable to provide explicitly a generating family for the polar cone K'.

However, notice that the characterization of $\hat{g}$ as the cost of the cheapest buy-and-hold strategy in Theorem 4.2 can also be used for the explicit computation of the value function of the superreplication problem; see Section 10.4.

10.2. *Call and put options.* Let $\kappa_1 \geq 0$ and $\kappa_2 \geq 0$ be two arbitrary constants and consider the real payoff function,

$$\begin{aligned} g^0(s^1, s^2) &= -\kappa_1 \mathbf{1}_{s^1 \geq \kappa_1} + \kappa_2 \mathbf{1}_{s^2 < \kappa_2}, \\ g^1(s^1, s^2) &= +s^1 \mathbf{1}_{s^1 > \kappa_1}, \\ g^2(s^1, s^2) &= -s^2 \mathbf{1}_{s^2 \leq \kappa_2} . \end{aligned}$$

Then it is easily checked that

$$v\left(0, S^1(0), S^2(0)\right) = (1 + \lambda^{01}) S^1(0) + \kappa_2.$$

Notice that there is no compensation between the two options : the value function is equal to the sum of the superreplication costs of each option. The superreplicating strategy consists in buying one unit of stock 1 and keeping in cash κ_2.

10.3. *Spread option.* Let $\kappa \geq 0$ be an arbitrary constant and consider the payoff function

$$\begin{aligned} g^0(s^1, s^2) &= -\kappa \mathbf{1}_{s^1 - s^2 \geq \kappa}, \\ g^1(s^1, s^2) &= +s^1 \mathbf{1}_{s^1 - s^2 > \kappa}, \\ g^2(s^1, s^2) &= -s^2 \mathbf{1}_{s^1 - s^2 \geq \kappa} . \end{aligned}$$

Notice that we have defined g^2 in order to insure lower-semicontinuity of the payoff function and that only the sign of the inequality matters. Then, it is easily checked that

$$v\left(0, S^1(0), S^2(0)\right) = (1 + \lambda^{01}) S^1(0).$$

Hence, the cheapest superreplicating strategy consists in buying $(1+\lambda^{10})$ units of stock S^1.

10.4. *Index call option.* Let (κ, d) be an arbitrary constant of $\mathbb{R}_+ \times \mathbb{N}\backslash\{0\}$, and consider the stock index $I(\cdot)$ defined as

$$I(t) = \sum_{k=1}^{d} \alpha^k S^k(t) \quad \text{for all } t \in [0, T],$$

$$\sum_{k=1}^{d} \alpha^k = 1, \; \alpha^k > 0 \quad \text{for all } k = 1, \dots, d,$$

and the European call option with pay-off function $[I(T) - \kappa]^+$. Then, the minimal superreplication cost is given by

$$v\,(0, S(0)) = \sum_{k=1}^{d} \alpha^k (1 + \lambda^{0k})(1 + \lambda^{k0}) S^k(0),$$

which is the cost of replication of the index.

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