The Annals of Applied Probability
2000, Vol. 10, No. 2, 592–615

ASYMPTOTIC EXPANSIONS FOR A STOCHASTIC MODEL OF QUEUE STORAGE[1]

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We consider an $M/M/\infty$ queue with servers ranked as $\{1, 2, 3, \ldots\}$. The Poisson arrival stream has rate λ and each server works at rate μ. A new arrival takes the lowest ranked available server. We let S be the set of occupied servers and $|S|$ is the number of elements of S. We study the distribution of $\max(S)$ in the asymptotic limit of $\rho = \lambda/\mu \to \infty$. Setting $P(m) = \Pr[\max(S) > m]$ we find that the asymptotic structure of the problem is different according as $m = O(1)$ or $m \to \infty$, at the same rate as ρ. For the latter it is furthermore necessary to distinguish the cases $m/\rho < 1$, $m/\rho \approx 1$ and $m/\rho > 1$. We also estimate the average amount of wasted storage space, which is defined by $E(\max(S)) - \rho$. This is the average number of idle servers that are ranked below the maximum occupied one. We also relate our results to those obtained by probabilistic approaches. Numerical studies demonstrate the accuracy of the asymptotic results.

1. Introduction. We consider the following model of queue storage. There is an $M/M/\infty$ queue with a Poisson arrival rate $= \lambda$ and each of the identical servers works at rate $= \mu$. We then define $\rho = \lambda/\mu$. We can clearly scale time so that $\mu = 1$ and then $\rho = \lambda$. The servers are ranked $\{1, 2, 3, \ldots\}$. Let S be the set of occupied servers. For example, if at a particular instant of time servers 1,4,7 and 11 occupied, then $S = \{1, 4, 7, 11\}$. We also denote by $|S|$ the number of elements, that is, cardinality, in the set. For the example we have $|S| = 4$. Furthermore, an arriving customer takes the lowest ranked available server. We denote by $\max(S)$ the highest ranked occupied server and its (steady-state) probability distribution by

$$P(m) = \Pr\big[\max(S) > m\big]; \quad m = 0, 1, 2, 3, \ldots \, . \tag{1.1}$$

We also set $p(m) = P(m-1) - P(m) = \Pr\big[\max(S) = m\big]$ for $m \geq 1$ with $p(0) = 1 - P(0)$.

The model with ranked servers has many applications. For example, the arrivals may represent requests for memory space in a computer system, with each server representing, say, a page of memory. The servers may also represent numbered parking spaces near a restaurant, with the lower numbered spaces being closest to the restaurant, so an arriving car will naturally choose the closest parking space.

This model is formulated and studied by Kosten [1], Coffman, Kadota and Shepp [2], Newell [3] and Coffman et. al. [4]. To compute the distribution of

Received March 1999; revised September 1999.
[1]Supported in part by NSF Grant DMS-99-71656.
AMS 1991 *subject classifications*. 60K30, 34E20.
Key words and phrases. Asymptotics, queue storage, first-fit allocation.

$\max(S)$ we first break up the servers into two groups, with the "primary" servers being those ranked $\{1, 2, \dots, m\}$ and the "secondary" servers being ranked $\{m+1, m+2, m+3, \dots\}$. The joint steady state distribution of N_1 and N_2, the number of occupied primary and secondary servers, will be denoted

$$\pi(k, r) = \pi(k, r; m, \rho) = \Pr[N_1 = k, N_2 = r]; \quad 0 \le k \le m, r \ge 0. \tag{1.2}$$

In [2] it is shown that $\pi(k, r)$ satisfies a two-dimensional difference equation in the lattice strip $\{0 \le k \le m, r \ge 0\}$. This is explicitly solved using generating functions and, in particular, the authors obtain the distribution of $\max(S)$, which is related to $\pi(k, r)$ in (1.2) via

$$\Pr\big[\max(S) > m\big] = 1 - \Pr\left[\max(S) \le m\right] = 1 - \sum_{k=0}^{m} \pi(k, 0).$$

The main result in [2] is the expression

$$P(m) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} \rho^{n+1} \left[\sum_{k=0}^{m} \binom{m}{k} \frac{(n+k)!}{\rho^k} \right]^{-1}. \tag{1.3}$$

The purpose of this paper is to study the distribution $P(m) = P(m; \rho)$ for $\rho \to \infty$ and various ranges of m. We shall show that the asymptotic expansion of $P(m)$ is different for the four cases (1) $m = O(1)$, (2) $m \to \infty$ with $0 < m/\rho < 1$, (3) $m \to \infty$ with $m - \rho = O(\sqrt{\rho})$ and (4) $m \to \infty$ with $m/\rho > 1$. The asymptotic structure of $P(m)$ is not immediately clear from the representation (1.3), which is partly due to the presence of an alternating sum. Throughout the paper we shall obtain several alternate representations of $P(m)$, which prove more useful for the asymptotic analysis.

Of particular interest for this storage model is to estimate the amount of "wasted space." Letting N be the total number of occupied servers, the amount of wasted space is defined as the difference between the maximum ranked occupied server and the total number of occupied servers, that is, as $\max(S) - N$. In view of the $M/M/\infty$ model, $N(= N_1 + N_2)$ follows a Poisson distribution with parameter ρ, and hence

$$E\big(\max(S)\big) - \rho = \sum_{m=0}^{\infty} P(m) - \rho = \text{average wasted space}. \tag{1.4}$$

Coffman and Leighton [5] gave a simple probabilistic proof that the mean wasted space is $\Theta\big(\sqrt{\rho \log\log \rho}\big)$ [i.e., there exist constants c_1 and c_2 such that $c_1\sqrt{\rho \log\log \rho} < E\big(\max(S)\big) - \rho < c_2\sqrt{\rho \log\log \rho}$ for $\rho \to \infty$]. Aldous [6] used more refined probabilistic arguments to show that

$$E\big(\max(S)\big) - \rho \sim \sqrt{2\rho \log\log \rho}, \quad \rho \to \infty. \tag{1.5}$$

Here we will show that (1.5) can easily be obtained directly from the sum in (1.3). We will also give a refinement of (1.5) [cf. (2.6)].

Newell [3] gives a variety of asymptotic results for the joint distribution $\pi(k, r)$, for $\rho \to \infty$. While here we only study the distribution of $\max(S)$, we believe that the methods used here can also be used to obtain asymptotic results for $\pi(k, r)$. From the point of view of computer memory allocation, a more realistic queueing model is to have a single processor-sharing server rather than an infinite number of servers. This would better reflect the system's finite resources. Such a processor-sharing model is discussed in [4], where some approximations (involving lower and upper bounds) are analyzed. A good summary of estimates of wasted space for these types of models can be found in [7].

In this paper we use mostly classical methods of applied mathematics, such as Watson transformations and methods for asymptotically evaluating integrals [8], such as the Laplace method, the method of steepest descent, and special techniques that treat coalescing saddle points and saddles coalescing with algebraic singularities of integrands. Our final answers are quite simple; they involve elementary functions and well-studied special functions such as parabolic cylinder functions, Airy functions and Hermite polynomials [9].

The paper is organized as follows. In Section 2 we summarize the main results. They are derived in Sections 3–5 and in Section 6 we include numerical studies and comparisons.

2. Summary of results. Below we assume that $\rho \to \infty$ and give our main results, which are the asymptotic expansions of $P(m) = \Pr[\max(S) > m]$ or $1 - P(m) = \text{Prob}[\max(S) \le m]$ for various ranges of m.

THEOREM 1. *For $\rho \to \infty$, the distribution of $\max(S)$ has the following asymptotic behaviors, listed in increasing size of m/ρ.*

(a) $\rho \to \infty$, $m = O(1)$,

$$1 - P(m) \sim \exp\{-\rho\}\rho^{m/2} \frac{1}{\sqrt{2\pi i}} \int_{\text{Br}} \frac{\exp\{\xi^2/2\}}{He_m(\xi)} d\xi,$$

where $He_m(\cdot)$ is the mth Hermite polynomial and the contour Br *is a vertical contour in the ξ-plane, which lies to the right of all the zeros of $He_m(\xi)$.*

(b) *$m, \rho \to \infty$ with $m/\rho = X \in (0, 1)$,*

$$1 - P(m) \sim \rho^{-1/3} K(X) \exp\{\rho\Phi(X)\} \exp\{\rho^{1/3}\Phi_1(X)\},$$

$$\Phi(X) = -1 + \sqrt{X} - \frac{1}{2} X \log X + \left(1 - \sqrt{X}\right)^2 \log\left(1 - \sqrt{X}\right),$$

$$\Phi_1(X) = -r_0 X^{-1/6} \left(1 - \sqrt{X}\right)^{2/3} \log\left(1 - \sqrt{X}\right),$$

$$K(X) = \left[Ai'(r_0)\right]^{-1} X^{-1/3} \left(1 - \sqrt{X}\right)^{-2/3} \exp\left[-\frac{1}{2\sqrt{X}} \log\left(1 - \sqrt{X}\right)\right].$$

Here $r_0 = \max\{z : Ai(z) = 0\} = -2.33810741\ldots$ is the largest (negative) root of the Airy function $Ai(\cdot)$.

