ON STATIONARY STOCHASTIC FLOWS AND PALM PROBABILITIES OF SURFACE PROCESSES

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We consider a random surface Φ in $\mathbb{R}^d$ tessellating the space into cells and a random vector field u which is smooth on each cell but may jump on Φ. Assuming the pair (Φ, u) stationary we prove a relationship between the stationary probability measure P and the Palm probability measure P_Φ of P with respect to the random surface measure associated with Φ. This result involves the flow of u induced on the individual cells and generalizes a well-known inversion formula for stationary point processes on the line. An immediate consequence of this result is a formula for certain generalized contact distribution functions of Φ, and as first application we prove a result on the spherical contact distribution in stochastic geometry. As another application we prove an invariance property for P_Φ which again generalizes a corresponding property in dimension $d = 1$. Under the assumption that the flow can be defined for all time points, we consider the point process N of sucessive crossing times starting in the origin 0. If the flow is volume preserving, then N is stationary and we express its Palm probability in terms of P_Φ.

1. Introduction. The framework within which we work in this paper is a probability space $(\Omega, \mathscr{F}, P)$ and a pair (Φ, u), where Φ is a random surface of $\mathbb{R}^d$ defined on Ω and $u: \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a random vector field. The pair (Φ, u) is assumed to be stationary, that is, distributionally translation invariant, under P. For any given $\omega \in \Omega$ the surface $\Phi(\omega)$ is supposed to be such that its complement, that is, $\mathbb{R}^d \setminus \Phi(\omega)$, is a union of at most countably many open connected sets (cells), the boundary of this union being $\Phi(\omega)$. Any bounded subset of $\mathbb{R}^d$ is supposed to intersect with only a finite number of cells, and the corresponding intersection with $\Phi(\omega)$, if not empty, is assumed to be, essentially, a union of finitely many $(d-1)$-dimensional smooth manifolds. Typical examples are shown in Figures 1 and 2 for the case $d = 2$.

The vector field u is assumed to be smooth on $\mathbb{R}^d \setminus \Phi$ but will typically be discontinuous on Φ. Suppressing dependence on ω in the notation, let G denote a given open cell generated by Φ as described above. Then we assume that the restriction u_G of u to G is divergence free on G and is obtained from a flow on G as follows. There is a G-valued function $\alpha_G(t, x)$, $x \in G$, where t runs through some finite "time interval" $(\sigma_G(x), \tau_G(x))$ containing the time $t = 0$, such that $\alpha_G(0, x) = x$, $\alpha_G(t + s, x) = \alpha_G(t, \alpha_G(s, x))$ for all possible values of s, t and $(\partial/\partial t)\alpha_G(t, x) = u_G(\alpha_G(t, x))$. We assume, moreover, that $\alpha_G(t, x)$ is smooth in (t, x) and that the limits $\lim_{t \to \sigma_G(x)} \alpha_G(t, x)$ and
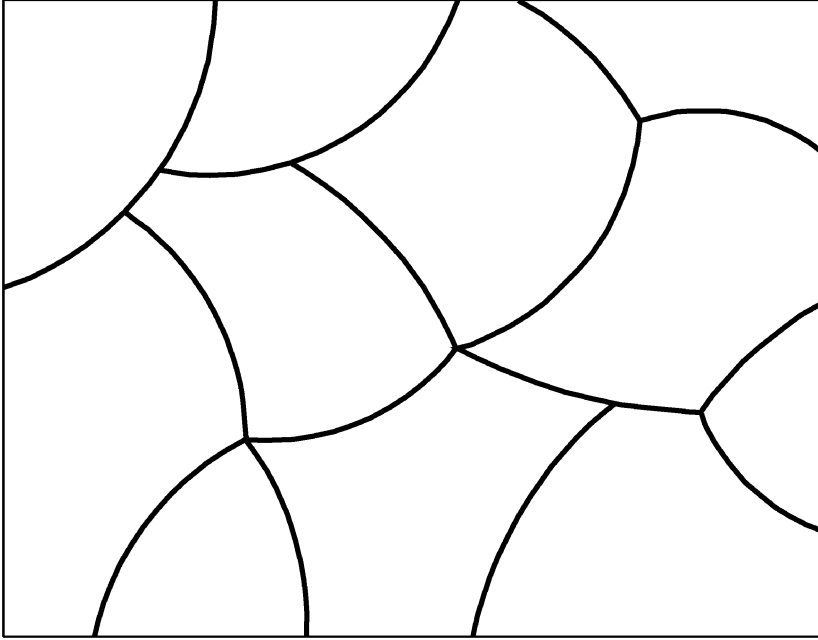
FIG. 1.   *A tessellation of a part of $\mathbb{R}^2$.*
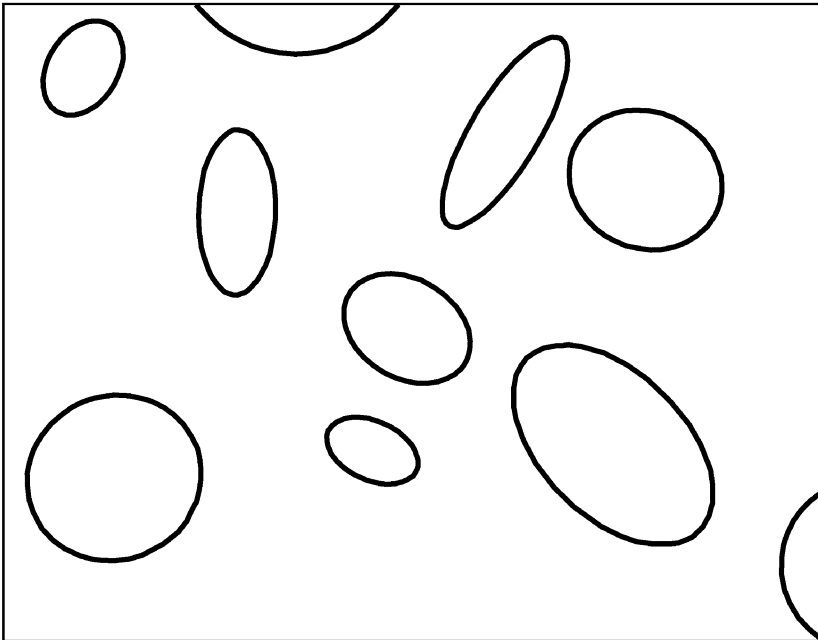

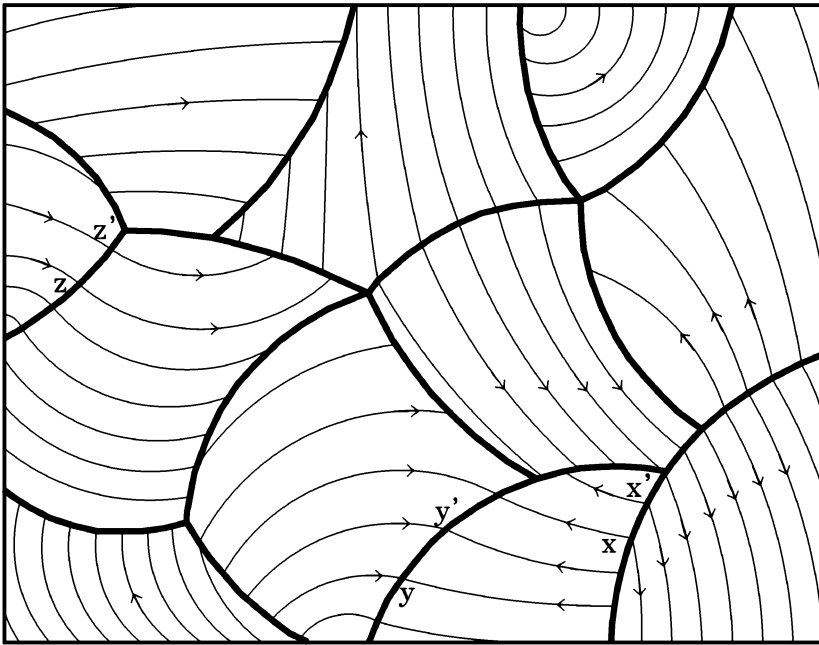FIG. 2.   *Window showing part of $\mathbb{R}^2$ consisting of one "void" cell and "solid" cells.*

FIG. 3. *Cells of Figure* 1 *endowed with directed flow lines.*

$\lim_{t \to \tau_G(x)} \alpha_G(t, x)$ exist and are boundary points of the cell containing x. Thus we may speak of the flow traversing the cell in finite time, and of flow lines, one exactly through each $x \in G$, directed in the sense of increasing time. An illustration is given in Figure 3 for the tessellation depicted in Figure 1. It will be convenient to view α_G as restriction of a certain measurable function $\alpha(\omega, t, x)$, $\omega \in \Omega$, $x \in \mathbb{R}^d$, $t \in \mathbb{R}$.

Although we will have nothing new for the case $d = 1$ it is instructive to look at it briefly. If $d = 1$, then Φ is a stationary point process (on the real line), the cells G are the open intervals between successive points, u_G is identically equal to a constant, and, if G is the interval (a, b), then $\alpha_G(t, x) = u_G t + x$, $x \in G$, where t runs through all values such that $a < u_G t + x < b$. For this case, a fundamental relationship is the so-called Palm inversion formula [see, e.g., Baccelli and Brémaud (1994)]. In its standard form it does not involve u and is given by

$$(1.1) \qquad E[h(0)] = \lambda_\Phi E_\Phi \left[\int_0^{\tau^+(0)} h(s)\, ds \right],$$

where $h: \Omega \times \mathbb{R} \to \mathbb{R}$ is jointly stationary with Φ, E (E_Φ) denote expectation with respect to P (P_Φ), λ_Φ is the intensity of the point process Φ, and $\tau^+(0)$ is the first point of Φ strictly to the right of the origin. Note that the origin, under P_Φ, is itself a point of Φ. Involving u, we may slightly generalize this formula. Define $u^+(0) := \lim_{x \to 0+} u(x)$. If $u^+(0) > 0$, then we let $\tau^+(0)$ now

denote the *time* it takes to travel at the speed $|u^+(0)|$ from the point 0 to its nearest neighbor to the right. If $u^+(0) \leq 0$, we define $\tau^+(0) := 0$. Then

$$E[h(0)] = \lambda_\Phi E_\Phi \left[|u^+(0)| \int_0^{\tau^+(0)} h(\alpha^+(s,0)) \, ds + |u^-(0)| \int_0^{\tau^-(0)} h(\alpha^-(s,0)) \, ds \right],$$

where $\alpha^+(s,0) = \lim_{x \to 0+} \alpha(s,x) = u^+(0)s$, $u^-(0) := \lim_{x \to 0-} u(x)$, $\tau^-(0)$ is the *time* it takes to travel at the speed $|u^-(0)|$ from the point 0 to its nearest neighbor to the left (taken 0 if $u^-(0) \geq 0$) and $\alpha^-(s,0) = \lim_{x \to 0-} \alpha(s,x) = u^-(0)s$. If $u \equiv 1$, then this formula simplifies to (1.1).

The purpose of the present paper is a generalization of the latter formula to the case $d > 1$, and to give some applications of this new inversion formula. Generally speaking, such an inversion formula relates a quantity as seen from a point in space which is "typical" under P to a quantity as seen from a "randomly chosen" point of Φ, that is, a "typical point" of Φ. In practical situations it may be possible to take random samples of observations at typical points in space but not at typical points on Φ, or vice versa. In such situations the inversion formula helps to do statistics for the unobservable part. For example, for transport phenomena modelled in porous media [see, e.g., Bear and Bachmut (1990)], a structure such as depicted in Figure 2 is common. There are solid objects (e.g., clay lentils) strewn about the "void" (e.g., a soil of sand). There is a flow throughout all or part of it, not necessarily of a material nature but, for instance, of a physical quantity such as momentum or energy. Smoothness of the flow is violated at the boundaries between the void and the solid objects. It may be possible to take measurements in the "void," but not on the boundaries between void and solid objects. In such a situation, an inversion formula may be a valuable tool for doing statistics. Indeed, material science is an obvious field for the application of our inversion formula, but our program does not aim at this. Other potential applications of our results are in stochastic geometry. In fact, our model for Φ comprises random tessellations of $\mathbb{R}^d$ or random surface processes in $\mathbb{R}^d$. The monograph by Stoyan, Kendall and Mecke (1995) does not only contain some fundamental results about these concepts but also numerous applications. Palm probabilities play an important role in this theory.

