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ORDER OF DECAY OF THE WASTED SPACE FOR A STOCHASTIC PACKING PROBLEM [1]

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A packing of a collection of rectangles contained in $[0, 1]^2$ is a disjoint subcollection; the *wasted space* is the measure of the area of the part of $[0, 1]^2$ not covered by the subcollection. A *simple* packing has the further restriction that each vertical line meets at most one rectangle of the packing. Given a collection of N independent uniformly distributed subrectangles of $[0, 1]$, we proved in a previous work that there exists a number K such that the wasted space W_N in an optimal simple packing of these rectangles satisfies for all N

$$EW_N \leq \frac{K}{\sqrt{N}} \exp K\sqrt{\log N}.$$

We prove here that

$$\frac{1}{K\sqrt{N}} \exp \frac{1}{K}\sqrt{\log N} \leq EW_N.$$

1. Introduction. Packing problems are of fundamental importance, in particular in computer science. The reader is referred to the survey [1] for an in depth discussion of this importance. A description of the main stochastic packing problems, reflecting the knowledge of about 1990, will be found there. Since then several new directions have been investigated, some of which will be mentioned here.

A specific feature of the rectangle packing problem considered here is that, not only the size of the rectangles, but also their positions are of importance. In this paper, we consider only rectangles with sides parallel to the axis. Such a rectangle is a product of two intervals. There is an obvious bijection between the set of subintervals of [0, 1] and

$$D = \{(x, y) : 0 \leq x \leq y \leq 1\}.$$

A (uniformly distributed) random interval is a random interval such that the corresponding point is uniformly distributed over D. A (uniformly distributed) random rectangle is the product of two independent random intervals. Remarkable results about the packings (i.e., disjoint subcollections) of random intervals [3] provide motivation for studying the packings (i.e., disjoint subcollection) of random rectangles. The combinatorial difficulties related to the structure of two dimensional packings are better addressed by first studying

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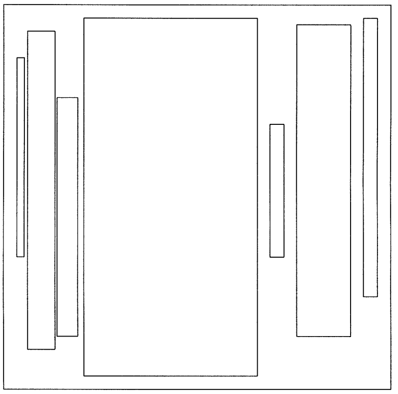

FIG. 1. *A simple packing.*

packings with a simpler structure. The ones considered here will be called *simple* packings, and are defined by the property that any vertical line meets at most one rectangle of the packing. Such a simple packing is shown on Figure 1. (This packing happens to be somewhat representative of a typical optimal simple packing.)

Given a rectangle $I \times J$, only I and $X_I = 1 - |J|$ are relevant towards the behavior of this rectangle $I \times J$ in a simple packing. The random variable X_I is independent of I, and

$$P(X_I \leq t) = t^2. \tag{1}$$

Thus, it seems appropriate to reformulate the question of studying simple packings of rectangles as follows. Given N intervals $I_1, \ldots, I_N$, with associated "weights" $(X_{I_\ell})_{\ell \leq N}$, $0 \leq X_{I_\ell} \leq 1$, we are interested in the following. For a subfamily $\mathscr{I}$ of *disjoint* intervals, consider

$$C_{\mathscr{I}} = \left(1 - \sum_{I \in \mathscr{I}} |I|\right) + \sum_{I \in \mathscr{I}} X_I |I| \tag{2}$$

and define W_N as the infimum of $C_{\mathscr{I}}$ over all possible choices of $\mathscr{I}$. When $I_1, \ldots, I_N$ are independent uniformly distributed, and X_{I_ℓ} are independent satisfying (1), W_N is a random variable; the problem is to evaluate EW_N.