(c) *$m, \rho \to \infty$ with $m = \rho + \beta\sqrt{\rho}$ and β fixed,*

$$1 - P(m) \sim \rho^{-z_0/2} \exp\{-\beta^2/4\}\Gamma(z_0)/\Delta(\beta)$$

where $z_0 = z_0(\beta) = \min\{z > 0 : D_z(-\beta) = 0\}$ *is the smallest positive root of the parabolic cylinder function* $D_z(-\beta)$ (*viewed as a function of the index*) *and*

$$\Delta(\beta) = -\frac{d}{dz} D_z(-\beta)\bigg|_{z=z_0(\beta)}.$$

(d) $m, \rho \to \infty$ *with* $m/\rho = X \in (1, \infty)$

$$P(m) \sim \sqrt{\rho} L(X) e^{\rho \Psi(X)},$$
$$\Psi(X) = -1 + X - X \log X,$$
$$L(X) = \frac{1}{\sqrt{2\pi}} \frac{X-1}{\sqrt{X}} \log\left(\frac{X}{X-1}\right).$$

We observe that for cases (a)-(c), $P(m)$ is close to one, while in case (d) it is exponentially small in ρ, as $\Psi(X) < 0$ for $X > 1$. In cases (a) and (b) $1 - P(m)$ is exponentially small but for a fixed β [i.e., case (c)], this quantity is only algebraically small, due to the factor $\rho^{-z_0/2}$.

We can gain further insight into the structure of the distribution by giving results that apply on scales intermediate to those in Theorem 1. For example we refer to case (ab) as the asymptotic matching region between cases (a) and (b). The corresponding formula can be obtained by expanding case (a) for $m \to \infty$ or case (b) for $X = m/\rho \to 0^+$. Below we give the three "intermediate" results.

(ab) $\rho \to \infty$, $m \to \infty$, $m/\rho \to 0$,

$$1 - P(m) \sim \exp\{-\rho\}\rho^{m/2} \exp\{3m/2\} m^{-m/2} m^{-1/3} \big[Ai'(r_0)\big]^{-1} \exp\{r_0 m^{1/3}\}\sqrt{e}.$$

(bc) $m, \rho \to \infty$ with $m/\rho \to 1^-$ and $\beta = (m - \rho)/\sqrt{\rho} \to -\infty$,

$$1 - P(m) \sim \left(\frac{2}{-\beta}\right)^{7/6} \big[Ai'(r_0)\big]^{-1} \exp\left[-\frac{3}{8}\beta^2 + \frac{1}{4}\beta^2 \log\left(\frac{-\beta}{2\sqrt{\rho}}\right)\right]$$
$$\times \rho^{1/4} \exp\left[-r_0 \left(-\frac{\beta}{2}\right)^{2/3} \log\left(-\frac{\beta}{2\sqrt{\rho}}\right)\right].$$

(cd) $m, \rho \to \infty$ with $m/\rho \to 1^+$ and $\beta = (m - \rho)/\sqrt{\rho} \to +\infty$,

$$P(m) \sim \frac{\beta}{\sqrt{2\pi}} \exp\{-\beta^2/2\} \log\left(\frac{\sqrt{\rho}}{\beta}\right).$$

Asymptotically, for $\rho \to \infty$, most of the probability mass is concentrated in that range of m where $P(m)$ undergoes the transition from being approximately equal to one, to being asymptotically small. Our results show that this occurs precisely in the matching region (cd). In fact, the result in (cd) may be recast as the following limit law. For $\rho > 1$ let $\beta_0(\rho)$ be the positive solution to the equation

$$\tfrac{1}{2}\beta_0^2 - \log \beta_0 + \log\log \beta_0 = \log\log \rho - \log\left(2\sqrt{2\pi}\right). \tag{2.1}$$

Then for

$$\beta = \beta_0(\rho) + \nu/\beta_0(\rho) \tag{2.2}$$

with $\rho \to \infty$ and a fixed $\nu > 0$ we have

$$P(m) \longrightarrow e^{-\nu}. \tag{2.3}$$

From (2.1) we clearly have

$$\beta_0(\rho) \sim \sqrt{2\log\log\rho} \quad \text{as } \rho \to \infty \tag{2.4}$$

and this may be easily refined to the estimate

$$\beta_0(\rho) = A + \frac{\log A}{A} - \frac{\log\log A}{A} - \frac{\log(2\sqrt{2\pi}\,)}{A} + o(A^{-1}) \tag{2.5}$$

where $A \equiv \sqrt{2\log\log\rho}$. The scaling (2.2) also shows that for $\rho \to \infty$ most of the mass is concentrated in the range $m = \rho + \beta_0\sqrt{\rho} + o(\sqrt{\rho}\,)$. Also, our analysis leads to the following leading order approximation for the amount of wasted storage space:

$$E\big(\max(S)\big) - \rho \approx \sqrt{\rho}\beta_0(\rho). \tag{2.6}$$

We also note that for small values of m, the result in part (a) of Theorem 1 can be made more explicit. We have $1 - P(0) \sim e^{-\rho}$, which is not only asymptotic but also exact. Furthermore, by contour integration we obtain

$$1 - P(1) \sim \sqrt{\frac{\pi}{2}}\sqrt{\rho}\,\exp\{-\rho\}, \tag{2.7}$$

$$1 - P(2) \sim \rho\exp\{-\rho\}\int_0^\infty \sinh(t)\exp\{-t^2/2\}dt$$

$$= \rho\exp\{-\rho\}\sqrt{e}\left[\sqrt{\frac{\pi}{2}} - \int_1^\infty \exp\{-t^2/2\}dt\right] \tag{2.8}$$

and

$$1 - P(3) \sim \rho^{3/2}\exp\{-\rho\}\frac{\sqrt{2\pi}}{6}(\exp\{3/2\} - 1). \tag{2.9}$$

Apparently the case of m odd leads to somewhat simpler results than that of m even, and for general $m = 2N+1$, $N \geq 1$ we obtain

$$1 - P(2N+1) \sim \rho^{N+1/2}\exp\{-\rho\}\sqrt{2\pi}\left[\frac{1}{2}\frac{1}{He'_{2N+1}(0)} + \sum_{\ell=1}^{N}\frac{\exp\{\xi_\ell^2/2\}}{He'_{2N+1}(\xi_\ell)}\right] \tag{2.10}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_N$ are the positive roots of $He_m(\xi) = 0$.

Finally, we note that the root $z_0(\beta)$ of $D_z(-\beta) = 0$ that appears in Theorem 1(c) satisfies $z_0(\beta) > 0$ with

$$z_0(\beta) \sim \frac{\beta^2}{4} - \left(\frac{-\beta}{2}\right)^{2/3} r_0 - \frac{1}{2}, \quad \beta \to -\infty,$$

$$z_0(0) = 1,$$

$$z_0(\beta) \sim \frac{\beta}{\sqrt{2\pi}}\exp\{-\beta^2/2\}, \quad \beta \to +\infty.$$

Also, we have ([9], page 687)

$$D_z(0) = \frac{\sqrt{\pi}2^{z/2}}{\Gamma\big((1-z)/2\big)} \tag{2.11}$$

which may be used to simplify $\Delta(\beta)$ in Theorem 1 for the case $\beta = 0$. Expanding (2.11) as $z \to 1$ we obtain

$$D_z(0) \sim \sqrt{\frac{\pi}{2}}(1-z), \quad z \to 1$$

and thus $\Delta(0) = \sqrt{\pi/2}$. Since $\Gamma\big(z_0(0)\big) = 1$, the result in Theorem 1(c) reduces to

$$1 - P(m) \sim \sqrt{\frac{2}{\pi\rho}}, \quad \beta = 0 \ (m = \rho).$$

3. Elementary methods. We show that some asymptotic information may be obtained directly from the representation of $P(m)$ in (1.3). In this section we use only elementary methods (i.e., ones that do not involve complex variables and contour integration). In particular, we give a simple intuitive derivation of the leading order estimate of Aldous [6] for the expected amount of wasted space, as well as the refinement in (2.6) and (2.1).

First we consider the limit $\rho \to \infty$ with $m = O(1)$. We clearly have $P(0) = 1 - e^{-\rho}$ and for $m = 1$ (1.3) yields

$$\begin{aligned}
P(1) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \frac{\rho^{n+1}}{1+\varepsilon(n+1)}, \quad \varepsilon = \rho^{-1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \rho^{n+1} \int_0^{\infty} \exp\{-[1+\varepsilon(n+1)]t\} dt \\
&= \int_0^{\infty} \exp\{-t\} \left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} (\rho \exp\{-\varepsilon t\})^{n+1} \right] dt \\
&= \int_0^{\infty} [1 - \exp\{-\rho \exp\{-\varepsilon t\}\}] \exp\{-t\} dt \\
&= 1 - \rho \int_0^{\infty} \exp\{-\rho(u + \exp\{-u\})\} du.
\end{aligned}$$

The last integral is easily evaluated using the Laplace method [8]. The major contribution comes from small u where we may approximate $u + e^{-u} = 1 + u^2/2 + \cdots$, and we ultimately obtain (2.7). For $m = 2$ we note that $1 + 2\varepsilon(n+1) + \varepsilon^2(n+1)(n+2) = (1+Z)^2 + \varepsilon Z$ where $Z = \varepsilon(n+1)$. For $\varepsilon \to 0$ the roots C_1, C_2 of the quadratic

$$(1+Z)^2 + \varepsilon Z = 0$$

satisfy $C_1 = -1 + \sqrt{\varepsilon} + O(\varepsilon)$ and $C_2 = -1 - \sqrt{\varepsilon} + O(\varepsilon)$. We use the partial fractions expansion

$$\frac{1}{(1+Z)^2 + \varepsilon Z} \equiv \frac{1}{\mathscr{P}_2(Z;\varepsilon)} = \frac{A_1}{Z - C_1} + \frac{A_2}{Z - C_2} \tag{3.1}$$

and represent $(Z - C_i)^{-1}$ as integrals:

$$\frac{1}{Z - C_i} = \int_0^\infty \exp\{(C_i - \varepsilon(n+1))t\}dt. \tag{3.2}$$