An important feature of our formula is its generality, allowing, apart from joint stationarity, for quite arbitrary relationships between Φ, α and the quantity of interest. Another aspect of our model is that some of the cells generated by Φ might actually represent "empty" space or "holes" carrying no flow. The union of all such holes will be denoted by Ξ. For an illustration, see Figure 4. Our formula is related to a general inversion formula due to Mecke (1967), but does not seem to be a consequence of Mecke's result.

After describing, in Section 2, our basic model in all detail and presenting the inversion formula in Theorem 2.1, we turn to a first application in Section 3, concerning the so-called spherical contact distribution in a germ-grain model. This may be viewed as an application within the field of stochastic geometry, but the latter has, of course, other real applications. In Section 4 it is assumed that at most one flow line starts from a given point $x \in \Phi$ (up
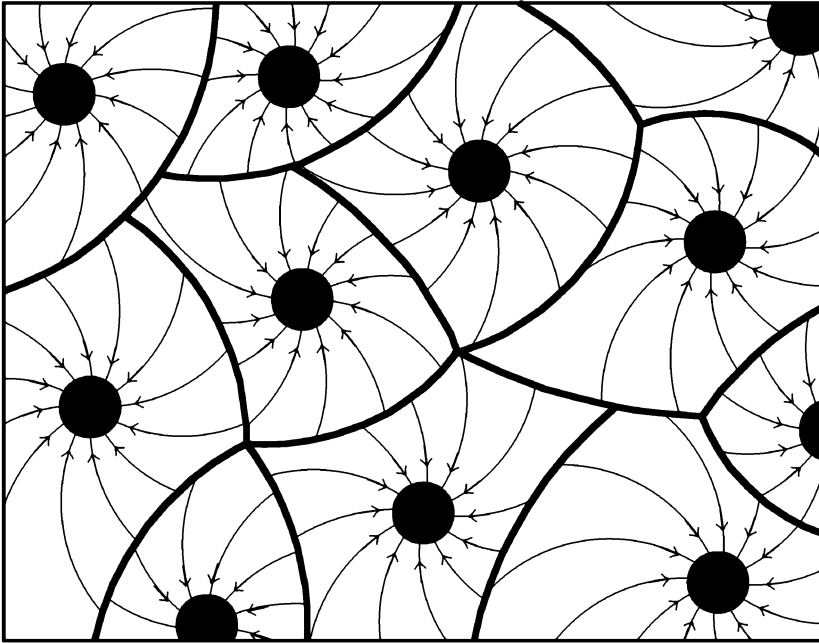
FIG. 4.   *This figure shows the tessellation of Figure* 1, *each cell containing a kernel and a flow directed towards the kernel. The union of the kernels is* Ξ.

to a negligible set of points). This allows following the flow across cell boundaries either ad infinitum or until getting stopped by a flow parallel to the boundary or by empty space (see Figures 7 and 8). The section is devoted to establishing a version of Theorem 2.1 (Theorem 4.1) in which the successive times of such crossings are prominent. Using this theorem, we extend a result in Neveu (1977) and Pitman (1987) from $d = 1$ to general d. In Section 5 we use Theorem 4.1 to generalize an important invariance property of Palm probabilites of stationary point processes on the line [see, e.g., (3.2.3), Chapter 1, in Baccelli and Brémaud (1994)] to our present more general situation. In Section 6 it is assumed that these successive times (starting in the origin 0, say) form a point process N "unbounded at both ends," that is, the flow can be followed into past and future ad infinitum. If the flow is volume preserving, then N becomes stationary and we express its Palm probability P_N in terms of P_Φ. Volume preserving flows are, for example, important in many models of material science [see, e.g., Bear and Bachmut (1990)] or in statistical fluid mechanics [see, e.g., Monin and Yaglom (1971)]. The paper closes with an Appendix summarizing facts about the Palm calculus.

2. Framework and main result.   We consider a *random closed surface* Φ in $\mathbb{R}^d$ defined on the probability space $(\Omega, \mathscr{F}, P)$. This is a random closed set in $\mathbb{R}^d$ [see Matheron (1975) and the Appendix], enjoying some additional

features [see Last and Schassberger (1996)] which we now describe. First we assume that $\Phi = \partial(\mathbb{R}^d \setminus \Phi)$ and $H^{d-1}(\Phi \cap B) < \infty$, for bounded measurable $B \subseteq \mathbb{R}^d$, where ∂G denotes the boundary of a set $G \subseteq \mathbb{R}^d$ and H^{d-1} is the $(d-1)$-dimensional Hausdorff measure on $\mathbb{R}^d$. [If $S \subset \mathbb{R}^d$ is a smooth $(d-1)$-dimensional surface, then $H^{d-1}(S)$ is the surface content of S.] Let $\mathscr{I}(\Phi)$ denote the system of connected components of $\mathbb{R}^d \setminus \Phi$. The elements of $\mathscr{I}(\Phi)$ are mutually disjoint, open and connected subsets of $\mathbb{R}^d \setminus \Phi$ and

$$(2.1) \qquad \mathbb{R}^d \setminus \Phi = \bigcup_{G \in \mathscr{I}(\Phi)} G.$$

Each connected subset of $\mathbb{R}^d \setminus \Phi$ is contained in one element of $\mathscr{I}(\Phi)$ and we assume that each compact $B \subseteq \mathbb{R}^d$ is hit by only a finite number of the cells $G \in \mathscr{I}(\Phi)$. Together with (2.1) this implies that

$$(2.2) \qquad \Phi = \bigcup_{G \in \mathscr{I}(\Phi)} \partial G.$$

With the definition

$$(2.3) \qquad \Phi(B) := H^{d-1}(\Phi \cap B),$$

we may look at Φ as a random measure on $\mathbb{R}^d$ [see Zähle (1982)]. We let $R(\Phi)$ denote the set of all *regular* boundary points of $\mathbb{R}^d \setminus \Phi$, that is, the set of all points $x \in \Phi$ which have an open neighborhood U, such that $\Phi \cap U$ is a smooth (i.e., of class C^1) $(d-1)$-dimensional manifold. Then $R(\Phi)$ is a smooth $(d-1)$-dimensional manifold while the set $S(\Phi) := \Phi \setminus R(\Phi)$ of all singular boundary points of $\mathbb{R}^d \setminus \Phi$ is closed and is assumed to satisfy $H^{d-1}(S(\Phi)) = 0$ and to depend measurably on $\omega \in \Omega$. We further assume that $\nu = \{\nu(x): x \in \mathbb{R}^d\}$ is a *normal field* of Φ, that is, a random vector field of unit vectors such that $\nu(x)$ is normal to Φ whenever $x \in R(\Phi)$. We can and will assume that both ν and $S(\Phi)$ are measurable with respect to the σ-field $\sigma(\Phi)$ generated by Φ.

To equip our model with additional structure, we let $\mathscr{I}'(\Phi)$ denote a subset of $\mathscr{I}(\Phi)$ and assume that $u = \{u(x): x \in \mathbb{R}^d\}$ is a random field of $\mathbb{R}^d$-vectors such that its restriction u_G to any given $G \in \mathscr{I}'(\Phi)$ is the velocity field of a uniquely defined flow α_G on G (see below) and that $u(x) = 0$ for $x \in \Xi$, where Ξ is the random closed set,

$$\Xi := \bigcup_{G \in \mathscr{I}(\Phi) \setminus \mathscr{I}'(\Phi)} G^{\mathrm{cl}},$$

and G^{cl} denotes the closure of G.

Now, before going into details about the flow α_G, let us state our basic assumption, namely that the quintuple $(\Phi, \nu, S(\Phi), u, \Xi)$ be *stationary*. Stationarity is expressed in terms of an abstract measurable flow $\{\theta_x: x \in \mathbb{R}^d\}$ of isomorphisms on the basic probability space $(\Omega, \mathscr{F}, P)$ such that $P \circ \theta_x = P$ for all $x \in \mathbb{R}^d$ (see Appendix for more details). For instance we have $\Phi \circ \theta_y = \{x - y: x \in \Phi\}$, $y \in \mathbb{R}^d$, and $u(\omega, x) = u(\theta_x \omega, 0)$ for all

$(\omega, x) \in \Omega \times \mathbb{R}^d$. Without risk of ambiguity we shall henceforth use Φ as a shorthand for the triple $(\Phi, \nu, S(\Phi))$.

We now introduce the flow generated by u in detail. Suppressing dependence on ω we assume that for each $G \in \mathscr{G}'(\Phi)$, there is an open subset A_G of $\mathbb{R} \times G$ and a *flow* $\alpha_G \colon A_G \to G$. This is to say that, for $x \in G$, the set $A_G(x) := \{t \colon (t, x) \in A_G\}$ is an open finite interval $(\sigma_G(x), \tau_G(x))$ containing 0 and that

(2.4) $$\alpha_G(0, x) = x \quad \text{and} \quad \alpha_G(s, \alpha_G(t, x)) = \alpha_G(s + t, x),$$

for all arguments where both sides are defined. The flow α_G is assumed to be continuously differentiable in both t and x. Furthermore, we assume that the corresponding *velocity field* is independent of t and given as the restriction u_G of u to G. Hence

(2.5) $$\frac{\partial}{\partial t} \alpha_G(t, x) = u_G(\alpha_G(t, x)), \qquad (t, x) \in A_G.$$

Moreover, u_G is assumed to be smooth on G and *divergence-free*. For a possible generalization of the latter condition of *incompressibility* we refer to Remark 2.4. We assume that the limits

$$\alpha_G(\sigma_G(x), x) := \lim_{t \to \sigma_G(x)} \alpha_G(t, x), \qquad \alpha_G(\tau_G(x), x) := \lim_{t \to \tau_G(x)} \alpha_G(t, x)$$

exist for all $x \in G$ and pertain to the boundary ∂G of G. We assume that the vector field u_G can be smoothly extended to $G \cup R(G)$, where $R(G)$ is the set of all regular boundary points of G. This means that one can find, for any $x \in R(G)$, an open neighborhood U and a smooth function $\tilde{u} \colon U \to \mathbb{R}^d$ such that u and $\tilde{u}$ coincide on $U \cap G$. The continuous (and smooth) extension of u to $G \cup R(G)$ is still denoted by u_G. Henceforth, we refer to the set $\{\alpha(t, x) \colon \sigma_G(x) \leq t \leq \tau_G(x)\}$ as the *flow line* through x and view this line as directed in the sense of increasing "time" t. The points $\alpha(\sigma_G(x), x)$ and $\alpha(\tau_G(x), x)$ are the corresponding *starting* and *end points*. If $x \in \partial G$ is a starting point of a unique flow line, then we extend the definitions of $\tau_G(x)$ and $\alpha_G(t, x)$, $0 \leq t \leq \tau_G(x)$, in the natural way. For all other $x \in \partial G$ we let $\tau_G(x) = 0$. If $x \in \partial G$ is the endpoint of a unique flow line, then we can define $\sigma_G(x)$ and $\alpha_G(t, x)$, $\sigma_G \leq t \leq 0$, in the natural way. Our final assumption on α_G is that the union of all flow lines starting in regular boundary points covers G up to a set of Lebesgue measure 0. That is to say that

(2.6) $$H^d(G \setminus G_+) = 0,$$

where H^d denotes Lebesgue measure on $\mathbb{R}^d$, and

$$G_+ := \{\alpha_G(t, x) \colon x \in R(G), \ 0 < t < \tau_G(x)\}.$$

Note that this assumption excludes fields where all flow lines start from a single point, but allows fields where all flow lines end in a single point. We make use of this second case in the proof of Theorem 3.5.

Still suppressing the dependence on $\omega \in \Omega$, we now define, for $x \in G \in \mathscr{G}'(\Phi)$, $\alpha(t, x) := \alpha_G(t, x)$ if $\sigma_G(x) \leq t \leq \tau_G(x)$ and $(\sigma(x), \tau(x)) :=$

$(\sigma_G(x), \tau_G(x))$. All these mappings are of course assumed to be measurable in all their arguments. In the next section we will extend the definition of $\sigma(x)$ $(\tau(x))$ also to such $x \in \Phi$ which are endpoints (starting points) of a unique flow line. Note that, for $x \notin (\Xi \cup \Phi)$,

$$\tau(x) = \inf\{t > 0: \alpha(t, x) \in \Phi\},$$

that is, $\tau(x)$ is the time (assumed to be finite) it takes a particle starting in $x \notin (\Phi \cup \Xi)$ to reach Φ when traveling on the flow.