THEOREM 1. *There exists a constant K such that for all N,*

$$\frac{1}{K\sqrt{N}} \exp \frac{1}{K}\sqrt{\log(N+1)} \leq EW_N \leq \frac{K}{\sqrt{N}} \exp K\sqrt{\log(N+1)}.$$

It should be self apparent from this formula that the packing problem under study is rather non-trivial. The problem of the minimum expected wasted space among (non necessarily simple) packings of N random subrectangles of $[0, 1]^2$ is probably much harder, and we have no conjecture to offer. There seems to be no obvious relationship between the problem of minimizing the space wasted by the packing, and the problem of maximizing the cardinality of the packing, problem that was recently solved in [2].

It turns out that it is technically convenient in order to prove Theorem 1 to replace it by a Poissonized version. To do this, we consider a parameter Λ, $\Lambda > 0$, and a homogeneous Poisson point process Π in D of intensity Λ. (Thus, $E \text{ card } \Pi = \Lambda/2$.) This generates a random family $\mathscr{G}$ of intervals. To each such interval I, we associate a random weight X_I with $P(X_I \le t) = t^2$. We then define the random variable W_Λ as the infimum of $C_{\mathscr{I}}$, where $\mathscr{I}$ is now a subfamily of $\mathscr{G}$. We will deduce Theorem 1 from the following.

THEOREM 2. *There exists a constant K such that if $\Lambda \ge 2$, we have*

$$\frac{1}{K\sqrt{\Lambda}} \exp \frac{1}{K}\sqrt{\log \Lambda} \le EW_\Lambda \le \frac{K}{\sqrt{\Lambda}} \exp K\sqrt{\log \Lambda}.$$

The upper bound was proved in [5], and the aim of the present paper is to prove the lower bound. The upper bound of [5] is obtained by a recursion procedure that amounts to the analysis of a specific packing strategy. The lower bound, which amounts to proving that no strategy can produce an essentially better result, is more difficult.

2. Proof. We first explain the relationship between Theorem 1 and 2. [This is routine, and the creative part of the proof starts with equation (4).] Let us start by observing that EW_N decreases as N increases, since having more intervals gives us the choice of more packings. Next, the fundamental property of the Poisson point process is conditionally on card $\Pi = N$, we can view Π as $\{Z_1, \dots, Z_N\}$ where $(Z_i)_{i \le N}$ are i.i.d. uniform over D. Thus

$$E(W_\Lambda) = \sum_{M \ge 0} P(\text{card } \Pi = M) EW_M.$$

Thus, given integers N, N', and since $W_M \le 1$,

$$P(\text{card } \Pi \le N') EW_{N'} \le EW_\Lambda \le EW_N + P(\text{card } \Pi \le N).$$

If we recall that card Π is a Poisson random variable of expectation $\Lambda/2$, we see that if we take $\Lambda = 4N$, $N' = 3N$, for large N, we have

$$\frac{1}{2} EW_{3N} \le EW_\Lambda \le EW_N + \exp\left(-\frac{N}{K'}\right),$$

for a certain constant K', so that Theorem 2 follows from Theorem 1.

We now prove Theorem 2. We will fix the value of Λ. For p integer, $p \ge 0$, we will denote by $W_{t,p}$ the minimum wasted space when the interval $[0, 1]$ is replaced by $[0, t]$ ($t \le 1$), and when one restricts the families $\mathscr{I}$ of (2) to those consisting only of intervals I for which $X_I \ge 2^{-p}$. Thus, if for $x_1 < x_2$ and a family $\mathscr{I}$ of disjoint intervals, contained in $[x_1, x_2]$, we set

$$C_{\mathscr{I}}([x_1, x_2]) = x_2 - x_1 - \sum_{I \in \mathscr{I}} |I| + \sum_{I \in \mathscr{I}} X_I |I|,$$

then $W_{t,p}$ is the minimum of $C_{\mathscr{I}}([0,t])$ when $\mathscr{I}$ is a disjoint subfamily of random intervals I contained in $[0,t]$ and for which $X_I \geq p$. The main purpose of the parameter p is to allow induction over p.

