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# STOCHASTIC WAVE EQUATIONS WITH POLYNOMIAL NONLINEARITY[1]

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This paper is concerned with a class of nonlinear stochastic wave equations in $\mathbb{R}^d$ with $d \leq 3$, for which the nonlinear terms are polynomial of degree $m$. As an example of the nonexistence of a global solution in general, it is shown that there exists an explosive solution of some cubically nonlinear wave equation with a noise term. Then the existence and uniqueness theorems for local and global solutions in Sobolev space $H_1$ are proven with the degree of polynomial $m \leq 3$ for $d = 3$, and $m \geq 2$ for $d = 1$ or 2.

**1. Introduction.** Wave motion is one of the most commonly observed physical phenomena. The progression of water waves and the propagation characteristics of light and sound are familiar everyday experiences. As mathematical models, wave motions are usually described by partial differential equations of hyperbolic type. In the deterministic case, they have been studied extensively for many years due to their wide-spread applications to engineering and sciences. In recent years, nonlinear wave or hyperbolic equations have attracted a great deal of attention, spurred by modern problems such as sonic booms, bottleneck in traffic flows, nonlinear optics and quantum field theory (see, e.g., Whitham [13] and Reed [11]). For many problems, such as wave propagation through the atmosphere or the ocean, the properties of media fluctuate randomly due to the presence of turbulence. More realistic models must take the random fluctuation into account. Such cosideration led to the introduction of stochastic wave equations in 1960's. Many problems in linear stochastic wave propagation and applications can be found in [1]. However there are relatively few papers dealing with nonlinear stochastic wave equations.

In the deterministic case, solutions to nonlinear wave equations with certain polynomial nonlinearity tend to develop singularities in finite time, physically manifested as shock waves or explosion (see, e.g., Reed [11] and John [3]). This means that these solutions exist only locally. It is therefore of interest to study the effects of random perturbation on the solution behavior of such equations. The existence of explosive solutions, or lack of it, to some nonlinear stochastic wave equation was first investigated by Mueller [7]. He considered the wave equation perturbed by a power-law type of state-dependent white noise. Regarded as a stochastic equation of Itô type, the existence of a long-time or global solution

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was proved under a nearly linear growth condition on the state dependence in the noise term. For a stronger nonlinear noise term, though unproven, it is plausible to anticipate an explosive solution. Motivated by Mueller [7], we raise a different kind of question: For a wave equation with a polynomial nonlinearity, how does a random perturbation affect the solution behavior. In general there exists only a local solution. So it is of interest to find suitable conditions to ensure the existence of a global solution, or no explosion in finite time.

To be specific, the paper is mainly concerned with the existence of local and global solutions to a class of stochastic wave equations in $\mathbb{R}^d$ with dimension $d \leq 3$. In particular let us consider the stochastic equation of the form

$$\begin{cases} \partial_t^2 u = \nabla^2 u + f(u) + \sigma(u)\partial_t W(t,x), & x \in \mathbb{R}^d,\ t > 0, \\ u(0,x) = g(x), \quad \partial_t u(0,x) = h(x), \end{cases} \tag{1.1}$$

where $\partial_t$ denotes the partial derivative in $t$, $\nabla^2$ the Laplacian; $W(t,\cdot)$ is a Wiener random field, which will be defined precisely later, and the initial data $g$ and $h$ are given functions. The nonlinear terms $f(u)$ and $\sigma(u)$ are assumed to be polynomials in $u$. When $\sigma \equiv 0$, equation (1.1) reduces to a classical wave equation with a polynomial nonlinearity

$$\partial_t^2 u = \nabla^2 u + f(u), \tag{1.2}$$

which is known to have an explosive solution when, for instance, $f(u)$ is cubic in $u$ (see Reed and Simon [11], page 311). Under a mild random perturbation, one can show the persistence of such an explosion in a probabilistic sense (see Section 3). In the light of the above result for equation (1.2), we were led to the consideration of the polynomially nonlinear equation (1.1). To avoid an explosive solution, similar to the deterministic case (see Reed [10], page 9), it is necessary for equation (1.1) to satisfy a certain energy inequality. The main results of the paper are given in Theorems 4.1 and 4.2 concerning the existence and uniqueness of local and global solutions, respectively, to a somewhat more general form of equation (1.1). Instead of pointwise solutions as considered by Mueller [7], we shall seek solutions in a Sobolev space.

The paper is organized as follows: In Section 2 we give some basic definitions and a key lemma pertaining to the energy equation for a linear stochastic wave equation, a proof of which is provided in the appendix. To exhibit the non-existence, in general, of a global solution, we present a cubically nonlinear stochastic wave equation in Section 3 and show that it has an explosive solution in a probabilistic sense (see Theorem 3.1). In Section 4 we consider a class of stochastic wave equations with a polynomial type of nonlinear terms. The existence and uniqueness of local and global solutions are proved in Theorem 4.1 and Theorem 4.2, respectively. The proofs rely on a $H_1$-Lipschitz truncation technique and the aforementioned energy inequality.

**2. Preliminaries.** For $x \in \mathbb{R}^d$, $g$ and $h \in C^\infty(\mathbb{R}^d)$, denote the gradient of $g$ by

$$(Dg)(x) = D_x g(x) = (\partial_{x_1} g, \dots, \partial_{x_d} g)(x),$$

with $\partial_{x_i} = \frac{\partial}{\partial x_i}$, and set

$$Jh = (h, \partial_{x_1} h, \dots, \partial_{x_d} h).$$

For any integer $k \geq 1$, the Euclidean norm on $\mathbb{R}^k$ is denoted by $|\cdot|$. In particular we have

$$|Dg| = \left\{ \sum_{i=1}^{d} |\partial_{x_i} g|^2 \right\}^{1/2}$$

and

$$|Jh| = \{|h|^2 + |Dh|^2\}^{1/2}.$$

The Laplacian $\nabla^2$ is defined as usual by

$$\nabla^2 g = \sum_{i=1}^{d} \partial_{x_i}^2 g.$$

Introduce $H = L^2(\mathbb{R}^d)$ with inner product

$$(g, h) = \int g(x) h(x)\, dx$$

and norm $\|g\| = (g, g)^{1/2}$. Let $H_1 = H_1(\mathbb{R}^d)$ be a Sobolev subspace of $H$ with norm

$$\|h\|_1 = \left\{ \int |Jh|^2\, dx \right\}^{1/2} = \left\{ \int (|h|^2 + |Dh|^2)\, dx \right\}^{1/2},$$

which will also be written as $\|Jh\|$. For $g \in L^p(\mathbb{R}^d)$ with $p \geq 2$, the $L^p$-norm of $g$ will be denoted by $|g|_p$ with $|g|_2 = \|g\|$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space for which a filtration $\{\mathcal{F}_t,\ t \in [0, \infty)\}$ of increasing sub $\sigma$-fields $\mathcal{F}_t$ is given. Let $W(t, x)$, $t \geq 0$, $x \in \mathbb{R}^d$, be a continuous Wiener random field defined in this space with mean zero and covariance function $r(x, y)$ defined by

$$EW(t, x)W(s, y) = (t \wedge s) r(x, y), \qquad t, s \geq 0,\ x, y \in \mathbb{R}^d,$$

where $(t \wedge s) = \min\{t, s\}$, and conditions on $r(x, y)$ will be given later. Let $\sigma(t, x) = \sigma(t, x, \omega)$, for $t \geq 0$, $x \in \mathbb{R}^d$ and $\omega \in \Omega$, be a continuous predictable random field such that

$$\int_0^T \sigma^2(t, x)\, dt < \infty \qquad \text{for each } x \in \mathbb{R}^d \text{ a.s.}$$