If $x \in R(\Phi)$, that is, if x is a regular boundary point, then we may use the normal field to distinguish between the two parts separated by $R(\Phi)$ in a neighborhood of x. Hence we define, for $x \in \mathbb{R}^d$,

$$u^-(x) := \lim_{t \to 0-} u(x + t\nu(x)), \qquad u^+(x) := \lim_{t \to 0+} u(x + t\nu(x)),$$

$$\tau^-(x) := \lim_{t \to 0-} \tau(x + t\nu(x)), \qquad \tau^+(x) := \lim_{t \to 0+} \tau(x + t\nu(x)),$$

$$\alpha^-(s, x) := \lim_{t \to 0-} \alpha(s, x + t\nu(x)), \qquad \alpha^+(s, x) := \lim_{t \to 0+} \alpha(s, x + t\nu(x)),$$

$$s \in \mathbb{R},$$

whenever these limits exist. Otherwise we define these expressions as 0. Formally, the last two definitions require a measurable extension of the flow α and of τ. However, the results of this section do not depend on the way this extension is actually performed.

The main result of this paper provides a relationship between the stationary probability measure P and the Palm probability P_Φ associated with the random surface measure generated by Φ:

$$P_\Phi(A) := \lambda_\Phi^{-1} \iint \mathbf{1}\{\theta_x \omega \in A, \ x \in [0, 1]^d\} \Phi(\omega)(dx) P(d\omega), \qquad A \in \mathcal{F},$$

where the intensity

$$\lambda_\Phi := E\Phi([0, 1]^d)$$

is assumed to satisfy $0 < \lambda_\Phi < \infty$. The number $P_\Phi(A)$ can be interpreted as the probability of A given that 0 is a "randomly chosen" point on the surface Φ.

Throughout the paper we use the notation

$$Z_+(x) := \langle u^+(x), \nu(x) \rangle, \qquad Z_-(x) := \langle u^-(x), \nu(x) \rangle, \qquad x \in \mathbb{R}^d.$$

Since $u \equiv 0$ on the interior of Ξ, we have $Z_+(x) = 0$ if $x \in R(\Phi)$ satisfies $x + t\nu(x) \in \Xi$ for all sufficiently small $t > 0$.

THEOREM 2.1. *Let (Φ, Ξ, u) be as described above and let $h = \{h(x): x \in \mathbb{R}^d\}$ be a nonnegative random field which is stationary jointly with Φ and vanishes on Ξ. Then*

$$(2.7) \quad \begin{aligned} Eh(0) = \lambda_\Phi E_\Phi \Big[&|Z_+(0)| \int_0^{\tau^+(0)} h(\alpha^+(s, 0))\, ds \\ &+ |Z_-(0)| \int_0^{\tau^-(0)} h(\alpha^-(s, 0))\, ds \Big], \end{aligned}$$

where E_Φ denotes expectation with respect to P_Φ.

PROOF. Let $G \in \mathcal{G}'(\Phi)$. We will see below that the set $G' := \{x \in G : \sigma_G(x) \in R(G)\}$ is open. From assumption (2.6) we have $H^d(G \setminus G') = 0$. Let $x_0 \in G'$ and let be V an open neighborhhood of $\alpha_G(\sigma_G(x_0), x_0)$ such that $R(G) \cap V$ admits a smooth parametrization $(v_1, \ldots, v_{d-1}) \mapsto z(v_1, \ldots, v_{d-1})$, where $(v_1, \ldots, v_{d-1})$ varies in an open subset W of $\mathbb{R}^d$. We claim that there is an open and bounded neighborhood $U \subseteq G'$ of x_0 such that $\alpha_G(\sigma_G(x), x) \in R(G) \cap V$, $x \in U$. To show this we take a smooth function $\tilde{u} \colon G \cup V \to \mathbb{R}^d$ such that $u = \tilde{u}$ on G. Let $\tilde{\alpha}$ be the maximal flow associated with $\tilde{u}$ and let $\tilde{A} \subseteq \mathbb{R} \times (G \cup V)$ denote its open domain [see Lang (1995)]. We have $A_G \subseteq \tilde{A}$ and $\alpha_G = \tilde{\alpha}$ on A_G. Since $(0, z_0) \in \tilde{A}$ and $\tilde{A}$ is open it is clear that $(t_0, x_0) \in \tilde{A}$, where $t_0 := \sigma_G(x_0)$. By continuity of $\tilde{\alpha}$ we find an $\varepsilon > 0$ as well as an open and bounded neighborhood $U_0 \subseteq G$ of x_0 such that $(t_0 - \varepsilon, t_0 + \varepsilon) \times U_0 \subseteq \tilde{A}$ and

$$(2.8) \qquad \tilde{\alpha}(t, x) \in \tilde{V}, \qquad (t, x) \in (t_0 - \varepsilon, t_0 + \varepsilon) \times U_0,$$

where $\tilde{V}$ is an open neighborhood of z_0 with $\tilde{V}^{\mathrm{cl}} \subseteq V$. A point $x \in R(G)$ is called a *one-sided* boundary point of G if it admits arbitrarily small neighborhoods whose intersection with G is connected. Otherwise it is called a *two-sided* boundary point. We may assume that the points in $R(G) \cap V$ are either all one-sided or all two-sided boundary points. For simplicity we consider only the first case. The second case can be treated similarly. Since z_0 is the starting point of an unique flow line in G, we can conclude that $\tilde{\alpha}(t^-, z_0) \notin G^{\mathrm{cl}}$, where $t_0 - \varepsilon < t^- < t_0$ and $(t^-, z_0) \in \tilde{A}$. Hence we find a nonempty open interval $I^- \subseteq (t_0 - \varepsilon, t_0)$ and an open neighborhood $U^- \subseteq U_0$ of x_0 such that $\tilde{\alpha}(t, x) \notin G^{\mathrm{cl}}$ for all $(t, x) \in I^- \times U^-$. Similarly we can find a nonempty open interval $(t_1, t_2) \subseteq (t_0, t_0 + \varepsilon)$ and an open neighborhood $U \subseteq U^-$ of x_0 such that $\tilde{\alpha}(t, x) \in G$ for all $(t, x) \in (t_1, t_2) \times U$. Since, by definition of t_0, each point in the compact set $\{\tilde{\alpha}(t, x_0) : t_2 \leq t \leq 0\}$ pertains to G we may even assume that $\tilde{\alpha}(t, x) \in G$ for all $t_2 \leq t \leq 0$ and $x \in U$. Any $x \in U$ hence satisfies $\tilde{\alpha}(t, x) \notin G^{\mathrm{cl}}$ for $t \in I^-$ and $\tilde{\alpha}(t, x) \in G$ for $t_1 < t \leq 0$. Combining this with (2.8) shows that the flow line $\tilde{\alpha}(\cdot, x)$ crosses the boundary ∂G for the first time in the interval $[t_0 - \varepsilon, t_0 + \varepsilon]$ and that the limit

$$\lim_{t \to \sigma_G(x)+} \alpha_G(t, x) = \lim_{t \to \sigma_G(x)+} \tilde{\alpha}(t, x)$$

belongs to $R(G) \cap V$, as desired.

For $(v_1, \ldots, v_{d-1}) \in W$ and $s > 0$ such that $\alpha_G(s, z(v_1, \ldots, v_{d-1})) \in U$, we define

$$\psi(v_1, \ldots, v_{d-1}, s) := \alpha(s, z(v_1, \ldots, v_{d-1}))$$

to obtain a smooth parametrization of U. We denote by $\partial_i z$ the partial derivative of z with respect to v_i, $i = 1, \ldots, d-1$, and use a similar notation for ψ.

It is a well-known algebraic fact that the Jacobian $J\psi$ of this transformation satisfies

$$|J\psi(v_1, \ldots, v_{d-1}, s)|$$
$$= \langle \partial_d \psi(v_1, \ldots, v_{d-1}, s), m(v_1, \ldots, v_{d-1}) \rangle \sqrt{H(s, v_1, \ldots, v_{d-1})},$$

where $m(v_1, \ldots, v_{d-1})$ is one of the two unit vectors orthogonal to $\partial_i \psi(v_1, \ldots, v_{d-1}, s)$, $i = 1, \ldots, d-1$, and $H(\cdot)$ is the determinant of the matrix $(\langle \partial_i \psi, \partial_j \psi \rangle)_{i,j=1,\ldots,d-1}$. Since the flow α_G is incompressible, it is easy to show that $J\psi$ does not depend on s. Letting $s \to 0$ on the right-hand side of the above formula, we obtain that

$$(2.9) \quad |J\psi(v_1, \ldots, v_{d-1}, s)|$$
$$= \langle u_G(z(v_1, \ldots, v_{d-1})), \nu_G(z(v_1, \ldots, v_{d-1})) \rangle \sqrt{H_0(v_1, \ldots, v_{d-1})},$$

where $H_0(\cdot)$ is the determinant of the matrix $(\langle \partial_i z, \partial_j z \rangle)_{i,j=1,\ldots,d-1}$ and $\nu_G(x)$, $x \in R(G)$, denotes a unit normal of $R(G)$ at x satisfying $\langle \nu_G(x), u_G(x) \rangle \geq 0$. Let $h: \mathbb{R}^d \to [0, \infty)$ be a measurable function vanishing outside U. Since $\sqrt{H_0(v_1, \ldots, v_{d-1})} \, dv_1 \cdot \cdots \cdot dv_{d-1}$ is the surface element of $R(G) \cap V$ in the coordinates $(v_1, \ldots, v_{d-1})$, we obtain from 2.9 that

$$\int_U h(x) H^d(dx)$$
$$= \int_{\psi^{-1}(U)} h(\psi(v_1, \ldots, v_{d-1}, s)) |J\psi(v_1, \ldots, v_{d-1}, s)| \, dv_1 \cdot \cdots \cdot dv_{d-1} \, ds$$
$$= \iint \mathbf{1}\{\alpha_G(s, z) \in U\} h(\alpha_G(s, z)) \langle u_G(z), \nu_G(z) \rangle H^{d-1}(dz) \, ds.$$

Since $x_0 \in G'$ was arbitrarily chosen, it follows by a standard covering argument that

$$\int_{G'} h(x) H^d(dx) = \iint \mathbf{1}\{\alpha_G(s, z) \in G'\} h(\alpha_G(s, z)) \langle u_G(z), \nu_G(z) \rangle H^{d-1}(dz) \, ds$$

for all measurable $h: \mathbb{R}^d \to [0, \infty)$. Since $\alpha(s, z) \in G'$ iff $z \in R(G)$ and $0 < s < \tau_G(z)$ we obtain that

$$(2.10) \quad \int_G h(x) H^d(dx) = \iint \mathbf{1}\{z \in R(G), 0 < s \leq \tau_G(z)\} h(\alpha_G(s, z)) \langle u_G(z),$$
$$\nu_G(z) \rangle H^{d-1}(dz) \, ds.$$

In the sequel we replace the (deterministic) function h by a nonnegative, random and stationary vector field $h = \{h(x)\}$. Let $B \subset \mathbb{R}^d$ be a measurable set

with $H^d(B) = 1$. Assuming that $h(x) = 0$ for $x \in \Xi$ and applying (2.10) with $\mathbf{1}\{x \in B\}h(x)$ instead of $h(x)$ we obtain

$$
\int_B h(x)H^d(dx)
$$

(2.11)
$$
= \sum_{G \in \mathscr{S}'(\Phi)} \iint \langle u_G(z), \nu_G(z) \rangle \mathbf{1}\{z \in R(G), 0 \le s \le \tau_G(z)\}
$$

$$
\times \mathbf{1}\{\alpha_G(s, z) \in B\}h(\alpha_G(s, z))H^{d-1}(dz)\,ds.
$$

If $z \in R(\Phi)$, then either $z \in R(G) \cap R(G')$ for two different $G, G' \in \mathscr{S}(\Phi)$ or $z \in R(G)$ for exactly one $G \in \mathscr{S}(\Phi)$ [see Lemma 5.4 in Last and Schassberger (1996)]. (In the latter case, z is a two-sided boundary point of G.) Assume for instance that $z \in R(G)$ for some $G \in \mathscr{S}'(\Phi)$. Then z can be a starting point of a flow line in G, an endpoint of a flow line, or the flow could be parallel to the boundary (see also Figure 3). In the second case we have $\tau_G(z) = 0$ while in the third case $\tau_G(z) = 0$ and $\langle u_G(z), \nu_G(z) \rangle = 0$. Taking into account all possible cases, and using (2.2) as well as our assumption $H^{d-1}(S(\Phi)) = 0$, (2.11) leads to

(2.12)
$$
\int_B h(x)H^d(dx) = \iint (Y_+(z, s) + Y_-(z, s))\Phi(dz)\,ds,
$$

where

$$
Y_+(z, s) := |Z_+(z)|\mathbf{1}\{0 \le s \le \tau^+(z)\}\mathbf{1}\{\alpha^+(s, z) \in B\}h(\alpha^+(s, z)),
$$

and $Y_-(z, s)$ is defined similarly. Below we will show that

(2.13)
$$
\tau^+(\omega, z) = \tau^+(\theta_z\omega, 0), \qquad z \in \Phi(\omega),
$$

(2.14)
$$
\alpha^+(\omega, s, z) = \alpha^+(\theta_z\omega, s, 0) + z, \qquad 0 \le s \le \tau^+(\omega, z).
$$

Since h is stationary this implies

$$
h(\omega, \alpha^+(\omega, s, z)) = h(\theta_z\omega, \alpha^+(\omega, s, 0)), \qquad 0 \le s \le \tau^+(\omega, z).
$$

Applying the refined Campbell formula (A.2) we obtain that

$$
E\left[\iint Y_+(z, s)\Phi(dz)\,ds\right]
$$

$$
= \lambda_\Phi E_\Phi\left[\iint |Z_+(0)|\mathbf{1}\{0 \le s \le \tau^+(0)\}\right.
$$

$$
\left. \times \mathbf{1}\{\alpha^+(s, 0) + z \in B\}h(\alpha^+(s, 0))H^d(dz)\,ds\right]
$$

$$
= \lambda_\Phi E_\Phi\left[|Z_+(0)| \int_0^{\tau^+(0)} h(\alpha^+(s, 0))\,ds\right].
$$

Inserting this into (2.12) and using a similar relationship involving Y_-, gives the assertion (2.7).