For $x \geq 1$, we set $f(x) = \sqrt{\log x}$. We consider two parameters L, L' and the function

$$\varphi(x) = \frac{1}{L'} \exp\left(\frac{1}{L} f(x)\right). \tag{3}$$

The parameters L, L' will be adjusted later. We consider the following statement:

$$\text{Given any } t \leq 1 \text{ with } t^2\Lambda \geq 1 \text{ and any } 1/16 \leq u < 1, \text{ we have}$$
$$P\left(W_{t,p} \leq \frac{u}{\sqrt{\Lambda}}\varphi(t^2\Lambda)\right) \leq u^{f(t^2\Lambda)}. \tag{4}$$

We will show that if L and L' are large enough (but of course *independent* of p), we can prove this statement by induction over p. Taking $t = 1, u = 1/2$, $p \to \infty$ will then prove the result.

Before the proof starts, we should comment on some remarkable features of (4). It will be obvious during the proof that it would not work to use as induction hypothesis such a statement for a fixed value of u (say $u = 1/2$). On the other hand, a statement such as (4) is wrong for large p and small u. For $u \to 0$, the dependence of the left hand side of (4) is in u^2. It is in fact quite unexpected that a statement like (4), involving a limited range for u, can be proved by induction. Most of the difficulty of the present proof lies in discovering a suitable form of the induction hypothesis, rather than in proving it works.

In the case $p = 0$, there is nothing to prove since with probability 1, there are no intervals I with $X_I = 1$. We now assume that we have proved (4) for all $t \leq 1$ with $t^2\Lambda \geq 1$, and all integers $p_1 < p$, and we prove it for p, and all $t \leq 1$ with $t^2\Lambda \geq 1$. We fix t and we assume $f(t^2\Lambda) \geq 3L$. The case $f(t^2\Lambda) \leq 3L$ will be treated separately later. First we observe that we can assume that

$$\frac{\varphi(t^2\Lambda)}{t\sqrt{\Lambda}} > 2^{-p}. \tag{5}$$

Indeed, otherwise if $\mathscr{I}$ consists of intervals I for which $X_I \geq 2^{-p}$, we have

$$C_{\mathscr{I}}([0,t]) = t - \sum_{I \in \mathscr{I}} |I| + \sum_{I \in \mathscr{I}} X_I |I| \geq 2^{-p} t \geq \frac{\varphi(t^2\Lambda)}{\sqrt{\Lambda}} \tag{6}$$

so that, if $u < 1$,

$$P\left(W_{t,p} \leq \frac{u}{\sqrt{\Lambda}}\varphi(t^2\Lambda)\right) = 0,$$

and the proof is complete. Consider the largest integer p_1 such that

$$2^{-p_1} \geq \frac{\varphi(t^2\Lambda)}{t\sqrt{\Lambda}}.$$

Thus by (5), $p_1 < p$ and

$$2^{-p_1} \le \frac{2\varphi(t^2\Lambda)}{t\sqrt{\Lambda}}. \tag{7}$$

Let us denote by $\mathscr{G}_0$ (resp. $\mathscr{G}_1$) the collection of random intervals I for which $X_I < 2^{-p_1}$ (resp. $X_I \ge 2^{-p_1}$). We will study $W_{t,p}$ conditionally on $\mathscr{G}_0$. Given $0 < x_1 < x_2 < t$, define the event Ω_{x_1,x_2} as follows:

(8) There exist two subfamilies $\mathscr{I}_1, \mathscr{I}_2$ of $\mathscr{G}_1$ such that $C_{\mathscr{I}_1}([0,x_1]) + C_{\mathscr{I}_2}([x_2,t]) \le u\Lambda^{-1/2}\varphi(t^2\Lambda)$.