Then we can evaluate the sum over n to get

$$\begin{aligned} P(2) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \frac{\rho^{n+1}}{1 + 2\varepsilon(n+1) + \varepsilon^2(n+1)(n+2)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} \rho^{n+1} \left[\frac{A_1}{\varepsilon(n+1) - C_1} + \frac{A_2}{\varepsilon(n+1) - C_2} \right] \\ &= \sum_{i=1}^{2} A_i \int_0^\infty \exp\{C_i t\}\left[1 - \exp(-\rho \exp\{-\varepsilon t\})\right] dt \\ &= -\left(\frac{A_1}{C_1} + \frac{A_2}{C_2} \right) - \rho \sum_{i=1}^{2} A_i \int_0^\infty \exp\left[\rho(C_i u - \exp\{-u\})\right] du \\ &= 1 - \rho \sum_{i=1}^{2} A_i \int_0^\infty \exp\left[\rho(C_i u - \exp\{-u\})\right] du \end{aligned} \tag{3.3}$$

where we have used (3.1) with $Z = 0$. Now we use the expansions of the roots C_i as $\varepsilon = \rho^{-1} \to 0$ and note that $A_1 \sim 1/(2\sqrt{\varepsilon})$ and $A_2 \sim -1/(2\sqrt{\varepsilon})$. We thus obtain from (3.3)

$$1 - P(2) \sim \tfrac{1}{2}\rho \exp\{-\rho\} \int_0^\infty (\exp\{v\} - \exp\{-v\}) \exp\{-v^2/2\} dv$$

which is precisely (2.8). Here we set $u = \sqrt{\varepsilon} v$ and expanded the integrand(s) for $\varepsilon \to 0$.

The same partial fractions expansion applies for arbitrary m. For $m = 3$ we define

$$\begin{aligned} \mathscr{P}_3(Z;\varepsilon) &= 1 + 3\varepsilon(n+1) + 3\varepsilon^2(n+1)(n+2) + \varepsilon^3(n+1)(n+2)(n+3) \\ &= (1+Z)^3 + 3\varepsilon Z(1+Z) + 2\varepsilon^2 Z. \end{aligned} \tag{3.4}$$

The three roots of the cubic $\mathscr{P}_3$ satisfy, as $\varepsilon \to 0$,

$$C_1 = -1 + \sqrt{3\varepsilon} + O(\varepsilon), \ C_2 = -1 - \sqrt{3\varepsilon} + O(\varepsilon), \ C_3 = -1 + O(\varepsilon). \tag{3.5}$$

We again obtain the right side of (3.3) except that now there are three terms in the sum and the corresponding A_i satisfy, for $\varepsilon \to 0$,

$$A_1 \sim \frac{1}{6\varepsilon}, \ A_2 \sim \frac{1}{6\varepsilon}, \ A_3 \sim -\frac{1}{3\varepsilon}.$$

We thus obtain

$$1 - P(3) \sim \tfrac{1}{6}\rho^{3/2}\exp\{-\rho\}\int_0^\infty \Big(\exp\{\sqrt{3}\,v\} + \exp\{-\sqrt{3}\,v\} - 2\Big)\exp\{-v^2/2\}dv$$

which may be explicitly evaluated to give (2.9).

For general m the partial fractions expansion leads to the following alternate representation for $1 - P(m)$:

$$1 - P(m) = \rho\sum_{i=1}^{m} A_i \int_0^\infty \exp\big[\rho(C_i u - \exp\{-u\})\big]du$$

where $C_1, C_2, \dots, C_m$ are the roots of

$$\mathscr{P}_m(Z;\varepsilon) = 1 + \varepsilon m(n+1) + \varepsilon^2\binom{m}{2}(n+1)(n+2) + \cdots + \varepsilon^m(n+1)\dots(n+m)$$

$$= 1 + mZ + \binom{m}{2}Z(Z+\varepsilon) + \cdots + Z(Z+\varepsilon)\dots\big[Z + \varepsilon(m-1)\big]$$

and $A_j = 1/\mathscr{P}_m'(C_j;\varepsilon)$. However, the case of general m is easier to treat using the methods in the next two sections.

Next we consider the case where m and ρ are both large. Representing the sum in (1.3) as an integral we obtain

$$P(m) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}e^{-\rho}\left[\int_1^\infty t^m e^{-\rho t}(t-1)^n dt\right]^{-1}. \tag{3.6}$$

Now suppose that $m, \rho \to \infty$ with m/ρ fixed and $m/\rho > 1$. Then we evaluate the integral in (3.6) by Laplace's method, with the major contribution coming from $t = t_0 \equiv m/\rho > 1$. We thus obtain

$$\begin{aligned} P(m) &\sim e^{-\rho}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}\left[e^{-m}\left(\frac{m}{\rho}\right)^m\frac{\sqrt{2\pi m}}{\rho}\left(\frac{m}{\rho}-1\right)^n\right]^{-1} \\ &= e^{m-\rho}\left(\frac{\rho}{m}\right)^m\frac{m-\rho}{\sqrt{2\pi m}}\log\left(\frac{m}{m-\rho}\right). \end{aligned} \tag{3.7}$$

The sum in (3.7) only converges for $m/\rho > 2$, but we argue that the right side of (3.7) is valid for all $m/\rho > 1$ (this is discussed more precisely in the next section). But (3.7) is precisely the formula in Theorem 1(d), after we set $m = \rho X$.

For any fixed $m/\rho > 1$ the expression in (3.7) is exponentially small in ρ. We again argue that for $\rho \to \infty$ the mass is concentrated in that range of m where $P(m)$ changes from $P(m) \approx 1$ to $P(m) \approx 0$. For $m/\rho \downarrow 1$ we can set $m = \rho + \sqrt{\rho}\beta$ and then (3.7) can be approximated by

$$\frac{\beta}{\sqrt{2\pi}}\exp\{-\beta^2/2\}\log\left(\frac{\sqrt{\rho}}{\beta}\right). \tag{3.8}$$

For the above to be $O(1)$, β must be sufficiently large so that $e^{-\beta^2/2}$ compensates for the large factor $\log(\sqrt{\rho})$, and this occurs for $\beta \sim \sqrt{2\log\log\rho}$. This shows that the average value of max(S) is about $\rho+\sqrt{2\rho\log\log\rho}$ and refined estimates are possible by solving (3.8) more precisely as $\rho\to\infty$.

In the next sections we treat other ranges of m and establish the remainder of Theorem 1. It seems difficult to obtain the results for $m/\rho<1$ and $m/\rho=1+O(\rho^{-1/2})$ directly from (3.6), and we need to establish alternate representations for $P(m)$.

4. Alternative integral representations. We convert (1.3) to a form more useful for asymptotic analysis. We consider the mth degree polynomial

$$\tag{4.1} \begin{aligned} Q_m(z;\rho) = 1 - \binom{m}{1}\frac{1}{\rho}z + \binom{m}{2}\frac{1}{\rho^2}z(z-1) - \binom{m}{3}\frac{1}{\rho^3}z(z-1)(z-2) \\ + \cdots + (-1)^m\binom{m}{m}\frac{1}{\rho^m}z(z-1)\cdots(z-m+1). \end{aligned}$$

In Appendix A we show that Q_m can also be represented as the integral

$$\tag{4.2} Q_m(z;\rho) = \frac{m!}{\rho^m}\frac{1}{2\pi i}\int_{C_1} t^{-m-1}\exp\{\rho t\}\exp\{z\log(1-t)\}dt$$

where C_1 is a closed loop on which $0<|t|<1$.

We next show that

$$\tag{4.3} P(m) = -\frac{1}{2\pi i}\int_{\mathrm{Br}'}\frac{\Gamma(z)\rho^{-z}}{Q_m(z;\rho)}dz$$

and

$$\tag{4.4} P(m) = 1 - \frac{1}{2\pi i}\int_{\mathrm{Br}}\frac{\Gamma(z)\rho^{-z}}{Q_m(z;\rho)}dz.$$

Here Br$'$ is a vertical contour in the z-plane on which $-1<\mathrm{Re}(z)<0$ and Br is another vertical contour on which $0<\mathrm{Re}(z)<z_*(\rho)$ where z_* is the smallest root of $Q_m(z;\rho)$.

To establish the equivalence of (1.3) and (4.3) we close the contour Br$'$ in the left-half plane. The only singularities of the integrand in this domain are the poles of the Gamma function $\Gamma(z)$, and these occur at $z=-1,-2,-3,\dots$ ($z=0$ lies to the right of Br$'$). By evaluating the residues and noting that

$$Q_m(-n-1;\rho) = \sum_{k=0}^{m}\binom{m}{k}\rho^{-k}\frac{(n+k)!}{n!};\quad n=0,1,2,\dots$$

we see that (4.3) is equivalent to (1.3). It is also equivalent to (4.4), since we can shift Br to Br$'$ and pick up the residue at $z=0$ of the integrand, which is equal to one since $Q_m(0;\rho)=1$ and $\Gamma(z)\sim 1/z$ as $z\to 0$.

To evaluate $P(m)$ in (4.3) or (4.4) asymptotically, we need to approximate the integrand for $\rho\to\infty$ and then find the singularity and/or saddle point that governs the asymptotics. We shall show that the behavior of the integrand is different for various ranges of ρ, m and z. We proceed to study Q_m asymptotically for these various cases.