It remains to check (2.13) and (2.14). To this end we prove

$$(2.15) \qquad \tau(\theta_x\omega, 0) = \tau(\omega, x), \qquad x \notin (\Xi(\omega) \cup \Phi(\omega))$$

and

$$(2.16) \qquad \begin{aligned} &\alpha(\theta_x\omega, t, 0) = \alpha(\omega, t, x) - x, \\ &\qquad x \notin (\Xi(\omega) \cup \Phi(\omega)), \qquad \sigma(\omega, x) < t < \tau(\omega, x). \end{aligned}$$

To prove this we fix ω and $x \notin (\Xi(\omega) \cup \Phi(\omega))$, where we note that the latter holds if and only if $0 \notin (\Xi(\theta_x\omega) \cup \Phi(\theta_x\omega))$. Denoting $\beta(t) := \alpha(\omega, t, x) - x$, $\gamma(t) := \alpha(\theta_x\omega, t, 0)$ we have $\beta(0) = \gamma(0) = 0$ and

$$\frac{d}{dt}\beta(t) = u(\omega, \alpha(\omega, t, x)) = u(\theta_x\omega, \beta(t)), \qquad \sigma(\omega, x) < t < \tau(\omega, x),$$

$$\frac{d}{dt}\gamma(t) = u(\theta_x\omega, \gamma(t)), \qquad \sigma(\theta_x\omega, 0) < t < \tau(\theta_x\omega, 0),$$

where we have used stationarity of the vector field u to get the first relationship. Assume now that $x \in G \in \mathscr{G}'(\Phi)$. Then $G - x \in \mathscr{G}'(\Phi(\theta_x(\omega))$ and $\gamma(t) \in G - x$ for $\sigma(\theta_x\omega, 0) < t < \tau(\theta_1\omega, 0)$. Since β is a maximal flow [see Lang (1995)], we see that $(\sigma(\theta_x\omega, 0), \tau(\theta_x\omega, 0)) \subseteq (\sigma(\omega, x), \tau(\omega, x))$ and $\beta = \gamma$ on the smaller interval. Since γ is also a maximal flow, we obtain also the other conclusion and (2.16), (2.15) follow. Using this and stationarity of the normal field, yields (2.13) and (2.14). $\square$

Theorem 2.1 generalizes the inversion formula for Palm probabilities on the line [see Mecke (1975), Franken, König, Arndt and Schmidt (1981), Baccelli and Brémaud (1994)]. Clearly, of particular interest will be the distribution of the time a particle situated in a typical point of $\mathbb{R}^d \setminus \Xi$ will have to ride on the flow in order to reach the boundary Φ, that is, the distribution $P(\tau(0) \in dr | 0 \notin \Xi)$. Theorem 2.1 allows expressing this hitting time distribution in terms of P_Φ. Denoting by p the *volume fraction* of Ξ, that is, letting $p := P(0 \in \Xi) = EH^d(\Xi \cap [0,1]^d)$ and assuming that $p < 1$, an expression for the density of this distribution is obtained.

COROLLARY 2.2.   *We have*

$$(1 - p)P(\tau(0) \in dr | 0 \notin \Xi)$$
$$= \lambda_\Phi E_\Phi\big[\mathbf{1}\{r \le \tau^+(0)\}|Z_+(0)| + \mathbf{1}\{r \le \tau^-(0)\}|Z_-(0)|\big]\, dr.$$

PROOF.   We apply Theorem 2.1 with $h(x) = g(\tau(x))$, where $g: [0, \infty) \to \mathbb{R}$ is measurable and nonnegative. Because $\tau(\alpha^+(s, x)) = \tau^+(x) - s$ and $\tau(\alpha^-(s, x)) = \tau^-(x) - s$ for $x \notin (\Phi \cup \Xi)$, the result is a straightforward consequence of Fubini's theorem. $\square$

For a special case of the formula just derived, take $\Xi = \varnothing$ and $u(x) \equiv w$, where w is a constant unit vector. Choose ν in such a way that $\langle \nu(x), w \rangle \ge 0$. Then one can interpret $\tau(0)$ as the distance of 0 from Φ in the direction of

w and, in the terminology of stochastic geometry, may call the distribution of $\tau(0)$, a linear contact distribution. This special case of Corollary 2.2 was derived in Last and Schassberger (1996). Another special case of a geometrical nature will be discussed in Section 3. In the case $d = 1$, Φ is a stationary point process, and letting $\nu \equiv 1, \Xi \equiv \varnothing, u \equiv 1$, the formula of Corollary 2.2 is, of course, the well-known formula for the distribution of the forward recurrence time in a stationary point process on the real line.

REMARK 2.3. For stationary point processes on the line, the inversion formula provides the possibility of constructing the stationary probability measure P from P_Φ [see, e.g., Remark 4.4.1, Chapter 1, in Baccelli and Brémaud (1994)]. Likewise, Theorem 2.1 provides such an algorithm in the case of our more general model. Consider the probability space $(\Omega, \mathscr{F}, P_\Phi)$ and let U^+ (U^-) be a random variable uniformly distributed on $[0, \tau^+(0)]$ ($[0, \tau^-(0)]$) and being independent from "everything else." Let h be a nonnegative random variable and $h(x) := h \circ \theta_x$. Then (2.7) can be written as

$$Eh = \lambda_\Phi E_\Phi \left[|Z_+(0)| \tau^+(0) h \circ \vartheta^+_{U^+} + |Z_-(0)| \tau^-(0) h \circ \vartheta^-_{U^-} \right],$$

where, for $t \in \mathbb{R}$, the mapping (shift) $\vartheta^+_t \colon \Omega \to \Omega$ is defined by $\vartheta^+_t(\omega) := \theta_{\alpha^+(\omega, s, 0)}(\omega)$ and ϑ^-_t is defined analogously. If $p^+ := \lambda_\Phi E_\Phi[|Z_+(0)| \tau^+(0)] > 0$, then we let P^+ denote the probability measure which is absolutely continuous with respect to P with density $|Z_+(0)| \tau^+(0)/p^+$. Defining p^- and P^- similarly, we obtain

$$Eh = p^+ E^+[h \circ \vartheta^+_{U^+}] + p^- E^-[h \circ \vartheta^-_{U^-}],$$

where E^+ (E^-) denotes expectation with respect to P^+ (P^-).

REMARK 2.4. Theorem 2.1 can also be formulated for a field u which is not divergence free if there exist a stationary scalar field $K = \{K(x) \colon x \in \mathbb{R}^d\}$ such that $\operatorname{div} Ku = 0$ on $\mathbb{R}^d \setminus \Phi$ and such that our model assumptions are satisfied with Ku instead of u (see Section 3 for an example). Still defining τ and α in terms of u, the result (2.7) takes the following form:

$$
\begin{aligned}
(2.17) \quad E[h(0)K(0)] = \lambda_\Phi E_\Phi \bigg[&|K^+(0)||Z_+(0)| \int_0^{\tau^+(0)} h(\alpha^+(s, 0))\, ds \\
&+ |K^-(0)||Z_-(0)| \int_0^{\tau^-(0)} h(\alpha^-(s, 0))\, ds \bigg].
\end{aligned}
$$

REMARK 2.5. Our analytic assumptions on the flow $\alpha_G, G \in \mathscr{G}'(\Phi)$, can be weakened. For example, it is not necessary that the vector field u_G be continuous on the two-sided regular boundary points of G. To keep things simple, we have tried to avoid additional technicalities.

3. An application to the spherical contact distribution in a germ-grain model. In this section we specify our model in such a way that the flow (which is defined outside Ξ) is directed towards Ξ (see Figure 4). We formulate a version of Theorem 2.1 and then apply it to a germ-grain model. This results in an inversion formula between the basic probability measure P and its Palm probability measure $P_{\Phi'}$ with respect to the so-called *exoskeleton* Φ' of the germ-grain model (Theorem 3.2). From a position outside the grains one can ask for the distance to the nearest grain. Under the basic probability measure P its distribution, the so-called spherical contact distribution, is the one which governs the distance as seen from a typical point in space (outside the grains). In Corollary 3.3 we specialize the result of Theorem 3.2 to obtain a formula which expresses the spherical contact distribution in terms of $P_{\Phi'}$, that is, in terms of quantities as they are seen from a typical point on the exoskeleton. Finally, we consider a point process Ξ and its *Voronoi tessellation* Φ' [see, e.g., Stoyan, Kendall and Mecke (1995)] and express the spherical contact distribution in terms of $P_{\Phi'}$.

The model (Φ, Ξ, u) is as in the previous section. We let Φ' be the stationary random closed set defined as the closure of $\Phi \setminus \partial\Xi$ and assume that

$$(3.1) \qquad\qquad H^{d-1}(\Phi' \cap \Xi) = 0, \qquad P\text{-a.s.}$$

Furthermore we assume that, outside $S(\Phi)$, no point of $\partial\Xi$ is a starting point of a flow line and each point of Φ' is a starting point of exactly two flow lines ending on $\partial\Xi$.

In the theorem below, $P_{\Phi'}$ and $\lambda_{\Phi'}$ denote the Palm probability and the intensity of the random measure $H^{d-1}(\Phi' \cap \cdot)$. In this section we need only assume $\lambda_{\Phi'} < \infty$ but not $\lambda_\Phi < \infty$.

THEOREM 3.1.   *Let (Φ, Ξ, u) be as described above and let $h = \{h(x) : x \in \mathbb{R}^d\}$ be a nonnegative and stationary random field. Then*

$$(3.2) \quad \begin{aligned} Eh(0) = \lambda_{\Phi'} E_{\Phi'} \Bigg[&|Z_+(0)| \int_0^{\tau^+(0)} h(\alpha^+(s, 0))\, ds \\ &+ |Z_-(0)| \int_0^{\tau^-(0)} h(\alpha^-(s, 0))\, ds \Bigg], \end{aligned}$$

where $E_{\Phi'}$ denotes expectation with respect to $P_{\Phi'}$.