We claim that if $W_{t,p} \le u\Lambda^{-1/2}\varphi(t^2\Lambda)$, there must exist $x_1, x_2 \in [0,t]$ with the following properties:

(9) Each of x_1, x_2 is the end point of an interval of $\mathscr{G}_0$

and

(10) Ω_{x_1,x_2} occurs.

Observe that in (9), we *do not* require that x_1, x_2 are endpoints of the same interval of $\mathscr{G}_0$. To prove the claim, by definition of $W_{t,p}$ there exists a disjoint family $\mathscr{I}$ of random intervals I, each of which satisfies $X_I \ge 2^{-p}$ and for which

$$C_{\mathscr{I}}([0,t]) \le u\Lambda^{-1/2}\varphi(t^2\Lambda). \tag{11}$$

As (6) shows, at least one of these intervals I must satisfy $X_I < 2^{-p}$. These might be several such intervals. Denote by x_1 (resp. x_2) the smallest (resp. largest) number x such that x is the endpoint of an interval $I \in \mathscr{I}$ for which $X_I < 2^{-p}$. Denote by $\mathscr{I}_1$ (resp. $\mathscr{I}_2$) those intervals I of $\mathscr{I}$ that are contained in $[0,x_1]$ (resp. $[x_2,t]$). Since trivially

$$C_{\mathscr{I}_1}([0,x_1]) + C_{\mathscr{I}_2}([x_2,t]) \le C_{\mathscr{I}}([0,t]),$$

we have proved (8).

We claim that

$$P\left(\Omega_{x_1,x_2}|\mathscr{G}_0\right) = P(X_1 + X_2 \le u\Lambda^{-1/2}\varphi(t^2\Lambda)) \tag{12}$$

where X_1, X_2 are independent copies of W_{x_1,p_1} and W_{t-x_2,p_1} respectively. This is because the family $\mathscr{G}_{1,1}$ of the intervals of $\mathscr{G}_1$ (and their weights) contained in $[0,x_1]$ is independent of the family $\mathscr{G}_{1,2}$ of intervals of $\mathscr{G}_1$ contained in $[x_2,t]$. Moreover, conditionally on $\mathscr{G}_0$, we can generate these families of intervals in the following manner. We generate the intervals and their weights according to the original process, and we throw away those with weights $\le 2^{-p_1}$ (which makes it harder to construct packings).

We denote by P_{x_1,x_2} the probability (12); we thus have

$$P\left(W_{t,p} < \frac{u}{\sqrt{\Lambda}}\varphi(t^2\Lambda)|\mathscr{G}_0)\right) \le \sum P_{x_1,x_2} \tag{13}$$

where the summation is over all $x_1 \leq x_2$, x_1, x_2 endpoints of (possibly different) intervals of $\mathscr{I}_0$.

In order to estimate the expected value of the right hand side of (13), in a first stage, we will rule out the case where x_1, x_2 can be "too close to the endpoints" of $[0, t]$ (event Ω_1 below), where "too close to the endpoints" is quantified by a number s defined by

$$\varphi(s^2\Lambda) = \tfrac{3}{4}\varphi(t^2\Lambda), \tag{14}$$

or equivalently

$$f(s^2\Lambda) = f(t^2\Lambda) - L\log\tfrac{4}{3}. \tag{15}$$

The existence of s follows from the assumption that $f(t^2\Lambda) \geq 3L$. Of course $s^2\Lambda \geq 1$ and moreover

$$f(s^2L) \geq \tfrac{2}{3}f(t^2L). \tag{16}$$

LEMMA 1. *If we assume that*

$$L' \geq \tfrac{4}{3}, \tag{17}$$

we have

$$\frac{s}{t} \leq \left(\frac{1}{\varphi(t^2\Lambda)}\right)^{L^2/K_0}.$$

Here, as well as in the sequel, $K_0, K_1, \ldots$ denote universal constants, that is, numbers (independent of Λ, L, etc.).