Let us first consider $\rho \to \infty$ with $z = O(1)$ and $m \to \infty$. Then the behavior (4.2) is determined by the singularity of the integrand that is closest to the origin, which is the branch point at $t = 1$, or by a saddle point, which satisfies

$$\frac{d}{dt}[\rho t - m \log t] = \rho - \frac{m}{t} = 0. \tag{4.5}$$

For $m/\rho < 1$ the saddle point lies to the left of the branch point and a standard steepest descent calculation [8] shows that

$$\begin{aligned} Q_m \sim & \frac{m!}{\rho^m} \exp\{z \log(1 - m/\rho)\} \exp\{m - m \log(m/\rho)\} \\ & \times \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \exp\left(\frac{\rho^2}{2m}\zeta^2\right) d\zeta \\ \sim & \frac{m}{\rho} \exp\{z \log(1 - m/\rho)\}. \end{aligned} \tag{4.6}$$

Here we have used Stirling's formula to approximate $m!$ for m large. When $m/\rho > 1$ the branch point at $t = 1$ dominates and we obtain

$$\begin{aligned} Q_m &\sim \frac{m!}{\rho^m} e^\rho \frac{-\sin(\pi z)}{\pi} \int_0^\infty u^z e^{(\rho - m)u} du \\ &= \frac{m!}{\rho^m} e^\rho (m - \rho)^{-z-1} \frac{-\sin(\pi z)}{\pi} \Gamma(z + 1). \end{aligned} \tag{4.7}$$

Note that the dominant contribution from the branch cut along $\{\mathrm{Im}(t) = 0, \mathrm{Re}(t) \geq 1\}$ for $\rho \to \infty$ comes from the range $t - 1 = O(\rho^{-1})$ and we have approximated the integrand in (4.2) near $t = 1$. While some of these calculations only apply for $\mathrm{Re}(z) > -1$, it is easy to verify that (4.7) holds for all $z = O(1)$.

Next we consider the case $m/\rho \approx 1$, where the branch point and saddle are close to one another. This is a standard problem in the asymptotic evaluation of integrals, which is discussed, for example, in [8], Chapter 9. We expand the integrand in (4.2) about $t = 1$ (since the singular points are both in this neighborhood):

$$\begin{aligned} t^{-1} \exp\{\rho t - m \log t\} \sim\ & \exp\{\rho\} \exp\{(\rho - m)(t - 1)\} \\ & \times \exp\{m(t - 1)^2/2 + O\big(m(t - 1)^3\big)\}. \end{aligned}$$

Scaling $m = \rho + \beta\sqrt{\rho}$ and $t = 1 - \omega/\sqrt{\rho}$ we ultimately obtain

$$Q_m \sim \frac{m!}{\rho^m} \exp\{\rho\} \rho^{-z/2} \frac{1}{\sqrt{\rho}} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \exp\{\beta\tau\} \exp\{\tau^2/2\} \tau^z d\tau. \tag{4.8}$$

But, in view of [9], page 688, a standard integral representation for parabolic cylinder functions is

$$D_z(-\beta) \exp\{-\beta^2/4\} \frac{1}{\sqrt{2\pi}} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \exp\{\beta\tau\} \exp\{\tau^2/2\} \tau^z d\tau. \tag{4.9}$$

We also note that for $\rho, m \to \infty$ with β fixed

$$\exp\{\rho - m + m \log(m/\rho)\} \sim \exp\{\beta^2/2\}. \tag{4.10}$$

We expand $m!$ in (4.7) and (4.8) using Stirling's formula and combine our results in (4.6)-(4.10) into the following lemma.

LEMMA 1. *For $m, \rho \to \infty$ and $z = O(1)$, the function $Q_m(z;\rho)$ has the following aysmptotic expansions.*

(i) $0 < m/\rho < 1$,

$$Q_m(z;\rho) \sim \frac{m}{\rho}\left(1 - \frac{m}{\rho}\right)^z.$$

(ii) $m/\rho = 1 + \beta/\sqrt{\rho} = 1 + O(\rho^{-1/2})$,

$$Q_m(z;\rho) \sim \rho^{-z/2}\exp\{\beta^2/4\}D_z(-\beta).$$

(iii) $m/\rho > 1$,

$$\begin{aligned} Q_m(z;\rho) &\sim \left(\frac{m}{\rho}\right)^m e^{\rho-m}\sqrt{2\pi m}(m-\rho)^{-z-1}\frac{-\sin(\pi z)}{\pi}\Gamma(z+1) \\ &= \left(\frac{m}{\rho}\right)^m e^{\rho-m}\sqrt{2\pi m}(m-\rho)^{-z-1}\frac{1}{\Gamma(-z)}. \end{aligned}$$

Next we consider $m, \rho \to \infty$ and also z large. For now we take z real and positive. A large z changes the equation locating the saddle points from (4.5) to

$$\frac{d}{dt}\big[\rho t - m\log t + z\log(1-t)\big] = 0. \tag{4.11}$$

This has two solutions, which we call t_0 and t_1:

$$\begin{aligned} t_0 = t_0(z;m,\rho) &= \frac{1}{2\rho}\left[\rho + m - z - \sqrt{(\rho+m-z)^2 - 4\rho m}\right], \\ t_1 = t_1(z;m,\rho) &= \frac{1}{2\rho}\left[\rho + m - z + \sqrt{(\rho+m-z)^2 - 4\rho m}\right]. \end{aligned} \tag{4.12}$$

Note that $t_1 > t_0$ and both saddles are real for $0 < z < m + \rho - 2\sqrt{\rho m}$. Also, we have $t_1(0) = 1$ and $t_0(0) = m/\rho$. We shall show that the case of z large will be only important to the asymptotics for $m/\rho < 1$. The two saddles coalesce at $t_0 = t_1 = \sqrt{m/\rho} < 1$ when $z = m + \rho - 2\sqrt{\rho m}$.

For $0 < z < m + \rho - 2\sqrt{\rho m}$, the saddle at t_0 determines the asymptotic behavior of Q_m. Again a standard saddle point calculation yields

$$\begin{aligned} Q_m &\sim \frac{m!}{\rho^m}\exp\{\rho t_0\}\exp\{-m\log t_0\}\exp\{z\log(1-t_0)\}\frac{1}{t_0} \\ &\quad\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\exp\left[\frac{1}{2}\left(\frac{m}{t_0^2} - \frac{z}{(1-t_0)^2}\right)(t-t_0)^2\right]dt \\ &\sim \exp\{-m\}\left(\frac{m}{\rho}\right)^m\sqrt{m}\left[m - \frac{zt_0^2}{(1-t_0)^2}\right]^{-1/2} \\ &\quad\times\exp\{\rho t_0\}\exp\{-m\log t_0\}\exp\{z\log(1-t_0)\}. \end{aligned} \tag{4.13}$$

Now suppose z is close to $m+\rho-2\sqrt{\rho m}$ so that the two saddles are close to each other. This is also a classic problem in the asymptotic evaluation of integrals, which is discussed in [8], Chapter 9. We define

$$\phi(t)=\rho t-m\log t+z\log(1-t)$$

and expand the integrand in (4.2) near $t=\sqrt{m/\rho}$. We have

$$\phi'\left(\sqrt{\frac{m}{\rho}}\right)=\rho-\sqrt{\rho m}-\frac{z}{1-\sqrt{m/\rho}}=\frac{m+\rho-2\sqrt{\rho m}-z}{1-\sqrt{m/\rho}},$$

$$\phi''\left(\sqrt{\frac{m}{\rho}}\right)=\frac{\rho(1-\sqrt{m/\rho}\,)^2-z}{(1-\sqrt{m/\rho}\,)^2}$$

and

$$\phi'''\left(\sqrt{\frac{m}{\rho}}\right)=-2m\left(\frac{\rho}{m}\right)^{3/2}-\frac{2z}{(1-\sqrt{m/\rho}\,)^3}.$$

Now scale

$$z=m+\rho-2\sqrt{\rho m}+\rho^{1/3}\left(1-\sqrt{m/\rho}\right)^{2/3}\eta$$

and

$$t=\sqrt{m/\rho}+s\rho^{-1/3}\left(1-\sqrt{m/\rho}\right)^{1/3},$$

so that $z-(\sqrt{m}-\sqrt{\rho}\,)^2=O(\rho^{1/3})$ and $t-\sqrt{m/\rho}=O(\rho^{-1/3})$. With this scaling

$$\phi'\left(\sqrt{\frac{m}{\rho}}\right)\left(t-\sqrt{\frac{m}{\rho}}\right)\sim -s\eta,\quad \phi''\left(\sqrt{\frac{m}{\rho}}\right)\left(t-\sqrt{\frac{m}{\rho}}\right)^2=O(\rho^{-1/3}),$$

$$\phi'''\left(\sqrt{\frac{m}{\rho}}\right)\left(t-\sqrt{\frac{m}{\rho}}\right)^3\sim -2\sqrt{\frac{\rho}{m}}s^3$$

and $\phi^{(j)}(\sqrt{m/\rho}\,)(t-\sqrt{m/\rho}\,)^j=o(1)$ for $j\geq 4$. We thus approximate the integrand in (4.2) to obtain

$$Q_m\sim\frac{m!}{\rho^m}\exp\left\{\phi\left(\sqrt{m/\rho}\right)\right\}\frac{(1-\sqrt{m/\rho}\,)^{1/3}}{\rho^{1/3}\sqrt{m/\rho}}\times\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\exp\{-s\eta\}\exp\left\{-\sqrt{\rho/m}s^3/3\right\}ds. \tag{4.14}$$

But, the Airy function $Ai(\cdot)$ has the integral representation(s) ([9], page 447)

$$\begin{aligned}Ai(y)&=\frac{1}{\pi}\int_0^\infty\cos\left(vy+\frac{1}{3}v^3\right)dv\\&=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\{ivy\}\exp\{iv^3/3\}dv\\&=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\exp\{uy\}\exp\{-u^3/3\}du.\end{aligned}$$

We use the above to express Q_m in (4.14) in terms of Ai and below summarize our results [combining (4.13 and (4.14)] as a second lemma.