PROOF.   The proof of Theorem 2.1 yields in fact that

$$\begin{aligned} Eh(0) = \int \Bigg[&|Z_+(0)| \int_0^{\tau^+(0)} h(\alpha^+(s, 0))\, ds \\ &+ |Z_-(0)| \int_0^{\tau^-(0)} h(\alpha^-(s, 0))\, ds \Bigg] \, dQ_\Phi, \end{aligned}$$

where Q_Φ is the Palm measure of the random measure $H^{d-1}(\Phi \cap \cdot)$. The boundary of Ξ does not contribute to the right-hand side because either $\tau^+(0)$ and $\tau^-(0)$ are both equal to 0 or else, if $\tau^+(0)$ $(\tau^-(0))$ is positive, then $Z_+(0)$ $(Z_-(0))$

equals 0. Using (3.1) and the definition of Palm probability the assertion easily follows. □

Now we turn to our example of a stationary germ-grain model [see, e.g., Stoyan, Kendall and Mecke (1995) and the Appendix]. In this case, specifying (Φ, Ξ, u) begins with specifying Ξ rather than Φ. Our Ξ is assumed to be of the form

$$\Xi = \bigcup_{i=1}^{\infty} (\Xi_i + \xi_i),$$

where the point process $\{\xi_i : i \in \mathbb{N}\}$ of *germs* is assumed to have a positive and finite intensity while the *primary grains* Ξ_i are assumed to be *convex* with boundary $\partial \Xi_i$ being a $(d-1)$-dimensional *manifold* of class C^2. This is our germ-grain model. The volume fraction $p = P(0 \in \Xi)$ of Ξ is positive and we assume that $p < 1$. For the sake of simplicity we exclude the possibility that the boundaries of two different grains $\Xi_i + \xi_i$ coincide on a set of positive $(d-1)$-dimensional Hausdorff measure. For the purpose of specifying Φ we introduce $r(x)$, $x \in \mathbb{R}^d$, as the Euclidean distance from $x \in \mathbb{R}^d$ to Ξ and, letting

$$\Pi(x) := \{y \in \Xi : \|y - x\| = r(x)\},$$

where $\|\cdot\|$ denotes Euclidean norm, we put

(3.3) $$\Phi = \partial \Xi \cup \{x \in \mathbb{R}^d \setminus \Xi : \operatorname{card} \Pi(x) \geq 2\}.$$

This implies that Φ', as defined above, is the *exoskeleton* of Ξ [see Serra (1982)]. Figure 5 illustrates this important notion. Note that $S(\Phi)$ contains the points of the set $\{x \in \Phi' : \operatorname{card} \Pi(x) > 2\}$.

For $x \notin (\Xi \cup \Phi')$ the convexity of the grains implies $\Pi(x) = \{p(x)\}$ for a unique $p(x) \in \partial \Xi$ satisfying $p(x) \in R(\Xi)$ (the set of regular boundary points of Ξ). We proceed to specify our flow α. We want it to have the velocity field

(3.4) $$u(x) = \begin{cases} K(x)v(x), & \text{if } x \notin (\Xi \cup \Phi'), \\ 0, & \text{otherwise,} \end{cases}$$

where $v(x) = (p(x) - x)/r(x)$ and $K(x)$ is positive and chosen in such a way as to render u divergence free (see below).

Regarding the function $K(x)$ let, for $z \in R(\Xi)$ and $d \geq 2$, $c(z) = (c_1(z), \ldots, c_{d-1}(z))$ denote the vector of the *principal curvatures* of $R(\Xi)$ at z defined with respect to the orientation induced by the outer normal to Ξ at z. The components of this vector are nonnegative and unique up to
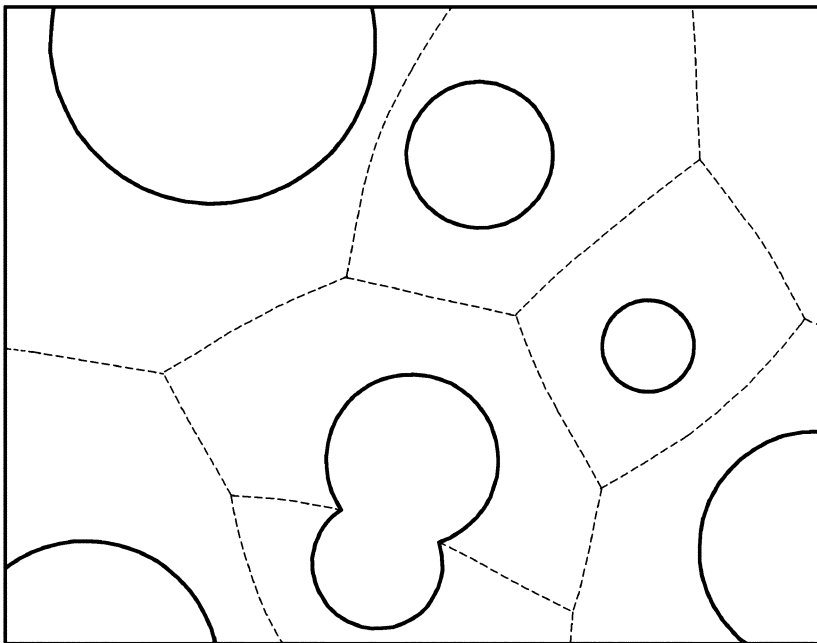
Fig. 5.   *A germ-grain model and its exoskeleton.*

their order. We can assume that the process $(\mathbf{1}\{x \in R(\Xi)\}, c(x)), x \in \mathbb{R}^d$, is stationary jointly with Ξ. Let q denote the function given by

$$q(r, c_1, \ldots, c_{d-1}) := \prod_{i=1}^{d-1} (1 + rc_i), \qquad r, c_1, \ldots, c_{d-1} \in \mathbb{R},$$

if $d \geq 2$, and by $q(r) := 1$ if $d = 1$. Now define

$$K(x) := \begin{cases} (q(r(x), c(p(x)))^{-1}, & \text{if } x \notin (\Xi \cup \Phi'), \\ 0, & \text{otherwise.} \end{cases}$$

Well-known facts about the distance function $r(x)$ [see, e.g., Gilbarg and Trudinger (1977)] can be used to show that indeed $\operatorname{div} Kv = 0$ on $\mathbb{R}^d \setminus (\Xi \cup \Phi')$.

If $x \in \Phi' \setminus S(\Phi)$, then $p^+(x), p^-(x) \in R(\Xi)$, where

$$(p^+(x), p^-(x)) := \begin{cases} (\alpha^+(\tau^+(x), x), \alpha^-(\tau^-(x), x)), & \text{if } x \in \Phi' \setminus S(\Phi), \\ (0, 0), & \text{otherwise.} \end{cases}$$

We let $\beta(x)$ denote half of the angle between the vectors $v^+(x)$ and $v^-(x)$, where $v^+(x) := (p^+(x) - x)/r(x)$ and $v^-(x)$ is defined similarly. These definitions are illustrated in Figure 6. We now use Theorem 3.1 to derive an apparently new relationship between the Palm probability $P_{\Phi'}$ and the stationary probability measure P.
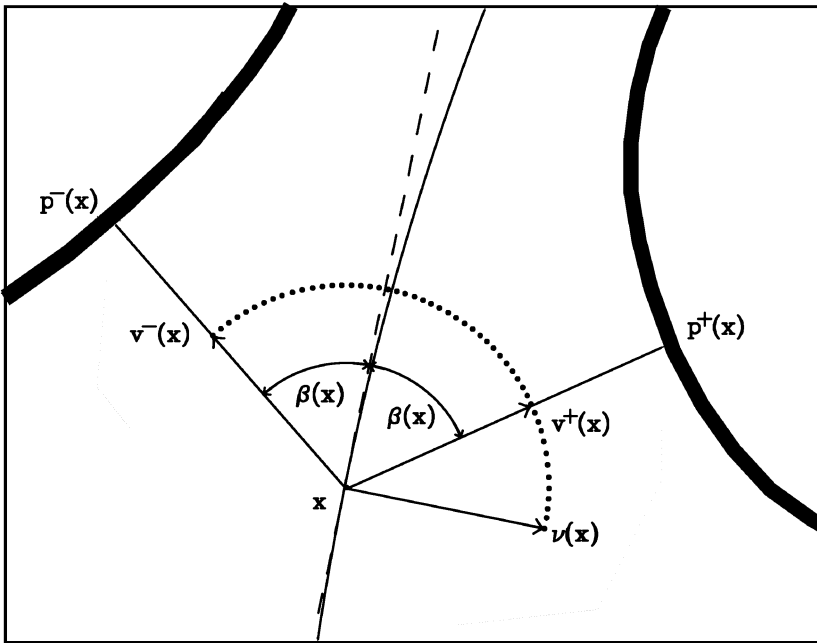
FIG. 6. *A detail of Figure 5 illustrating Theorem 3.2.*

THEOREM 3.2. *Consider the above described germ-grain model for $d \geq 2$. Let $\{h(x): x \in \mathbb{R}^d\}$ be a nonnegative stationary field, vanishing on Ξ. Then*

(3.5)
$$Eh(0) = \lambda_{\Phi'} E_{\Phi'} \left[\sin \beta(0) \int_0^{r(0)} \left(h(sv^+(0)) \frac{q(r(0) - s, c(p^+(0)))}{q(r(0), c(p^+(0)))} \right. \right.$$
$$\left. \left. + h(sv^-(0)) \frac{q(r(0) - s, c(p^-(0)))}{q(r(0), c(p^-(0)))} \right) ds \right].$$

PROOF. Theorem 3.1 yields the formula

$$Eh(0) = \lambda_{\Phi'} E_{\Phi'} \left[\langle v^+(0), \nu(0) \rangle K^+(0) \int_0^{\tau^+(0)} h(\alpha^+(s, 0)) \, ds \right.$$
$$\left. - \langle v^-(0), \nu(0) \rangle K^-(0) \int_0^{\tau^-(0)} h(\alpha^-(s, 0)) \, ds \right].$$

We need to do some computations. It is easy to check that

$$\alpha(t, x) = x + f(t, x)v(x),$$

where

$$\int_0^{f(t,x)} q(r(x) - s, c(p(x))) \, ds = t, \qquad 0 \leq t \leq \tau(x),$$

and where $\alpha(\tau(x), x) = x + f(\tau(x), x)v(x)$, that is, in view of the definition of v, $r(x) = f(\tau(x), x)$ or, equivalently,

$$\int_0^{r(x)} q(r(x) - t, c(p(x))) \, dt = \tau(x).$$

This yields, for $0 \notin (\Xi \cup \Phi')$,

$$\int_0^{\tau(0)} h(\alpha(s, 0)) \, ds = \int_0^{r(0)} h(tv(0))q(r(0) - t, c(p(0))) \, dt.$$

Next we turn to the scalar products $\langle v^+(0), \nu(0) \rangle$ and $\langle v^-(0), \nu(0) \rangle$. A simple geometric argument shows that one of the two possible choices for $\nu(x)$, $x \in (\Phi' \setminus S(\Phi))$, is given by

$$\nu(x) = \|v^-(x) - v^+(x)\|^{-1}(v^-(x) - v^+(x)).$$

With this choice we obtain, by definition of $\beta(0)$, $\langle v^+(0), \nu(0) \rangle = -\langle v^-(0), \nu(0) \rangle = \sin \beta(0)$. Noting that $\|p^+(0)\| = \|p^-(0)\|$ for $0 \in \Phi' \setminus S(\Phi)$, formula (3.5) follows directly. $\square$

Choosing $h(x) = g(r(x))$ in (3.5) we obtain the corollary.

COROLLARY 3.3.   *For all nonnegative, measurable functions g on $[0, \infty)$,*

$$(1 - p)E[g(r(0))|0 \notin \Xi]$$

(3.6)
$$= \lambda_{\Phi'} E_{\Phi'} \left[\sin \beta(0) \int_0^{r(0)} g(s) \left(\frac{q(s, c(p^+(0)))}{q(r(0), c(p^+(0)))} \right. \right.$$
$$\left. \left. + \frac{q(s, c(p^-(0)))}{q(r(0), c(p^-(0)))} \right) ds \right].$$

We illustrate the preceding results by an example with spherical grains.

EXAMPLE 3.4.   Assume that all the grains Ξ_n, $n \in \mathbb{N}$, are balls. If $z \in R(\Xi)$, then $q(r, c(z)) = (1 + r/R(z))^{d-1}$, where $R(z)$ is the radius of the unique ball Ξ_n with $z \in \partial\Xi_n$. Formula (3.5) can be written as

$$Eh(0)$$

(3.7)
$$= \lambda_{\Phi'} E_{\Phi'} \left[\sin \beta(0) \int_0^{r(0)} \left(h(sv^+(0)) \frac{(R(p^+(0)) + r(0) - s)^{d-1}}{(R(p^+(0)) + r(0))^{d-1}} \right. \right.$$
$$\left. \left. + h(sv^-(0)) \frac{(R(p^-(0)) + r(0) - s)^{d-1}}{(R(p^-(0)) + r(0))^{d-1}} \right) ds \right],$$

and (3.6) can be simplified accordingly. If, moreover, all balls have the same deterministic radius R, then (3.7) becomes

$$Eh(0) = \lambda_{\Phi'} E_{\Phi'} \left[\sin \beta(0) \int_0^{r(0)} (h(sv^+(0)) \right.$$

(3.8)
$$\left. + h(sv^-(0))) \frac{(R + r(0) - s)^{d-1}}{(R + r(0))^{d-1}} ds \right].$$

The results in Theorem 3.2 and Corollary 3.3 are new. In Last and Schassberger (1998) a result is obtained expressing the distribution of the spherical contact vector $p(0)$ in terms of quantities as seen from a typical point on the grain surface, rather than on the exoskeleton as above. This result is valid for general convex, not necessarily smoothly bounded, grains, and the derivation is by means of integral geometric arguments. For smoothly bounded grains, as assumed in the present section, this result has a particularly appealing look and can be derived from the formula (3.5) by employing a purely analytic relationship between the surface elements of Φ' and $\partial\Xi$. In fact, rather than dealing with $p(x) - x$, one can consider an arbitrary nonnegative and stationary field $\{h(x)\colon x \in \mathbb{R}^d\}$. We skip the details of the proof and just state the formula. For this purpose we let $P_{\partial\Xi}$ denote the Palm probability of the random measure $H^{d-1}(\partial\Xi \cap \cdot)$ whose intensity $\lambda_{\partial\Xi}$ is assumed to be finite. Then

$$(3.9) \qquad (1 - p)E[h(0)|0 \notin \Xi] = \lambda_{\partial\Xi} E_{\partial\Xi}\left[\int_0^{\sigma(0)} h(s\nu(0))q(s, c(0))\,ds\right],$$

where $E_{\partial\Xi}$ denotes expectation with respect to $P_{\partial\Xi}$, $\nu(z)$ is chosen as the outer normal to Ξ at a regular boundary point $z \in R(\Xi)$ and $\sigma(z) := r(x)$ if there is a unique $x \in \Phi'$ with $z \in \Pi(x)$.

