The Annals of Applied Probability
2001, Vol. 11, No. 4, 1330–1352

DUBINS–FREEDMAN PROCESSES AND RC FILTERS

BY CHRISTIAN MAZZA AND DIDIER PIAU

Université Claude Bernard Lyon-I

We use McFadden's integral equations for random RC filters to study the average distribution of Dubins–Freedman processes. These distributions are also stationary probability measures of Markov chains on $[0, 1]$, defined by the iteration of steps to the left $x \to u\,x$, and of steps to the right $x \to v + (1 - v)\,x$, where u and v are random from $[0, 1]$. We establish new algorithms to compute the stationary measure of these chains.

Turning to specific examples, we show that, if the distributions of u and $1 - v$ are $\mathrm{Beta}(a, 1)$, or $\mathrm{Beta}(a, 2)$, or if u and $1 - v$ are the exponential of $\mathrm{Gamma}(a, 2)$ distributed random variables, then the stationary measure is a combination of various hypergeometric functions, which are often ${}_3F_2$ functions. Our methods are based on a link that we establish between these Markov chains and some RC filters. We also determine the stationary distribution of RC filters in specific cases. These results generalize recent examples of Diaconis and Freedman.

1. Introduction. In [3], Diaconis and Freedman investigate the properties of iterated random functions. As a motivating example, they study a case of the following Markov chain on $[0, 1]$. Let $p \in (0, 1)$ and μ and ν be two probability measures on $[0, 1]$. Starting from x, the chain moves to the left with probability p and to the right with probability $q := 1 - p$. If the move is to the left, the chain picks u from μ and goes to $u\,x$. If the move is to the right, the chain picks v from ν and goes to $v + (1 - v)\,x$. We call this process the (p, μ, ν) DF chain. When $\mu = \nu$, Dubins and Freedman [4] introduced a version of the DF chain where p may be random as well. From [4] (see also [3]), there exists a unique stationary measure π, and π is the average distribution of a random distribution function, which we do not describe here.

The arguments in Section 9 of [4] can be adapted to the case where μ and ν may be different, yielding the following results: if μ and ν put no mass at 0 or 1, π has no atom, and π is purely singular or purely absolutely continuous; if μ and ν have C^{k+1} densities, π has a C^k density (Theorems 9.20, 9.25 and Lemma 9.18 of [4]). As regards specific examples, if $\mu = \nu$ is uniform, then $\pi = \mathrm{Beta}(q, p)$ (Theorem 9.21 of [4]). Hence, if $\mu = \nu$ is uniform and $p = \frac{1}{2}$, π has the arcsine density. Also, if $\mu = \nu$, π is uniform if and only if $\mu = \nu = \delta_p$ (Theorem 9.28 of [4]). Finally, from [3] if the DF chain is a Beta walk, that is, if $\mu = \nu = \mathrm{Beta}(a, a)$ and $p = \frac{1}{2}$, then π is not a Beta distribution, except when $a = 1$; that is, when $\mu = \nu$ is uniform (see above). To our knowledge, these are about the only cases where π is known.

Received October 2000; revised February 2001.

AMS 2000 *subject classifications*. Primary 60J05, 60F05.

Key words and phrases. Random affine system, iterated random functions, hypergeometric functions, Gauss's hypergeometric equation, Dubins–Freedman process, RC filter.

In this paper, we imbed DF chains in random RC filters, as defined by McFadden [5] and Pawula and Rice [7, 8], for example. Various stationary distributions related to RC filters are known to satisfy integro-differential equations, and we show how to recover π from some of these distributions. This yields two new algorithms to compute π explicitly. While the existence and unicity of a stationary distribution for random compositions of functions is a most studied problem (for DF chains, one has existence and unicity of π), less frequent is the explicit determination of this distribution in specific examples. However, see Chamayou and Letac [2] for explicit computations in a similar setting.

Our new results about examples of DF chains include the following. We determine the value of π if $\mathscr{D}(u) = \mathrm{Beta}(a, 1)$ and $\mathscr{D}(v) = \mathrm{Beta}(1, b)$, where $\mathscr{D}(X)$ is the distribution of X. If $p = \frac{1}{2}$ and if u and $1-v$ are the exponential of $\mathrm{Gamma}(\lambda, 2)$ random variables, we show that the density of π is the integral of various hypergeometric functions. We obtain explicit formulas when $\lambda = \frac{1}{2}$, $\lambda = \frac{1}{\sqrt{2}}$ and $\lambda = 1$. In these special cases, the density of π, as a function of $y \in [0, 1]$, is an affine combination of $w^n\, {}_3F_2(w)$ functions, where $w := 4\, y\,(1-y)$, and where ${}_3F_2$ is an hypergeometric function of type $(3, 2)$. Similar results hold if $p = \frac{1}{2}$ and $\mathscr{D}(u) = \mathscr{D}(1-v) = \mathrm{Beta}(a, 2)$. We provide a new proof of the fact, mentioned above and taken from [3], that the only case when $p = \frac{1}{2}$, $\mathscr{D}(u) = \mathscr{D}(1-v)$ is Beta and π is Beta, is when u and v are uniform.