LEMMA 2. *For $m, \rho \to \infty$ with $m/\rho < 1$ and $z \to \infty$ with $0 < z \le (\sqrt{m} - \sqrt{\rho}\,)^2$, we have the following asymptotic expansions for $Q_m(z;\rho)$.*

(i) $0 < z/\rho < (1 - \sqrt{m/\rho}\,)^2$,

$$Q_m(z;\rho) \sim \exp\{-m\}\left(\frac{m}{\rho}\right)^m \sqrt{m}\left[m - \frac{zt_0^2}{(1-t_0)^2}\right]^{-1/2}$$
$$\times \exp\{\rho t_0\}\exp\{-m\log t_0\}\exp\{z\log(1-t_0)\},$$
$$t_0 = \frac{1}{2\rho}\left[\rho + m - z - \sqrt{(\rho+m-z)^2 - 4\rho m}\,\right].$$

(ii) $z/\rho = (1-\sqrt{m/\rho}\,)^2 + \rho^{-2/3}(1-\sqrt{m/\rho}\,)^{2/3}\eta = (1-\sqrt{m/\rho}\,)^2 + O(\rho^{-2/3})$,

$$Q_m(z;\rho) \sim \exp\{-m\}\left(\frac{m}{\rho}\right)^m \sqrt{2\pi m}\left(1 - \sqrt{\frac{m}{\rho}}\right)^{1/3} m^{-1/3}$$
$$\times \exp\{\phi(\sqrt{m/\rho}\,)\}Ai(-\eta(m/\rho)^{1/6}),$$
$$\exp\{\phi(\sqrt{m/\rho}\,)\} = \exp\{\sqrt{m\rho}\}\left(\frac{\rho}{m}\right)^{m/2}$$
$$\times \exp\left\{\left[\left(\sqrt{\rho}-\sqrt{m}\,\right)^2 + \eta\rho^{1/3}\left(1-\sqrt{m/\rho}\right)^{2/3}\right]\right.$$
$$\left.\times \log\left(1-\sqrt{m/\rho}\right)\right\}.$$

We comment that when $z/\rho > (1-\sqrt{m/\rho}\,)^2$ the saddle points at t_0 and t_1 move off the real axis in the t-plane as complex conjugate pairs. Now both saddles contribute to the asymptotic development of Q_m. However, this range of z is not important to the asymptotic evaluation of $P(m)$, which is our ultimate goal. Thus, we omit analysis of the case $z/\rho > (1-\sqrt{m/\rho}\,)^2$.

The approximation in (ii) ceases to be useful if η is such that Ai vanishes. Then we need to consider the contribution from higher order terms in the asymptotic series. In Appendix B we explicitly calculate the next term, which leads to a refined approximation, where we replace $Ai(\cdot)$ in (ii) by [setting $u = -\eta(m/\rho)^{1/6}$]

$$Ai(u) - \rho^{-1/3}\left[\frac{(\rho/m)^{1/3}}{2(1-\sqrt{m/\rho}\,)^{2/3}}Ai'(u) + \mathscr{F}(u)\right] + O(\rho^{-2/3})$$

where $\mathscr{F}(u)$ vanishes whenever $Ai(u)$ does. Even if we omit $\mathscr{F}$ the above is a uniform approximation in that it applies for all u, including when u is close to a zero of Ai. Since the poles of the integrand in (4.4) in the right half-plane are precisely the zeros of Q_m, we need to carefully approximate the integrand near these zeros.

Finally we consider cases where $\rho \to \infty$ but with $m = O(1)$. If $z \to \infty$ with $z/\rho < 1$ we scale $t = u/\rho$ in (4.2) and obtain

$$\begin{aligned} Q_m &\sim \frac{m!}{\rho^m} \frac{1}{2\pi i} \int_{C_1} \frac{\exp(u) \exp\left[-\frac{z}{\rho} u + O(z\rho^{-2})\right]}{(u/\rho)^m} \frac{du}{u} \\ &\sim \left(1 - \frac{z}{\rho}\right)^m \end{aligned}$$

and this remains valid for $z = O(1)$. If $z/\rho \approx 1$ we scale $z = \rho + \sqrt{\rho}\xi$ and $t = w/\sqrt{\rho}$, and use

$$\rho t + z \log(1-t) = (\rho - z)t - \frac{z}{2}t^2 + O(zt^3) \sim -\xi w - \frac{1}{2}w^2.$$

Hence, in this limit,

$$\begin{aligned} Q_m &\sim \frac{m!}{\rho^{m/2}} \frac{1}{2\pi i} \int_{C_1} \frac{\exp\{-\xi w\} \exp\{-w^2/2\}}{w^{m+1}} dw \\ &= \rho^{-m/2} \mathscr{D}_w^{(m)} \left(\exp\{-\xi w\} \exp\{-w^2/2\}\right) \Big|_{w=0} \end{aligned} \tag{4.15}$$

where $\mathscr{D}_w^{(m)}$ is the mth derivative with respect to w. Noting that the Hermite polynomials $H_k(\cdot)$ have the generating function

$$\exp\{-t^2\} \exp\{2tx\} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x),$$

we see that

$$\mathscr{D}_w^{(m)} \left(\exp\{-\xi w\} \exp\{-w^2/2\}\right) \Big|_{w=0} = 2^{-m/2} H_m\left(\frac{-\xi}{\sqrt{2}}\right) = He_m(-\xi).$$

Below we summarize the results for $m = O(1)$.

LEMMA 3. *For $\rho \to \infty$ and $m = O(1)$, $Q_m(z;\rho)$ has the following asymptotic expansions:*

(i) $0 \le z/\rho < 1$,

$$Q_m(z;\rho) \sim \left(1 - \frac{z}{\rho}\right)^m.$$

(ii) $z = \rho + \sqrt{\rho}\xi = \rho + O(\sqrt{\rho})$,

$$Q_m(z;\rho) \sim \rho^{-m/2} He_m(-\xi).$$

This completes the analysis. We omit consideration of the case $m = O(1)$, $z/\rho > 1$ since this will play no role in the asymptotics of $P(m)$. In the next section we use Lemmas 1–3 to obtain the asymptotic expansions of the integrals in (4.3) and (4.4).

5. Asymptotic evaluation of the distribution of max (S). We consider (4.3) and (4.4) and first assume that ρ, $m \to \infty$ with $m/\rho > 1$. For $z = O(1)$ we use Lemma 1 (iii) to approximate the integrand in (4.3). Noting that $\Gamma(z+1) = z\Gamma(z)$ we thus obtain

$$P(m) = \frac{\rho^m}{m!}e^{-\rho}(m-\rho)\frac{1}{2\pi i}\int_{\mathrm{Br}'}\left(\frac{m}{\rho}-1\right)^z \frac{\pi}{z\sin(\pi z)}[1+O(\rho^{-1})]dz. \tag{5.1}$$

The above integrand has a double pole at $z = 0$ and simple poles at $z = \pm 1, \pm 2, \ldots$. The integral in (5.1) is easily evaluated. For $m/\rho > 2$ we close the contour Br' in the left half-plane, picking up the residues from $z = -j$, $j \geq 1$. This yields

$$\begin{aligned} P(m) &\sim \frac{\rho^m}{m!}e^{-\rho}(m-\rho)\sum_{\ell=1}^{\infty}\left(\frac{m}{\rho}-1\right)^{-\ell}\frac{\pi}{-\ell(-1)^\ell \pi} \\ &= \frac{\rho^m}{m!}e^{-\rho}(m-\rho)\log\left(\frac{m}{m-\rho}\right), \qquad \frac{m}{\rho} > 2. \end{aligned} \tag{5.2}$$

For $1 < m/\rho < 2$ we close the contour in the right half-plane. Noting that the residue at $z = 0$ is given by

$$\mathrm{res}_{z=0}\left\{\left(\frac{m}{\rho}-1\right)^z\frac{\pi}{z\sin(\pi z)}\right\} = \log\left(\frac{m}{\rho}-1\right),$$

we obtain

$$\begin{aligned} P(m) &\sim \frac{\rho^m}{m!}e^{-\rho}(m-\rho)(-1)\left[\log\left(\frac{m}{\rho}-1\right)+\sum_{\ell=1}^{\infty}\left(\frac{m}{\rho}-1\right)^{\ell}\frac{(-1)^\ell}{\ell}\right] \\ &= \frac{\rho^m}{m!}e^{-\rho}(m-\rho)(-1)\log\left(1-\frac{\rho}{m}\right), \qquad 1 < \frac{m}{\rho} < 2 \end{aligned} \tag{5.3}$$

which is the same as (5.2). Thus, (5.3) holds for all $m/\rho > 1$. Setting $m = \rho X$ and using Stirling's formula for $m!$, we have established part (d) of Theorem 1.