In the remainder of this section we assume that $\Xi = \{\xi_i\colon i \in \mathbb{N}\}$ is a stationary point process. The distances $r(x)$ and the sets $\Pi(x)$, $x \in \mathbb{R}^d$, as well as the exoskeleton Φ' are defined exactly as before. The latter is the Voronoi tessellation generated by Φ. If $x \in \Phi' \setminus S(\Phi)$, then $\Pi(x) = \{p^+(x), p^-(x)\}$ for exactly two $p^+(x), p^-(x) \in \Xi$. For such x we define $v^+(x)$, $v^-(x)$ and $\beta(x)$ exactly as before. The following result is another interesting example for Theorem 3.1.

THEOREM 3.5. *Consider a stationary point process Ξ and its Voronoi tessellation Φ' as described above and assume that $\lambda_{\Phi'} < \infty$. Let $\{h(x)\colon x \in \mathbb{R}^d\}$ be a nonnegative stationary field. Then*

$$Eh(0) = \lambda_{\Phi'} E_{\Phi'}\left[\sin\beta(0)\int_0^{r(0)}(h(sv^+(0))\right.$$

$$(3.10)$$

$$\left. + h(sv^-(0)))\left(\frac{r(0) - s}{r(0)}\right)^{d-1}\,ds\right].$$

In particular,

$$(3.11) \qquad P(r(0) \in dr) = 2\lambda_{\Phi'} E_{\Phi'}\left[\mathbf{1}\{r \le r(0)\}\sin\beta(0)\frac{r^{d-1}}{r(0)^{d-1}}\right]\,dr.$$

PROOF. An informal proof can be obtained from (3.7) by letting there $R \to 0$. A formal proof proceeds as the proof of Theorem 3.2, using a vector field u defined by (3.4), where $v(x), x \notin \Phi'$ is defined as above and $K(x) := r(x)^{-(d-1)}$. It can be easily shown that u is divergence free and $\tau(x) = d^{-1}r(x)^d$. $\square$

4. Excursions. In the present section we continue to work within the model of Section 2. The aim is to establish a fairly straightforward modification of Theorem 2.1, formulated as Theorem 4.1. It will be used to extend a result in Neveu (1977) and Pitman (1987) (Theorem 4.2) and will be the basis of the developments in Sections 5 and 6. The idea is to follow the flow across boundaries between adjacent cells $G \in \mathscr{I}'(\Phi)$. For this purpose we define successive crossing times and crossing points as follows.

Let Φ^+ (Φ^-) denote the set of all $x \in \Phi$, which are starting points (endpoints) of exactly one flow line. First we extend the definitions of σ, τ and α. For $x \in \Phi^+$, we set

$$\tau(x) := \sup\{\tau(y)\colon y \notin (\Phi \cup \Xi), x = \alpha(\sigma(y), y)\},$$

$\alpha(0, x) := x$, and, for $0 < t \le \tau(x), \alpha(t, x) := y$ if $-\sigma(y) = t$ and $\alpha(-t, y) = x$. For $x \in \Phi^-$ we define

$$\sigma(x) := \inf\{\sigma(y)\colon y \notin (\Phi \cup \Xi), x = \alpha(\tau(y), y)\},$$

$\alpha(0, x) := x$, and, for $\sigma(x) \le t < 0, \alpha(t, x) := y$ if $-\tau(y) = t$ and $\alpha(-t, y) = x$.

For $x \in \mathbb{R}^d$ we now inductively define a sequence $(\tau_n(x), \pi_n(x)), n \in \mathbb{Z}$, as follows. First we let

$$(\tau_1(x), \pi_1(x)) := \begin{cases} (\tau(x), \alpha(\tau(x), x)), & \text{if } x \notin (\Phi \cup \Xi) \text{ or } x \in \Phi^+, \\ (0, x), & \text{otherwise.} \end{cases}$$

For $n \ge 1$ we define

$$(\tau_{n+1}(x), \pi_{n+1}(x)) := \begin{cases} (\tau_n(x) + \tau(\pi_n(x)), \\ \quad \alpha(\tau(\pi_n(x)), \pi_n(x))), & \text{if } \pi_n(x) \in \Phi^+, \\ (\tau_n(x), \pi_n(x)), & \text{otherwise.} \end{cases}$$

Further, we define

$$(\tau_0(x), \pi_0(x)) := \begin{cases} (0, x), & \text{if } x \in (\Phi \cup \Xi), \\ (\sigma(x), \alpha(\sigma(x), x)), & \text{otherwise} \end{cases}$$

and for $n \le 0$,

$$(\tau_{n-1}(x), \pi_{n-1}(x)) := \begin{cases} (\tau_n(x) + \sigma(\pi_n(x)), \\ \quad \alpha(\sigma(\pi_n(x), x), \pi_n(x))), & \text{if } \pi_n(x) \in \Phi^-, \\ (\tau_n(x), \pi_n(x)), & \text{otherwise.} \end{cases}$$

Figure 7 is an illustration of these definitions.

Note that we adopt here the convention to keep stepping on place whenever the flow cannot be followed any further. In particular we start stepping on place whenever we get to a point of Φ where two flow lines start or two flow lines end. We want to exclude this by imposing in addition the condition that

$$(4.1) \qquad \Phi \setminus S(\Phi) \subseteq \{x\colon Z_+(x) \ge 0, Z_-(x) \ge 0\}.$$
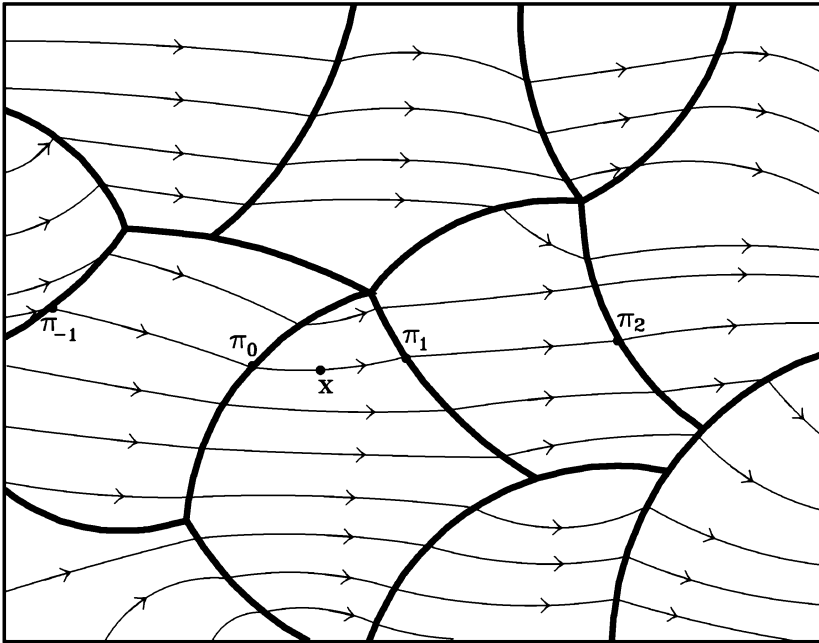
FIG. 7. *An illustration of the crossing points $\pi_n(x) = \alpha(\tau_n(x), x)$ as defined in Section* 4.

In fact, it can be directly checked that this is equivalent to requiring that

$$(4.2) \qquad \Phi \setminus S(\Phi) \subseteq (\Phi^+ \cup \{Z_+ = 0\}) \cap (\Phi^- \cup \{Z_- = 0\})$$

and

$$(4.3) \quad Z_+(x) \geq 0 \quad \text{if } x \in (\Phi^+ \setminus S(\Phi)), \qquad Z_-(x) \geq 0 \quad \text{if } x \in (\Phi^- \setminus S(\Phi)).$$

We further extend the definition of the flow α to satisfy

$$(4.4) \qquad \alpha(t, x) = \alpha(t - \tau_n(x), \pi_n(x)), \qquad \tau_n(x) \leq t < \tau_{n+1}(x).$$

This determines $\alpha(\cdot, x)$ on the interval $(\tau_{-\infty}(x), \tau_{+\infty}(x))$, where

$$\tau_{+\infty}(x) := \sup\{\tau_n(x): n \geq 1\}, \qquad \tau_{-\infty}(x) := \inf\{\tau_n(x): n \leq 0\}.$$

If $t \geq \tau_{+\infty}(x)$ or $t \leq \tau_{-\infty}(x)$ then we set $\alpha(t, x) := x$. Hence α is a measurable function of all its arguments.

THEOREM 4.1. *Let (Φ, Ξ, u) be as described above and let $h = \{h(x): x \in \mathbb{R}^d\}$ be a nonnegative stationary random field which vanishes on Ξ. Then*

$$(4.5) \qquad Eh(0) = \lambda_\Phi E_\Phi \left[Z_+(0) \int_0^{\tau_1} h(\alpha(s, 0)) \, ds \right],$$

where $\tau_n := \tau_n(0)$, $n \in \mathbb{Z}$. Assume, in addition, that the union of all flow lines ending in regular boundary points of $G \in \mathscr{G}'(\Phi)$ covers G up to a set of Lebesgue measure 0. Then

$$Eh(0) = \lambda_\Phi E_\Phi\left[Z_-(0) \int_{\tau_{-1}}^0 h(\alpha(s,0))\,ds \right].$$

Moreover, in this case we have for all $n \in \mathbb{Z}$ that

$$(4.6) \qquad Eh(0) = \lambda_\Phi E_\Phi\left[Z_+(0) \int_0^{\tau_n} h(\alpha(s,0))\,ds - Z_-(0) \int_0^{\tau_{n-1}} h(\alpha(s,0))\,ds \right],$$

provided that $E_\Phi\big[Z_+(0) \int_0^{\tau_n} h(\alpha(s,0))\,ds\big] < \infty$ for $n \geq 1$ or $E_\Phi\big[Z_+(0) \int_{\tau_n}^0 h(\alpha(s,0))\,ds\big] < \infty$ for $n \leq 0$.

PROOF. In analogy to (2.15) and (2.16) one can prove by induction that $\tau_n(x) = \tau_n(0) \circ \theta_x$ and

$$(4.7) \qquad \alpha(\theta_x\omega, t, 0) = \alpha(\omega, t, x) - x, \qquad (\omega, x, t) \in \Omega \times \mathbb{R}^d \times \mathbb{R},$$

if $\tau_{-\infty}(x) < t < \tau_{+\infty}(x)$. Since $\tau_{+\infty}(x) = \tau_{+\infty}(0) \circ \theta_x$ and $\tau_{-\infty}(x) = \tau_{-\infty}(0) \circ \theta_x$, (4.7) even holds for all (ω, x, t). The assumptions of this section imply that $P_\Phi(Z_+(0) \geq 0) = 1$. Therefore the first assertion is a special case of Theorem 2.1, while the proof of the second equality requires only a slight modification of the proof given there. To avoid repetition we will prove the more general equation (4.6) as an application of Theorem 5.1 in the next section. □

The Ω-valued process $\{\theta_{\alpha(s,0)}\colon 0 < s \leq \tau_1\}$ describes an *excursion* of a particle starting in the typical point 0. Adapting the terminology of Pitman (1987), we could call $Q_\Phi = \lambda_\Phi P_\Phi$ the *equilibrium excursion law*. We now extend a formula for point processes on the line [see Neveu (1977), Pitman (1987)] to our model. For simplicity we assume that $\Xi = \varnothing$.