We also give explicit formulas for the stationary distributions of RC filters in specific cases. Hypergeometric functions are ubiquitous in all these results, a fact which has yet to be explained by structural reasons. Finally, we mention that we plan to apply the methods of this paper to generalized DF chains on $[0, 1]$, where x moves to $w + u\,(x - w)$, and where (u, w) follows a given distribution on $[0, 1]^2$. Thus, w is the random fixed point of the step. In usual DF chains, $\mathscr{D}(w)$ is concentrated on $\{0, 1\}$. This model is related to Dirichlet distributions, as explained for instance in Section 7 of [3].

Organization. Section 2 defines RC filters and proves that DF chains are special cases of RC filters. In Section 3, we prove relations holding between various stationary distributions associated naturally to RC filters, and does the same thing for the distributions of DF chains. The part of Section 3 dealing with RC filters follows the work of McFadden [5], but our proofs are self-contained. Section 4 gives two algorithms to compute π, which draw on the decomposition in Theorem B of Section 3.

Turning to specific examples, Section 5 studies (after, e.g., Pawula and Rice [7] or [3]) exponentially distributed σ and τ, where $u =: e^{-\sigma}$ and $1 - v =: e^{-\tau}$. Gauss's hypergeometric differential equation appears. Even in this relatively simple case, our Theorem C generalizes Theorem 9.21 of [4]. Section 7 deals with $\mathrm{Gamma}(\lambda, 2)$ distributed σ and τ, and is prepared by Section 6, which reviews some technical tools. Section 8 states a result about RC filters with $\mathrm{Gamma}(\lambda, 2)$ distributed time intervals. Section 9 studies DF chains such that $\mathscr{D}(u) = \mathscr{D}(1-v) = \mathrm{Beta}(a, 2)$. All these results generalize examples of [3].

2. DF chains as cases of RC filters. In this section, we define RC filters and we embed DF chains in RC filters.

RC filters are defined by independent sequences $(S_n)_n$ and $(T_n)_n$ of i.i.d. positive random variables. The random telegraph wave $(x(t))_{t\geq 0}$ takes values in $\{0, 1\}$ as follows. First, $x(t) = 0$ on $t \in [0, S_1)$, then $x(t) = 1$ during a time interval of length T_1, then $x(t) = 0$ during a time interval of length S_2, and so on. If $x(t)$ is the input of an RC filter, the output $y(t)$ solves

$$y'(t) + y(t) = x(t), \tag{1}$$

where $y'(\cdot)$ is the derivative of $y(\cdot)$. Finally, $y(0) \in [0, 1]$ is independent of $(x(t))_t$. Thus, $y(t) \in [0, 1]$ for every t, and the sample functions of $(y(t))_{t\geq 0}$ are made of a succession of rising and decaying exponentials.

REMARK 1. If the distributions of S_n or T_n put mass on 0, one can realize the same system with different distributions having no atom at 0. Also, beginning with the value $x(0+) = 1$ does not change the results.

DEFINITION 2. Set $u_n =: e^{-\sigma_n}$ and $v_n =: 1 - e^{-\tau_n}$. Let $\mathscr{D}(X)$ denote the distribution of X. Introduce the functions which are sampled by the DF chain:

$$\varphi_u(x) := u\,x, \qquad \psi_v(x) := v + (1 - v)\,x.$$

THEOREM A. *For any parameters (p, μ, ν), there exists distributions of S and T such that the local extrema of the DF chain and the local extrema of the RC filter follow the same distribution $(\pi_0, \pi_1) = (P_0, P_1)$, in the stationary regime.*

Furthermore, (π_0, π_1) is uniquely determined by equation (4) *in Lemma* 3 *below. Starting from any distribution, the local extrema of the DF chain converge in distribution to (π_0, π_1). The same holds for the RC filter, when S and T follow the adequate distributions.*

PROOF. Let $R_0 := 0$ and $R_n := S_1 + T_1 + \cdots + S_n + T_n$. Since $x(t) = 0$ for $t \in [R_{n-1}, R_{n-1} + S_n)$ and $x(t) = 1$ for $t \in [R_{n-1} + S_n, R_n)$, one has

$$\begin{aligned} y(R_{n-1} + S_n) &= e^{-S_n}\, y(R_{n-1}), \\ y(R_n) &= e^{-T_n}\,(y(R_{n-1} + S_n) - 1) + 1. \end{aligned} \tag{2}$$

Hence, the values of $y(t)$ at the moments when the value of $x(t)$ changes are obtained by the alternated application of $\varphi_{e^{-S_n}}$ and $\psi_{1-e^{-T_n}}$. However, the DF chain applies φ functions or ψ functions at random, while the RC filter applies alternately φ functions and ψ functions. The following randomization procedure deals with this difference.