Next consider the scaling $m - \rho = \beta\sqrt{\rho} = O(\sqrt{\rho}\,)$. Now we use the asymptotic result in Lemma 1 (ii) to obtain

$$P(m) \sim -\frac{1}{2\pi i}\int_{\mathrm{Br}'}\frac{\Gamma(z)\rho^{-z/2}\exp\{-\beta^2/4\}}{D_z(-\beta)}[1+O(\rho^{-1/2})]dz. \tag{5.4}$$

For $\rho \to \infty$ we shift the contour to the right. The first pole encountered is that in $\Gamma(z)$ at $z = 0$ and, noting that $D_0(-\beta) = \exp\{-\beta^2/4\}$, the corresponding residue is equal to one. The next singularity is the smallest positive root of $D_z(-\beta)$, which we denote by $z_0(\beta)$. By computing the residue at this pole we are led to part (c) of Theorem 1. To obtain higher order terms in the asymptotic series for $m - \rho = O(\sqrt{\rho}\,)$ we need to (1) refine Lemma 1 (ii) to an asymptotic series in powers of $\rho^{-1/2}$ and (2) evaluate the contribution to (5.4) from the other positive roots of $D_z(-\beta) = 0$. The jth root $z_j(\beta)$ contributes a term of order $\rho^{-z_j/2}$ to the asymptotic series.

Next we consider m, $\rho \to \infty$ with $0 < m/\rho < 1$. Now we will shift the contour in (4.4) further to the right, and we shall need the asymptotic behavior of Q_m for z large. This is given in Lemma 2. For $0 < z/\rho < (1 - \sqrt{m/\rho}\,)^2$, we may use Lemma 2 (i) to approximate Q_m. This leads to an integrand whose most rapidly growing factor is

$$\exp\{z \log z - z - z \log \rho - \rho t_0(z) + m \log(t_0(z)) - z \log[1 - t_0(z)]\}.$$

Here we also approximated $\Gamma(z)$ for $z \to \infty$ with $|\arg(z)| < \pi/2$ (i.e., in the right half-plane) using Stirling's formula. Using the fact that t_0 satisfies the quadratic equation $\rho - m/t_0 - z/(1 - t_0) = 0$, we see that the saddle points of the integrand in (4.4) satisfy

$$\log z - \log \rho - \log[1 - t_0(z)] = 0 \tag{5.5}$$

or $z/\rho = 1 - t_0(z)$. Using the definition of t_0 in (4.12) we can easily show that (5.5) has no solutions for $m/\rho < 1$. It follows that we must shift the contour further to the right, until z/ρ is close to $(1 - \sqrt{m/\rho}\,)^2$. But then Lemma 2 (i) no longer applies and we must use the expression in Lemma 2 (ii).

After changing $\eta \to -\eta$, scaling $\eta = (\rho/m)^{1/6} u$ and using Lemma 2 (ii) and Appendix B in (4.4), we are led to

$$\begin{aligned} 1 - P(m) \sim \frac{1}{2\pi i} \int_{\mathrm{Br}'} & \frac{\exp\left[-\log(1 - \sqrt{m/\rho}\,)(1 - \sqrt{m/\rho}\,)^{2/3} \sqrt{\rho} m^{-1/6} u\right]}{Ai(u) - m^{-1/3}(1 - \sqrt{m/\rho}\,)^{-2/3} Ai'(u)/2} du \\ & \times \left(\frac{\rho}{m}\right)^{m/2} \exp\{-\rho\} \frac{m^{-1/3}}{(1 - \sqrt{m/\rho}\,)^{2/3}} \\ & \times \exp\left[\sqrt{\rho m} + \rho(1 - \sqrt{m/\rho}\,)^2 \log(1 - \sqrt{m/\rho}\,)\right]. \end{aligned} \tag{5.6}$$

Here we have again approximated $\Gamma(z)$ by $\sqrt{2\pi/z} z^z e^{-z}$ and changed integration variables from z to η and then to $-\eta$ and u. The contour Br in (5.6) is a vertical contour with $\mathrm{Re}(u) \geq 0$. The integral may be further simplified by noting that for ρ, $m \to \infty$ with $m/\rho \in (0, 1)$, we have $\sqrt{\rho} m^{-1/6} = O(\rho^{1/3})$. As is well known the zeros of Ai are along the negative real axis. We can thus write the integral in (5.6) as a residue series, that is, as

$$\begin{aligned} & \sum_{\ell=0}^{\infty} \frac{\exp\{-\rho^{1/3} r_\ell \Phi_0(m/\rho)\}}{Ai'(r_\ell)} \exp\left[-\frac{1}{2\sqrt{X}} \log(1 - \sqrt{X}\,)\right], \\ & \Phi_0(X) = X^{-1/6} \left(1 - \sqrt{X}\right)^{2/3} \log\left(1 - \sqrt{X}\right). \end{aligned} \tag{5.7}$$

The dominant term corresponds to $\ell = 0$ and this is precisely [combining (5.6) and (5.7)] the result in part (b) of Theorem 1. We also note that $\Phi_1(X) = -r_0 \Phi_0(X)$ and that $Ai'(r_0) > 0$.

Next we examine the asymptotic matching between parts (b) and (c) of Theorem 1. As is well known (see, e.g., [9], page 689) for $z \to \infty$ and $\beta \to -\infty$ the parabolic cylinder function $D_z(-\beta)$ may be approximated by Airy functions. The precise scaling is $z \to \infty$, $|\beta| \to \infty$ with $|\beta| = 2\sqrt{z}\big(1+O(z^{-2/3})\big)$. Then we obtain

$$D_z(-\beta) \sim 2^{z/2}\Gamma\left(\frac{z+1}{2}\right)z^{1/6}\left[Ai\left(z^{1/6}\left(|\beta|-2\sqrt{z}\right)\right) - \frac{1}{2}z^{-1/3}Ai'\left(z^{1/6}\left(|\beta|-2\sqrt{z}\right)\right)\right]. \tag{5.8}$$

From (5.8) it follows that as $\beta \to -\infty$ the smallest root $z_0(\beta)$ satisfies

$$z_0(\beta) \sim \frac{\beta^2}{4} - \left(\frac{-\beta}{2}\right)^{2/3} r_0 - \frac{1}{2}, \quad \beta \to -\infty \tag{5.9}$$

and the higher roots $z_j(\beta)$ satisfy (5.9) with r_0 replaced by the jth root r_j of the Airy function Ai, with $0 > r_0 > r_1 > r_2 > \cdots$. From (5.8) we also find that as $\beta \to -\infty$,

$$\Delta(\beta) = -\frac{d}{dz}D_z(-\beta)\bigg|_{z=z_0} \sim z_0^{z_0/2}\exp\left(\frac{-z_0}{2}\right)\left(\frac{2}{-\beta}\right)^{1/3}\sqrt{2\pi}Ai'(r_0). \tag{5.10}$$

We combine (5.9) and (5.10) and use the result in Theorem 1 (c). Then we can easily verify that the $\beta \to -\infty$ behavior of Theorem 1 (c) agrees with the $X \uparrow 1$ behavior of Theorem 1 (b). This verifies the asymptotic matching and yields the intermediate formula (bc) that is given in Section 2.

The matching between Theorem 1, parts (c) and (d) follows immediately from the fact that [9]

$$D_z(-\beta) \sim (-\beta)^z \exp\{-\beta^2/4\} + \frac{\sqrt{2\pi}}{\Gamma(-z)}\exp\{\beta^2/4\}\beta^{-z-1}, \quad \beta \to \infty. \tag{5.11}$$

From (5.11) we see that $z_0 \to 0$ as $\beta \to +\infty$ and more precisely

$$z_0(\beta) \sim \frac{\beta}{\sqrt{2\pi}}\exp\{-\beta^2/2\}, \quad \beta \to +\infty. \tag{5.12}$$

With (5.12) we can easily expand Theorem 1 (c) as $\beta \to \infty$ and show that the result agrees with part (d), as $X \downarrow 1$.

Finally, we consider the case $m = O(1)$. We use Lemma 3 to approximate Q_m in (4.4). For $z/\rho < 1$, Lemma 3 (i) shows that there are no saddle points of (4.4) in the range $0 < z/\rho < 1$. We must thus shift the Br contour sufficiently far to the right until we reach the zeros of Q_m. Lemma 3 (ii) shows that the appropriate scaling is $z = \rho + O(\sqrt{\rho})$. Expanding $\Gamma(z)\rho^{-z}$ by Stirling's formula, setting $z = \rho + \sqrt{\rho}\xi$ with $\xi = O(1)$ and using Lemma 3 (ii), we obtain

$$
\begin{aligned}
1 - P(m) \sim \sqrt{2\pi}\frac{\rho^{m/2}}{m!}\exp\{-\rho\} \\
\times \int_{\mathrm{Br}'} \frac{\exp\{\xi^2/2\}}{\int_{C_1}\exp\{-\xi w\}\exp\{-w^2/2\}w^{-m-1}dw}d\xi.
\end{aligned} \tag{5.13}
$$

Here Br is taken to the left of the singularities of the integrand. Changing $\xi \to -\xi$ and expressing the integrand in (5.13) as a Hermite polynomial we obtain precisely part (a) of Theorem 1.

