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APPROXIMATION OF AMERICAN PUT PRICES BY EUROPEAN PRICES VIA AN EMBEDDING METHOD

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In mathematical finance, the price of the so-called “American Put option” is given by the value function of the optimal-stopping problem with the option payoff $\psi\colon x \to (K-x)^+$ as a reward function. Even in the Black–Scholes model, no closed-formula is known and numerous numerical approximation methods have been specifically designed for this problem.

In this paper, as an application of the theoretical result of B. Jourdain and C. Martini [*Ann. Inst. Henri Poincaré Anal. Nonlinear* **18** (2001) 1–17], we explore a new approximation scheme: we look for payoffs as close as possible to ψ, the American price of which is given by the European price of another claim. We exhibit a family of payoffs $\widehat{\varphi}_h$ indexed by a measure h, which are continuous, match with $(K-x)^+$ outside of the range $]K^*, K[$ (where K^* is the perpetual Put strike), are analytic inside with the right derivative (-1) at both ends. Moreover a numerical procedure to select the best h in some sense yields nice results.

1. Introduction. Consider the classical Black–Scholes model

$$dX_t^x = \rho X_t^x\,dt + \sigma X_t^x\,dB_t, \qquad X_0^x = x > 0, \qquad \rho \in \mathbb{R}, \tag{1.1}$$

where B is a standard Brownian motion, ρ the instantaneous interest rate and σ the volatility of X and denote by

$$\mathcal{A} f(x) = \frac{\sigma^2 x^2}{2} f''(x) + \rho x f'(x) - \rho f(x)$$

the Black–Scholes infinitesimal generator. Given a continuous function $\psi\colon \mathbb{R}_+^* \to \mathbb{R}_+$ satisfying some growth assumptions, the price of the so-called American option with payoff ψ, time to maturity $t > 0$ and spot x is given by the expression

$$v_\psi^{\mathrm{am}}(t,x) = \sup_{\tau \in \mathcal{T}(0,t)} \mathbb{E}[e^{-\rho\tau}\psi(X_\tau^x)] \tag{1.2}$$

where τ runs across the set of stopping times of the Brownian filtration such that $\tau \le t$ almost surely. For $x > 0$, the function $t \to v_\psi^{\mathrm{am}}(t,x)$ is nondecreasing. Moreover, it is greater than $\psi(x)$ and typically the space $]0,\infty[\times \mathbb{R}_+^*$ splits into two regions, the so-called Exercise region where by definition $v_\psi^{\mathrm{am}} = \psi$ and its complement the Continuation region where $v_\psi^{\mathrm{am}} > \psi$.

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In this paper, we are interested in the price $v_{\mathrm{Put}}^{\mathrm{am}}(t, x)$ of the American Put option given by $\psi(x) = (K - x)^+$, where K is some positive constant (the strike of the option). In case $\rho \leq 0$, it is obvious by a convexity argument that the optimal stopping time is $\tau = t$ and $v_{\mathrm{Put}}^{\mathrm{am}}(t, x)$ is equal to the price of the European Put option. From now on, we suppose that $\rho > 0$. Even if there is no closed-form expression for $v_{\mathrm{Put}}^{\mathrm{am}}(t, x)$, its limit as $t \to +\infty$, the price of the so-called perpetual Put option, can be computed explicitly as

$$\begin{aligned} v_{\mathrm{Put}}^{\mathrm{am}}(\infty, x) &= (K - K^*)\left(\frac{x}{K^*}\right)^{-\alpha} \mathbb{1}_{\{x \geq K^*\}} + (K - x)\mathbb{1}_{\{x < K^*\}}, \\ \alpha &= \frac{2\rho}{\sigma^2} \quad \text{and} \quad K^* = \frac{\alpha K}{1 + \alpha}. \end{aligned} \tag{1.3}$$

K^* is called the perpetual strike. Moreover there is a continuous nonincreasing function $\tilde{t}$: $]0, +\infty[\to [0, +\infty]$ with $\tilde{t}(x) = +\infty$ if $x \leq K^*$ and $\tilde{t}(x) = 0$ if $x \geq K$, such that the Exercise region of the American Put option is given by $\{(t, x) : 0 < t \leq \tilde{t}(x)\}$. In the (t, x) plane the situation is sketched in Figure 1. The reader may consult [8] for a nice exposition of the properties of he free boundary $\tilde{t}$, where it is also shown to solve a one-dimensional integral equation.

The purpose of the paper is to construct an approximation of $v_{\mathrm{Put}}^{\mathrm{am}}(t, x)$ thanks to the following embedding result obtained in a previous work [5]: let φ: $\mathbb{R}_+^* \to \mathbb{R}_+$ be a continuous function such that $\sup_{x>0} \varphi(x)/(x + x^\alpha) < +\infty$ and $v_\varphi(t, x) = \mathbb{E}[e^{-\rho t}\varphi(X_t^x)]$ denote the price of the European option with payoff φ. If the function $x \to \widehat{\varphi}(x) = \inf_{t \geq 0} v_\varphi(t, x)$ is continuous and if there is a continuous function $\widehat{t}$: $]0, +\infty[\to [0, +\infty]$ such that $\forall x > 0$, $\widehat{\varphi}(x) = v_\varphi(\widehat{t}(x), x)$

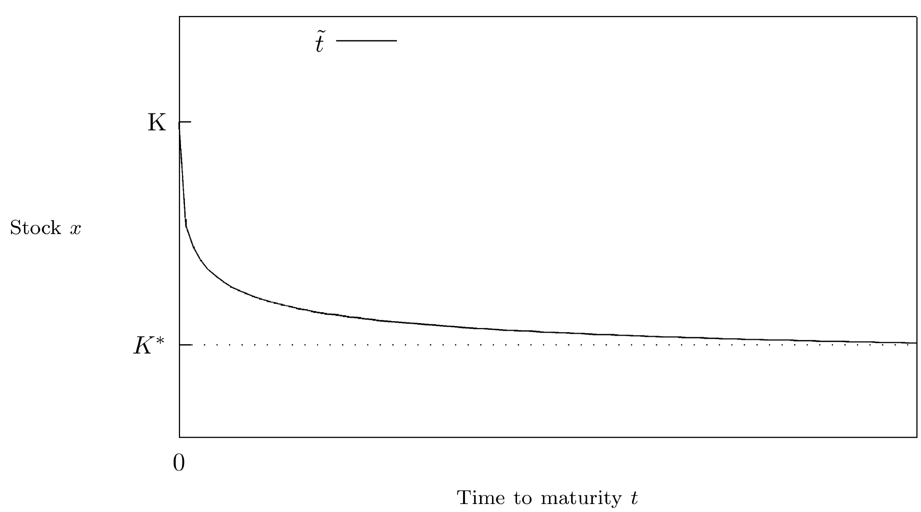

FIG. 1.

[convention: $v_\varphi(\infty, x) = \liminf_{t\to+\infty} v_\varphi(t, x)$], then the price of the American option with payoff $\widehat{\varphi}$ is embedded in the function $v_\varphi(t, x)$ in the following sense:

$$\forall (t, x) \in [0, +\infty[\times]0, +\infty[, \qquad v_{\widehat{\varphi}}^{\mathrm{am}}(t, x) = v_\varphi\big(t \vee \widehat{t}(x), x\big).$$

As an easy consequence, the set $\{(t, x) : 0 < t \le \widehat{t}(x)\}$ is included in the Exercise region of the American option.

The main drawback of the above result is that we do not know, at the moment, how to design a function φ such that $\widehat{\varphi}$ matches a given target payoff of interest. Even in the special Put case, despite many attempts, we could not find any European payoff φ with associated American payoff $\widehat{\varphi}(x) = (K - x)^+$. Nevertheless we rely on the above theoretical result to design closed-form prices for a large class of payoffs very close to the Put payoff. This is done in three steps.

First, in Section 2 we design a family of European payoffs which verify very crude necessary conditions for $\widehat{\varphi}(x) = (K - x)^+$ to have any chance to hold. This is the main step, it relies on the parameterization of φ by a measure h related to $\mathcal{A}\varphi$. Then we focus on the Continuation region. Among our family we find necessary and sufficient conditions which grant that the equation $\inf_{t\ge 0} v_\varphi(t, x) = v_\varphi(\widehat{t}(x), x)$ defines a curve which displays the same qualitative features as the free boundary of the American Put (Section 3).

Unfortunately, it is easy to see that for any function among our family $\widehat{\varphi}(x) = (K - K^*)(x/K^*)^{-\alpha}\mathbb{1}_{\{x\ge K^*\}}$ below K^*, which is not satisfactory. The third step is to prove that the price of the American option with modified payoff $(K - x)^+\mathbb{1}_{\{x\le K^*\}} + \widehat{\varphi}(x)\mathbb{1}_{\{x>K^*\}}$, denoted by $\widehat{\varphi}_h$ to emphasize the dependence on the parameter h, and matching $(K - x)^+$ both for $x \ge K$ and for $x \le K^*$ is still embedded in $v_\varphi(t, x)$: $v_{\widehat{\varphi}_h}^{\mathrm{am}}(t, x) = (K - x)^+\mathbb{1}_{\{x\le K^*\}} + v_\varphi(t \vee \widehat{t}(x), x)\mathbb{1}_{\{x>K^*\}}$. This is done in Section 4.

Since we show that $\widehat{\varphi}_h$ cannot be equal to the Put payoff everywhere [indeed $\widehat{\varphi}_h''(K^{*+}) > 0$], we believe that at this stage there is little to get from further calculations. The last stage is to select among our family the point h^* so that, in some sense, $\widehat{\varphi}_{h^*}$ is the closest payoff to $(K - x)^+$. We choose the criterion

$$\sup_x \big|\widehat{\varphi}_h(x) - (K - x)^+\big|.$$

This is done in a numerical manner which is explained in detail in the previous section (Section 5): choosing φ in a particular low-dimensional subclass, we compute a discretized version of $\widehat{\varphi}$ and then minimize the above criterion. The numerical results seem very good.

2. A first set of tentative payoffs φ.

Let us now look for a class of initial payoffs φ for which there is some hope that $\widehat{\varphi}(x) = (K - x)^+$ holds, at least for x between K^* and K.

Notice first that the European price of φ should match the American Put price in the Continuation region. In particular it should increase from 0 to $(K-K^*)(x/K^*)^{-\alpha}$ as t goes from 0 to ∞ for $x \geq K$. This gives at once $\varphi(x)=0$ for $x \geq K$. Another condition is that the European price of φ decreases to $\widehat{\varphi}(x)$, for x between K^* and K, as t goes from 0 to $\widehat{t}(x)$ (the tentative free boundary). This should also hold for x below K^* with $\widehat{t}(x)=\infty$. Note that these conditions are necessary only if we restrict ourselves to the simple case of a single curve where $\inf_{t\geq 0} v_\varphi(t,x)$ is attained which splits the (t,x) plane in two regions where respectively $\partial_t v_\varphi \leq 0$ and $\partial_t v_\varphi \geq 0$. Thanks to the Black–Scholes PDE this gives that $\mathcal{A}\varphi(x)$ (defined in any reasonable sense) should be non-positive between 0 and K. Now a natural way to proceed is to parameterize φ by $\mathcal{A}\varphi$, or, in other words, to solve the ODE

$$\mathcal{A}\varphi = m.$$

The solutions of $\mathcal{A}\varphi=0$ are the functions $x \to ax+bx^{-\alpha}$ for 2 reals (a,b). By a straightforward integration this gives

$$\varphi(x) = ax + bx^{-\alpha} - \frac{2}{\sigma^2} x^{-\alpha} \int_0^x y^\alpha \int_y^\infty \frac{m(dr)}{r^2} \tag{2.1}$$

or yet by Fubini's theorem, since m should be supported in $]0,K]$ to ensure $\varphi=0$ above K,

$$\varphi(x) = ax + bx^{-\alpha} - \frac{2}{\sigma^2(\alpha+1)} x^{-\alpha} \int_0^K (r \wedge x)^{\alpha+1} \frac{m(dr)}{r^2}, \tag{2.2}$$

as soon as the measure m satisfies $\int_0^K r^{\alpha-1}|m|(dr) < \infty$.