Then the stochastic integral

$$M(t,x)=\int_0^t \sigma(s,x)\,dW(s,x), \qquad t>0,\ x\in\mathbb{R}^d, \tag{2.1}$$

is well defined, in the sense of Kunita [4], as a continuous martingale with spatial parameter $x$. Alternatively the above integral can be defined as in [12]. The mutual variation of $M(t,x)$ is given by

$$\langle M(\cdot,x), M(\cdot,y)\rangle_t=\int_0^t q(s,x,y)\,ds, \qquad t>0,\ x,y\in\mathbb{R}^d, \tag{2.2}$$

where $q$ will be called the covariation function given by

$$q(t,x,y)=r(x,y)\sigma(t,x)\sigma(t,y). \tag{2.3}$$

Let $Q_t$ be the covariation operator with kernel $q$ so that

$$(Q_t g)(x)=\int q(t,x,y)g(y)\,dy.$$

Assume that the trace of $Q_t$ satisfies

$$\operatorname{Tr} Q_t=\int q(t,x,x)\,dx=\int r(x,x)\sigma^2(t,x)\,dx<\infty. \tag{2.4}$$

Then $M_t=M(t,\cdot)$ can be regarded as a continuous $H$-valued martingale with its covariation given by (see Metivier and Pellaumail [5], Chapter 6)

$$\langle\langle M\rangle\rangle_t=\int_0^t Q_s\,ds, \tag{2.5}$$

or, for $g,h\in H$,

$$\langle (M_\cdot,g),(M_\cdot,h)\rangle_t=\int_0^t (Q_t g,h)\,ds.$$

Now consider the Cauchy problem for a linear stochastic wave equation

$$\begin{cases}\partial_t^2 u=c^2\nabla^2 u-\gamma^2 u+f(t,x)+\partial_t M(t,x),\\ u(0,x)=g(x), \quad \partial_t u(0,x)=h(x),\end{cases} \tag{2.6}$$

where $c$ and $\gamma$ are real parameters, $f(t,x)=f(t,x,\omega)$ is a predictable random field; $M(t,x)$ is defined by (2.1) and the initial functions $g$ and $h$ are given. In a Hilbert space setting, let $u_t=u(t,\cdot)$, $f_t=f(t,\cdot)$ and so on, and rewrite the equation (2.6) as a system of Itô equations:

$$\begin{cases}du_t=v_t\,dt,\\ dv_t=(c^2\nabla^2-\gamma^2)u_t\,dt+f_t\,dt+dM_t,\\ u_0=g, \quad v_0=h.\end{cases} \tag{2.7}$$

As in the deterministic case, the energy equation and inequality will play an important role for stochastic wave equations. Unlike the parabolic case, the general *Itô* formula does not hold for equation (2.7). However the energy equation for system (2.7) is known to be valid, as stated in the following lemma.

LEMMA 2.1 (Energy equation). *Suppose that $f_t$ is a continuous predictable $H$-valued process, and $M_t$ is a continuous $H$-valued martingale with covariation operator $Q_t$ such that*

$$E \sup_{t \le T} \operatorname{Tr} Q_t < \infty.$$

*If $g \in H_1$, and $h \in H$, then the system* (2.7) *has a unique pair of solutions $u_t$ and $v_t$, which are continuous processes on $[0, T]$ with values in $H_1$ and $H$, respectively. Moreover the following equation holds*:

$$\begin{aligned} e(u_t) = e(u_0) + 2\int_0^t (v_s, f_s)\,ds + \int_0^t \operatorname{Tr} Q_s\,ds \\ + 2\int_0^t (v_s, dM_s), \qquad t \in [0, T], \end{aligned} \tag{2.8}$$

*where*

$$e(u_t) = \|v_t\|^2 + c^2\|Du_t\|^2 + \gamma^2\|u_t\|^2. \tag{2.9}$$

This lemma follows from a more general result first proved by Pardoux [8]. His proof is based on a nonlinear regularization technique by regarding equation (2.7) as the limiting case of a nonlinear stochastic wave equation with a monotone dissipation. It seems of some technical interest to give a more direct proof based on the method of smoothing. Such a proof will be provided in the Appendix.

**3. Example of explosive solution.** As an example of explosive solution, consider the cubic nonlinear stochastic wave equation in $\mathbb{R}^d$ for $d \le 3$:

$$\begin{cases} \partial_t^2 u = c^2\nabla^2 u - \gamma^2 u + \lambda u^3 + \sigma(t, x)\partial_t W(t, x), & t > 0, \\ u(x, 0) = g(x), \quad \partial_t u(0, x) = h(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.1}$$

where $\lambda > 0$ and the remaining parameters and functions are similar to that in equation (2.6). Here the martingale $M(t, x)$ is given by

$$M(t, x) = \int_0^t \sigma(s, x)\,dW(s, x)$$

with the covariation function

$$q(t, x, y) = r(x, y)\sigma(t, x)\sigma(t, y).$$

For simplicity, $\sigma$, $r$, $g$ and $h$ are all assumed to be sufficiently smooth nonrandom functions (see the conditions to be given in Theorem 3.1). Rewrite equation (3.1) as a system of Itô equations in $\mathcal{H} = H_1 \times H$:

$$\begin{cases} du_t = v_t\,dt, \\ dv_t = c^2\nabla^2 u_t\,dt - \gamma^2 u_t\,dt + \lambda u_t^3\,dt + dM_t, \\ u_0 = g, \quad v_0 = h. \end{cases} \tag{3.2}$$

As to be shown later on, the Cauchy problem (3.2) has a unique continuous (local) solution $u_t \in H_1$ and $v_t \in H$. The following lemma will be needed later on.

LEMMA 3.1. *Suppose that the system* (3.2) *has a regular solution for* $t < T$ *as mentioned above*. *Define*

$$\phi(t) = \tfrac{1}{2} E\|u_t\|^2. \tag{3.3}$$

*Then* $\phi$ *is twice continuously differentiable and its first and second order derivatives are given by*

$$\begin{aligned} \phi'(t) &= E(u_t, v_t) \\ &= (g, h) + E\int_0^t \{\|v_s\|^2 - \gamma^2\|u_s\|^2 - c^2\|Du_s\|^2 + \lambda\|u_s^2\|^2\}\, ds \end{aligned} \tag{3.4}$$

*and*

$$\phi''(t) = E\{\|v_t\|^2 - \gamma^2\|u_t\|^2 - c^2\|Du_t\|^2 + \lambda\|u_t^2\|^2\}. \tag{3.5}$$

PROOF. From the equations in (3.2), we have

$$d\|u_t\|^2 = d(u_t, u_t) = 2(u_t, v_t)\, dt$$

and

$$(u_t, v_t) = (g, h) + \int_0^t (u_s, dv_s) + \int_0^t \|v_s\|^2 ds,$$

which yield

$$\begin{aligned} \phi'(t) &= E(u_t, v_t) \\ &= (g, h) + E\int_0^t \|v_s\|^2\, ds + E\int_0^t (u_s, dv_s). \end{aligned} \tag{3.6}$$

The last integral can be written as

$$\begin{aligned} E\int_0^t (u_s, dv_s) &= E\int_0^t \{c^2\langle u_s, \nabla^2 u_s\rangle - \gamma^2\|u_s\|^2 + \lambda\|u_s^2\|^2\}\, ds \\ &= E\int_0^t \{-c^2\|Du_s\|^2 - \gamma^2\|u_s\|^2 + \lambda\|u_s^2\|^2\}\, ds, \end{aligned} \tag{3.7}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H_1$ and its dual $H_1^*$, and use was made of an integration by parts, which can be justified as in the deterministic case (see Mizohata [6], page 318). Upon substituting (3.7) into (3.6), formula (3.4) follows. A simple differentiation of (3.4) yields equation (3.5). □

With the aid of this lemma, we will show that, under certain conditions, the solution of the nonlinear problem (3.1) or (3.2) may become unbounded in finite time in a probabilistic sense. To state the result as a theorem, let $C_0^k$ denote the set of $C^k(\mathbb{R}^d)$-functions with compact support and let $e(u_t)$ be the linearized energy function as defined in (2.9).