From the semigroup property of the orbits of (1), or from a direct computation, the composition of $\varphi_{e^{-\sigma}}$ and $\varphi_{e^{-\sigma'}}$ is $\varphi_{e^{-(\sigma+\sigma')}}$, and the composition of $\psi_{1-e^{-\tau}}$ and $\psi_{1-e^{-\tau'}}$ is $\psi_{1-e^{-(\tau+\tau')}}$. The DF chain applies ϕ functions a random number of times, say M times, before switching to ψ functions for a random number N of times, and so on. Then, $\mathscr{D}(M)$ is geometric of mean q^{-1}; that is,

$$\mathscr{D}(M) = \sum_{k\geq 0} q\, p^{k-1}\, \delta_k, \tag{3}$$

and $\mathscr{D}(N)$ is geometric of mean p^{-1}. Thus, if

$$\mathscr{D}(S) = \mathscr{D}(\sigma_1 + \cdots + \sigma_M) \quad \text{and} \quad \mathscr{D}(T) = \mathscr{D}(\tau_1 + \cdots + \tau_N),$$

the local extrema of the DF chain and of the RC filter follow the same dynamics. Hence, Lemma 3 below implies the theorem. □

LEMMA 3. *The solution* (P_0, P_1) *of the system* $P_0 = \mathscr{D}(y_0)$, $P_1 = \mathscr{D}(y_1)$,

$$\mathscr{D}(y_0) = \mathscr{D}(\varphi_{e^{-S}}(y_1)), \qquad \mathscr{D}(y_1) = \mathscr{D}(\psi_{1-e^{-T}}(y_0)), \tag{4}$$

is unique, where S, T, y_1 *and* y_0 *are independent.*

PROOF. Such recurrence relations yield explicit formulas for y_0 and y_1, as well as for their moments (see Munford [6]). For instance, $\mathscr{D}(y_0) = \mathscr{D}(y_0^n + z_0^n)$, with

$$y_0^n := \sum_{k=1}^{n} e^{-R_{k-1}-S_k}\,(1 - e^{-T_k}) \quad \text{and} \quad z_0^n := e^{-R_n - S_{n+1}}\, y_1^n,$$

where all the random variables involved are independent, and where $\mathscr{D}(y_1^n) = P_1$. Since $R_k \to \infty$ as $k \to \infty$ and $|y_1^n| \leq 1$, the error term z_0^n converges to 0 in distribution. On the other hand, y_0^n converges a.s., since $E(y_0^n)$ is uniformly bounded as soon as $E(e^{-S-T}) < 1$. Hence, y_0^n converges in distribution to

$$y_0^\infty := \sum_{k\geq 1} e^{-R_{k-1}-S_k}\,(1 - e^{-T_k}),$$

and any solution of the system of the lemma must satisfy $P_0 = \mathscr{D}(y_0^\infty)$. The same applies to P_1. □

3. Stationary renewal theory.

3.1. *RC filters.* We use the following notations.

DEFINITION 4. Let P_- or P_+ be the stationary distribution (or its density) of $y(t)$ conditioned by the fact that $\{x(t) = 0\}$ or $\{x(t) = 1\}$. Let P_0 or P_1 be the stationary distribution (or its density) of the local minima or maxima of $y(\cdot)$. Let y_i follow the distribution P_i for $i \in \{0, 1, +, -\}$.

Let f or g be the density of the distribution of σ or τ. Let F or G be the density of the distribution of S or T. Let μ_0 or μ_1 be the mean of S or T. Finally, for any density H on $s \geq 0$ of finite mean μ_H, introduce

$$H_*(s) := \mu_H^{-1} \int_s^{+\infty} H(s')\, ds'.$$

Hence, if H is the density of $\mathscr{D}(X)$, H_* is the density of $\mathscr{D}(X_*)$, where

$$X_* := \widehat{X} \cdot \text{Unif}.$$

Here, $\mathscr{D}(\text{Unif})$ is uniform on $(0,1)$, Unif is independent of $\widehat{X}$ and $\mathscr{D}(\widehat{X})$ is a size-biased version of $\mathscr{D}(X)$. That is, $\mathscr{D}(\widehat{X})$ satisfies, for any nonnegative function A,

$$E(A(\widehat{X})) = E(X\,A(X))/E(X).$$

The distributions of $\widehat{X}$ and X_* appear naturally as stationary distributions in renewal theory. We recall without proof the following fact (see, e.g., Section 10 of Chapter 2 in Thorisson [9]).