Now by the Lebesgue theorem, it is easy to see that $a = \lim_{x\to\infty}(\varphi(x)/x)$ which gives for us $a=0$. Then $\varphi(x)=0$ for $x \geq K$ gives the condition

$$b = \frac{2}{\sigma^2(\alpha+1)} \int_0^K r^{\alpha-1} m(dr). \tag{2.3}$$

Observe next that since $\lim_{x\to\infty}(\varphi(x)/x) = a = 0$ and by Lebesgue's theorem $\lim_{x\to 0^+}(\varphi(x)/x^{-\alpha}) = b$, according to Section A.2 in the Appendix, $\lim_{t\to\infty} v_\varphi(t,x) = ax+bx^{-\alpha} = bx^{-\alpha}$. This gives the value of b: $b=(K-K^*)/K^{*-\alpha}$.

We have not yet used the fact that m should be nonpositive on $]0,K[$. Obviously for (2.3) to hold, since b is positive, m should be of the form:

$$m(dr) = c\delta_K(dr) - \mathbb{1}_{]0,K[}(r) \frac{\sigma^2(\alpha+1)K^*}{2} h(dr),$$

where h is a positive measure on $]0,K[$ [we wrote the indicator function for clarity's sake; also the factor $(\sigma^2(\alpha+1)K^*)/2$ before h will lead to easier calculations later on] and c a strictly positive number.

By the way, c is related to the left derivative of φ at K; accordingly to (2.1)

$$\left(\varphi(x)x^{\alpha}\right)' = -\frac{2}{\sigma^2}x^{\alpha}\int_x^K \frac{m(dr)}{r^2},$$

whence by $\varphi(K) = 0$,

$$c = \frac{-\sigma^2\varphi'(K_-)K^2}{2}.$$

As soon as φ has a few regularity properties on the left of K, since $\widehat{t}(x)$ goes to 0 as x goes to K from below, $\widehat{\varphi}'(x)$ should go to $\varphi'(K_-)$. But $\widehat{\varphi}'(x)$ should be -1, so we get the value of c: $c = \sigma^2K^2/2$.

The last point to check is that this is compatible with (2.3). This rewrites now:

$$K^*\int_0^K r^{\alpha-1}h(dr) = \frac{K^2}{(\alpha+1)}K^{\alpha-1} - (K-K^*)K^{*\alpha} = K^*\frac{K^{\alpha}-K^{*\alpha}}{\alpha}.$$

In particular this is a positive quantity.

So far we have reached the following:

LEMMA 1. *Let $\varphi(x)$ be a continuous payoff satisfying $\mathcal{A}\varphi = m$ where m is a measure on* $]0,+\infty[$ *such that* $\int_0^{+\infty} r^{\alpha-1}|m|(dr) < +\infty$.

Then the four conditions

(i) $\varphi(x) = 0$ *for* $x \geq K$,
(ii) *for every* $x \geq K$, $v_{\varphi}(t,x) \to (K-K^*)(x/K^*)^{-\alpha}$ *as* $t \to \infty$,
(iii) *in a weak sense* $\mathcal{A}\varphi \leq 0$ *below* K,
(iv) $\varphi'(K^-) = -1$,

hold if and only if, $m(dr) = \frac{1}{2}\sigma^2K^2\delta_K(dr) - \frac{1}{2}\sigma^2(\alpha+1)K^*h(dr)$, *where* h *is a positive measure on* $]0,K[$ *such that* $\int_0^K r^{\alpha-1}h(dr) = (K^{\alpha}-K^{*\alpha})/\alpha$ *and*

$$\varphi(x) = (K-K^*)\left(\frac{x}{K^*}\right)^{-\alpha} - x^{-\alpha}\frac{(K\wedge x)^{\alpha+1}}{\alpha+1} + K^*\int_0^K x^{-\alpha}\frac{(r\wedge x)^{\alpha+1}}{r^2}h(dr). \tag{2.4}$$

An additional calculation (cf. Section A.1 in the Appendix) gives also:

LEMMA 2. *The function φ in* (2.4) *is nonnegative*.

2.1. *Computing the corresponding price.* From now on we suppose that φ is given by (2.4). Let

$$e_r(x) = x^{-\alpha}(r\wedge x)^{\alpha+1} = r^{\alpha+1}x^{-\alpha}\mathbb{1}_{\{x>r\}} + x\mathbb{1}_{\{x\leq r\}}.$$

Then, after (2.4), since the function $x \mapsto x^{-\alpha}$ is invariant, also using $K/K^* = 1 + 1/\alpha$:

$$\frac{v_\varphi(t,x)}{K^*} = \frac{(x/K^*)^{-\alpha}}{\alpha} - \frac{1}{\alpha K} v_{e_K}(t,x) + \int_0^K v_{e_r}(t,x) \frac{h(dr)}{r^2},$$

where

$$v_{e_r}(t,x) = e^{-\rho t} \mathbb{E}\Big[e_r\Big(x \exp\Big(\Big(\rho - \frac{\sigma^2}{2}\Big)t + \sigma B_t\Big)\Big)\Big],$$

which gives, after straightforward calculations (cf. Section A.3 in the Appendix).

LEMMA 3. *One has*

$$v_{e_r}(t,x) = r^{\alpha+1} x^{-\alpha} N\Big(-\Big(\frac{\ln(r/x) + ((\alpha+1)/2)\sigma^2 t}{\sqrt{\sigma^2 t}}\Big)\Big) + x N\Big(\frac{\ln(r/x) - ((\alpha+1)/2)\sigma^2 t}{\sqrt{\sigma^2 t}}\Big),$$

where $N(z) = \int_{-\infty}^z e^{-y^2/2}/\sqrt{2\pi}\, dy$ *denotes the cumulative distribution function of the Normal law*.

Setting $a = \ln(K^*)$, $b = \ln(K)$, $y = \ln(x)$, $u = \ln(r)$, also $\lambda = 1/(\sigma^2 t)$ and denoting the image of the measure $h(dr)$ by the function $r \to \ln(r)$ by $dh(e^u)$, we thus get

$$\begin{aligned} e^{-a} v_\varphi(\lambda, y) &= \frac{e^{\alpha(a-y)}}{\alpha} - \frac{e^{\alpha(b-y)}}{\alpha} N\Big(-(b-y)\sqrt{\lambda} - \Big(\frac{\alpha+1}{2}\Big)\frac{1}{\sqrt{\lambda}}\Big) \\ &\quad - \frac{e^{(y-b)}}{\alpha} N\Big((b-y)\sqrt{\lambda} - \Big(\frac{\alpha+1}{2}\Big)\frac{1}{\sqrt{\lambda}}\Big) \\ &\quad + e^{-\alpha y} \int_{-\infty}^b e^{\alpha u} \frac{dh(e^u)}{e^u} N\Big(-(u-y)\sqrt{\lambda} - \Big(\frac{\alpha+1}{2}\Big)\frac{1}{\sqrt{\lambda}}\Big) \\ &\quad + e^y \int_{-\infty}^b e^{-u} \frac{dh(e^u)}{e^u} N\Big((u-y)\sqrt{\lambda} - \Big(\frac{\alpha+1}{2}\Big)\frac{1}{\sqrt{\lambda}}\Big). \end{aligned}$$

In terms of the measure $\widetilde{h}(du) = \alpha e^{(\alpha-1)(u-b)/2}(dh(e^u)/e^u$, we get:

LEMMA 4. *Let* $a = \ln(K^*)$, $b = \ln(K)$, $y = \ln(x)$, $u = \ln(r)$, $\lambda = 1/(\sigma^2 t)$, *also*

$$\widetilde{h}(du) = \alpha e^{(\alpha-1)(u-b)/2} \frac{dh(e^u)}{e^u}.$$

Then one has

$$\begin{aligned}
&\alpha e^{-a} v_\varphi(\lambda, y) \\
&\quad = e^{\alpha(a-y)} - e^{\alpha(b-y)} N\left(-(b-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\qquad - e^{(y-b)} N\left((b-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\qquad + e^{(\alpha-1)(b-y)/2} \int_{-\infty}^{b} e^{(\alpha+1)(u-y)/2} \widetilde{h}(du) N\left(-(u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right) \\
&\qquad + e^{(\alpha-1)(b-y)/2} \int_{-\infty}^{b} e^{-(\alpha+1)(u-y)/2} \widetilde{h}(du) N\left((u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right).
\end{aligned}$$

3. Tentative φ's with reasonable theta-zero curve. As we are interested in $\widehat{t}(x)$ such that $\inf_{t\geq 0} v_\varphi(t,x) = v_\varphi(\widehat{t}(x), x)$, we are going to study the so-called theta-zero points solution of $\partial_t v_\varphi(t,x) = 0$. More precisely we look for conditions on the measure h which ensure that

$$\text{(3.1)} \quad \widehat{t}(x) \text{ is continuous,} \qquad \widehat{t}^{-1}(0) = [K, +\infty[, \qquad \widehat{t}^{-1}(+\infty) =]0, K^*].$$

3.1. *The theta-zero curve.* Since the price of the European option with payoff φ satisfies the Black–Scholes partial differential equation $\partial_t v_\varphi(t,x) = \mathcal{A} v_\varphi(t,x)$ for $t, x > 0$, in order to find the theta-zero points, we compute $\mathcal{A} v_\varphi(t,x)$.

One main advantage of our parameterization of φ by $\mathcal{A}\varphi = m$ is the simplicity of the following computations. Indeed by the semi-group property $\mathcal{A} v_\varphi(t,x) = v_{\mathcal{A}\varphi}(t,x)$. Since $v_{\mathcal{A}\varphi}$ solves the Black–Scholes partial differential equation

$$\forall t, x > 0, \qquad \partial_t v_{\mathcal{A}\varphi}(t,x) = \mathcal{A} v_{\mathcal{A}\varphi}(t,x), \qquad v_{\mathcal{A}\varphi}(0,\cdot) = m,$$

by the Feynman–Kacs representation formula

$$\forall t, x > 0, \qquad v_{\mathcal{A}\varphi}(t,x) = e^{-\rho t} \int_0^K p_t^X(x,r) m(dr)$$

where $p_t^X(x,r)$ is the transition density of the Black–Scholes process. If $n_{\alpha^2}(z) = e^{-z^2/(2\alpha^2)}/\sqrt{2\pi\alpha^2}$ denotes the Gaussian density, an easy calculation yields $p_t^X(x,r) = \frac{x}{r} n_{\sigma^2 t}(\ln(r/x) - (\rho - \sigma^2/2)t)$.

As a conclusion:

LEMMA 5. *We have*

$$\mathcal{A} v_\varphi(t,x) = e^{-\rho t} \int_0^K n_{\sigma^2 t}\left(\ln\left(\frac{r}{x}\right) - \left(\rho - \frac{\sigma^2}{2}\right)t\right) \frac{m(dr)}{r}.$$

We recall that $m(dr) = \frac{\sigma^2 K^2}{2}\delta_K(dr) - \frac{\sigma^2(\alpha+1)K^*}{2}h(dr)$. Changing notation by setting

$$y = \ln(x), \qquad u = \ln(r), \qquad \lambda = \frac{1}{\sigma^2 t}, \qquad a = \ln(K^*), \qquad b = \ln(K),$$

we obtain that

$$\partial_t v_\varphi(t,x) = \mathcal{A} v_\varphi(t,x) = C(\lambda, y) F(\lambda, y),$$

where $C(\lambda, y) = \sigma^2\sqrt{\lambda}e^{-(1+\alpha)^2/8\lambda}e^{(\alpha-1)(b-y)/2}e^b/(2\sqrt{2\pi}) > 0$ for $\lambda > 0$ and

$$F(\lambda, y) = e^{-(\lambda/2)(b-y)^2} - \int_{-\infty}^{b} e^{-(\lambda/2)(u-y)^2}\tilde{h}(du). \tag{3.2}$$

Thus we are interested in the solutions of

$$F(\lambda, y) = 0. \tag{3.3}$$

From now on, we suppose that

$$\forall x < K, \qquad h(]x, K[) > 0 \quad \text{and} \quad \int_0^K \ln^2(r) r^{(\alpha-3)/2} h(dr) < +\infty. \tag{3.4}$$

LEMMA 6. *The function F is C^1 on $[0,+\infty) \times \mathbb{R}$. Moreover, for $y \in \mathbb{R}$, the function $\lambda \geq 0 \to F(\lambda, y)$ vanishes at most twice. Finally, $\forall y \geq b$ (resp. $y < b$), $F(\lambda, y)$ is positive (resp. negative) for λ big enough.*

PROOF. The integrability assumption in (3.4) is equivalent to the convergence of $\int_{-\infty}^{b} u^2\tilde{h}(du)$. By Lebesgue theorem, we easily deduce that F is C^1.