THEOREM 3.1. *For the Cauchy problem* (3.1) *in* $\mathbb{R}^d$ *with* $d \le 3$, *suppose that the following conditions hold true*:

(i) $u_0 = g \in C^1 \cap L^4$ *and* $v_0 = h \in H$ *such that* $(u_0, v_0) > 0$.

(ii) *The covariance function* $r(x, y)$ *and the function* $\sigma(t, x)$, *for* $t > 0$, $x, y \in \mathbb{R}^d$, *are given such that*

$$q_\infty := \int_0^\infty \int q(t, x, x)\,dx\,dt = \int_0^\infty \int r(x, x)\sigma^2(t, x)\,dx\,dt < \infty.$$

(iii) $\lambda \ge 2[q_\infty + e(u_0)]/\|g^2\|^2$, *where* $e(u_0) = \|h\|^2 + \gamma^2\|g\|^2 + c^2\|Dg\|^2$.

*Then either the solution* $u_t$ *explodes in finite time with a positive probability or else the mean-square solution* $E\|u_t\|^2$ *tends to infinity within a finite time interval.*

PROOF. Introduce an increasing sequence of stopping times $\tau_n$ defined by

$$\tau_n = \inf\{t > 0 : \|u_t\| > n\},$$

and let

$$\tau_\infty = \lim_{n\to\infty} \tau_n$$

which may be infinite.

First consider the case $P\{\tau_\infty = \infty\} = 1$ or $P\{\tau_\infty \le T\} = 0$ for any $T > 0$. Define the function

$$F_\alpha(t) = \phi^{-\alpha}(t) \quad \text{for } \alpha > 0. \tag{3.8}$$

Compute its first two derivatives:

$$F'_\alpha(t) = -\alpha\phi^{-(\alpha+1)}(t)\phi'(t) \tag{3.9}$$

and

$$F''_\alpha(t) = \alpha\phi^{-(\alpha+1)}(t)\left\{(\alpha + 1)\frac{[\phi'(t)]^2}{\phi(t)} - \phi''(t)\right\}. \tag{3.10}$$

By Lemma 3.1,

$$\phi'(t) = E(u_t, v_t) \le \{E\|u_t\|^2 E\|v_t\|^2\}^{1/2}$$

or

$$\frac{[\phi'(t)]^2}{\phi(t)} \le 2E\|v_t\|^2. \tag{3.11}$$

In view of (3.5) and (3.11), equation (3.10) yields

$$F''_\alpha(t) \le \alpha\phi^{-(\alpha+1)}(t)G_\alpha(t), \tag{3.12}$$

where

$$G_\alpha(t) = (2\alpha+1)E\|v_t\|^2 + E\{\gamma^2\|u_t\|^2 + c^2\|Du_t\|^2 - \lambda\|u_t^2\|^2\}. \tag{3.13}$$

By applying Lemma 2.1 to equation (3.2) with $f_t = \lambda u_t^3$, we obtain

$$\begin{aligned} E\|v_t\|^2 = e(u_0) - \frac{\lambda}{2}\|u_0^2\|^2 + E\left\{\frac{\lambda}{2}\|u_t^2\|^2 - c^2\|Du_t\|^2 - \gamma^2\|u_t\|^2\right\} \\ + \int_0^t \operatorname{Tr} Q_s\, ds. \end{aligned} \tag{3.14}$$

After substituting (3.14) into (3.13) and simplifying the resulting equation, it gives

$$\begin{aligned} G_\alpha(t) = -(2\alpha+1)\left\{\frac{\lambda}{2}\|u_0^2\|^2 - e(u_0) - \int_0^t \operatorname{Tr} Q_s\, ds\right\} \\ - 2\alpha E\big(c^2\|Du_t\|^2 + \gamma^2\|u_t\|^2\big) - \lambda\left(\frac{1}{2} - \alpha\right)E\|u_t^2\|^2. \end{aligned}$$

Therefore, if we choose $\alpha \in (0, \frac{1}{2}]$, then, by conditions (ii) and (iii) of the theorem, the above equation and (3.12) imply that

$$F''_\alpha(t) \le 0 \quad \text{for } t \ge 0. \tag{3.15}$$

Also, setting $t = 0$ in equations (3.8) and (3.9), and invoking condition (i), we see that

$$F_\alpha(0) = \phi^{-\alpha}(0) = \big(\tfrac{1}{2}\|u_0\|^2\big)^{-\alpha} > 0 \tag{3.16}$$

and

$$F'_\alpha(0) = -\alpha\phi^{-(\alpha+1)}(0)\phi'(0) = -\alpha\big(\tfrac{1}{2}\|u_0\|^2\big)^{-(\alpha+1)}(u_0, v_0) < 0. \tag{3.17}$$

The inequalities (3.15)–(3.17) show clearly that the function $F_\alpha(t)$ is monotonically decreasing to zero at time $T_0 \le T_\alpha$, with $T_\alpha = -F_\alpha(0)/F'_\alpha(0)$. Hence, by definition (3.8), we conclude that

$$\lim_{t\to T_0} \phi(t) = \tfrac{1}{2}\lim_{t\to T_0} E\|u_t\|^2 = \infty.$$

Now consider the case: $P\{\tau_\infty = \infty\} < 1$, this means, of course, that $P\{\tau_\infty < \infty\} > 0$ or the solution $u_t$, as a process in $H$, becomes unbounded in finite time with a positive probability as asserted. □

This example shows clearly that, for stochastic wave equations with a polynomial nonlinearity, a global solution does not exist in general. It will be seen that, as in the deterministic case, the energy inequality is the key to the existence of a global solution.

**4. Local and global solutions.** We consider the following initial-value problem

$$\begin{cases} \partial_t^2 u = c^2\nabla^2 u - \gamma^2 u + f(x,u) + \sigma(t,x,Ju)\partial_t W(t,x), \\ u(0,x) = g(x), \quad \partial_t u(0,x) = h(x), \end{cases} \tag{4.1}$$

where $f(x,\cdot)$ and $\sigma(t,x,\cdot)$ are some nonlinear functions of $u$ and $Ju = (u, \partial_{x_1} u, \dots, \partial_{x_d} u)$, respectively.