The asymptotic matching between parts (a) and (b) of Theorem 1 is readily verified. We use the fact that $He_m(\xi) = \exp\{\xi^2/4\}D_m(\xi)$ and let m and $\xi \to \infty$ with $\xi \approx 2\sqrt{m}$. We set

$$
\xi = 2\sqrt{m} + m^{-1/6}s, \quad s = O(1) \tag{5.14}
$$

and use (5.8) to obtain, in the limit (5.14),

$$
He_m(\xi) \sim \exp\{\xi^2/4\}m^{m/2}\exp\{-m/2\}m^{1/6}\sqrt{2\pi}\left[Ai(s) - \frac{1}{2}m^{-1/3}Ai'(s)\right]. \tag{5.15}
$$

With (5.14) and (5.15) Theorem 1 (a) becomes, for $m \to \infty$,

$$
\begin{aligned}
\exp\{-\rho\}\rho^{m/2}m^{-m/2}\exp\{3m/2\}m^{-1/3}\frac{1}{2\pi i}\int_{\mathrm{Br}'}\frac{\exp\{m^{1/3}s\}}{Ai(s) - m^{-1/3}Ai'(s)/2}ds \\
\sim \exp\{-\rho\}\rho^{m/2}m^{-m/2}\exp\{3m/2\}m^{-1/3}\frac{\exp\{m^{1/3}r_0\}}{Ai'(r_0)}\sqrt{e}.
\end{aligned} \tag{5.16}
$$

But, (5.16) agrees precisely with the expansion of part (b) of Theorem 1 as $X = m/\rho \to 0$. This completes the proof of Theorem 1 and the derivation of the intermediate-scale expansions. □

6. Numerical results. In Figures 1 and 2 we plot the (exact) probabilities $p(m) = P(m-1) - P(m)$; $m = 0, 1, 2, \ldots$ (with $P(-1) \equiv 1$) for the respective cases $\rho = 5$ and $\rho = 10$. We see that most of the mass is concentrated when m is slightly larger than ρ, and that the tails of the distribution are highly asymmetrical. Thus, the qualitative structure of $p(m)$ is certainly consistent with our asymptotic analysis.

Next we discuss the accuracy of the asymptotic formlae in Theorem 1. We first consider the right tail of the distribution, where $m/\rho > 1$ and hence Theorem 1 (d) applies. In Table 1 we consider first $\rho = 5$ and then increase ρ to 20. When $\rho = 5$ we compare exact and asymptotic results for $m = 6$, 10, 15 and 20. The exact value was obtained by using MAPLE to evaluate the sum (1.3) to 50 decimal places. We denote by Asy(a)–(d) the values of the formulas in parts (a)–(d) of Theorem 1. When $m = 6$, Table 1 shows that the exact and Asy (d) results are not close. This is not surprising, since $m = 6$ is more in

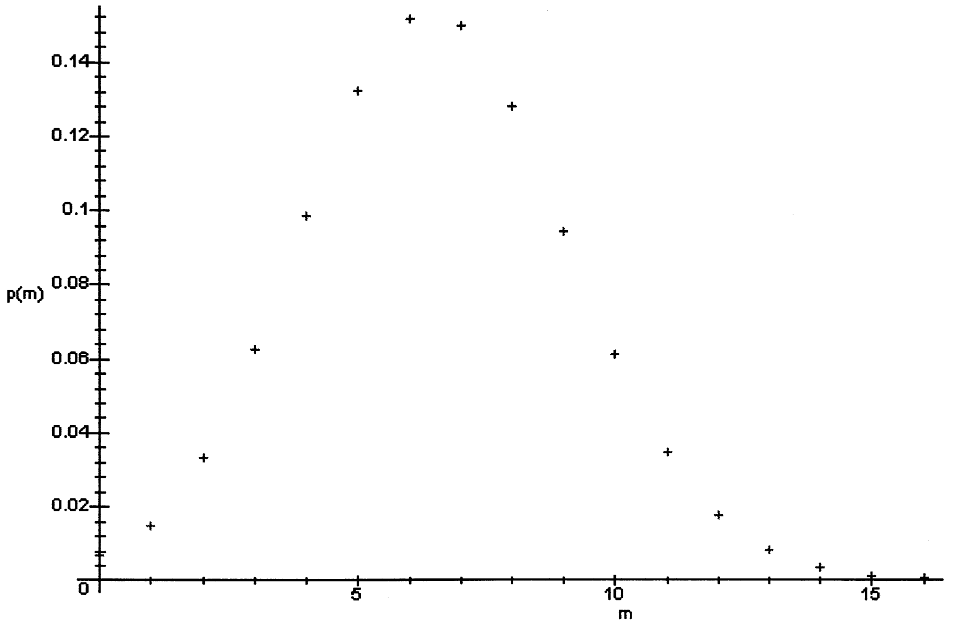

FIG. 1. *A graph of the distribution for* $\rho = 5$.

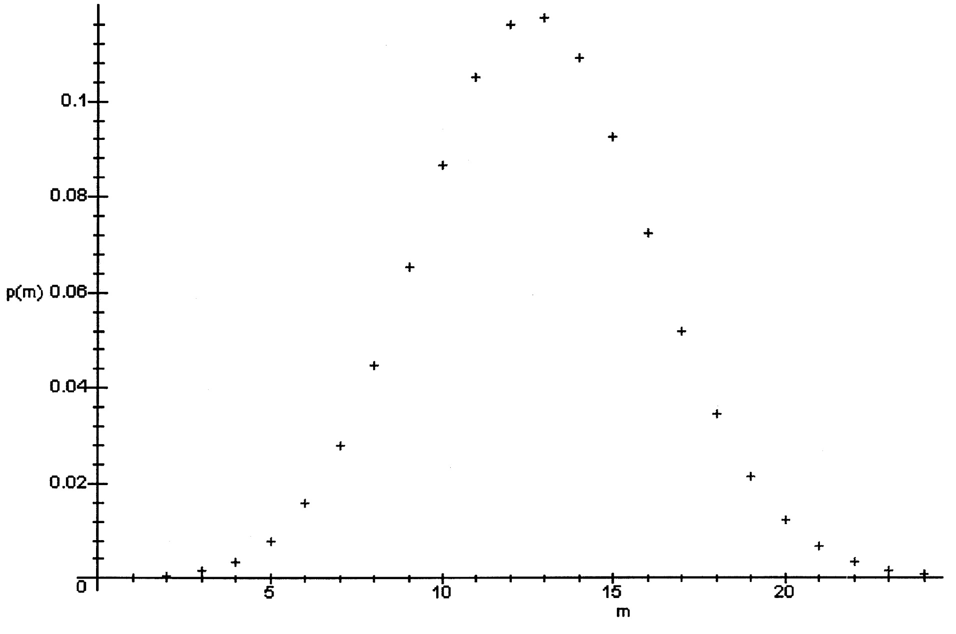

FIG. 2. *A graph of the distribution for* $\rho = 10$.

TABLE 1

ρ	m	P(m)	Asy(d)
5	6	.49909	.26566
	10	6.5584(10^{-2})	6.3369(10^{-2})
	15	6.4516(10^{-4})	6.4113(10^{-4})
	20	1.1473(10^{-6})	1.1445(10^{-6})
20	21	.75206	.25861
	25	.41171	.36000
	30	9.2286(10^{-2})	9.1918(10^{-2})
	40	3.8676(10^{-4})	3.8586(10^{-4})
	50	1.1739(10^{-7})	1.1712(10^{-7})
	60	4.6453(10^{-12})	4.6382(10^{-12})

the range $m - \rho = O(\sqrt{\rho})$ (where Asy (c) applies) than in the range $m/\rho > 1$. However, for $m = 10,15$ and 20 (corresponding to $X = 2$, 3 and 4), Asy (d) is quite close to the exact value. When $\rho = 20$ we consider $m = 21$, 25, 30, 40, 50 and 60. Asy (d) and the exact result are not close when $m = 21$, which is clearly in the range $m - \rho = O(\sqrt{\rho})$. While $m = 25$ is also in this range, Table 2 shows that Asy (d) is within about 12% of the exact value. For $m \geq 30$ (i.e., $X \geq 1.5$) the asymptotic result is extremely accurate.

Next we test our approximation to the left tail, where $m/\rho < 1$. In Table 2 we give the $m = O(1)$ result [i.e., Asy (a)] for $m = 0$, 1, 2 and 3 for $\rho = 5$, 20 and 50. Note that when $m = 0$, Asy (a) is in fact exact. The table shows that the exact and asymptotic results are within about 20% when $\rho = 5$ and the accuracy is very good when $\rho = 20$ or 50.

Finally we test Asy (c), which applies in the range where most of the mass is concentrated. For simplicity we take $m = \rho$ so that $\beta = 0$. We recall that for $\beta = 0$, Asy (c) reduces to $1 - P(m) \sim \sqrt{2/(\pi\rho)}$. In Table 3 we compare the exact $P(m)$ to 1−Asy (c) for various ρ in the range $5 \leq \rho \leq 100$. The two

TABLE 2

ρ	m	1 − P(m)	Asy(a)
5	0	6.7379(10^{-3})	6.7379(10^{-3})
	1	2.1485(10^{-2})	1.8883(10^{-2})
	2	5.5018(10^{-2})	4.7526(10^{-2})
	3	.11782	.10957
20	0	2.0612(10^{-9})	2.0612(10^{-9})
	1	1.2291(10^{-8})	1.1553(10^{-8})
	2	6.1919(10^{-8})	5.8153(10^{-8})
	3	2.7202(10^{-7})	2.6815(10^{-7})
50	0	1.9287(10^{-22})	1.9287(10^{-22})
	1	1.7093(10^{-21})	1.7766(10^{-21})
	2	1.3604(10^{-20})	1.4126(10^{-20})
	3	9.9188(10^{-20})	9.9714(10^{-20})

TABLE 3

$\rho = m$	$P(m)$	$1 - Asy(c)$
5	.65095	.64318
10	.74581	.74769
15	.79076	.79399
20	.81823	.82159
30	.85131	.85433
40	.87122	.87384
50	.88486	.88716
60	.89494	.89699
70	.90279	.90463
80	.90912	.91079
90	.91436	.91590
100	.91880	.92021

answers agree to within 1% even when $\rho = 5$ and when $\rho = 100$ the error improves to about 0.1%.