Equation (3.3) gives

$$-\frac{\lambda}{2}(b-y)^2 = \ln\left(\int_{-\infty}^{b} e^{-(\lambda/2)(u-y)^2}\tilde{h}(du)\right).$$

Hence for fixed $y \in \mathbb{R}$, the solutions are given by the intersection of a straight line and the Log-Laplace transform of a positive measure which is strictly convex under (3.4). We conclude that $\lambda \geq 0 \to F(\lambda, y)$ vanishes at most twice. The last assertion is a consequence of the first part of (3.4). $\square$

Let us now derive necessary conditions on h for (3.1) to hold.

If (3.1) holds then $t \to v_\varphi(t \vee \widehat{t}(x), x)$ is nondecreasing. As a consequence, when $x \in]K^*, K[$, $\partial_t v_\varphi(t,x) \geq 0$ for $t \geq \widehat{t}(x)$, that is, when $y \in]a, b[$, $F(\lambda, y) \geq 0$ for λ positive and small. For $x \leq K^*$, $\widehat{t}(x) = +\infty$, that is, $\inf_{t \geq 0} v_\varphi(t,x) = \liminf_{t \to +\infty} v_\varphi(t,x)$. Since $\lambda \geq 0 \to F(\lambda, y)$ vanishes at most twice, so does $t > 0 \to \partial_t v_\varphi(t,x)$. Hence when $x \leq K^*$, $\partial_t v_\varphi(t,x) < 0$ for t big enough, that is, when $y \leq a$, $F(\lambda, y) < 0$ for λ positive and small.

Since F is continuous, to get the previous sign conditions, we need $F(0,a)=0$, that is, $\tilde{h}$ is a probability measure

$$\int_{-\infty}^{b} \tilde{h}(du) = 1. \tag{3.5}$$

As $F(0,y)$ is independent of y, the sign conditions then imply respectively $\partial_\lambda F(0,y) \geq 0$ for $y \in]a,b[$ and $\partial_\lambda F(0,y) \leq 0$ for $y \leq a$. Since F is C^1, $\partial_\lambda F(a,0)=0$, that is,

$$\int_{-\infty}^{b} (u-a)^2 \tilde{h}(du) = (b-a)^2. \tag{3.6}$$

The necessary conditions (3.5) and (3.6) will turn out to be sufficient for (3.1) to hold:

PROPOSITION 7. *If* $\forall y < b$, $\tilde{h}(]y,b[) > 0$ *and* (3.5) *and* (3.6) *hold*, *then*

$$\begin{aligned} \forall y \in]a,b[,\ \exists! \lambda^*(y) > 0 \quad & \textit{such that } F(\lambda^*(y), y) = 0, \\ F(\lambda, y) > 0 \quad & \textit{for } \lambda \in]0, \lambda^*(y)[, \\ F(\lambda, y) < 0 \quad & \textit{for } \lambda > \lambda^*(y), \end{aligned} \tag{3.7}$$

$$\forall y \geq b,\ \forall \lambda > 0, \quad F(\lambda, y) > 0, \tag{3.8}$$

$$\forall y \leq a,\ \forall \lambda > 0, \quad F(\lambda, y) < 0. \tag{3.9}$$

PROOF. By (3.5), $\forall y \in \mathbb{R}$, $F(0,y) = 0$. It is easy then to deduce (3.8) from (3.2).

Next, $\forall y \in \mathbb{R}$, writing $(u-y)^2 = (u-a)^2 + (a-y)^2 - 2(y-a)(u-a)$, developing $(b-y)^2$ in a similar way and using (3.5) and (3.6) we get

$$\partial_\lambda F(0,y) = \frac{1}{2}\int_{-\infty}^{b} (u-y)^2 \tilde{h}(du) - \frac{1}{2}(b-y)^2 = (y-a)\int_{-\infty}^{b} (b-u)\tilde{h}(du).$$

Hence $\partial_\lambda F(0,y)$ is positive (resp. negative) for $y > a$ (resp. $y < a$), which implies that $F(\lambda,y)$ is positive (resp. negative) for λ positive and small when $y > a$ (resp. $y < a$). By Lemma 6, when $y < b$, $F(\lambda,y)$ is negative for λ big enough. Moreover, as $\lambda \to F(\lambda,y)$ vanishes at $\lambda = 0$, this function vanishes at most for one $\lambda(y) > 0$ and then $\partial_\lambda F(\lambda(y),y) \neq 0$. By the intermediate value property, we deduce (3.7) and (3.9) for $y < a$. As $F(0,a) = \partial_\lambda F(0,a) = 0$, the function $F(\lambda,a)$ does not vanish for $\lambda > 0$ and (3.9) also holds for $y = a$. $\square$

Setting $\lambda^*(y) = 0$ for $y \leq a$ and $\lambda^*(y) = +\infty$ for $y \geq b$, then $\forall y \in \mathbb{R}$, $\widehat{\varphi}(e^y) = v_\varphi(\lambda^*(y), y)$. It is enough to check that $\widehat{\varphi}$ is continuous and that λ^* is continuous and nondecreasing to conclude that (3.1) holds. Let us now turn to a detailed study of λ^* and $\widehat{\varphi}$.

3.2. *Behavior of* $\lambda^*(y)$ *for* $y \in]a,b[$.

PROPOSITION 8. *Under the assumptions of Proposition* 7, *the function* λ^* *is analytic and increasing from* $]a,b[$ *to* $\mathbb{R}_+^*$ *and satisfies*

$$\lim_{y\to a^+} \lambda^*(y) = 0, \qquad \lim_{y\to b^-} \lambda^*(y) = \infty.$$

More precisely,

$$\lambda^*(y)(b-y)^2 \to_{y\to b^-} \infty. \tag{3.10}$$

If we suppose moreover that $d\tilde{h}$ *is absolutely continuous in a neighborhood of* b, *that is, for some* $b_* \in]a,b[$ $\tilde{h}(du) = \tilde{h}(u)du$ *on* $]b_*,b[$ *and that* $\lim_{u\to b^-} \tilde{h}(u) = \tilde{h}(b^-) > 0$ *exists, then*

$$\lim_{y\to b^-} \frac{\ln(b-y)}{\lambda^*(y)(b-y)^2} = -\frac{1}{2}. \tag{3.11}$$

Finally, the following equivalent holds for $\lambda^*(y)$ *as* $y \to a^+$:

$$\lambda^*(y) \sim_{y\to a^+} \frac{8(y-a)\int_{-\infty}^b (b-u)\tilde{h}(du)}{\int_{-\infty}^b (u-a)^4 \tilde{h}(du) - (b-a)^4}. \tag{3.12}$$

In case $\int_{-\infty}^b (u-a)^4\tilde{h}(du) = +\infty$ $(\Leftrightarrow \int_0^K \ln^4(r) r^{(\alpha-3)/2} m(dr) = +\infty)$, (3.12) *means that* $\lambda^*(y) = o(y-a)$.

Before coming to the proof of the proposition let us notice that (3.11) is equivalent to the equivalent of Barles, Burdeau, Romano and Samson [1] and Lamberton [7].

LEMMA 9. *Let* $\lambda^*(y) \to \infty$ *as* $y \to b^-$. *Then* (3.11) *holds if and only if*

$$\lim_{y\to b^-} \frac{\lambda^*(y)(b-y)^2}{\ln(\lambda^*(y))} = 1. \tag{3.13}$$

PROOF. If (3.13) holds, then $\ln(\lambda^*) + 2\ln(b-y) - \ln(\ln(\lambda^*)) \to 0$. By dividing by $\lambda^*(b-y)^2$, which is far from zero since it goes to infinity by (3.13), we get

$$\frac{\ln(\lambda^*)}{\lambda^*(b-y)^2} + 2\frac{\ln(b-y)}{\lambda^*(b-y)^2} - \frac{\ln(\ln(\lambda^*))}{\lambda^*(b-y)^2} \to 0$$

which gives (3.11) since $\ln(\ln(\lambda^*))/\ln(\lambda^*) \to 0$.

Conversely we get from (3.11) $\ln(-\ln(b-y)) - \ln(\lambda^*) + 2\ln(b-y) \to -\ln(2)$, whence if (3.11) holds,

$$\frac{\ln(-\ln(b-y))}{\lambda^*(b-y)^2} - \frac{\ln(\lambda^*)}{\lambda^*(b-y)^2} - 2\frac{\ln(b-y)}{\lambda^*(b-y)^2} \to 0,$$

then (3.13) since $\ln(-\ln(b-y))/\ln(b-y) \to 0$. $\square$

Let us now prove Proposition 8.

PROOF. We first compute the first order derivatives of F:

$$\begin{aligned}\partial_y F(\lambda, y) &= \lambda\biggl((b-y)\exp\Bigl(-\frac{\lambda}{2}(b-y)^2\Bigr) \\ &\qquad - \int_{-\infty}^{b}(u-y)\exp\Bigl(-\frac{\lambda}{2}(u-y)^2\Bigr)\tilde{h}(du)\biggr), \\ \partial_\lambda F(\lambda, y) &= -\frac{1}{2}(b-y)^2\exp\Bigl(-\frac{\lambda}{2}(b-y)^2\Bigr) \\ &\qquad + \frac{1}{2}\int_{-\infty}^{b}(u-y)^2\exp\Bigl(-\frac{\lambda}{2}(u-y)^2\Bigr)\tilde{h}(du).\end{aligned}$$

Let $y \in]a, b[$. Applying Jensen's inequality to the strictly convex function $z\ln(z)$ and the moment equality $F(\lambda^*(y), y) = 0$, we get $\partial_\lambda F(\lambda^*(y), y) < 0$. Moreover, using $F(\lambda^*(y), y) = 0$, we get

$$\partial_y F\bigl(\lambda^*(y), y\bigr) = \lambda^*(y)\int_{-\infty}^{b}(b-u)\exp\Bigl(-\frac{\lambda^*(y)}{2}(u-y)^2\Bigr)\tilde{h}(du) > 0.$$

Now the price $v_\varphi(t, x)$ of the European option is analytic on $\mathbb{R}_+^* \times \mathbb{R}_+^*$, therefore $\partial_t v_\varphi(\lambda, y)$ is analytic on $\mathbb{R}_+^* \times \mathbb{R}$. Since for $y \in]a, b[$, $\lambda^*(y)$ is the unique $\lambda > 0$ solution of $\partial_t v_\varphi(\lambda, y) = 0$ and

$$\begin{aligned}&\partial_\lambda\bigl(\partial_t v_\varphi\bigl(\lambda^*(y), y\bigr)\bigr) = C\bigl(\lambda^*(y), y\bigr)\partial_\lambda F\bigl(\lambda^*(y), y\bigr) < 0, \\ &\partial_y\bigl(\partial_t v_\varphi\bigl(\lambda^*(y), y\bigr)\bigr) > 0\end{aligned}$$

by the implicit functions theorem for analytic functions λ^* is analytic with a positive derivative on $]a, b[$.