As before, we rewrite equation (4.1) as a system in $\mathscr{H} = H_1 \times H$:

$$\begin{cases} du_t = v_t\,dt, \\ dv_t = (c^2\nabla^2 - \gamma^2)u_t\,dt + f(u_t)\,dt + \sigma_t(Ju_t)\,dW_t, \\ u_0 = g, \quad v_0 = h, \end{cases} \tag{4.2}$$

in which we set $f(\cdot,u) = f(u_t)$ and $\sigma(t,\cdot,Ju) = \sigma_t(Ju_t)$. To recast system (4.2) as an Itô equation in $\mathscr{H}$, let $Z_t = (u_t, v_t)$, $F(Z_t) = (0, f(u_t))$ and $\Sigma_t(Z_t) = (0, \sigma_t(Ju_t))$, as column matrices, and set

$$A = \begin{bmatrix} 0 & I \\ c^2\nabla^2 - \gamma^2 I & 0 \end{bmatrix} \tag{4.3}$$

which is a $2 \times 2$ matrix with $I$ as an identity operator. Then system (4.2) can be written as

$$\begin{cases} dZ_t = AZ_t\,dt + F(Z_t)\,dt + \Sigma_t(Z_t)\,dW_t, \\ Z_0 = (g,h). \end{cases} \tag{4.4}$$

We know that $A$, with domain $D(A)$ dense in $\mathscr{H}$, generates a $C_0$-semigroup of operators $e^{tA}$ on $H$ (see Pazy [9], page 220). $Z_t$ is said to be a mild solution of (4.2) or (4.3) in the sense of Da Prato and Zabczyk [2], if it is a continuous $\mathscr{H}$-valued process that satisfies the integral equation

$$Z_t = e^{tA}Z_0 + \int_0^t e^{(t-s)A}F(Z_s)\,ds + \int_0^t e^{(t-s)A}\Sigma_s(Z_s)\,dW_s \quad \text{a.s.} \tag{4.5}$$

Suppose that $f(\cdot)\colon H_1 \to H$ and $\sigma_t(\cdot)\colon H^{d+1} \to L(H)$, where $H^{d+1} = H \times \overset{(d+1)}{\cdots} \times H$, and $L(H)$ denotes the set of all bounded linear operators on $H$. For $Ju \in H^{d+1}$ and $h \in H$, we define

$$[\sigma_t(Ju)h](x) = \sigma\big(t,x,Ju(x)\big)h(x), \qquad x \in \mathbb{R}^d.$$

If $f$ and $\sigma_t$ are uniformly Lipschitz continuous and of linear growth, then the following lemma follows from the standard existence theorem (Da Prato and Zabczyk [2], Theorem 7.4) for a stochastic evolution equation. In what follows, $C$ and $K$ with or without subscripts, denote some generic constants of which the meaning may vary from line to line.

LEMMA 4.1. *Suppose that the following conditions hold*:

(i) *Let* $f(\cdot)\colon H_1 \to H$ *be such that*

$$\|f(u)\|^2 \le C_1\big(1 + \|u\|_1^2\big) \tag{4.6}$$

*and*

$$\|f(u) - f(u')\| \le C_2\|u - u'\|_1 \qquad \forall u, u' \in H_1, \tag{4.7}$$

*for some constants* $C_1, C_2 > 0$.

(ii) *For any* $Ju \in H^{d+1}$, *the map* $\sigma.(Ju)\colon [0,T] \to L(H)$ *is continuous. There exist positive constants* $C_3$ *and* $C_4$ *such that*

$$\operatorname{Tr}[\sigma_t(Ju)R\sigma_t^*(Ju)] \le C_3\big(1 + \|u\|_1^2\big) \tag{4.8}$$

*and*

$$\operatorname{Tr}\big\{[\sigma_t(Ju) - \sigma_t(Ju')]R[\sigma_t(Ju) - \sigma_t(Ju')]^*\big\} \le C_4\|u - u'\|_1^2, \tag{4.9}$$

*for any* $u, u' \in H_1$, $t \in [0,T]$, *where* $*$ *denotes the adjoint*.

(iii) $W_t$ *is a* $H$-*valued Wiener process with covariance operator* $R$ *satisfying* $\operatorname{Tr} R < \infty$.

*Then for* $u_0 \in H_1$ *and* $v_0 \in H$, *the system* (4.2) *has a unique* (*mild*) *solution* $(u_t, v_t)$ *on* $[0,T]$ *with* $u. \in C([0,T], H_1)$ *and* $v. \in C([0,T], H)$. *Moreover the following energy equation holds*:

$$\begin{aligned} e(u_t) = e(u_0) &+ 2\int_0^t \big(v_s, f_s(u_s)\big)\,ds + 2\int_0^t \big(v_s, \sigma_s(Ju_s)\,dW_s\big) \\ &+ \int_0^t \operatorname{Tr}[\sigma_s(Ju_s)R\sigma_s^*(Ju_s)]\,ds. \end{aligned} \tag{4.10}$$

PROOF. For $Z = (u, v) \in \mathscr{H} = H_1 \times H$, without confusion, the norm of $Z$ will be denoted by

$$\|Z\| = \big\{\|u\|_1^2 + \|v\|^2\big\}^{1/2}.$$

In view of conditions (4.6)–(4.9), it is easy to verify that

$$\|F(Z)\|^2 = \|f(u)\|^2 \le C_1(1 + \|u\|^2) \le C_1(1 + \|Z\|^2), \tag{4.11}$$

$$\|F(Z) - F(Z')\|^2 = \|f(u) - f(u')\|^2 \le C_2\|u - u'\|_1^2 \le C_2\|Z - Z'\|^2, \tag{4.12}$$

$$\begin{aligned} \operatorname{Tr}[\Sigma_t(Z)R\Sigma_t^*(Z)] &= \operatorname{Tr}[\sigma_t(Ju)R\sigma_t^*(Ju)] \\ &\le C_3\big(1 + \|u\|_1^2\big) \le C_3(1 + \|Z\|^2) \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} &\operatorname{Tr}\big\{[\Sigma_t(Z) - \Sigma_t(Z')]R[\Sigma_t(Z) - \Sigma_t(Z')]^*\big\} \\ &\qquad = \operatorname{Tr}\big\{[\sigma_t(Ju) - \sigma_t(Ju')]R[\sigma_t(Ju) - \sigma_t(Ju')]^*\big\} \\ &\qquad \le K_2\|u - u'\|_1^2 \le C_4\|Z - Z'\|^2. \end{aligned} \tag{4.14}$$

Therefore the conditions for the existence theorem (Da Prato and Zabcyzk [2], Theorem 7.4), are satisfied for equation (4.5). So it has a unique solution $Z_t = (u_t, v_t)$ as claimed. Now the energy equation (4.10) can be obtained easily by making use of Lemma 2.1. □

The next lemma quotes some well-known Sobolev inequalities (see Reed [10], page 21), which will be needed later on. Recall that $|\cdot|_p$ denotes the $L^p$-norm and $C_0^\infty$, the set of $C^\infty$-functions on $\mathbb{R}^d$ with compact support.

LEMMA 4.2. *For $u, v \in C_0^\infty$ and $1 \le k \le m$, we have*

$$|u|_{2k} \le C_1 \|u\|_1,$$
$$\|u^{k-1} v\|^2 \le C_2 \|u\|_1^{2(k-1)} \|v\|_1^2,$$

*for some constants $C_1, C_2 > 0$, where $m = 3$ for $d = 3$, and $m \ge 1$ for $d = 1$ or 2.*

In what follows, we shall state and prove the local existence theorem for equation (4.1) with polynomial nonlinearities.

THEOREM 4.1. *Consider the Cauchy problem for the stochastic wave equation (4.1) in $\mathbb{R}^d$ with $d \le 3$. Assume that:*

(i) *$f(x, u)$, $x \in \mathbb{R}^d$, $u \in \mathbb{R}$, is a polynomial of the form*

$$f(x, u) = \sum_{j=1}^{m} a_j(x) u^j, \tag{4.15}$$

*where $m \le 3$ for $d = 3$ and $m \ge 1$ for $d = 1$ or 2, and $a_j(x)$ is bounded and continuous for $j = 1, \dots, m$.*

(ii) *The function $\sigma(t, x, s, y)$, for $t \ge 0$, $s \in \mathbb{R}$, $x$ and $y \in \mathbb{R}^d$, is continuous. There exist positive constants $C_1$ and $C_2$ such that with $p \le m$,*

$$|\sigma(t, x, s, y)|^2 \le C_1(1 + |s|^{2p} + |y|^2) \quad \forall t \in [0, T],\ x, y \in \mathbb{R}^d \tag{4.16}$$