PROOF OF LEMMA 1. Let $r = t/s$. Thus,

$$f(t^2\Lambda) = \sqrt{\log s^2\Lambda + 2\log r}.$$

Use of the elementary inequality

$$\sqrt{a+b} \leq \sqrt{a} + \frac{b}{2\sqrt{a}}$$

for $a, b > 0$ together with (15) then yields

$$0 \leq \frac{\log r}{f(s^2\Lambda)} - L\log\frac{4}{3}$$

and thus

$$\begin{aligned} r \geq \exp\left(L\log\frac{4}{3}f(s^2\Lambda)\right) &= (L'\varphi(s^2\Lambda))^{L^2\log(4/3)} \\ &= \left(L'\frac{3}{4}\varphi(t^2\Lambda)\right)^{L^2\log(4/3)} \\ &\geq (\varphi(t^2\Lambda))^{L^2\log(4/3)}. \end{aligned}$$

$\square$

Next, we consider the event Ω_1 defined as follows:

$$\Omega_1 : \text{There is a random interval } I \subset [0,t] \text{ with } X_I \leq 2^{-p_1} \text{ that has one endpoint either in } [0,s] \text{ or in } [t-s,t]. \tag{18}$$

LEMMA 2. *Under* (17), *we have*

$$P(\Omega_1) \leq 8\left(\varphi(t^2\Lambda)\right)^{2-L^2/K_0}. \tag{19}$$

PROOF. It is good to observe the following, that will be used again. If A is a subset of D, the number of intervals I, the endpoints of which belong to A (when one identifies D with the sets of intervals) and for which $X_I \leq v$ is a Poisson random variable with expectation $\Lambda v^2 \text{Area } A$.

It then follows that the probability of Ω_1 is at most

$$2st2^{-2p_1}\Lambda \leq \frac{8s}{t}\varphi^2(t^2\Lambda) \leq 8\left(\varphi(t^2\Lambda)\right)^{2-L^2/K_0} \tag{20}$$

where we have used (7) and Lemma 1. $\square$

We consider $M \geq \lceil 6\varphi^2(t^2\Lambda)\rceil$ and the event Ω_2 defined as follows:

The number of intervals $I \subset [0,t]$ with $X_I < 2^{-p_1}$ is at most M.

LEMMA 3.

$$P(\Omega_2) \leq \exp(-M).$$

PROOF. This number of intervals is a Poisson random variable with expectation

$$\tfrac{1}{2}t^2\Lambda 2^{-2p_1} \leq 2\varphi^2(t^2\Lambda). \qquad \square$$

It follows from (13) that

$$P\left(W_{t,p} \leq \frac{u}{\sqrt{\Lambda}}\varphi(t^2\Lambda)\right) \leq P(\Omega_1) + P(\Omega_2) + M^2 Q \tag{21}$$

where

$$Q = Q(u) = \sup_{s_1,s_2 \geq s} P\left(X_1 + X_2 \leq \frac{u}{\sqrt{\Lambda}}\varphi(t^2\Lambda)\right)$$

where X_1, X_2 are independent copies of W_{s_1,p_1}, W_{s_2,p_1} respectively. We turn to the task of bounding Q. By induction hypothesis, we have, for $1/16 < v \leq 1$ and $j = 1, 2$,

$$P\left(W_{s_j,p_1} \leq \frac{v}{\sqrt{\Lambda}}\varphi(s_j^2\Lambda)\right) \leq v^{f(s_j^2\Lambda)} \leq v^{f(s^2\Lambda)}$$

since $s_j \geq s$ and $s^2\Lambda \geq 1$. We recall that

$$\varphi(s^2\Lambda) \leq \varphi(s_j^2\Lambda) \leq \varphi(t^2\Lambda) = \tfrac{4}{3}\varphi(s^2\Lambda)$$

so that if $v \geq 1/12$, then

$$v' = v\frac{\varphi(s^2\Lambda)}{\varphi(s_j^2\Lambda)}$$

satisfies $1/16 \leq v' \leq v$, and thus

$$P\left(W_{s_j,p_1} \leq \frac{v}{\sqrt{\Lambda}}\varphi(s^2\Lambda)\right) = P\left(W_{s_j,p_1} \leq \frac{v'}{\sqrt{\Lambda}}\varphi(s_j^2\Lambda)\right) \leq v'^{f(s^2\Lambda)} \leq v^{f(s^2\Lambda)} \tag{22}$$