We deduce that $\lambda^*(y)$ has a limit when $y \to a^+$. Since F is continuous, $F(\lim_{y\to a^+}\lambda^*(y), a) = 0$. Now the unique $\lambda \geq 0$ such that $F(\lambda, a) = 0$ is 0. Hence $\lim_{y\to a^+}\lambda^*(y) = 0$. By a similar reasoning, we check that $\lim_{y\to b^-}\lambda^*(y) = +\infty$.

To make precise the speed of convergence, we recall that $\lambda^*(y)$ is given by

$$\exp\Bigl(-\frac{\lambda^*(y)}{2}(b-y)^2\Bigr) = \int_{-\infty}^{b}\exp\Bigl(-\frac{\lambda^*(y)}{2}(u-y)^2\Bigr)\tilde{h}(du). \tag{3.14}$$

As $y \to b^-$, $\lambda^*(y) \to +\infty$ and $\forall u < b$, $e^{-(\lambda^*(y)/2)(u-y)^2} \to 0$. Hence by Lebesgue theorem the right-hand side of (3.14) goes to 0 and $\lambda^*(y)(b-y)^2 \to +\infty$.

Let us now turn to (3.11). By Lebesgue's theorem,

$$\begin{aligned}&\exp\Bigl(\frac{\lambda^*(y)}{2}(b-y)^2\Bigr)\int_{-\infty}^{2y-b}\exp\Bigl(-\frac{\lambda^*(y)}{2}(u-y)^2\Bigr)\tilde{h}(du) \\ &\quad = \int_{-\infty}^{2y-b}\exp\Bigl(-\frac{\lambda^*(y)}{2}(b-u)(2y-b-u)\Bigr)\tilde{h}(du) \to_{y\to b^-} 0.\end{aligned}$$

We now suppose that $\tilde{h}(du)$ has a density $\tilde{h}$ on $]b_*, b[$ and that $\lim_{u\to b^-} \tilde{h}(u) = \tilde{h}(b^-) > 0$. Setting $u = y + \beta(b-y)$, we get from the above remark

$$\begin{aligned}
1 &\sim_{y\to b^-} \int_{2y-b}^{b} \exp\Big(-\frac{\lambda^*(y)}{2}\big((u-y)^2-(b-y)^2\big)\Big)\widetilde{h}(du)\\
&= (b-y)\int_{-1}^{1} \exp\Big(-\frac{\lambda^*(y)}{2}(b-y)^2(\beta^2-1)\Big)\widetilde{h}\big(y+\beta(b-y)\big)\,d\beta\\
&\sim (b-y)\widetilde{h}(b^-)\int_{-1}^{1} \exp\Big(-\frac{\lambda^*(y)}{2}(b-y)^2(\beta^2-1)\Big)\,d\beta.
\end{aligned}$$

Therefore, by the Laplace method,

$$\begin{aligned}
&\Big(\frac{1}{b-y}\Big)^{2/(\lambda^*(y)(b-y)^2)}\\
&\quad\sim \Big(\widetilde{h}(b^-)\int_{-1}^{1} \exp\Big(-\frac{\lambda^*(y)}{2}(b-y)^2(\beta^2-1)\Big)\,d\beta\Big)^{2/(\lambda^*(y)(b-y)^2)}\\
&\quad\to \sup_{\beta\in]-1,1[} e^{-(\beta^2-1)} = e,
\end{aligned}$$

which gives (3.11).

To make precise the behavior of $\lambda^*(y)$ as $y \to a^+$, we make Taylor's expansions in (3.14)

$$\begin{aligned}
&1-\frac{\lambda^*(y)}{2}(b-y)^2+\frac{\lambda^*(y)^2}{8}(b-y)^4+o\big(\lambda^*(y)^2\big)\\
&\quad=\int_{-\infty}^{b}\Big(1-\frac{\lambda^*(y)}{2}(u-y)^2\\
&\qquad\qquad+\frac{\lambda^*(y)^2}{4}(u-y)^4\int_0^1(1-\theta)\exp\Big(-\frac{\theta\lambda^*(y)}{2}(u-y)^2\Big)\,d\theta\Big)\widetilde{h}(du),
\end{aligned}$$

which simplifies after (3.5) and (3.6), writing $(b-y)^2 = (b-a)^2 + (y-a)^2 + 2(b-a)(a-y)$, developing $(u-y)^2$ and also $(b-y)^4$ in a similar way, to

$$\begin{aligned}
&\lambda^*(y)(y-a)\int_{-\infty}^{b}(b-u)\widetilde{h}(du)+\frac{\lambda^*(y)^2(b-a)^4}{8}+o\big(\lambda^*(y)^2\big)\\
&\quad=\lambda^*(y)^2\int_0^1\frac{1-\theta}{4}\Big(\int_{-\infty}^{b}(u-y)^4\exp\Big(-\frac{\theta\lambda^*(y)}{2}(u-y)^2\Big)\tilde{h}(du)\Big)d\theta.
\end{aligned}$$

In case $\int_{-\infty}^{b}(u-a)^4\tilde{h}(du) < +\infty$, the right-hand side is equivalent to

$$\lambda^*(y)^2\int_{-\infty}^{b}(u-a)^4\tilde{h}(du)/8.$$

Since $\widetilde{h}$ is not a Dirac mass, by Jensen's inequality,

$$\int_{-\infty}^{b}(u-a)^4\widetilde{h}\,(du) > \left(\int_{-\infty}^{b}(u-a)^2\widetilde{h}(du)\right)^2 = (b-a)^4$$

according to (3.6), and we deduce (3.12).

This assertion still holds in case $\int_{-\infty}^{b}(u-a)^4\tilde{h}(du) = +\infty$: indeed, by Fatou's lemma,

$$\int_0^1 \frac{1-\theta}{4}\left(\int_{-\infty}^{b}(u-y)^4\exp\left(-\frac{\theta\lambda^*(y)}{2}(u-y)^2\right)\tilde{h}(du)\right)d\theta \to +\infty. \qquad \square$$

3.3. *The price along the theta-zero curve.* The interesting price is obtained by setting $\lambda = \lambda^*(y)$:

PROPOSITION 10. *Under the assumptions of Proposition* 7, *the payoff* $\widehat{\varphi}$ *is given for x between K^* and K (y between a and b) by*

$$\begin{aligned}\alpha e^{-a}\widehat{\varphi}(e^y) = {} & e^{\alpha(a-y)} - e^{\alpha(b-y)}N\left(-(b-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)\\ & - e^{(y-b)}N\left((b-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)\\ & + e^{(\alpha-1)(b-y)/2}\int_{-\infty}^{b} e^{(\alpha+1)(u-y)/2}\widetilde{h}(du)\\ & \qquad \times N\left(-(u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)\\ & + e^{(\alpha-1)(b-y)/2}\int_{-\infty}^{b} e^{-(\alpha+1)(u-y)/2}\widetilde{h}(du)\\ & \qquad \times N\left((u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)\end{aligned}$$

where $\lambda = \lambda^*(y) > 0$ is given by $F(\lambda^*(y), y) = 0$.

3.4. *Computation of $\widehat{\varphi}'$ for $K^* < x < K$.* By derivation of $\widehat{\varphi}(e^y)$ with respect to y (see Section A.4 in the Appendix), we obtain:

LEMMA 11. *For $y \in]a, b[$,*

$$\begin{aligned}e^{-a}\widehat{\varphi}'(e^y) = {} & -e^{-y}e^{\alpha(a-y)} + e^{-y}e^{\alpha(b-y)}N\left(-(b-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)\\ & - \frac{e^{-b}}{\alpha}N\left((b-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)\\ & - e^{-(\alpha+1)y}e^{(\alpha-1)b/2} \qquad (3.15)\end{aligned}$$

$$\times \int_{-\infty}^{b} e^{(\alpha+1)u/2}\widetilde{h}(du) N\left(-(u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)$$
$$+ \frac{e^{(\alpha-1)b/2}}{\alpha}\int_{-\infty}^{b} e^{-(\alpha+1)u/2}\widetilde{h}(du) N\left((u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)$$

where $\lambda = \lambda^*(y) > 0$ *is given by* $F(\lambda^*(y), y) = 0$.

3.5. *Behavior of* $\widehat{\varphi}$ *as* $x \to K^{*+}$.

PROPOSITION 12. *Under the assumptions of Proposition* 7,

$$\lim_{x\to K^{*+}} \widehat{\varphi}(x) = K - K^*.$$

Moreover, $\lim_{x\to K^{*+}} \widehat{\varphi}'(x) = -1$ *and*

$$\lim_{x\to K^{*+}} \frac{\widehat{\varphi}'(x)+1}{x-K^*} = \frac{\alpha+1}{K^*} > 0,$$

that is, *the behavior of* $\widehat{\varphi}(x)$ *when* $x \to K^{*+}$ *is similar to the one of the perpetual Put price and* $\widehat{\varphi}$ *cannot be equal to* $K - x$ *on* $[K^*, K]$.

PROOF. We recall that $\lim_{y\to a^+} \lambda^*(y) = 0$. Hence, in the expression of $e^{-a}\widehat{\varphi}(e^y)$ given by Proposition 10, when $y \to a^+$, the first term has a limit equal to $1/\alpha$ and the second and third terms go to 0. The fourth and the fifth terms also vanish according to Lebesgue theorem and the following upper bounds: $\forall u \le b,\ \forall y \ge a$,

$$\begin{aligned}
&e^{(\alpha+1)(u-y)/2} N\left(-(u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)\\
&\qquad \le e^{-(\alpha+1)^2/(8\lambda)} \mathbb{1}_{\{u-y\le -(\alpha+1)/(4\lambda)\}}\\
&\qquad\quad + e^{(\alpha+1)(b-a)/2} N\left(-\frac{\alpha+1}{4\sqrt{\lambda}}\right)\mathbb{1}_{\{u-y\ge -(\alpha+1)/(4\lambda)\}}\\
&\qquad \le e^{-(\alpha+1)^2/(8\lambda)} + e^{(\alpha+1)(b-a)/2} N\left(-\frac{\alpha+1}{4\sqrt{\lambda}}\right),\\
&e^{-(\alpha+1)(u-y)/2} N\left((u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)\\
&\qquad \le e^{-(\alpha+1)(\alpha+1-2\sqrt{\lambda})/(4\lambda)} \mathbb{1}_{\{(u-y)\sqrt{\lambda}-(\alpha+1)/(2\sqrt{\lambda})\ge -1\}}\\
&\qquad\quad + \frac{1}{\sqrt{2\pi}} e^{-(\alpha+1)(u-y)/2}
\end{aligned}$$

$$\times \exp\left(-\frac{1}{2}\left((u-y)\sqrt{\lambda} - \left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)^2\right)\mathbb{1}_{\{(u-y)\sqrt{\lambda}-(\alpha+1)/(2\sqrt{\lambda})\leq -1\}}$$
$$\leq e^{-(\alpha+1)(\alpha+1-2\sqrt{\lambda})/(4\lambda)} + e^{-(\alpha+1)^2/(8\lambda)}.$$

Hence $\lim_{x\to K^{*+}} \widehat{\varphi}(x) = e^a/\alpha = K^*/\alpha = K - K^*$.