*and*

$$\begin{aligned} &|\sigma(t, x, s, y) - \sigma(t, x, s', y')|^2 \\ &\quad \le C_2\{(1 + |s|^{2(p-1)} + |s'|^{2(p-1)})|s - s'|^2 + |y - y'|^2\} \\ &\qquad\qquad \forall t \in [0, T],\ x \in \mathbb{R}^d;\ s, s' \in \mathbb{R};\ y, y' \in \mathbb{R}^d. \end{aligned} \tag{4.17}$$

(iii) *$W(t, x)$, $t \ge 0$, $x \in \mathbb{R}^d$, is a continuous Wiener random field with covariance function $r(x, y)$ such that*

$$\operatorname{Tr} R = \int r(x, x)\, dx < \infty \quad \text{and} \quad r_0 = \sup_{x \in \mathbb{R}^d} r(x, x) < \infty.$$

*Then, for $u_0 \in H_1$ and $v_0 \in H$, the Cauchy problem has a unique continuous local solution $u(t, \cdot) \in H_1$ with $\partial_t u(t, \cdot) \in H$.*

PROOF. The main idea of the proof is to show that conditions (4.6)–(4.9) for Lemma 4.1 are satistified locally, where the constants $C_i$ are replaced by some $H_1$-bounded functions of $u$ and $u'$. Then the existence and uniqueness of a local solution will be proved by the method of truncation. For clarity the proof will be given in the several steps.

*Step* 1. $f\colon H_1 \to H$ is bounded. First rewrite equation (4.1) as the system (4.2), where

$$f(u) = f(\cdot, u) = \sum_{j=1}^{m} a_j u^j.$$

Then, by Lemma 4.2, we get

$$\|f(u)\| \le a_0 \sum_{j=1}^{m} \|u^j\| \le a_0 c_1 \sum_{j=1}^{m} \|u\|_1^j,$$

where $a_0 = \sup_{x\in\mathbb{R}^d, 1\le j\le m} |a_j(x)|$.

Hence $f\colon H_1 \to H$ is bounded and, for $u \in H_1$,

$$\|f(u)\|^2 \le b_1(\|u\|_1)\|u\|_1^2 \le b_1(\|u\|_1)\big(1 + \|u\|_1^2\big), \tag{4.18}$$

where $b_1(\|u\|_1) = (a_0c_1)^2(\sum_{j=1}^m \|u\|_1^{j-1})^2$.

*Step* 2. $f\colon H_1 \to H$ is locally Lipschitz-continuous. Similar to Step 1, we can show that, for $u, u' \in H_1$,

$$\|f(u) - f(u')\|^2 \le b_2(\|u\|_1, \|u'\|_1)\|u - u'\|_1^2, \tag{4.19}$$

where $b_2(r, s)$ is a polynomial of degree $2(m-1)$ in $r$ and $s$ with positive coefficients. For instance, consider the case $d = 3$ with $m = 3$. From (4.15) we have

$$\|f(u) - f(u')\|^2 \le \sum_{j=1}^{3} \|u^j - u'^j\|^2. \tag{4.20}$$

By invoking Lemma 4.2,

$$\begin{aligned} \|u^2 - u'^2\|^2 &\le C_2\|u + u'\|_1^2\|u - u'\|_1^2 \\ &\le 2C_2(\|u\|_1^2 + \|u'\|_1^2)\|u - u'\|_1^2 \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} \|u^3 - u'^3\|^2 &\le \|(u^2 + uu' + u'^2)(u - u')\|^2 \\ &\le 8\big(\|u^2(u - u')\|^2 + \|u'^2(u - u')\|^2\big) \\ &\le C_3(\|u\|_1^4 + \|u'\|_1^4)\|u - u'\|_1^2. \end{aligned} \tag{4.22}$$

In view of (4.20)–(4.22), inequality (4.19) holds for $d = 3$. For $d = 1$ or 2, it can be verified in a similar fashion.

*Step* 3. For $Ju \in H^{d+1}$ and $t \in [0, T]$, there exists a polynomial $b_3(r)$ with positive coefficients such that

$$\mathrm{Tr}[\sigma_t(Ju)R\sigma_t^*(Ju)] \le b_3(\|u\|_1)\big(1 + \|u\|_1^2\big). \tag{4.23}$$

By making use of conditions (ii) and (iii) together with Lemma 4.2, we get

$$\begin{aligned}\mathrm{Tr}[\sigma_t(Ju)R\sigma_t^*(Ju)] &= \int r(x,x)\sigma^2(t,x,u,Du)\,dx\\ &\le C_1\int r(x,x)\{1 + |u|^{2p} + |Du|^2\}\,dx\\ &\le C_1\{\mathrm{Tr}\,R + r_0|u|_{2p}^{2p} + r_0\|u\|_1^2\}\\ &\le K_1\big[1 + \|u\|_1^{2(p-1)}\big]\big(1 + \|u\|_1^2\big) \quad \text{for } p \le m,\end{aligned}$$

which yields (4.23) if we set $b_3(r) = K_1[1 + r^{2(p-1)}]$, for some constant $K_1 > 0$.

*Step* 4. For $t \in [0, T]$, $u$ and $u' \in H_1$, there exists a polynomial $b_4(r, s)$ with positive coefficients such that

$$\begin{aligned}&\mathrm{Tr}[\sigma_t(Ju) - \sigma_t(Ju')]R[\sigma_t(Ju) - \sigma_t(Ju')^*]\\ &\qquad \le b_4(\|u\|_1, \|u'\|_1)\|u - u'\|_1.\end{aligned} \tag{4.24}$$

Similar to Step 3, by means of conditions (ii) and (iii), and Lemma 4.2, we deduce that

$$\begin{aligned}&\mathrm{Tr}\{\sigma_t(Ju) - \sigma_t(Ju')]R[\sigma_t(Ju) - \sigma_t(Ju')]^*\}\\ &\quad = \int r(x,x)[\sigma(t,x,u,Du) - \sigma(t,x,u',Du')]^2\,dx\\ &\quad \le C_2\int r(x,x)\{\big[1 + |u|^{2(p-1)} + |u'|^{2(p-1)}\big]|u - u'|^2 + |Du - Du'|^2\}\,dx\\ &\quad \le C_2 r_0\{\|u - u'\|_1^2 + \|u^{p-1}(u - u')\|^2 + \|u'^{p-1}(u - u')\|^2\}\\ &\quad \le K_2\big(1 + \|u\|_1^{2(p-1)} + \|u'\|_1^{2(p-1)}\big)\|u - u'\|_1^2,\end{aligned}$$

for some constant $K_2 > 0$. It yields (4.24) with $b_4(r, s) = K_2(1 + r^{2(p-1)} + s^{2(p-1)})$.

*Step* 5. Existence of local solutions by $H_1$-Lipschitz truncation.