PROPOSITION 5. *Consider the homogeneous renewal process on $t \geq 0$ defined by i.i.d. intervals of distribution $\mathscr{D}(X)$, in the stationary regime, or equivalently, around a large value of t.*

Then, the length of the overlapping interval follows the distribution $\mathscr{D}(\widehat{X})$. Furthermore, the location of t in this interval is uniform. Hence, the length of the part of this interval which lies before t follows the distribution $\mathscr{D}(X_*)$.

We now relate the various stationary distributions of the RC filter, in terms of the densities P_i or in terms of the random variables y_i.

REMARK 6. Most of what follows appears in Pawula and Rice [7], which draws on earlier work of McFadden [5]. The results of [7] hold for RC filters on $[-1,1]$, and we translate them to the case of RC filters on $[0,1]$. In order to give more tractable formulas, we sometimes change variables in the integrals. Finally, we use the fact that some densities are zero outside of $[0,1]$, to simplify the notations.

Assume that $y(t)$ is in the stationary regime. Then, $y(t)$ spends a fraction $\mu_0/(\mu_0+\mu_1)$ of its time in the $x(t)=0$ regime. Hence,

$$(\mu_0+\mu_1)\,P = \mu_0\,P_- + \mu_1\,P_+.$$

For $y \in [0,1]$, one can compute the mean amounts of mass which cross the level y during the time interval $[t, t+dt]$, upward and downward. Since the upward and downward crossings occur at rates $1-y$ and y, respectively, the conservation of the mass at stationarity reads

$$\mu_0\,y\,P_-(y) = \mu_1\,(1-y)\,P_+(y).$$

This in turn yields

$$\begin{aligned} \mu_0\,P_-(y) &= (\mu_0+\mu_1)\,(1-y)\,P(y), \\ \mu_1\,P_+(y) &= (\mu_0+\mu_1)\,y\,P(y). \end{aligned} \tag{5}$$

DEFINITION 7. For any nonnegative H on $(0, +\infty)$ and Q on $(0, 1)$, define

$$K_-(H, Q)(y) := \int Q(y\, e^s)\, e^s\, H(s)\, ds$$

and

$$K_+(H, Q)(y) := \int Q(1 - (1 - y)\, e^t)\, e^t\, H(t)\, dt.$$

If Q is the density of the distribution of the DF chain before a step to the left and if the distribution of σ has density H, where $u =: e^{-\sigma}$, then $K_-(H, Q)$ is the density of the distribution of the chain after this step. A similar interpretation holds for $K_+(H, Q)$ with respect to a step to the right and to the distribution of τ, where $1 - v =: e^{-\tau}$. The definition above is equivalent to

$$K_-(H, Q)(y) := \int_y^1 Q(x)\, H\left(\log\left(\frac{x}{y}\right)\right) \frac{dx}{y},$$

$$\text{and} \quad K_+(H, Q)(y) := \int_0^y Q(x)\, H\left(\log\left(\frac{1-x}{1-y}\right)\right) \frac{dx}{1-y}.$$

PROPOSITION 8.

$$P_0 = K_-(F, P_1), \qquad P_1 = K_+(G, P_0), \tag{6}$$

$$\text{and} \quad P_- = \mathrm{K}_-(F_*, P_1), \qquad P_+ = \mathrm{K}_+(G_*, P_0). \tag{7}$$

This follows from the identities in distribution

$$\mathscr{D}(y_0) = \mathscr{D}(e^{-S}\, y_1), \qquad \mathscr{D}(y_1) = \mathscr{D}(1 - e^{-T}\,(1 - y_0)), \tag{8}$$

$$\text{and} \quad \mathscr{D}(y_-) = \mathscr{D}(e^{-S_*}\, y_1), \qquad \mathscr{D}(y_+) = \mathscr{D}(1 - e^{-T_*}\,(1 - y_0)), \tag{9}$$

where all the random variables are independent.

PROOF. We establish the identities in distribution. The definition of the RC filter shows that a version of (8) relates the distributions of successive extrema. At stationarity, every minimum follows the distribution $\mathscr{D}(y_0)$ and every maximum follows the distribution $\mathscr{D}(y_1)$, hence (8) holds.