Denoting by $T_i(y)$, $1 \leq i \leq 5$ the terms of the right-hand side of (3.16), we have $T_1(a) = -e^{-a}$ and $T_1'(a) = (\alpha+1)e^{-a}$. We conclude the proof by checking that $\forall 2 \leq i \leq 5, \forall n \in \mathbb{N}, \lim_{y\to a^+} T_i(y)/(y-a)^n = 0$ thanks to (3.12) and the previous upper bounds. $\square$

3.6. *Behavior of $\widehat{\varphi}$ as $x \to K^-$.*

PROPOSITION 13. *If the assumptions of Proposition* 7 *are satisfied and*

$$\int_0^K r^{\alpha-1} h(dr) = \frac{K^\alpha - K^{*\alpha}}{\alpha},$$

which is equivalent to

$$\int_{-\infty}^b e^{(\alpha+1)u/2}\tilde{h}(du) = \frac{e^{(1-\alpha)b}}{2(e^{\alpha b} - e^{\alpha a})},$$

where

$$\widetilde{h}(du) = \alpha e^{(\alpha-1)(u-b)/2}\,\frac{dh(e^u)}{e^u},$$

then

$$\lim_{x\to K^-} \widehat{\varphi}(x) = 0 \quad \textit{and} \quad \lim_{x\to K^-} \widehat{\varphi'}(x) = -1.$$

PROOF. Since $\lim_{y\to b^-} \sqrt{\lambda}(b-y) = +\infty$, taking the limit $y \to b^-$ in the expression of $e^{-a}\widehat{\varphi}(e^y)$ given by Proposition 10,

$$\begin{aligned}\lim_{y\to b^-} e^{-a}\widehat{\varphi}(e^y) &= \frac{e^{\alpha(a-b)}}{\alpha} + 0 - \frac{1}{\alpha} + \frac{1}{\alpha}\int_{-\infty}^b e^{(\alpha+1)(u-b)/2}\tilde{h}(du) + 0\\ &= \left(e^{\alpha(a-b)} - 1 + e^{-\alpha b}(e^{\alpha b} - e^{\alpha a})\right)/\alpha = 0.\end{aligned}$$

Taking the limit in (3.16), we obtain

$$\begin{aligned}&\lim_{y\to b^-} e^{-a}\widehat{\varphi}'(e^y)\\ &\quad = -e^{\alpha(a-b)}e^{-b} + 0 - \frac{e^{-b}}{\alpha} - e^{-(\alpha+1)b}e^{(\alpha-1)b/2}\int_{-\infty}^b e^{(\alpha+1)u/2}\widetilde{h}(du) + 0\\ &\quad = -e^{\alpha(a-b)}e^{-b} - \frac{e^{-b}}{\alpha} - e^{-(\alpha+1)b}(e^{\alpha b} - e^{\alpha a}) = -e^{-b}\left(1 + \frac{1}{\alpha}\right) = -e^{-a}. \quad \square\end{aligned}$$

REMARK 14. In case $d\tilde{h}$ is absolutely continuous in a neighborhood of b with a density $\tilde{h}$ such that $\lim_{u\to b^-}\tilde{h}(u)=\tilde{h}(b^-)>0$ exists, it is possible to prove that the second order derivative of $\widehat{\varphi}$ at K^- depends on $\tilde{h}(b^-)$:

$$\lim_{x\to K^-}\frac{\widehat{\varphi}'(x)+1}{x-K}=e^{-b}\lim_{y\to b^-}\frac{\widehat{\varphi}'(e^y)+1}{y-b}=\frac{\alpha-\tilde{h}(b^-)}{K}.$$

Under the assumptions of the proposition, we have $\varphi'(K^-)=-1=\widehat{\varphi}'(K^-)$. If, moreover, the above assumption on $d\tilde{h}$ is satisfied, we can check that $\varphi''(K^-)=(\alpha-\tilde{h}(b^-))/K=\widehat{\varphi}''(K^-)$. The equality of the first and second derivatives of φ and $\widehat{\varphi}$ at K^- is not surprising since for $y\in]a,b[$, $\widehat{\varphi}(e^y)=v_\varphi(1/(\sigma^2\lambda^*(y)),e^y)$ and $1/(\sigma^2\lambda^*(y))=o((b-y)^2)$ as $y\to b^-$.

4. The main result. We are now ready to summarize all the properties of $\widehat{t}(x)=1/(\sigma^2\lambda^*(\ln(x))$ and $\widehat{\varphi}$ and to apply the embedding result of [5]. First we state a theorem which is a direct application of [5], then a modification well suited to the Put case.

Note that (3.5) and (3.6) rewrites into the two last conditions on h in the following theorem.

THEOREM 15. *Assume that*

$$\varphi(x)=(K-K^*)\left(\frac{x}{K^*}\right)^{-\alpha}-x^{-\alpha}\frac{(K\wedge x)^{\alpha+1}}{\alpha+1}+K^*\int_0^K x^{-\alpha}\frac{(r\wedge x)^{\alpha+1}}{r^2}h(dr),$$

where h is a positive measure on $]0,K[$ such that $\forall x<K,\ h(]x,K[)>0$ and

$$\int_0^K r^{\alpha-1}h(dr)=(K^\alpha-K^{*\alpha})/\alpha,$$

$$\int_0^K r^{(\alpha-3)/2}h(dr)=K^{(\alpha-1)/2}/\alpha,$$

$$\int_0^K \ln^2(r/K^*)r^{(\alpha-3)/2}h(dr)=K^{(\alpha-1)/2}\ln^2(K/K^*)/\alpha,$$

then $\widehat{\varphi}(x)=\inf_{t\geq 0}v_\varphi(t,x)$ is continuous equal to 0 for $x\geq K$, equal to $(K-K^*)(x/K^*)^{-\alpha}$ if $x\leq K^*$, satisfies $\widehat{\varphi}'(K^{*+})=\widehat{\varphi}'(K^-)=-1$ and $\widehat{\varphi}''(K^{*+})=(\alpha+1)/K^*$. Moreover $\widehat{\varphi}(x)=v_\varphi(\widehat{t}(x),x)$ where $\widehat{t}$ is continuous, nonincreasing, analytic on $]K^*,K[$, equal to 0 for $x\geq K$ and to $+\infty$ for $x\leq K^*$. The price of the American option with payoff $\widehat{\varphi}$ is $v_{\widehat{\varphi}}^{\mathrm{am}}(t,x)=v_\varphi(t\vee\widehat{t}(x),x)$.

Here now is the main result:

THEOREM 16. *Under the assumptions of the previous theorem, the payoff $\widehat{\varphi}_h(x)=(K-x)^+\mathbb{1}_{\{x\leq K^*\}}+\widehat{\varphi}(x)\mathbb{1}_{\{x>K^*\}}$ is continuous and its American price is given by*

$$(K-x)^+\mathbb{1}_{\{x\leq K^*\}}+v_\varphi(t\vee\widehat{t}(x),x)\mathbb{1}_{\{x>K^*\}}.$$

PROOF. It is easily seen that $\widehat{\varphi}_h(x) = (K-x)^+ \le (K-K^*)(x/K^*)^{-\alpha} = \widehat{\varphi}(x)$ for $x \le K^*$, therefore the American price $v^{\mathrm{am}}_{\widehat{\varphi}_h}(t,x)$ is smaller than $v^{\mathrm{am}}_{\widehat{\varphi}}(t,x)$. Now in the region $x > K^*$, the American price of $\widehat{\varphi}_h$ is greater than $v_\varphi(t \vee \widehat{t}(x), x)$: indeed the latter may be written as $\mathbb{E}[e^{-\rho\tau}\widehat{\varphi}_h(X^x_\tau)]$ where τ is the entrance time in the region $\{t \le \widehat{t}(x)\}$ [convention: $\tau = 0$ if $t \le \widehat{t}(x)$] and $\widehat{\varphi}_h(X^x_\tau) = \widehat{\varphi}(X^x_\tau)$. Therefore $v^{\mathrm{am}}_{\widehat{\varphi}_h}(t,x) = v_\varphi(t \vee \widehat{t}(x), x)$ for $x > K^*$ and also $x \ge K^*$ by continuity. In particular the line $x = K^*$ is contained in the Exercise region.

Take now a point (t,x) with $x < K^*$. By the optimal stopping representation of the American price, one has

$$v^{\mathrm{am}}_{\widehat{\varphi}_h}(t,x) = \sup_{\tau<\tau^*} \mathbb{E}[e^{-\rho\tau}\widehat{\varphi}_h(X^x_\tau)]$$

where τ runs across the set of stopping times of the Brownian filtration less than the crossing time τ^* of the boundary $\{(0,x),\ x < K^*\} \cup \{(t, K^*),\ t \ge 0\}$. In this area $\widehat{\varphi}_h$ is equal to the Put payoff, therefore this quantity is less than the American price of the Put. But by definition of K^* we lie in the Exercise region of the American Put, so $v^{\mathrm{am}}_{\widehat{\varphi}_h}(t,x) \le (K-x)^+$ and on another hand $(K-x)^+ = \widehat{\varphi}_h(x) \le v^{\mathrm{am}}_{\widehat{\varphi}_h}(t,x)$. □

REMARK 17. The same result holds for any continuous payoff obtained by replacing $\widehat{\varphi}(x)$ under K^* by a continuous function $\psi(x)$ smaller than $(K - K^*)(x/K^*)^{-\alpha}$ with $\psi(K^*) = (K - K^*)$ and such that the region $\{x \le K^*\}$ lies in the Exercise region of the modified payoff. For instance in case $k \le K^*$ it is easy to check by comparison with the Put option that the region $\{x \le K^*\}$ is included in the Exercise region of the American Put-Spread option with payoff $(K-x)^+ - (k-x)^+ = (K-k) \wedge (K-x)^+$. Hence the price of the American option with modified payoff $\widehat{\varphi}_k(x) = (K-k) \wedge (K-x)^+ \mathbb{1}_{\{x \le K^*\}} + \widehat{\varphi}(x)\mathbb{1}_{\{x > K^*\}}$ is

$$(K-k) \wedge (K-x)^+ \mathbb{1}_{\{x \le K^*\}} + v_\varphi\big(t \vee \widehat{t}(x), x\big)\mathbb{1}_{\{x > K^*\}}.$$

It is natural to wonder whether the payoff $\widehat{\varphi}_h$ is nonincreasing like the Put payoff. The answer is positive at least for values of α of practical interest since:

LEMMA 18. *There is a constant $\alpha_0 < 1/2$, such that when $\alpha \ge \alpha_0$, under the assumptions of Theorem* 15, *both $\widehat{\varphi}$ and $\widehat{\varphi}_h$ are nonincreasing*.

The proof of this lemma is postponed to Section A.5 in the Appendix.

5. Discretization. In this section we solve a discretized version of the program:

$$\inf_{h \in H} \sup_x \big|\widehat{\varphi}_h(x) - (K-x)^+\big|$$

where H is a low-dimensional subspace of the set of measures h which verify the moment conditions of the theorems.

5.1. *Normalization.* For practical purposes, it would be interesting to get a measure h^* which depend on as few parameters as possible. It will certainly depend on α, but we can design an approximation which will work for every value of K in the following way: we normalize the situation so that $K^* = 1$ (any other value would work!), therefore $K = k \stackrel{\text{def}}{=} 1 + 1/\alpha$.

This does not matter in the following sense: to emphasize the dependence on the strike K, we denote by $v^{\text{am}}_{\text{Put}}(t, x, K)$ the American Put price for the maturity t and the underlying value x. If we manage to design an approximation such that, for a given value of t:

$$\sup_x \left| v^{\text{am}}_{\text{Put}}(t, x, k) - \text{Approx}(t, x, k) \right| < \varepsilon$$

then since obviously $v^{\text{am}}_{\text{Put}}(t, x, k) = (K/k) v^{\text{am}}_{\text{Put}}(t, (k/K)x, k)$, the approximation by $(K/k)\text{Approx}(t, k/K, x, k)$ will satisfy

$$\sup_x \left| v^{\text{am}}_{\text{Put}}(t, x, K) - \frac{K}{k} \text{Approx}\left(t, \frac{k}{K} x, k\right) \right| < \frac{K}{k} \varepsilon.$$

In other words, the error we face in term of a percentage of the strike K is given by ε/k.

From now on we work thus with:

$$K^* = 1, \qquad k \stackrel{\text{def}}{=} K = 1 + \frac{1}{\alpha}, \qquad (K - k)K^{*-\alpha} = K - K^* = \frac{1}{\alpha}$$

and with the variables $y = \ln(x)$ and $\lambda = 1/(\sigma^2 t)$.