For $R > 0$, let $\eta_R(\cdot)$: $\mathbb{R}^+ = [0, \infty) \to \mathbb{R}^+$ be a $C_0^\infty$-function such that

$$\eta_N(s) = \begin{cases} 1, & \text{for } 0 \le s \le N/2,\\ 0, & \text{for } s > N,\end{cases} \tag{4.25}$$

and $0 \le \eta_N(s) \le 1$ for $N/2 < s \le N$. For $u \in H_1$, define $S_N u = \eta_N(\|u\|_1)u$, $f^N(u) = \eta_N(\|u\|_1)f(S_N u)$ and $\sigma_t^N(Ju) = \eta_N(\|u\|_1)\sigma_t(JS_N u)$. Instead of (4.2), consider the truncated system

$$\begin{cases} du_t = v_t\,dt,\\ dv_t = (c^2\nabla^2 - \gamma^2)u_t\,dt + f^N(u_t)\,dt + \sigma_t^N(Ju)\,dW_t,\\ u_0 = g, \quad v_0 = h.\end{cases} \tag{4.26}$$

Then, by (4.18), we have

$$f^N(u) = \eta_N(\|u\|_1) f(S_N u) \le b_1(\|S_N u\|_1)(1 + \|S_N u\|_1^2) \le b_1(N)\big(1 + \|u\|_1^2\big). \tag{4.27}$$

It can also be shown that, by (4.19) and (4.25), there exists constant $b_2(N) > 0$ such that

$$\|f^N(Ju) - f^N(Ju')\| \le b_2(N)\|u - u'\|_1 \quad \text{for } u, u' \in H_1. \tag{4.28}$$

Similarly, by taking (4.23), (4.24) and (4.25) into account, we can deduce that

$$\mathrm{Tr}[\sigma_t^N(Ju) R \sigma_t^{N*}(Ju)] \le b_3(N)\big[1 + \|u\|_1^2\big] \tag{4.29}$$

and

$$\mathrm{Tr}\big[\sigma_t^N(Ju) - \sigma_t^N(Ju')\big] R \big[\sigma_t^N(Ju) - \sigma_t^N(Ju')\big]^* \le b_4(N)\|u - u'\|_1^2 \tag{4.30}$$

for some $b_4(N) > 0$, $u, u' \in H_1$.

In view of (4.27)–(4.30) and condition (iii), Lemma 4.1 is applicable to the truncated system (4.26). Therefore there exists a unique continuous solution $Z_t^N = (u_t^N, v_t^N) \in \mathcal{H} = H_1 \times H$, for $t \in [0, T]$. Introduce a stopping time $\tau_N$ defined by

$$\tau_N = \inf\{t > 0 : \|u_t^N\|_1 > N/2\}.$$

Then, for $t < \tau_N$, $u_t = u_t^N$ is the solution of (4.1) with $\partial_t u_t = v_t^N$. As $\tau_N$ is increasing in $N$, let $\tau_\infty = \lim_{N\to\infty} \tau_N$. Define $u_t$, for $t < \tau_\infty \wedge T$, by $u_t = u_t^N$ if $t < \tau_N \le T$. Then $u_t$ is the unique local solution as claimed. □

To obtain a global solution, it is necessary to impose further conditions on $f$ and $\sigma$ so that a certain energy bound can be established to prevent the unlimited growth. To state the next theorem, it is convenient to introduce the function $G(x, u)$ defined by

$$G(x, u) = -2\int_0^u f(x, s)\, ds = -\sum_{j=1}^m \frac{2}{(j+1)} a_j(x) u^{j+1}. \tag{4.31}$$

THEOREM 4.2. *Suppose that conditions* (i)–(iii) *for Theorem* 4.1 *hold true. In addition to conditions* (i) *and* (ii), *we assume that*:

(i)′ *Let* $m = (2k+1)$ *be odd and there exist positive constants* $\alpha \ge 0$ *and* $\lambda > 0$ *such that*

$$G(x, s) \ge (\alpha + \lambda s^{2k}) s^2 \qquad \forall x \in \mathbb{R}^d,\ s \in \mathbb{R}. \tag{4.32}$$

(ii)′ *Condition* (ii) *holds with* $p = (k+1)$.

*Then, for* $u_0 \in H_1$, $v_0 \in H$ *and for any* $T > 0$, *the Cauchy problem* (4.1) *or* (4.2) *has a unique continuous solution* $u_t = u(t, \cdot) \in H_1$ *with* $v_t = \partial_t u(t, \cdot) \in H$ *in* $[0, T]$ *such that*

$$E \sup_{t \le T}\{e(u_t) + |u_t|_{m+1}^{m+1}\} < \infty. \tag{4.33}$$

Before proving this theorem, we first establish the following technical lemma.

LEMMA 4.2. *Suppose that the conditions for Theorems* 4.1 *and* 4.2 *hold true*. *Then the following local energy inequality holds*:

$$
\begin{aligned}
&Ee_\lambda(u_{t\wedge\tau_N}) + \alpha E\|u_{t\wedge\tau_N}\|^2 \\
&\quad \le \{e(u_0) + \hat{G}(u_0) + Ct\} + K\int_0^t Ee_\lambda(u_{s\wedge\tau_N})\,ds,
\end{aligned}
\tag{4.34}
$$

*for some constants* $C$ *and* $K > 0$, *where*

$$
e_\lambda(u) = e(u) + \lambda|u|_{2(k+1)}^{2(k+1)}.
\tag{4.35}
$$

PROOF. Recall that, for $t < \tau_N \le T$, $u_t = u_t^N$ is a solution of the system (4.2). By means of Lemma 4.1, the following energy equation holds:

$$
\begin{aligned}
e(u_{t\wedge\tau_N}) &= e(u_0) + 2\int_0^{t\wedge\tau_N} \big(v_s, f(u_s)\big)\,ds + 2\int_0^{t\wedge\tau_N} \big(v_s, \sigma_s(Ju_s)\,dW_s\big) \\
&\quad + \int_0^{t\wedge\tau_N} \mathrm{Tr}[\sigma_s(Ju_s)R\sigma_s^*(Ju_s)]\,ds,
\end{aligned}
$$

or, by noting definition (4.31),

$$
\begin{aligned}
e(u_{t\wedge\tau_N}) + \hat{G}(u_{t\wedge\tau_N}) &= e(u_0) + \hat{G}(u_0) + 2\int_0^{t\wedge\tau_N} \big(v_s, \sigma_s(Ju_s)\,dW_s\big) \\
&\quad + \int_0^{t\wedge\tau_N} \mathrm{Tr}[\sigma_s(Ju_s)R\sigma_s^*(Ju_s)]\,ds
\end{aligned}
\tag{4.36}
$$

where, by condition (i)$'$, (4.31) and Lemma 4.2,

$$
\hat{G}(u) = \int G\big(x, u(x)\big)\,dx \ge \int (\alpha + \lambda u^{2k})u^2\,dx = \alpha\|u\|^2 + \lambda|u|_{2(k+1)}^{2(k+1)}
\tag{4.37}
$$

and

$$
|\hat{G}(u_0)| \le \sum_{j=1}^m \frac{2}{j+1}\int |a_j||u_0|^{j+1}\,dx \le a_0\sum_{j=1}^m |u_0|_{j+1}^{j+1} \le C_0\sum_{j=1}^m \|u_0\|_1^{j+1} < \infty.
$$

By taking the expectation of (4.36), we obtain

$$
\begin{aligned}
Ee(u_{t\wedge\tau_N}) &= e(u_0) + \hat{G}(u_0) - E\hat{G}(u_{t\wedge\tau_N}) \\
&\quad + E\int_0^{t\wedge\tau_N} \mathrm{Tr}[\sigma_s(Ju_s)R\sigma_s^*(Ju_s)]\,ds.
\end{aligned}
\tag{4.38}
$$

Now, in view of conditions (ii) and (ii)$'$, we have

$$
\begin{aligned}
\mathrm{Tr}[\sigma_t(Ju)R\sigma_t^*(Ju)] &= \int r(x,x)\sigma^2(t,x,u,Du)\,dx \\
&\le C_1\int r(x,x)\big\{1 + |u|^{2(k+1)} + |Du|^2\big\}\,dx \\
&\le \big\{\mathrm{Tr}\,R + r_0|u|_{2(k+1)}^{2(k+1)} + r_0\|u\|_1^2\big\} \le C + Ke_\lambda(u),
\end{aligned}
\tag{4.39}
$$

for some constants $C$ and $K > 0$. By taking (4.37)–(4.39) into account, we deduce that

$$\begin{aligned} Ee_\lambda(u_{t\wedge\tau_N}) + \alpha E\|u_{t\wedge\tau_N}\|^2 &\le Ee(u_{t\wedge\tau_N}) + E\hat{G}(u_{t\wedge\tau_N}) \\ &= e(u_0) + \hat{G}(u_0) + E\int_0^{t\wedge\tau_N} \mathrm{Tr}[\sigma_s(Ju_s)R\sigma_s^*(Ju_s)]\,ds \\ &\le \{e(u_0) + \hat{G}(u_0) + Ct\} + K\int_0^t Ee_\lambda(u_{s\wedge\tau_N})\,ds. \end{aligned}$$

This completes the proof of the lemma. □

With the aid of this lemma, we are ready to prove the global existence Theorem 4.2.