THEOREM 4.2. *Let (Φ, u) be as described above and assume that $\Xi = \varnothing$. Then*

$$P(-\tau_0 \in ds, \theta_{\alpha(\tau_0,0)} \in d\omega) = \lambda_\Phi Z_+(0)(\omega)\mathbf{1}\{s \leq \tau_1(\omega)\}\,ds\,P_\Phi(d\omega).$$

PROOF. We employ (4.5) with $h(x) = \tilde{h}(-\tau_0, \theta_{\tau_0}) \circ \theta_x$, $x \in \mathbb{R}^d$, where $\tilde{h}\colon \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ is measurable. For $0 \in \Phi$ and $0 < s \leq \tau_1$ we have $\tau_0 \circ \theta_{\alpha(s,0)} = -s$ and $\theta_{\alpha(\tau_0,0)} \circ \theta_{\alpha(s,0)} = \theta_0$. Hence

$$E\tilde{h}(-\tau_0, \theta_{\tau_0}) = \lambda_\Phi E_\Phi\left[Z_+(0) \int_0^\infty \tilde{h}(s, \theta_0)\mathbf{1}\{s \leq \tau_1\}\,ds \right],$$

implying the result. □

5. An invariance property of Palm probabilities. An important property of the Palm probability P_Φ of a stationary point process $\Phi = \{T_n : n \in \mathbb{N}\}$ on $\mathbb{R}$ is its invariance under the shifts θ_{T_n} [see, e.g., Chapter 1, Section 3.2 in Baccelli and Brémaud (1994)]. It is the aim of the present section to formulate and prove a corresponding property for general dimension d. We let (Φ, Ξ, u) be as described in Section 2 and assume that (4.1) is satisfied. In addition we assume that the union of all flow lines ending in regular boundary points of $G \in \mathscr{G}'(\Phi)$ covers G up to a set of Lebesgue measure 0. We introduce the mapping $\vartheta \colon \Omega \to \Omega$ by

$$\vartheta(\omega) := \theta_{\alpha(\omega, \tau_1(\omega), 0)} \omega.$$

If $0 \in \Phi$ is a starting point of a unique flow line, then ϑ shifts the origin to the "next" point $\pi_1(0)$ of Φ. The next theorem shows how the Palm probability measure P_Φ is affected by this shift.

THEOREM 5.1. *Let h be a nonnegative random variable satisfying $h = 0$ on the event $\{0 \in \Xi\}$. Under the assumptions made above we have*

$$(5.1) \qquad E_\Phi Z_+(0) h \circ \vartheta = E_\Phi Z_-(0) h.$$

PROOF. It is convenient to introduce for all $t \in \mathbb{R}$ the mapping $\vartheta_t \colon \Omega \to \Omega$ by

$$(5.2) \qquad \vartheta_t \omega := \theta_{\alpha(\omega, t, 0)} \omega, \qquad \omega \in \Omega.$$

From Theorem 4.1 we obtain that

$$(5.3) \qquad Eh(0) = \lambda_\Phi E_\Phi \left[Z_+(0) \int_0^{\tau_1} h \circ \vartheta_s \, ds \right] = \lambda_\Phi E_\Phi \left[Z_-(0) \int_{\tau_{-1}}^0 h \circ \vartheta_s \, ds \right],$$

where we recall from (4.2) and (4.3) that $Z_+(0) \geq 0$, $Z_-(0) \geq 0$ P_Φ-almost surely. In particular we may choose here $h(x) = h_1(\tau_1(x))g(x)$, where $h_1 \colon \mathbb{R} \to [0, \infty)$ is measurable and $g(x) = h_2(\pi_1(x))$ for a nonnegative and stationary process $\{h_2(x) \colon x \in \mathbb{R}^d\}$. [Using (4.7), it is easy to check that $\{h(x) \colon x \in \mathbb{R}^d\}$ is stationary.] Therefore,

$$E_\Phi \left[Z_+(0) \int_0^{\tau_1} h_1(\tau_1 \circ \vartheta_s) g(0) \circ \vartheta_s \, ds \right]$$
$$= E_\Phi \left[Z_-(0) \int_{\tau_{-1}}^0 h_1(\tau_1 \circ \vartheta_s) g(0) \circ \vartheta_s \, ds \right].$$

For $0 \leq s < \tau_1$ we have $g(0) \circ \vartheta_s = h_2(0) \circ \vartheta$ and for $\tau_{-1} < s \leq 0$ we have $g(0) \circ \vartheta_s = h_2(0)$. Furthermore, $\tau_1 \circ \vartheta_s = -s$ for $\tau_{-1} \leq s < 0$ and $\tau_1 \circ \vartheta_s = \tau_1 - s$ for $0 \leq s < \tau_1$. Choosing $h_2(x) := h \circ \theta_x$, $x \in \mathbb{R}^d$, we obtain

$$E_\Phi \left[Z_+(0) h \circ \vartheta \int_0^{\tau_1} h_1(s) \, ds \right] = E_\Phi \left[Z_-(0) h \int_{\tau_{-1}}^0 h_1(-s) \, ds \right].$$

Hence

$$E_\Phi \left[\mathbf{1}\{0 \leq s < \tau_1\} Z_+(0) h \circ \vartheta \right] = E_\Phi \left[\mathbf{1}\{0 \leq s < -\tau_{-1}\} Z_-(0) h \right],$$

for H^1-a.e. $s \in \mathbb{R}$, where H^1 is Lebesgue measure. Letting $s \to 0$ gives

$$E_\Phi[\mathbf{1}\{\tau_1 > 0\}Z_+(0)h \circ \vartheta] = E_\Phi[\mathbf{1}\{\tau_{-1} < 0\}Z_-(0)h].$$

However, by (4.3) we have P_Φ-almost surely that $\{Z_+(0) > 0\} \subseteq \{\tau_1 > 0\}$ and $\{Z_-(0) > 0\} \subseteq \{\tau_{-1} < 0\}$ and the theorem is proved. $\square$

As a first application we prove (4.6) of Theorem 4.1. Let $n \in \mathbb{N}$ and $\{h(x)\}$ be a stationary nonnegative field. Using (4.7) (see also Lemma 6.1 in the next section) it is easy to see that

$$\int_{\tau_n}^{\tau_{n+1}} h(\alpha(s,0)) \, ds = \left[\int_{\tau_{n-1}}^{\tau_n} h(\alpha(s,0)) \, ds \right] \circ \vartheta.$$

Hence we obtain from Theorem 5.1 that

$$E_\Phi \left[Z_+(0) \int_{\tau_n}^{\tau_{n+1}} h(\alpha(s,0)) \, ds \right] = E_\Phi \left[Z_-(0) \int_{\tau_{n-1}}^{\tau_n} h(\alpha(s,0)) \, ds \right].$$

Using induction starting with (4.5), we obtain (4.6), provided that the random variable $Z_+(0) \int_0^{\tau_n} h(\alpha(s,0)) \, ds$ is integrable. The proof for $n \leq 0$ is similar.

REMARK 5.2. Assuming, for simplicity, that $Z_-(0) \circ \vartheta \neq 0$, (5.1) can be rewritten as

$$E_\Phi h = E_\Phi \left[\frac{Z_+(0)}{Z_-(0) \circ \vartheta} h \circ \vartheta \right].$$

In heuristic terms we can express this as follows. If 0 is a typical point of Φ, then $\alpha(\tau_1, 0)$ is a typical point under the probability measure $Z_+(0)/(Z_-(0) \circ \vartheta) \, dP_\Phi$.

EXAMPLE 5.3. Assume that $u(x) \equiv w$ for some unit vector $w \in \mathbb{R}^d$ and choose ν such that $\langle \nu(x), w \rangle \geq 0$ for all x. This defines a parallel deterministic flow $\alpha(s,x) = x + sw$ and $\tau_1(x)$ is the distance from x to the next point $\alpha(\tau_1(x), x)$ of Φ in the direction of w. The mapping ϑ shifts the (typical) point $0 \in \mathbb{R}^d$ to the next point $\alpha(\tau_1(0), 0)$ and (5.1) reads as

(5.4) $$E_\Phi[\langle w, \nu(0) \rangle h \circ \vartheta] = E_\Phi[\langle w, \nu(0) \rangle h].$$

Assume, in addition, that Φ is a process of parallel hyperplanes with normal v satisfying $\langle w, v \rangle \neq 0$. Then (5.4) simplifies to

(5.5) $$E_\Phi h \circ \vartheta = E_\Phi h,$$

which is "almost" the classical invariance property for Palm probability measures of point processes on the real line.
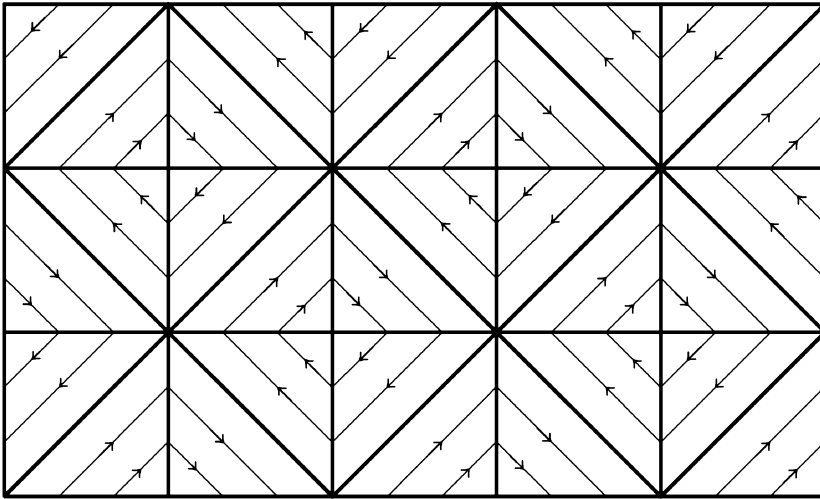
FIG. 8. *This figure shows a regular cell pattern. Starting at an (inner) point of a cell one can follow the flow ad infinitum but returns to the same point infinitely often.*

6. Volume-preserving flows. In this section we consider the model of Section 5; that is, we let (Φ, Ξ, u) be as described in Section 2, assume (4.1) and that the union of all flow lines ending in regular boundary points of $G \in \mathscr{G}'(\Phi)$ covers G up to a set of Lebesgue measure 0. For simplicity we also assume that $\Xi = \varnothing$. We study the point process

$$N(\cdot) := \mathbf{1}\{\tau_{-\infty} = -\infty, \tau_{+\infty} = \infty\} \sum_{n \in \mathbb{Z}} \mathbf{1}\{\tau_n \in \cdot\},$$

assuming that

(6.1) $$\tau_{-\infty} = -\infty, \qquad \tau_{+\infty} = \infty, \quad P\text{-a.s.},$$

where $(\tau_{-\infty}, \tau_{+\infty}) := (\tau_{-\infty}(0), \tau_{+\infty}(0))$. Note that models with loops are not excluded (see Figure 8). Under a natural assumption on α, the point process N becomes stationary and, using Theorem 4.1, we then will express its Palm probability P_N in terms of P_Φ.

The flow α is called *volume preserving* if, for all $t \in \mathbb{R}$, P-almost all ω and all measurable $g \colon \mathbb{R}^d \to \mathbb{R}$,

$$\int g(\alpha(\omega, t, x)) H^d(dx) = \int g(x) H^d(dx).$$

Because u is divergence free on $\mathbb{R}^d \setminus \Phi$, it is well known that α is volume preserving if the velocity field u is smooth everywhere on $\mathbb{R}^d$. In our more general case it is easy to prove that α is volume preserving if and only if

(6.2) $$Z_+(0) = Z_-(0), \quad P_\Phi\text{-a.s.}$$

This could be proved with the help of Theorem 4.1. However, this is a purely analytic fact which has nothing to do with stationarity. A short proof can

be based on Gauss's divergence theorem [cf. also Bear and Bachmut (1990)]. The results in Geman and Horowitz (1975) and Mecke (1975) show that the volume-preserving property is also equivalent to the invariance of P under the shifts $\vartheta_t: \Omega \to \Omega, t \in \mathbb{R}$, defined by (5.2). Indeed, we may apply Theorem 10(i) in Geman and Horowitz (1975) with $Z(\omega) := \alpha(\omega, t, 0)$ for a fixed $t \in \mathbb{R}$. Since by (4.7),

$$x + Z(\theta_x \omega) = x + \alpha(\theta_x \omega, t, 0) = \alpha(\omega, t, x),$$

this theorem yields that $\alpha(\omega, t, \cdot)$ is P-almost surely volume preserving iff P is invariant under ϑ_t.