5.2. *Choice of a special class of* $\widetilde{h}$. We further restrict ourselves to a particular class of measures $\widetilde{h}$ which lead to easy implementation. Whatever the measure $\widetilde{h}$ at hand there is a priori two steps to obtain $v^{\text{am}}_{\widehat{\varphi}_h}(\lambda, y)$ for given values of $y \in]a = 0, b = \ln(k)[$ and $\lambda > 0$: first compute the value of the theta-zero curve, that is, find $\lambda^*(y) \in]0, +\infty[$ solving $F(\lambda^*(y), y) = 0$ then compute the price $v_\varphi(\lambda \wedge \lambda^*(y), y) = v^{\text{am}}_{\widehat{\varphi}_h}(\lambda, y)$. In general both steps require numerical procedures, a dichotomy to find the zero of the time derivative (there is exactly one for every $y \in]a, b[$ after the above calculations), next a numerical (one-dimensional) integration (with respect to $\widetilde{h}$) to get the price. In case $y \geq b$, only the second step is required since $\lambda^*(y) = +\infty$ and in case $y \leq 0$, $v^{\text{am}}_{\widehat{\varphi}_h} = (k - e^y)$.

We choose to work with a low-dimensional family of combination of point measures. This allows the direct computation of the price at the second step.

Notice that the condition $\widetilde{h}(]y, b[) > 0$ for $y < b$ is not satisfied yet: so we add a uniform measure $\varepsilon \mathbb{1}_{]0,b[}\, du$, for which it is easily seen that the corresponding contribution to the price may be computed explicitly. We have implemented the case of 3-points measures, which gives already astonishing results. Our family may be parametrized in the following way:

$$\begin{aligned} \widetilde{h}(du) = {} & \varepsilon \mathbb{1}_{]0,b[}\, du + \beta\, \delta_{\log(r_1)}(du) + \gamma\, \delta_{\log(r_2)}(du) \\ & + (1 - \varepsilon b - \beta - \gamma)\delta_{\log(r_3)}(du) \end{aligned}$$

with $\varepsilon > 0,\ \varepsilon b < 1,\ \beta > 0,\ \gamma > 0$ and $\beta + \gamma < 1 - \varepsilon b$.

By convention we choose $\log(r_1) < \log(r_2) < \log(r_3)$.

Remember that the support of $\widetilde{h}$ should lie below b, so we further set

$$\log(r_3) = \mu\, b,$$

and also

$$\log(r_1) = x_1\, \mu\, b, \qquad \log(r_2) = x_2\, \mu\, b.$$

Therefore the parameter $(\varepsilon, \mu,\ x_1,\ x_2)$ should live in: $0 < \varepsilon < b,\ \mu \leq 1,\ x_1 \leq x_2 \leq 1$.

For a given value of $(\varepsilon, \mu,\ x_1,\ x_2)$ we compute the values of β and γ which fit the two remaining moment conditions:

$$\int_{-\infty}^{b} u^2 \widetilde{h}(du) = b^2, \qquad \int_{-\infty}^{b} e^{(\alpha+1)u/2}\, \widetilde{h}(du) = \frac{e^{\alpha b} - 1}{e^{(\alpha-1)b/2}}.$$

This translates as the 2×2 linear system

$$\begin{aligned}
&(1 - x_1^2)\beta + (1 - x_2^2)\gamma = \varepsilon b\left(\frac{1}{3\mu^2} - 1\right) + 1 - \frac{1}{\mu^2},\\
&(1 - e^{(\alpha+1)(x_1-1)\,\mu\, b/2})\beta + (1 - e^{(\alpha+1)(x_2-1)\,\mu\, b/2})\gamma\\
&\quad = \varepsilon\left(\frac{2e^{-(\alpha+1)\,\mu\, b/2}}{(\alpha+1)}(e^{(\alpha+1)b/2} - 1) - b\right) + 1 - e^{(\alpha+1)\,(1-\mu)\, b/2}(1 - e^{-\alpha b}),
\end{aligned}$$

which gives closed formula for β and γ. In case one of the conditions $\beta > 0,\ \gamma > 0$ and $\beta + \gamma < 1 - \varepsilon b$ is not satisfied the point $(\varepsilon, \mu,\ x_1,\ x_2)$ is rejected, otherwise we sample the range $]0, b[$ with n points, say $y_i = \frac{i}{n} b$ with $0 < i < n$ and for every y_i we proceed as follows.

5.3. *Calculation of* $\lambda^*(y)$. We find $\lambda^*(y_i)$ by a dichotomy algorithm making use of the closed formula for $F(\lambda, y)$. This is obviously very fast, although a little care is required when y_i is near 0 or b to deal with possibly very high or small values of $\lambda^*(y_i)$.

5.4. *Computation of the price.* This is also very fast since no numerical integration is required. We make use of the standard approximation of the normal cumulative distribution which relies on the classical series expansion.

5.5. *Selection of the optimal point.* Then for a given value of $(\varepsilon, \mu,\ x_1,\ x_2)$ we compute the error quantity

$$\mathrm{err}(\varepsilon, \mu,\ x_1,\ x_2) = \sup_i \left|\widehat{\varphi}(e^{y_i}) - (k - e^{y_i})^+\right|,$$

and next, after a clever or systematic scan of the domain, we pick the point which minimizes this criteria, with a value $\mathrm{err}^* = \mathrm{err}(\varepsilon^*, \mu^*, x_1^*, x_2^*)$. The corresponding American payoff is denoted by $\widehat{\varphi}^*$.

5.6. *Archiving the results.* The optimal point will depend on α. In practice we maintain an archive with 100 values of α equally sampled between 0.5 and 50.0 (for an annual interest rate of 5%, $\alpha = 0.4$ is $\sigma = 50\%$, $\alpha = 25.6$ is $\sigma = 6.25\%$). The computation of the archive is done once for all, the practical usage for the ambient value of α consists in picking up the closest value of the table or performing a linear interpolation since the optimal point, for our choice of the domain at least, depends "continuously" on α. Therefore the computation time is that of the dichotomy (typically ten iterations or so) and of the price, which is very fast.

5.7. *Numerical results.* We choose YAAAP (Yet Another Approximation for the American Put) for our methods's name.

The first graph (Figure 2) plots err* as a function of α, expressed in percentage of the strike K.

The fact that this plot is decreasing corresponds to the fact that the size of the range $]K^*, K[$ increases as α decreases, whereas our family of approximating payoffs does not get richer as α decreases. It seems that at least for values of α not too small, this error is relevant in practice.

Figures 3 and 4 give the difference $D(x) = \widehat{\varphi}^*(x) - (k - x)^+$ for $\alpha = 1$, in percentage of the strike $k = 2$ and next of the premium at maturity [i.e., $(k - x)^+$]:

The price error will be much smaller since err* is the maximal error over the underlying and since it will be smoothed by the probability law of the spot value at the time the free boundary is reached and reduced by the corresponding discounting factor. More precisely, if τ_{opt} and τ_* denote respectively the entrance times in the Exercise regions of the American Put option and of the American

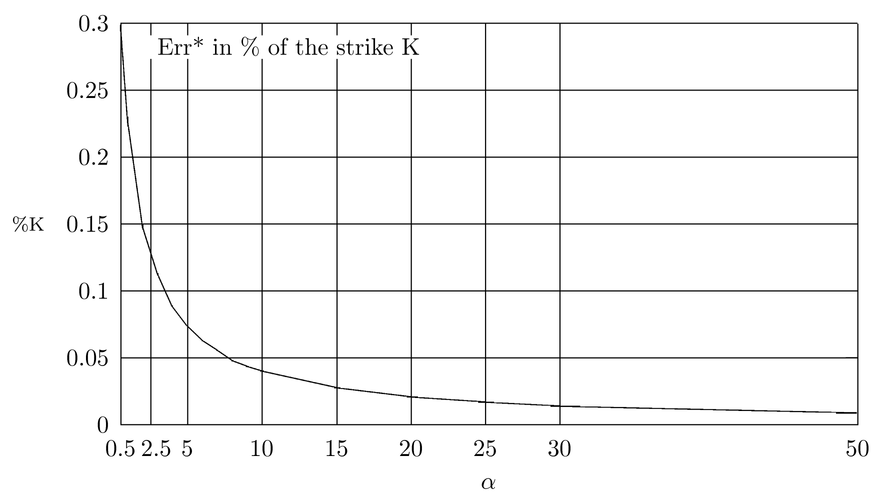

FIG. 2.

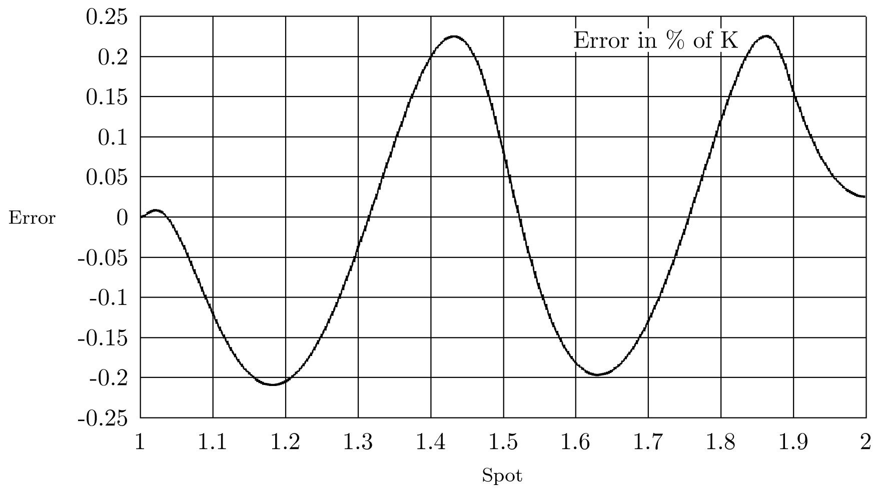

FIG. 3.

option with payoff $\widehat{\varphi}^*$, then

$$v_{\mathrm{Put}}^{\mathrm{am}}(t,x) = \mathbb{E}\big[e^{-\rho\tau_{\mathrm{opt}}}(k - X_{\tau_{\mathrm{opt}}}^x)^+\big] \quad \text{and} \quad v_{\widehat{\varphi}^*}^{\mathrm{am}}(t,x) = \mathbb{E}\big[e^{-\rho\tau_*}\widehat{\varphi}^*(X_{\tau_*}^x)\big]$$

and as τ_{opt} and τ_* are optimal stopping times, we easily check that

$$v_{\widehat{\varphi}^*}^{\mathrm{am}}(t,x) - \mathbb{E}\big[e^{-\rho\tau_*}D(X_{\tau_*}^x)\big] \le v_{\mathrm{Put}}^{\mathrm{am}}(t,x) \le v_{\widehat{\varphi}^*}^{\mathrm{am}}(t,x) + \mathbb{E}\big[e^{-\rho\tau_{\mathrm{opt}}}\big(-D(X_{\tau_{\mathrm{opt}}}^x)\big)\big].$$

The larger the maturity, the more effective the smoothing of the error.