PROOF OF THEOREM 4.2. For any $T > 0$, we will show that $u_{t\wedge\tau_N} \to u_t$ a.s. as $N \to \infty$ for any $t \le T$, so that the local solution becomes a global one. To this end, it suffices to show that $\tau_N \to \infty$ as $N \to \infty$ with probability one.

Recall that, for $t < \tau_N \le T$, $u_t = u_t^N$ is a solution of the system (4.2). By applying Lemma 4.2 with $\rho_N(t) = Ee_\lambda(u_{t\wedge\tau_N})$ and noting $\alpha \ge 0$, inequality (4.34) yields

$$\rho_N(t) \le \{e(u_0) + \hat{G}(u_0) + Ct\} + K\int_0^t \rho_N(s)\,ds$$

which, by the Gronwall inequality, implies that

$$\rho_N(T) \le \{e(u_0) + \hat{G}(u_0) + CT\}e^{KT} = C_T. \tag{4.40}$$

On the other hand, we have

$$\begin{aligned} \rho_N(T) = Ee_\lambda(u_{t\wedge\tau_N}) &\ge E\{\mathcal{I}(\tau_N \le T)e_\lambda(u_{\tau_N})\} \\ &\ge CE\{\|u_{\tau_N}\|_1^2\mathcal{I}(\tau_N \le T)\} \ge C\left(\frac{N}{2}\right)^2 P\{\tau_N \le T\}, \end{aligned}$$

where $\mathcal{I}$ is the indicator function and $C > 0$ is a constant. In view of (4.40), the above inequality gives

$$P\{\tau_N \le T\} \le 4\rho_N(T)/CN^2 \le 4C_T/CN^2$$

which, with the aid of the Borel–Cantelli lemma, implies that

$$P\{\tau_\infty \le T\} = 0,$$

or $\lim_{N\to\infty} \tau_N = \infty$ a.s. Now we let $u_t^N = u_{t\wedge\tau_N}$ and denote its limit: $\lim_{N\to\infty} u_t^N$ still by $u_t$. Then $u_t$ is the global solution as announced. To verify the energy bound (4.33), we take the limit (as $N \to \infty$) in (4.36) to get the energy equation:

$$e(u_t) + \hat{G}(u_t) = e(u_0) + \hat{G}(u_0) + \int_0^t \operatorname{Tr}[\sigma_s(Ju_s)R\sigma_s^*(Ju_s)]\,ds + 2\int_0^T \big(v_s, \sigma_s(Ju_s)\,dW_s\big).$$

Similar to (4.38), the above equation yields

$$E\sup_{\theta\le t} e_\lambda(u_\theta) \le \{e(u_0) + \hat{G}(u_0) + Ct\} + E\int_0^t \operatorname{Tr}[\sigma_s(Ju_s)R\sigma_s^*(Ju_s)]\,ds + 2E\sup_{\theta\le t}\int_0^\theta \big(v_s, \sigma_s(Ju_s)\,dW_s\big). \tag{4.41}$$

By means of the Birkholder–Davis–Gundy inequality (Kunita [4], page 66), we have

$$\begin{aligned} E\sup_{\theta\le t}&\int_0^\theta \big(v_s, \sigma_s(Ju_s)\,dW_s\big) \\ &\le C_1 E\left\{\int_0^t \big(\sigma_s(Ju_s)R\sigma_s^*(Ju_s)v_s, v_s\big)\,ds\right\}^{1/2} \\ &\le C_1 E\left\{\sup_{\theta\le t}\|v_\theta\|^2 \int_0^t \operatorname{Tr}[\sigma_s(Ju_s)R\sigma_s^*(Ju_s)]\,ds\right\}^{1/2} \\ &\le \frac{1}{4}E\sup_{\theta\le t}\|v_\theta\|^2 + C_2 E\int_0^t \operatorname{Tr}[\sigma_s(Ju_s)R\sigma_s^*(Ju_s)]\,ds \end{aligned} \tag{4.42}$$

for some constants $C_1, C_2 > 0$.

In view of (4.39), (4.41) and (4.42), there exist positive constants $C_3$ and $C_4$, depending on $\lambda$, $T$, etc., such that

$$E\sup_{\theta\le t} e_\lambda(u_\theta) \le C_3 + C_4\int_0^t E\sup_{\theta\le s} e_\lambda(u_\theta)\,ds.$$

By applying the Gronwall inequality, the above gives

$$E\sup_{t\le T} e_\lambda(u_t) \le C_3 e^{C_4 T},$$

which implies the energy bound (4.33). □

REMARKS. (i) In contrast to the example given in Section 3 for an explosive solution, suppose that $f(x,u) = -(\alpha u + \lambda u^3)$ with $\alpha, \lambda > 0$, and $\sigma = \alpha_0 + \alpha_1 u + \alpha_2 u^2$. Then conditions (i)$'$ and (ii)$'$ of Theorem 4.2 are met. The corresponding Cauchy problem with $d \le 3$ has a unique global solution as stated.

(ii) Throughout the paper, for simplicity, the noise consists of a simple term $\sigma W$. The theorems hold true for multiple noise terms $\sum_{i=1}^k \sigma_i W_i$ with independent Wiener fields $W_i' s$, provided that each $\sigma_i$ satisfies the conditions imposed on $\sigma$.

(iii) The existence theorems given here can be generalized to a class of second-order stochastic hyperbolic equations with nonlinear terms of polynomial growth. This type of problems will be treated in a separate paper. □

## APPENDIX

PROOF OF LEMMA 2.1. We shall only prove the energy equation, which is the key point of this lemma. The proof is based on smoothing equation (2.7) by means of the Friedrichs' mullifier $\rho_{\varepsilon *}$ defined as

$$g^{\varepsilon}(x) = (\rho_{\varepsilon *} g)(x) = \int \rho_{\varepsilon}(x - y) g(y)\, dy$$

where $\rho_{\varepsilon}$ is a certain positive, even $C^{\infty}$-function with compact support in a $\varepsilon$-neighborhood of the origin such that $\int \rho_{\varepsilon}\, dx = 1$. We apply $\rho_{\varepsilon *}$ to equation (2.7) to obtain the mullified system

$$\text{(A.1)} \qquad \begin{cases} du_t^{\varepsilon} = v_t^{\varepsilon}\, dt, \\ dv_t^{\varepsilon} = (c^2 \nabla^2 - \gamma^2) u_t^{\varepsilon} + f_t^{\varepsilon} + dM_t^{\varepsilon}, \\ u_0^{\varepsilon} = g, \quad v_0^{\varepsilon} = h. \end{cases}$$

Since the mullified functions $u_t^{\varepsilon}$, $v_t^{\varepsilon}$, etc., are smooth ($C^{\infty}$) in $x$, we can apply the *Itô* formula to (A.1) for each $x \varepsilon \mathbb{R}^d$ to get (Kunita [4], see page 92):

$$|v^{\varepsilon}(t,x)|^2 = |v_0^{\varepsilon}(x)|^2 + 2\int_0^t v^{\varepsilon}(s,x)(c^2\nabla^2 - \gamma^2) u^{\varepsilon}(s,x)\, ds$$
$$+ 2\int_0^t v^{\varepsilon}(s,v) f^{\varepsilon}(s,x)\, ds + 2\int_0^t v^{\varepsilon}(s,x)\, dM_s^{\varepsilon} + \int_0^t q^{\varepsilon}(s,x,x)\, ds,$$

where

$$q^{\varepsilon}(t,x,y) = [(\rho_{\varepsilon *}) \otimes (\rho_{\varepsilon *}) q_t](x,y) = \iint \rho_{\varepsilon}(x-\xi)\rho_{\varepsilon}(y-\eta) q(t,\xi,\eta)\, d\xi\, d\eta,$$

and $\otimes$ denotes the tensor product.