We first need to prove that the family $\{\vartheta_t: t \in \mathbb{R}\}$ is a flow on $(\Omega, \mathscr{F})$ and that the point process N is adapted to this flow.

LEMMA 6.1.   *For all $s, t \in \mathbb{R}$, $\vartheta_t \circ \vartheta_s = \vartheta_{t+s}$ and $N(\vartheta_s \omega, \cdot) = N(\omega, \cdot + s)$.*

PROOF.   From (4.7),

(6.3) $\alpha(\vartheta_s \omega, t, 0) = \alpha(\omega, t, \alpha(\omega, s, 0)) - \alpha(\omega, s, 0) = \alpha(\omega, s+t, 0) - \alpha(\omega, s, 0).$

Using this equation we obtain

$$\vartheta_t \circ \vartheta_s(\omega) = \theta_{\alpha(\theta_s \omega, t, 0)}(\vartheta_s \omega) = \theta_{\alpha(\omega, s+t, 0)}(\theta_{-\alpha(\omega, s, 0)}(\vartheta_s \omega))$$

$$= \theta_{\alpha(\omega, s+t, 0)} \omega = \vartheta_{t+s} \omega.$$

We define

(6.4) $A_\infty := \{\tau_{-\infty} = -\infty, \tau_{+\infty} = \infty\}$

and note that, for any $t \in \mathbb{R}$, $\omega \in A_\infty$ iff $\vartheta_t \omega \in A_\infty$. The second assertion is then equivalent to

(6.5) $\tau_n \circ \vartheta_s = \tilde{\tau}_n(s) - s, \qquad n \in \mathbb{Z} \quad \text{on } A_\infty,$

where $\tilde{\tau}_1(s) := \inf\{\tau_n: \tau_n > s\}, \tilde{\tau}_{n+1}(s) := \inf\{\tau_m: \tau_m > \tilde{\tau}_n(s)\}, n \geq 1$, and $\tilde{\tau}_n, n \leq 0$, is defined analogously. Equation (6.5) follows by induction, using (6.3). For instance, if $\tau_{+\infty}(\omega) = \infty$,

$$\tau_1(\vartheta_s \omega) = \inf\{t > 0: \alpha(\vartheta_s \omega, t, 0) \in \Phi(\vartheta_s \omega)\}$$

(6.6) $= \inf\{t > 0: \alpha(\omega, t + s, 0) - \alpha(\omega, s, 0) \in \Phi(\theta_{\alpha(\omega, s, 0)} \omega)\}$

$$= \inf\{t > 0: \alpha(\omega, t + s, 0) \in \Phi(\omega)\} = \tilde{\tau}_1(s) - s. \qquad \square$$

The preceding lemma justifies the definition of the Palm probability P_N of N with respect to the family of shifts $\{\vartheta_t\}$, provided that P is invariant under these shifts and the intensity $\lambda_N := E(N(0, 1])$ is finite. By (4.1) the numbers

$$\mu^+ := E_\Phi Z_+(0), \qquad \mu^- := E_\Phi Z_-(0)$$

are nonnegative and we assume in the remainder of the section that $\mu^+ < \infty$ and $\mu^- < \infty$.

THEOREM 6.2.   *Let* (Φ, u) *be as described above and assume that* α *is volume preserving. Then*

$$(6.7) \qquad\qquad\qquad\qquad \lambda_N = \lambda_\Phi \mu$$

and the Palm probability P_N *of* P *with respect to* N *satisfies*

$$(6.8) \qquad\qquad P_N(A) = \mu^{-1} E_\Phi \left[\mathbf{1}_A Z_+(0) \right], \qquad A \in \mathscr{F},$$

where $\mu := \mu^+$.

PROOF.   We have already seen that P is stationary under ϑ_t. By (6.1) and the inversion formula [see Mecke (1967)],

$$E h = \int \left[\int_0^{\tau_1} h \circ \vartheta_s \, ds \right] dQ_N,$$

where Q_N is the Palm measure of N (see Appendix). By Theorem 4.1,

$$\int \left[\int_0^{\tau_1} h \circ \vartheta_s \, ds \right] dQ_N = E_\Phi \left[Z_+(0) \int_0^{\tau_1} h \circ \vartheta_s \, ds \right]$$

and similarly as in the proof of (5.1) it follows that

$$(6.9) \qquad\qquad\qquad \int h \, dQ_N = \lambda_\Phi E_\Phi Z_+(0) h.$$

Taking $h \equiv 1$ yields (6.7) such that (6.8) follows from (6.9).   □

We illustrate with a simple example on the intersection of Φ with a line.

EXAMPLE 6.3.   Let u and α be as in Example 5.3. Theorem 6.2 implies for all measurable $h: \Omega \to [0, \infty)$ and all measurable $B \subset \mathbb{R}^d$ that

$$(6.10) \qquad \lambda_N E_N[h \mathbf{1}\{\nu(0) \in B\}] = \lambda_\Phi E_\Phi[\langle \nu(0), w \rangle h \mathbf{1}\{\nu(0) \in B\}].$$

In particular we obtain a relationship between the (directed) *rose of directions* $P_N(\nu(0) \in \cdot)$ of N and the *rose of directions* $P_\Phi(\nu(0) \in \cdot)$ of Φ:

$$(6.11) \qquad\qquad \lambda_N P_N(\nu(0) \in dv) = \lambda_\Phi \langle v, w \rangle P_\Phi(\nu(0) \in dv).$$

For $d = 2$ this is equation (9.3.5) in Stoyan, Kendall and Mecke (1995) and for $d = 3$ we refer to (9.5.10) in this book. Assume for instance that Φ is the union of m jointly stationary processes $\Phi_i, i = 1, \ldots, m$, of parallel hyperplanes with pairwise distinct unit normals $v_1, \ldots, v_m$ satisfying $\langle v_i, w \rangle > 0$. Choosing $B = \{v_i\}$ in (6.10) yields

$$(6.12) \qquad\qquad \lambda_{N_i} E_{N_i} h = \lambda_{\Phi_i} \langle v_i, w \rangle E_{\Phi_i} h, \qquad i = 1, \ldots, m,$$

where

$$N_i(\cdot) := \mathbf{1}\{\tau_{-\infty} = -\infty, \tau_{+\infty} = \infty, \nu(0) = v_i\} \sum_{n \in \mathbb{Z}} \mathbf{1}\{\tau_n \in \cdot\}.$$

The result of this section can be generalized to the case where $\Xi \neq \varnothing$ and where the divergence of u does not vanish on $\mathbb{R}^d \setminus (\Xi \cup \Phi)$, provided that there is a scalar field K such that $\operatorname{div} Ku = 0$ on $\mathbb{R}^d \setminus (\Xi \cup \Phi)$ (cf. Remark 2.4). We can skip the details here.

APPENDIX

Palm calculus.   All random elements are defined on the probability space $(\Omega, \mathscr{F}, P)$. Assume that $\theta_x \colon \Omega \to \Omega$, $x \in \mathbb{R}^d$, is a family of measurable isomorphisms such that $(\omega, x) \mapsto \theta_x \omega$ is measurable,

$$\theta_x \circ \theta_y = \theta_{x+y}, \qquad x, y \in \mathbb{R}^d$$

and

$$P \circ \theta_x = P, \qquad x \in \mathbb{R}^d.$$

A random measure Φ on $\mathbb{R}^d$ [see Kallenberg (1983)] is called *stationary* if

$$T_y \Phi(\omega) = \Phi(\theta_y \omega), \qquad \omega \in \Omega, \, y \in \mathbb{R}^d,$$

where $T_y \Phi$ is the random measure defined by

$$T_y \Phi(A) := \Phi(\{x + y \colon x \in A\}), \qquad A \in \mathscr{B}^d,$$

and $\mathscr{B}^d$ is the Borel σ-field on $\mathbb{R}^d$. The distribution of a stationary random measure Φ is invariant under the shifts T_y, $y \in \mathbb{R}^d$. In particular the *intensity measure* $\Lambda(A) := E\Phi(A)$, $A \in \mathscr{B}^d$, is given by

$$\Lambda(dx) = \lambda_\Phi H^d(dx),$$

where $\lambda_\Phi := E\Phi([0, 1]^d)$ is the *intensity* of Φ and H^d is the Lebesgue measure on $\mathbb{R}^d$. The measure

$$(\text{A.1}) \qquad Q_\Phi(F) := \iint \mathbf{1}\{\theta_x \omega \in F, \, x \in [0, 1]^d\} \Phi(\omega)(dx) P(d\omega), \qquad F \in \mathscr{F}$$

is called the *Palm measure* of Φ. It is σ-finite and satisfies the refined Campbell formula

$$(\text{A.2}) \qquad \iint f(\theta_x \omega, x) \Phi(\omega)(dx) P(d\omega) = \iint f(\omega, x) Q_\Phi(d\omega) H^d(dx)$$

for all measurable $f: \Omega \times \mathbb{R}^d \to \mathbb{R}^+$. If $0 < \lambda_\Phi < \infty$, then we can define the *Palm probability* $P_\Phi := \lambda_\Phi^{-1} Q_\Phi$ of Φ. We refer to Mecke (1967) for the definition of Q_Φ and P_Φ in a canonical framework. We may interpret P_Φ as the conditional probability given that 0 is a "*typical*" point of Φ. An example of a random measure is $\Phi(\cdot) := \text{card} \{i: \xi_i \in \cdot\}$, where $\{\xi_i: i \in \mathbb{N}\}$ is a (*simple*) *point process* of pairwise distinct points in $\mathbb{R}^d$ [see Kallenberg (1983)] that are not allowed to accumulate in bounded sets. Here and in the main text we tacitly assume that a point process has an infinite number of points. We will make no difference between a point process and its associated random measure.

A process $Z = \{Z(x): x \in \mathbb{R}^d\}$ with values in some arbitrary measurable space is called *stationary* if $Z(0)$ is measurable and $Z(\omega, x) = Z(\theta_x \omega, 0)$ for all $\omega \in \Omega$ and $x \in \mathbb{R}^d$, where 0 is the zero vector. If, in addition, Φ is a stationary random measure then we refer to $\{Z(x)\}$ as being stationary *jointly* with Φ because both objects are adapted to the same "flow" $\{\theta_x: x \in \mathbb{R}^d\}$. In this case it might be helpful to think in terms of the *marked* (or weighted) random measure $\int \mathbf{1}\{(x, Z(x)) \in \cdot\} \Phi(dx)$; see Stoyan, Kendall and Mecke (1995). If the intensity λ_Φ is finite then

$$E\left[\int \mathbf{1}\{(x, Z(x)) \in \cdot\} \Phi(dx)\right] = \lambda_\Phi \iint \mathbf{1}\{(x, z) \in \cdot\} H^d(dx) M(dz),$$

where $M := P_\Phi(Z(0) \in \cdot)$ is the *mark distribution* of the marked random measure. The Palm probability P_Φ itself is an example of a mark distribution: take $Z(\omega, x) = \theta_x \omega$. If Φ is defined by a point process as above then we can define the *mark* $Z_n := Z(\xi_n)$ of the point ξ_n. The set $\{(\xi_n, Z_n): n \in \mathbb{N}\}$ is called a *marked point process*. It is stationary in the sense that

$$\text{card} \{i: (\xi_i(\omega), Z_i(\omega)) \in (A + y) \times B\}$$
$$= \text{card} \{i: (\xi_i(\theta_y \omega), Z_i(\theta_y \omega)) \in A \times B\}.$$

If, conversely, $\{(\xi_n, Z_n): n \in \mathbb{N}\}$ is a stationary marked point process we can define $Z(x) := Z_n$ if $x = \xi_n$ for some $n \in \mathbb{N}$ and give $Z(x)$ an arbitrary fixed value otherwise. Then $\{Z(x): x \in \mathbb{R}^d\}$ is a stationary process and the mark distribution $M = P_\Phi(Z(0) \in \cdot)$ is the distribution of the mark of a typical point of Φ.

A random closed set Ξ [see Matheron (1975)] is called stationary if

$$T_y \Xi(\omega) = \Xi(\theta_y \omega), \qquad \omega \in \Omega, \, y \in \mathbb{R}^d,$$

where $T_y \Xi := \{x - y: x \in \Xi\}$. A special case is a stationary *germ-grain* model

$$\Xi = \bigcup_{i=1}^\infty (\Xi_i + \xi_i),$$

where $\{(\xi_i, \Xi_i): i \in \mathbb{N}\}$ is a stationary marked point process with marks in the set of all nonempty compact subsets of $\mathbb{R}^d$ and where each bounded set is intersected by only a finite number of the grains $\Xi + \xi_i$. The points ξ_n and the marks Ξ_n are called *germs* and *primary grains*, respectively.

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