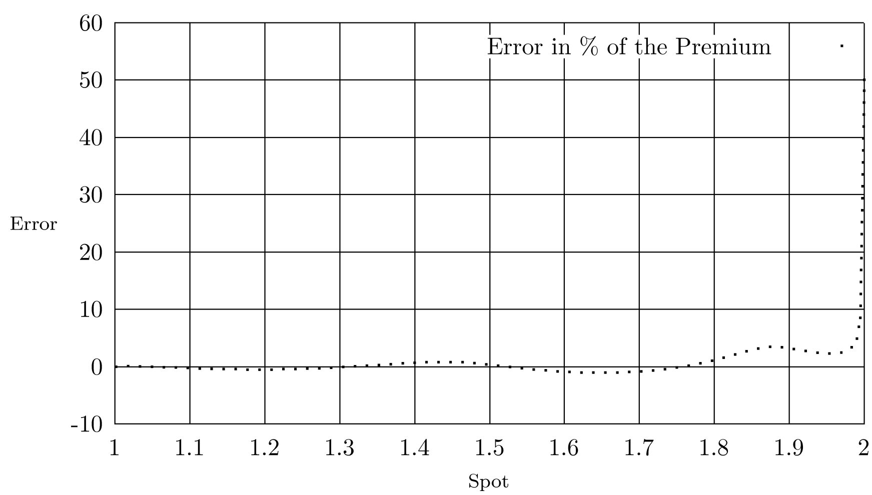

FIG. 4.

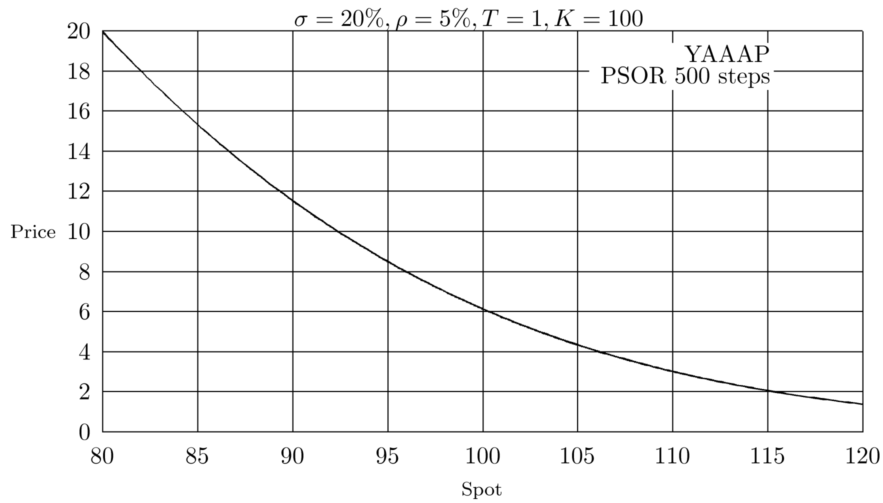

FIG. 5.

5.8. *Comparison with other methods.* Figures 5 and 6 show the comparison with a heavy finite-difference method (PSOR algorithm; cf. [2]) with a large number of steps (500), so that the yielded price may be considered as the right one, for two different values of the maturity.

Even if the scale is quite large, this shows that our approximation is good. Netherthless, we should compare it with the most accurate methods available at the moment. To our knowledge, these are the LUBA method of Broadie and Detemple [3], which is a nonconvergent method (like ours), and the method

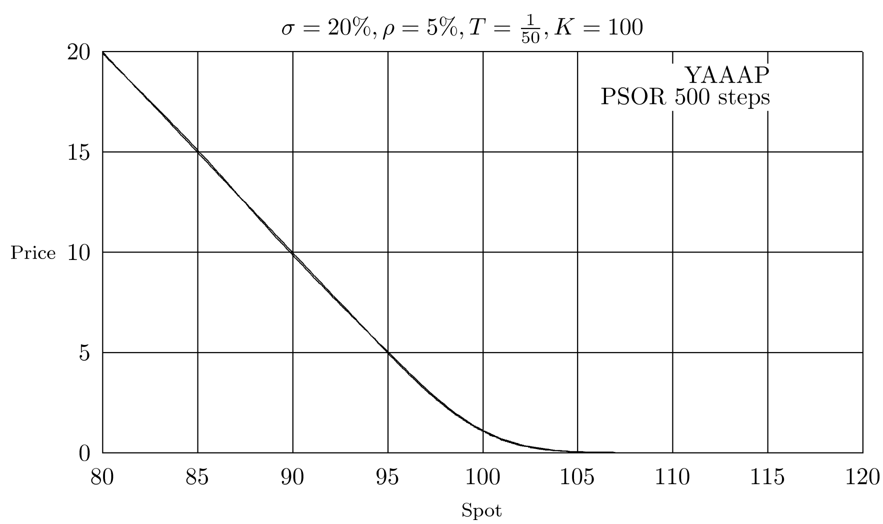

FIG. 6.

proposed by Ju [6] who computes the best exponential piece-wise approximation of the free boundary for a given time step. The method proposed in [8] also seems very fast and accurate. The advantage of the methods of [6] and [8] is to provide also an approximation of the delta. Moreover these are convergent methods.

Both Ju and Broadie and Detemple report an accuracy comparable to that of a Cox–Ross–Rubinstein (CRR) algorithm at 800 steps, which we could check. But we were not able to reproduce the execution times given in [3] and [6]. Our method gives the accuracy of a CRR algorithm at 100 steps, 10 times faster. Notice that we use an optimized binomial tree where the values of the payoff at the nodes are evaluated once for all at the beginning and store in an array. This is 2.5 times faster than a standard one. The type of tree used in [3] and [6] is not detailed.

6. Conclusion. In this paper, we apply the theoretical result in [5] to the pricing of the American Put in the Black–Scholes model. We get a closed-formula for a family of payoffs which are very close to the Put payoff. The numerical results are nice, even if the Ju and LUBA approximation methods perform better. Nevertheless we believe that the interest of our method lies in the new kind of procedure which is developed: the YAAAP prices and deltas are the exact Black–Scholes American prices and deltas of a contingent claim the payoff of which matches the Put payoff below K^* and above K, is analytic within the range $]K^*, K[$, has the right first derivative -1 at K^*_+ and K_-, and lastly which deviates at most of err* from the Put payoff within $]K^*, K[$. In particular, unlike numerous other approximation methods (like LUBA) there is a exact hedge ratio associated to our price, which can be computed through the same type of almost-closed formula. Therefore a safe way of making use of our approximation method is to trade the corresponding sub- and super-strategies with the YAAAP deltas and the selling price YAAAP price $+$ err*, buying price price $-$ err*, which leaving aside discrete-time hedging and model errors considerations will always yield a nonnegative Profit and Loss. Remember that err* is less than 0.15% of the strike as soon as $2\rho/\sigma^2$ is greater than 2.

We believe that the approach of this paper can be extended to deal with continuous dividends and can be applied to numerous other option payoffs (like Put-Spread, Call-Spread, ...). Also discretization schemes alternative to the one given in section 5 may be developed.

APPENDIX

A.1. Proof of Lemma 2. Indeed, by (2.1),

$$\varphi(x) = bx^{-\alpha} - \frac{2}{\sigma^2(\alpha+1)} x^{-\alpha} \int_0^K (r \wedge x)^{\alpha+1} \frac{m(dr)}{r^2}$$

where by (2.3) $b = \frac{2}{\sigma^2(\alpha+1)} \int_0^K r^{\alpha-1} m(dr)$. Therefore

$$\varphi(x) = \frac{2}{\sigma^2(\alpha+1)} x^{-\alpha} \int_0^K \left[r^{\alpha-1} - \frac{(r \wedge x)^{\alpha+1}}{r^2} \right] m(dr).$$

Now

$$m = \frac{\sigma^2 K^2}{2} \delta_K(dr) - \mathbb{1}_{]0,K[}(r) \frac{\sigma^2(\alpha+1)K^*}{2} h,$$

whence

$$\varphi(x) = \frac{x^{-\alpha}}{(\alpha+1)} [K^{\alpha+1} - (K \wedge x)^{\alpha+1}] - x^{-\alpha} K^* \int_0^K \left[r^{\alpha-1} - \frac{(r \wedge x)^{\alpha+1}}{r^2} \right] h(dr).$$

For $x < K$,

$$\frac{x^\alpha \varphi(x)}{K^{\alpha+1} - (K \wedge x)^{\alpha+1}} = \frac{1}{(\alpha+1)} - K^* \int_0^K \frac{r^{\alpha+1} - (r \wedge x)^{\alpha+1}}{K^{\alpha+1} - (K \wedge x)^{\alpha+1}} \frac{h(dr)}{r^2}.$$

Now $(r^{\alpha+1} - (r \wedge x)^{\alpha+1})/(K^{\alpha+1} - (K \wedge x)^{\alpha+1}) \leq (r^{\alpha+1})/(K^{\alpha+1})$, plugging $\int_0^K r^{\alpha-1} h(dr) = (K^\alpha - K^{*\alpha})/\alpha$, we get

$$\frac{x^\alpha \varphi(x)}{K^{\alpha+1} - (K \wedge x)^{\alpha+1}} \geq \frac{1}{(\alpha+1)} \left(\frac{\alpha}{\alpha+1} \right)^\alpha > 0$$

and φ is nonnegative.

A.2. Behavior of the European price as the maturity goes to $+\infty$. We prove here the following:

PROPOSITION 19. *Let φ: $\mathbb{R}_+^* \to \mathbb{R}$ a measurable function such that*

$$\sup_{x>0} |\varphi|(x)/(x + x^\alpha) < +\infty.$$

If $a = \lim_{x\to\infty} \varphi(x)/x$ and $b = \lim_{x\to 0^+} \varphi(x)/x^{-\alpha}$ exist and are finite, then

$$\lim_{t\to\infty} v_\varphi(t, x) = ax + bx^{-\alpha}.$$

PROOF. Indeed

$$\begin{aligned} v_\varphi(t,x) &= e^{-\rho t} \mathbb{E}[\varphi(X_t^x)] \\ &= e^{-\rho t} \mathbb{E}\left[\frac{\varphi(X_t^x)}{X_t^x + (X_t^x)^{-\alpha}} \left(X_t^x + (X_t^x)^{-\alpha} \right) \right] \\ &= x \mathbb{E}\left[\frac{\varphi(X_t^x)}{X_t^x + (X_t^x)^{-\alpha}} \, e^{-\rho t} X_t^1 \right] + x^{-\alpha} \mathbb{E}\left[\frac{\varphi(X_t^x)}{X_t^x + (X_t^x)^{-\alpha}} \, e^{-\rho t} (X_t^1)^{-\alpha} \right]. \end{aligned}$$

Since $e^{-\rho t}X_t^1 = e^{\sigma B_t - (\sigma^2/2)t}$, by Girsanov's theorem the first term is equal to

$$x\mathbb{E}^{\widetilde{P}}\left[\frac{\varphi(Y_t^x)}{Y_t^x + (Y_t^x)^{-\alpha}}\right]$$

where $Y_t^x = xe^{\rho t+\sigma(B_t-\sigma t)+\sigma^2 t-(\sigma^2/2)t} = xe^{\rho t+\sigma\widetilde{B}_t+(\sigma^2/2)t}$ and $\widetilde{B}$ is a $\widetilde{P}$ Brownian motion. In particular, $\widetilde{P}$ a.s., $e^{\rho t+\sigma\widetilde{B}_t+(\sigma^2/2)t} \to \infty$ as $t \to \infty$. Therefore, by Lebesgue's theorem,

$$\lim_{t\to\infty}\mathbb{E}^{\widetilde{P}}\left[\frac{\varphi(Y_t^x)}{Y_t^x + (Y_t^x)^{-\alpha}}\right] = \lim_{y\to\infty}\frac{\varphi(y)}{y}.$$

In the same way $e^{-\rho t}(X_t^1)^{-\alpha} = e^{-\alpha\sigma B_t-(\alpha^2\sigma^2/2)t}$ is a martingale and the second term is equal to

$$x^{-\alpha}\mathbb{E}^{\widetilde{P}}\left[\frac{\varphi((Z_t^x)^{-1/\alpha})}{Z_t^x + (Z_t^x)^{-1/\alpha}}\right] \quad \text{where } Z_t^x = x^{-\alpha}e^{\rho t+\alpha\sigma\widetilde{B}_t+(\alpha^2\sigma^2/2)t}.$$

Therefore it goes to

$$x^{-\alpha}\lim_{y\to\infty}\frac{\varphi(y^{-1/\alpha})}{y+y^{-1/\alpha}} = x^{-\alpha}\lim_{y\to 0}\frac{\varphi(y)}{y^{-\alpha}}. \qquad \square$$