After intergrating the above equation over $\mathbb{R}^d$, by parts if necessary, and evaluating integrals over $[0,t]$ when possible, we arrive at the following mullified energy equation:

$$\text{(A.2)} \qquad \begin{aligned} e(u_t^{\varepsilon}) = e(u_0) + 2\int_0^t (v_s^{\varepsilon}, f_s^{\varepsilon})\, ds + 2\int_0^t (v_s^{\varepsilon}, dM_s^{\varepsilon}) \\ + \int_0^t \operatorname{Tr} Q_s^{\varepsilon}\, ds, \end{aligned}$$

where $Q_t^{\varepsilon}$ is the covariation operator of $M_t^{\varepsilon}$ with kernel $q^{\varepsilon}(t,x,y)$. By some well-known properties of the mullifier $(\rho_{\varepsilon *})$ (see Chapter 1 of [6]), as $\varepsilon \to 0$, we show easily that $Ee(u_t^{\varepsilon}) \to Ee(u_t)$ for each $t$ a.s. We can also prove that

$$\text{(A.3)} \qquad E\left|\int_0^t (v_s^{\varepsilon}, f_s^{\varepsilon})\, ds - \int_0^t (v_s, f_s)\, ds\right| \to 0$$

and

$$E\left|\int_0^t \operatorname{Tr} Q_s^\varepsilon\, ds - \int_0^t \operatorname{Tr} Q_s\, ds\right| \to 0 \quad \text{as } \varepsilon \to 0. \tag{A.4}$$

To be brief, let us verify (A.4) only. Note that

$$\begin{aligned}
&\int_0^t \operatorname{Tr} Q_s^\varepsilon\, ds - \int_0^t \operatorname{Tr} Q_s\, ds \\
&\quad = \int_0^t \int [q^\varepsilon(s,x,x) - q(s,x,x)]\, dx\, ds \\
&\quad = \int_0^t \int \left\{ \iint \tilde{\rho}_\varepsilon(\xi,\eta)[q(s,x-\xi,x-\eta) - q(s,x,x)]\, d\xi\, d\eta \right\} dx\, ds,
\end{aligned} \tag{A.5}$$

where $\tilde{\rho}_\varepsilon(x,y) = \rho_\varepsilon(x)\rho_\varepsilon(y)$. Similar to the proof of Lemma 1.3 (d) in Mizohata [6], we can show that

$$\lim_{\varepsilon\to 0} \int \left\{ \iint \tilde{\rho}_\varepsilon(\xi,\eta)|q(s,x-\xi,x-\eta) - q(s,x,x)|\, d\xi\, dy \right\} dx = 0 \tag{A.6}$$

for each $s \in [0,t]$ a.s. In view of (A.5) and (A.6), with the aid of the bounded convergence theorem, we deduce that (A.4) holds true.

Now it remains to show that

$$\lim_{\varepsilon\to 0} E\left|\int_0^t (v_s^\varepsilon, dM_s^\varepsilon) - \int_0^t (v_s, dM_s)\right| = 0. \tag{A.7}$$

To this end, we start with

$$\begin{aligned}
&E\left|\int_0^t (v_s^\varepsilon, dM_s^\varepsilon) - \int_0^t (v_s, dM_s)\right| \\
&\qquad \le E\left|\int_0^t (v_s^\varepsilon - v_s, dM_s)\right| + E\left|\int_0^t \big(v_t^\varepsilon, d(M_s^\varepsilon - M_s)\big)\right|.
\end{aligned} \tag{A.8}$$

By means of Birkholder–Davis–Gundy inequality for martingales, we have

$$\begin{aligned}
E\left|\int_0^t (v_s^\varepsilon - v_s, dM_s)\right| &\le \left\{ E\left|\int_0^t (v_s^\varepsilon - v_s, dM_s)\right|^2 \right\}^{1/2} \\
&\le \sqrt{2}\left\{ E\int_0^t \big(Q_s(v_s^\varepsilon - v_s), v_s^\varepsilon - v_s\big)\, ds \right\}^{1/2} \\
&\le 2\left\{ E \sup_{t\le T} \operatorname{Tr} Q_t \right\}^{1/2} \left\{ E\int_0^t \|v_s^\varepsilon - v_s\|^2\, ds \right\}^{1/2}
\end{aligned}$$

which implies that

$$E\left|\int_0^t (v_s^\varepsilon - v_s, dM_s)\right| \to 0 \quad \text{as } \varepsilon \to 0. \tag{A.9}$$

Similarly we have

$$
\text{(A.10)}\qquad \begin{aligned} &E\left|\int_0^t \big(v_s^\varepsilon, d(M_s^\varepsilon - M_s)\big)\right| \\ &\quad \le 2\Big\{E \sup_{t\le T} \operatorname{Tr} \Delta Q_t^\varepsilon\Big\}^{1/2}\Big\{E\int_0^t \|v_s^\varepsilon\|^2\,ds\Big\}^{1/2}, \end{aligned}
$$

where $\Delta Q_t^\varepsilon = \frac{d}{dt}\langle\langle M^\varepsilon - M\rangle\rangle_t$ and

$$
\operatorname{Tr}\Delta Q_t^\varepsilon = \int [(\rho_\varepsilon * - I)\otimes(\rho_\varepsilon * - I)q_t](x,x)\,dx,
$$

where $\otimes$ denotes the tensor product and $I$ is the identity operator.

Therefore,

$$
\text{(A.11)}\qquad \begin{aligned} E\sup_{t\le T}\operatorname{Tr} Q_t^\varepsilon &\le E\sup_{t\le T}\left|\int [\rho_\varepsilon * \otimes(\rho_\varepsilon * - I)q_t](x,x)\,dx\right| \\ &\quad + E\sup_{t\le T}\left|\int [I\otimes(\rho_\varepsilon * - I)q_t](x,x)\,dx\right| \\ &= I_1 + I_2. \end{aligned}
$$

Explicitly we have

$$
I_1 = E\sup_{t\le T}\left|\int\left\{\iint \tilde\rho_\varepsilon(\xi,\eta)[q(t,x-\xi,x-\eta) - q(t,x-\xi,x)]\,d\xi\,d\eta\right\}dx\right|
$$

and

$$
I_2 = E\sup_{t\le T}\left|\int\left\{\int \rho(\eta)[q(t,x,x-\eta) - q(t,x,x)]\,d\eta\right\}dx\right|,
$$

each of which can be shown, similar to (A.6), to go to zero as $\varepsilon \to 0$. In view of (A.10) and (A.11), we get

$$
\lim_{\varepsilon\to 0} E\left|\int_0^t \big(v_s^\varepsilon, d(M_s^\varepsilon - M_s)\big)\right| = 0,
$$

which together with (A.8) and (A.9) verify (A.7). By taking (A.3), (A.4) and (A.7) into account, we set $\varepsilon \to 0$ in the truncated equation (A.2) to yield the energy equation (2.8). □

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