APPENDIX A

We show that (4.1) and (4.2) are equivalent. We can simply expand $e^{z\log(1-t)} = (1-t)^z$ about $t = 0$ and then it is easy to obtain (4.1) from (4.2). On the other hand we can start with (4.1) and obtain (4.2). To this end we recall that

$$z(z-1)(z-2)\cdots(z-m+1) = \sum_{p=0}^{m} S_m^{(p)} z^p \tag{A.1}$$

where $S_m^{(p)}$ are the Stirling numbers of the first kind ([7], page 824). These have the integral representation

$$S_m^{(p)} = \frac{m!}{p!}\frac{1}{2\pi i}\int_{C_0} \frac{[\log(1+t)]^p}{t^{m+1}}dt \tag{A.2}$$

where $0 < |t| < 1$ on the closed loop C_0. Using (A.1) and (A.2), (4.1) becomes

$$\begin{aligned} Q_m(z;\rho) &= \sum_{M=0}^{m} (-1)^M \binom{m}{M} \rho^{-M} \sum_{p=0}^{M} S_M^{(p)} z^p \\ &= \sum_{M=0}^{m} \frac{(-1)^M}{\rho^M} \frac{m!}{(m-M)!} \frac{1}{2\pi i}\int_{C_0} \frac{\exp\{z\log(1+t)\}}{t^{M+1}}dt. \end{aligned} \tag{A.3}$$

We note that we have explicitly performed the p-sum in (A.3) and that it is irrelevant whether we sum over $p = 0, 1, \ldots, M$ or $p = 0, 1, \ldots, \infty$, in view of the fact that the integral in (A.2) vanishes for $p > m$.

Next we represent

$$\frac{1}{(m-M)!} = \frac{1}{2\pi i}\int_{C_1} \frac{e^\tau}{\tau^{m-M+1}}d\tau \tag{A.4}$$

where C_1 is any closed loop in the τ-plane. Using (A.4) in (A.3) and evaluating the resulting (finite) geometric sums, we are led to

$$Q_m = \frac{m!}{(2\pi i)^2}\int_{C_0}\int_{C_1}\frac{\rho}{\tau+\rho t}\left[\left(\frac{1}{\tau}\right)^{m+1}+\left(\frac{-1}{\rho t}\right)^{m+1}\right] \times \exp\{\tau\}\exp\{z\log(1+t)\}dtd\tau \tag{A.5}$$

We can clearly choose the contours C_0 and C_1 in (A.5) so that the second part [involving $(-\rho t)^{-m-1}$] drops out and in the first part we evaluate the integral over t to get

$$Q_m = \frac{m!}{2\pi i}\int_{C_1}\tau^{-m-1}e^{\tau}\exp\left[z\log\left(1-\frac{\tau}{\rho}\right)\right]d\tau$$

from which (4.2) follows by setting $\tau \to \rho t$.

APPENDIX B

We compute the correction term for the approximation in Lemma 2 (ii). Again setting $\phi(t) = \rho t - m\log t + z\log(1-t)$ we easily obtain

$$\tfrac{1}{2}\phi''\left(\sqrt{m/\rho}\right)(t-\sqrt{m/\rho}\,)^2 \sim -\tfrac{1}{2}\rho^{-1/3}\left(1-\sqrt{m/\rho}\right)^{-2/3}s^2,$$
$$\tfrac{1}{6}\phi'''\left(\sqrt{m/\rho}\right)\left(t-\sqrt{m/\rho}\right)^3 = -\tfrac{1}{3}\sqrt{\rho/m}s^3 + O(\rho^{-2/3})$$

and

$$\tfrac{1}{24}\phi^{(\mathrm{iv})}\left(\sqrt{m/\rho}\right)\left(t-\sqrt{m/\rho}\right)^4 \sim \tfrac{1}{4}\rho^{-1/3}\sqrt{\rho/m}\left(\sqrt{\rho/m}-2\right)\left(1-\sqrt{m/\rho}\right)^{-2/3}s^4.$$

Here s and η are defined as in Section 4 [see below (4.13)]. From the above it follows that

$$\begin{aligned} t^{-1}&\exp\{\phi(t)\} \\ &= \frac{\exp\{\phi(\sqrt{m/\rho}\,)\}}{\sqrt{m/\rho}}\left[\exp\{-s\eta\}\exp\{-\sqrt{\rho/m}s^3/3\}\right] \\ &\quad\times\left\{1+\rho^{-1/3}\left[-\frac{(1-\sqrt{m/\rho}\,)^{1/3}s}{\sqrt{m/\rho}} - \frac{s^2}{2(1-\sqrt{m/\rho}\,)^{2/3}} + \frac{\sqrt{\rho/m}(\sqrt{\rho/m}-2)s^4}{4(1-\sqrt{m/\rho}\,)^{2/3}}\right] + O(\rho^{-2/3})\right\}. \end{aligned} \tag{B.1}$$

We use (B.1) to approximate Q_m in (4.4), which yields

$$Q_m = \left(\frac{m}{\rho}\right)^m \exp\{-m\}\sqrt{2\pi}\rho^{1/6}\left(1-\sqrt{m/\rho}\right)^{1/3} \times\exp\left\{\phi\left(\sqrt{m/\rho}\right)\right\}[I_0+\rho^{-1/3}I_1+O(\rho^{-2/3})] \tag{B.2}$$

where

$$I_0 = \frac{1}{2\pi i}\int_C \exp\{-s\eta\}\exp\left\{-\sqrt{\rho/m}s^3/3\right\}ds$$

and

$$I_1 = \frac{1}{2\pi i}\int_C \exp\{-s\eta\}\exp\left\{-\sqrt{\rho/m}s^3/3\right\} \times\left\{-\frac{(1-\sqrt{m/\rho}\,)^{1/3}s}{\sqrt{m/\rho}} - \frac{s^2}{2(1-\sqrt{m/\rho}\,)^{2/3}} + \frac{\sqrt{\rho/m}(\sqrt{\rho/m}-2)s^4}{4(1-\sqrt{m/\rho}\,)^{2/3}}\right\}ds.$$

Here the contour C is such that s goes from $\infty\ e^{-2\pi i/3}$ to $\infty\ e^{2\pi i/3}$. We set $s = (m/\rho)^{1/6}y$ and $-\eta(m/\rho)^{1/6} = u$. As previously noted,

$$Ai(u) = \frac{1}{2\pi i}\int_C \exp\{uy\}\exp\{-y^3/3\}dy$$

and hence the ℓth derivative of Ai has the integral representation

$$Ai^{(\ell)}(u) = \frac{1}{2\pi i}\int_C y^\ell \exp\{uy\}\exp\{-y^3/3\}dy. \tag{B.3}$$

We use (B.3) to evaluate the integrals in I_1 and also use the Airy equation $Ai''(u) = uAi(u)$, from which it follows that $Ai^{(\mathrm{iv})}(u) = 2Ai'(u) + u^2Ai(u)$. We thus obtain $I_0 = (m/\rho)^{1/6}Ai(u)$ and

$$I_1 = \left(\frac{m}{\rho}\right)^{1/6}\left\{-\left(1-\sqrt{\frac{m}{\rho}}\right)^{1/3}\left(\frac{\rho}{m}\right)^{1/3}Ai'(u) - \frac{(m/\rho)^{1/3}}{2(1-\sqrt{m/\rho}\,)^{2/3}}Ai''(u) + \frac{(\rho/m)^{1/3}(1-2\sqrt{m/\rho}\,)}{4(1-\sqrt{m/\rho}\,)^{2/3}}Ai^{(\mathrm{iv})}(u)\right\}$$

which leads to the following approximation for Q_m

$$\begin{aligned} Q_m = {} & \left(\frac{m}{\rho}\right)^m \exp\{-m\}\sqrt{2\pi}m^{1/6}\left(1-\sqrt{\frac{m}{\rho}}\right)^{1/3}\exp\left\{\phi(\sqrt{m/\rho}\,)\right\} \\ & \times\left\{Ai(u) + \rho^{-1/3}\left[\frac{-(\rho/m)^{1/3}}{2(1-\sqrt{m/\rho}\,)^{2/3}}Ai'(u) \right.\right. \\ & \left.\left. + \frac{1}{(1-\sqrt{m/\rho}\,)^{2/3}}\left(\frac{1}{2}\sqrt{\frac{m}{\rho}}\eta + \frac{1-2\sqrt{m/\rho}}{4}\eta^2\right)Ai(u)\right] + O(\rho^{-2/3})\right\}, \\ & u = -\eta\left(\frac{m}{\rho}\right)^{1/6}. \end{aligned} \tag{B.4}$$

If u is not close to a zero of the Airy function, (B.4) is a two-term asymptotic approximation. If u is close to a zero [more precisely, if $u = r_\ell + O(\rho^{-1/3})$ where

$Ai(r_\ell) = 0$], then (B.4) is a one-term asymptotic approximation. A uniform one-term approximation, which is valid for all u, may be obtained simply by dropping the part of the $O(\rho^{-1/3})$ correction term that is proportional to $Ai(u)$. This is sufficient for the calculations in Section 5.

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