A.3. Proof of Lemma 3. One has

$$\begin{aligned} e^{\rho t}v_{e_r}(t,x) &= r^{\alpha+1}x^{-\alpha}e^{-\alpha(\rho-\sigma^2/2)t}\mathbb{E}\big[\exp(-\alpha\sigma B_t)\mathbb{1}_{\{\alpha\sigma B_t>\alpha l(x)\}}\big] \\ &\quad + xe^{(\rho-\sigma^2/2)t}\mathbb{E}\big[\exp(\sigma B_t)\mathbb{1}_{\{\sigma B_t<l(x)\}}\big] \end{aligned}$$

where

$$l(x) = \ln\left(\frac{r}{x}\right) - \left(\rho - \frac{\sigma^2}{2}\right)t.$$

Since $\alpha\sigma^2/2 = \rho$ and $\alpha\rho = (\alpha\sigma)^2/2$,

$$\begin{aligned} v_{e_r}(t,x) &= r^{\alpha+1}x^{-\alpha}e^{((\alpha\sigma)^2/2)t}\,\mathbb{E}\big[\exp(-\alpha\sigma B_t)\mathbb{1}_{\{\alpha\sigma B_t>\alpha l(x)\}}\big] \\ &\quad + xe^{-(\sigma^2/2)t}\,\mathbb{E}\big[\exp(\sigma B_t)\mathbb{1}_{\{\sigma B_t<l(x)\}}\big]. \end{aligned}$$

Now,

$$\begin{aligned} e^{-(\gamma^2/2)t}\mathbb{E}\big[\exp(-\gamma B_t)\mathbb{1}_{\{\gamma B_t>\beta\}}\big] &= e^{-(\gamma^2/2)t}\int_\beta^\infty e^{-z}e^{-z^2/(2\gamma^2 t)}\frac{dz}{\sqrt{2\pi\gamma^2 t}} \\ &= \int_\beta^\infty e^{-(z+\gamma^2 t)^2/(2\gamma^2 t)}\frac{dz}{\sqrt{2\pi\gamma^2 t}} \\ &= N\left(-\left(\frac{\beta+\gamma^2 t}{\sqrt{\gamma^2 t}}\right)\right). \end{aligned}$$

In the same way

$$e^{-(\gamma^2/2)t}\mathbb{E}\big[\exp(\gamma B_t)\mathbb{1}_{\{\gamma B_t<\beta\}}\big] = N\left(\frac{\beta-\gamma^2 t}{\sqrt{\gamma^2 t}}\right),$$

whence

$$v_{e_r}(t,x) = r^{\alpha+1}x^{-\alpha}N\left(-\left(\frac{\alpha\sigma l(x)+(\alpha\sigma)^2 t}{\sqrt{(\alpha\sigma)^2 t}}\right)\right) + xN\left(\frac{\sigma l(x)-\sigma^2 t}{\sqrt{\sigma^2 t}}\right),$$

where $l(x)=\ln(r/x)-(\rho-\sigma^2/2)t=\ln(r/x)-((\alpha-1)/2)(\sigma^2 t)$ so that

$$\begin{aligned}
v_{e_r}(t,x) &= r^{\alpha+1}x^{-\alpha}N\left(-\left(\frac{\alpha\ln(r/x)-\alpha((\alpha-1)/2)(\sigma^2 t)+\alpha^2(\sigma^2 t)}{\sqrt{\alpha^2(\sigma^2 t)}}\right)\right)\\
&\qquad + xN\left(\frac{\ln(r/x)-((\alpha-1)/2)(\sigma^2 t)-(\sigma^2 t)}{\sqrt{\sigma^2 t}}\right)\\
&= r^{\alpha+1}x^{-\alpha}N\left(-\left(\frac{\ln(r/x)+((\alpha+1)/2)\sigma^2 t}{\sqrt{\sigma^2 t}}\right)\right)\\
&\qquad + xN\left(\frac{\ln(r/x)-((\alpha+1)/2)\sigma^2 t}{\sqrt{\sigma^2 t}}\right).
\end{aligned}$$

A.4. Computation of $\widehat{\varphi}'$ for $K^* < x < K$. For $y\in]a,b[$, $\widehat{\varphi}(e^y)=P\varphi(\lambda^*(y),y)$ is given in Proposition 10. Since

$$\partial_\lambda v_\varphi\big(\lambda^*(y),y\big) = -\frac{1}{(\sigma\lambda^*(y))^2}\partial_t v_\varphi\big(\lambda^*(y),y\big)=0,$$

derivation with respect to y yields

$$\begin{aligned}
&e^{-a}\widehat{\varphi}'(e^y)\\
&\quad = -e^{-y}e^{\alpha(a-y)} + e^{-y}e^{\alpha(b-y)}N\left(-(b-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)\\
&\qquad - \frac{e^{-y}e^{\alpha(b-y)}}{\alpha}\sqrt{\lambda}N'\left(-(b-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)\\
&\qquad - \frac{e^{-b}}{\alpha}N\left((b-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)\\
&\qquad + \frac{e^{-b}}{\alpha}\sqrt{\lambda}N'\left((b-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)\\
&\qquad - e^{-(\alpha+1)y}e^{(\alpha-1)b/2}\\
&\qquad\quad \times \int_{-\infty}^{b} e^{(\alpha+1)u/2}\widetilde{h}(du)N\left(-(u-y)\sqrt{\lambda}-\left(\frac{\alpha+1}{2}\right)\frac{1}{\sqrt{\lambda}}\right)
\end{aligned}$$

$$+\frac{e^{-(\alpha+1)y}}{\alpha}e^{(\alpha-1)b/2}$$
$$\times\int_{-\infty}^{b} e^{(\alpha+1)u/2}\widetilde{h}(du)\sqrt{\lambda}N'\biggl(-(u-y)\sqrt{\lambda}-\Bigl(\frac{\alpha+1}{2}\Bigr)\frac{1}{\sqrt{\lambda}}\biggr)$$
$$+\frac{e^{(\alpha-1)b/2}}{\alpha}\int_{-\infty}^{b} e^{-(\alpha+1)u/2}\widetilde{h}(du)N\biggl((u-y)\sqrt{\lambda}-\Bigl(\frac{\alpha+1}{2}\Bigr)\frac{1}{\sqrt{\lambda}}\biggr)$$
$$-\frac{e^{(\alpha-1)b/2}}{\alpha}\int_{-\infty}^{b} e^{-(\alpha+1)u/2}\widetilde{h}(du)\sqrt{\lambda}N'\biggl((u-y)\sqrt{\lambda}-\Bigl(\frac{\alpha+1}{2}\Bigr)\frac{1}{\sqrt{\lambda}}\biggr),$$

where, for simplicity of notation, λ stands for $\lambda^*(y)$.

Since

$$\sqrt{2\pi}N'\biggl(-(z-y)\sqrt{\lambda}-\Bigl(\frac{\alpha+1}{2}\Bigr)\frac{1}{\sqrt{\lambda}}\biggr)$$
$$=e^{(\alpha+1)(y-z)/2}\exp\biggl(-\frac{1}{2}\Bigl(\frac{\alpha+1}{2\sqrt{\lambda}}\Bigr)^2\biggr)\exp\Bigl(-\frac{\lambda}{2}(z-y)^2\Bigr),$$

using the definition of $\lambda^*(y)$, we obtain that the sum of the third and the seventh terms of the r.h.s. is nil. Similarly the sum of the fifth and the ninth terms is nil.

A.5. Proof of Lemma 18. If φ is nonincreasing then $\forall t\geq 0$, $x\to v_\varphi(t,x)$ is nonincreasing. Since $\widehat{\varphi}(x)=\inf_{t\geq 0}v_\varphi(t,x)$, the same property holds for $\widehat{\varphi}$ and for the modified payoff $\widehat{\varphi}_h$.

Therefore, we are going to study the monotony of φ. Let $x<K$. We recall that

$$\bigl(x^\alpha\varphi(x)\bigr)'=-\frac{2}{\sigma^2}x^\alpha\int_x^K\frac{m(dr)}{r^2}=-x^\alpha+x^\alpha\alpha K\int_x^K\frac{h(dr)}{r^2}.$$

As

$$\varphi(x)=(K-K^*)\Bigl(\frac{K^*}{x}\Bigr)^\alpha-\frac{x}{\alpha+1}+K^*\biggl(x\int_x^K\frac{h(dr)}{r^2}+x^{-\alpha}\int_0^x r^{\alpha-1}h(dr)\biggr),$$

and $1/(\alpha+1)=(K-K^*)/K$, we deduce that

$$x^\alpha\varphi'(x)=(K-K^*)\biggl(x^\alpha\alpha\int_x^K\frac{h(dr)}{r^2}-\frac{x^\alpha}{K}-\alpha\frac{K^{*\alpha}}{x}\biggr)-\alpha\frac{K^*}{x}\int_0^x r^{\alpha-1}h(dr).$$

We upper-bound $\int_x^K\frac{h(dr)}{r^2}$ thanks to the second moment assumption on h:

$$\int_x^K\frac{h(dr)}{r^2}\leq x^{-(\alpha+1)/2}\int_0^K r^{(\alpha-3)/2}h(dr)=\frac{x^{-(\alpha+1)/2}K^{(\alpha-1)/2}}{\alpha}.$$

Combining this inequality with $x^\alpha/K+\alpha K^{*\alpha}/x\geq 2x^{(\alpha-1)/2}\sqrt{\alpha K^{*\alpha}/K}$ we obtain

$$x^\alpha\varphi'(x)\leq (K-K^*)x^{(\alpha-1)/2}K^{(\alpha-1)/2}\bigl(1-2\sqrt{\alpha(K^*/K)^\alpha}\bigr).$$

Hence φ is nonincreasing as soon as $4\alpha(\alpha/(\alpha+1))^\alpha \geq 1$. It is easy to check that the function $\alpha \in]0,+\infty[\to f(\alpha) = 4\alpha(\alpha/(\alpha+1))^\alpha$ is increasing. Since $f(1/2) = \sqrt{4/3} > 1$ and $\lim_{\alpha\to 0} f(\alpha) = 0$, the equation $f(\alpha) = 1$ has a unique solution α_0. Moreover $\alpha_0 \leq 1/2$ and $\forall\alpha \geq \alpha_0$, φ is nonincreasing. $\square$

REFERENCES

[1] BARLES, G., BURDEAU, J., ROMANO, M. and SAMSON, N. (1993). Estimation de la frontière libre des options américaines au voisinage de l'échéance. *C. R. Acad. Sci. Paris Sér. I Math.* **316** 171–174.

[2] BRANDT, A. and CRYER, C. W. (1983). Multigrid algorithm for the solutions of linear complementarity problems arising from free boundary problems. *SIAM J. Sci. Statist. Comput.* **4** 655–684.

[3] BROADIE, M. and DETEMPLE, J. (1996). American option caluation: New bounds, approximations, and a comparison of existing methods. *Review of Finance Studies* **9** 1211–1250.

[4] CARR, P., JARROW, R. and MYNENI, R. (1992). Alternative characterizations of American put options. *Math. Finance* **2** 87–106.

[5] JOURDAIN, B. and MARTINI, C. (2001). American prices embedded in European prices. *Ann. Inst. H. Poincaré Anal. Nonlinear* **18** 1–17.

[6] JU, N. (1996). Pricing an American option by approximating its early exercise boundary as a multipiece exponential function. *Review of Finance Studies* **11** 627–646.

[7] LAMBERTON, D. (1995). Critical price for an American option near maturity. In *Seminar on Stochastic Analysis, Random Fields and Applications* 353–358. Birkhäuser, Basel.

[8] LITTLE, T., PANT, V. and HOU, C. (2000). An integral representation of the early exercise boundary for American put options. *J. Comput. Finance* **3**.

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