LEMMA 4. *There exist numbers a_0, K_2 with the following property. Consider $a \geq a_0$. Consider two independent random variables Y_1, Y_2 such that, for $1/12 \leq v \leq 1$ we have for $j = 1, 2$:*

$$P(Y_j \leq v) \leq v^a. \tag{23}$$

Then, for $1/16 \leq u \leq 1$, we have

$$P\left(Y_1 + Y_2 \leq \tfrac{4}{3}u\right) \leq \exp\{-a/K_2\}u^{3a/2}.$$

PROOF. Consider the measure μ_0 that has mass 12^{-a} at zero. Consider the measure μ_1 that has density av^{a-1} on $[1/12, 1]$, and zero elsewhere. It suffices to consider the case where Y_j has distribution $\mu = \mu_0 + \mu_1$. (Y_j is "the smallest possible.") Thus, if $B_t = \{(x, y); x \geq 0, y \geq 0, x + y \leq t\}$, we try to bound $\mu \otimes \mu(B_t)$. We have

$$\mu_0 \otimes \mu_0(B_t) \leq 12^{-2a},$$

$$\mu_0 \otimes \mu_1(B_t) = \mu_1 \otimes \mu_0(B_t) \leq 12^{-a}\int_0^t ax^{a-1}dx \leq 12^{-a}t^a$$

and

$$\mu_1 \otimes \mu_1(B_t) \leq \int\int_{B_t} a^2(xy)^{a-1}dxdy.$$

Now, since $xy \leq ((x + y)/2)^2$, this is at most

$$A = a^2\int\int_{B_t}\left(\frac{x+y}{2}\right)^{2a-2}dxdy.$$

We set $x + y = z$, $x - y = w$, so that $0 \leq z \leq t$, $-1 \leq w \leq 1$ and thus

$$A \leq a^2\int\int_{0\leq z\leq t,\ |w|\leq 1}\left(\frac{z}{2}\right)^{2a-2}\frac{1}{2}dzdw = \frac{2a^2}{2a-1}\left(\frac{t}{2}\right)^{2a-1}.$$

Collecting these estimates, for all $t > 0$ we have

$$\begin{aligned}P(Y_1 + Y_2 \leq t) &\leq \mu \otimes \mu(B_t)\\ &\leq 12^{-2a} + 2\cdot 12^{-a}t^a + \frac{2a^2}{2a-1}\left(\frac{t}{2}\right)^{2a-1}\end{aligned}$$

and thus, for all $u > 0$,

$$P(Y_1 + Y_2 \le \frac{4}{3}u) \le 12^{-2a} + 2 \cdot \left(\frac{1}{9}\right)^a u^a + \frac{2a^2}{2a-1}\left(\frac{2}{3}\right)^{2a-1} u^{2a-1}.$$

For $u \ge 1/16$, the right end side is at most (if $a \ge 2$)

$$\left(12^{-2a}4^{3a} + 2 \cdot \left(\frac{4}{9}\right)^a + \frac{2a^2}{2a-1}\left(\frac{2}{3}\right)^{2a-1}\right) u^{\frac{3a}{2}}.$$

The result is now obvious. □

COROLLARY 1. *If $u \ge 1/16$, $f(s^2\Lambda) \ge a_0$, then*

$$Q(u) \le \exp(-f(s^2\Lambda)/K_3) u^{3f(s^2\Lambda)/2}.$$

PROOF. If we set

$$Y_j = \frac{\sqrt{\Lambda}}{\varphi(s^2\Lambda)} X_j,$$

(22) means that (23) holds for $a = f(s^2\Lambda)$. Thus, if $f(s^2\Lambda) \ge a_0$, we get from Lemma 4 that for $1/16 \le u \le 1$,

$$P\left(Y_1 + Y_2 \le \frac{4}{3}u\right) \le \exp(-a/K_3) u^{3a/2}$$

and since $4/3\varphi(s^2\Lambda) = \varphi(t^2\Lambda)$,

$$P\left(X_1 + X_2 \le \frac{u}{\sqrt{\Lambda}}\varphi(t^2\Lambda)\right) \le \exp(-a/K_3) u^{3a/2}.$$ □

We fix $L \ge 2K_3$ such that

$$\frac{L^2}{K_0} - 2 \ge 2L \log 16 \tag{24}$$

so that (19) gives

$$P(\Omega_1) \le 8 \cdot 16^{-2Lf(t^2\Lambda)}. \tag{25}$$

We combine Corollary 1, (25) to get if $a \ge 1/16$,

$$\begin{aligned} P\left(W_{t,p} \le \frac{u}{\sqrt{\Lambda}}\varphi(t^2\Lambda)\right) &\le 8 \cdot 16^{-2f(t^2\Lambda)} + \exp\{-M\} \\ &\quad + M^2 \exp\{-f(s^2\Lambda)\} u^{3f(s^2\Lambda)/2}. \end{aligned} \tag{26}$$

Now using (16),

$$\exp\{-f(s^2\Lambda)/K_2\} u^{3f(s^2\Lambda)/2} \le \exp\{-f(t^2\Lambda)/K_3\} u^{f(t^2\Lambda)}.$$

Since $f(t^2\Lambda) \geq 3L$, (26) becomes, for $u \geq 1/16$, $M \geq 6\varphi^2(t^2\Lambda)$,

$$P\left(W_{t,p} \leq \frac{u}{\sqrt{\Lambda}}\varphi(t^2\Lambda)\right) \leq \frac{1}{4}u^{f(t^2\Lambda)} + \exp\{-M\} + M^2\exp\{-f(t^2\Lambda)/K_3\}u^{f(t^2\Lambda)}. \tag{27}$$

We take M the largest such that

$$M^2 \leq \tfrac{1}{4}\exp\{f(t^2\Lambda)/K_3\}.$$

Since $L \geq 2K_3$, we have $M \geq 6\varphi^2(t^2\Lambda)$ if $L'^2 \geq 24$. Moreover, there exists $a_1 \geq \mathrm{Max}(3L, a_0)$ such that

$$f(t^2\Lambda) \geq a_1 \Rightarrow e^{-M} \leq \tfrac{1}{4}16^{f(t^2\Lambda)}.$$

Thus the right hand side of (27) is $\leq u^{f(t^2\Lambda)}$, that is we have proved the induction hypothesis provided $f(t^2\Lambda) \geq a_1$, and $L'^2 \geq 24$. To prove (4) when $f(t^2\Lambda) \leq a_1$, it suffices to prove that if $1 \leq t^2\Lambda \leq f^{-1}(a_1) = \exp\{a_1^2\} =: a_2$, then

$$P\left(W_{t,p} \leq \frac{b}{\sqrt{\Lambda}L'}\right) \leq 16^{-a_1} \tag{28}$$

where $b = \exp(a_1/L)$. Let us now indicate the value of Λ in $W_{t,p}$ by writing $W_{t,p} = W_{t,p}^{\Lambda}$. For $u > 0$, by homogeneity $W_{t,p}^{\Lambda}$ is distributed like $tW_{t,1}^{t^2\Lambda}$. Thus (28) amount to say that if $1 \leq t^2\Lambda \leq a_2$, we have

$$P\left(W_{1,p}^{t^2\Lambda} \leq \frac{b}{t\sqrt{\Lambda}L'}\right) \leq 16^{-a_1}.$$

But this follows from the fact that

$$P\left(W_{1,p}^{a_2} \leq \frac{b}{L'}\right) \leq 16^{-a_1}$$

if L' is large enough. □

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