As concerns (9), conditioning, for instance, on $\{x(t) = 0\}$ cancels the T_n intervals. Hence, one considers the renewal process defined by $(S_n)_n$. From Proposition 5, the last change from $x(\cdot) = 1$ to $x(\cdot) = 0$ occurred at time $t_* := t - S_*$, hence $y(t) = e^{-S_*}\, y(t_*)$ and $y(t_*)$ is a local maximum, that is the first part of (9). The proof of the second part is similar. □

DEFINITION 9. For any density H on $(0, +\infty)$, let $L(H)$ denote the Laplace transform of H, and $R(H)$ be the function or the measure such that

$$L(R(H))(s) := \frac{s\, L(H)(s)}{1 - L(H)(s)}.$$

PROPOSITION 10.

$$P_0 = \mu_0 \, K_-(R(F), P_-), \qquad P_1 = \mu_1 \, K_+(R(G), P_+). \tag{10}$$

PROOF. Definition 9 amounts to asking that

$$F = \mu_0 \, F_* * R(F) \quad \text{and} \quad G = \mu_1 \, G_* * R(G).$$

Hence, (6) and (7) and the semigroup properties of the operators K with respect to the convolution yield the proposition. □

LEMMA 11. *The densities P_0 and P_1 are related to P through*

$$(\mu_0 + \mu_1) \frac{d}{dy}(y \, (1-y) \, P(y)) = P_0(y) - P_1(y).$$

PROOF. Proposition 8 gives P_- as an integral of P_1. Write this result as

$$y \, P_-(y) = \int_y^1 P_1(x) \, F_*(\log(x/y)) \, dx.$$

Since $F_*'(s) = -\mu_0^{-1} \, F(s)$ and $F_*(0) = \mu_0^{-1}$, the differentiation yields

$$\begin{aligned}(y \, P_-(y))' &= -P_1(y) \, \mu_0^{-1} + y^{-1} \, \mu_0^{-1} \int_y^1 P_1(x) \, F(\log(x/y)) \, dx \\ &= \mu_0^{-1}(-P_1(y) + P_0(y)),\end{aligned}$$

where the last equality is a consequence of (6). Finally, (5) shows that $y \, P_-(y)$ is a multiple of $y \, (1-y) \, P(y)$. □

When the RC filter is induced by a DF chain in the way of Section 2, one can bypass the computation of F and G from f and g, thanks to the following.

LEMMA 12. *$R(F) = q \, R(f)$ and $R(G) = p \, R(g)$.*

PROOF. Direct from $L(F) = q \, L(f)/(1 - p \, L(f))$ and from the similar expression of $L(G)$. □

PROPOSITION 13. *Let $P_R(y) := y \, P(y)$ and $P_L(y) := (1-y) \, P(y)$. Then, if F and G are the randomizations of f and g,*

$$\frac{d}{dy}(y \, (1-y) \, P(y)) = q \, K_-(R(f), P_R) - p \, K_+(R(g), P_L). \tag{11}$$

PROOF. The proof follows from Lemma 11 and from the fact that P_0 and P_1 are images of P_- and P_+ by operators K, that P_- and P_+ are multiples of P_L and P_R, and that $R(F)$ and $R(G)$ are multiples of $R(f)$ and $R(g)$. □

3.2. *DF chains.* Assume that the DF chain is imbedded in the RC filter (S, T). We use the following notations.

DEFINITION 14. Recall that π_0 or π_1 is the stationary distribution (or its density) of the local minima or maxima of the DF chain. Let π_- or π_+ be the distribution π conditioned by the fact that the last move was to the left or to the right. Let Y or Y_i follow the distribution π or π_i for $i \in \{0, 1, +, -\}$.

THEOREM B. $\pi = p\,\pi_0 + q\,\pi_1 = p\,P_0 + q\,P_1$.

Recall that $P_0 = \pi_0$ and $P_1 = \pi_1$. Hence, (P_0, P_1) solves

$$\mathscr{D}(Y_0) = \mathscr{D}(e^{-S}\,Y_1), \quad \mathscr{D}(Y_1) = \mathscr{D}(1 - e^{-T}\,(1 - Y_0)).$$

This shows that Theorem B is a direct consequence of Proposition 15.

PROPOSITION 15. (i) $\mathscr{D}(Y) = p\,\mathscr{D}(Y_-) + q\,\mathscr{D}(Y_+)$.

(ii) $\mathscr{D}(Y_0) = \mathscr{D}(Y_-)$ *and* $\mathscr{D}(Y_1) = \mathscr{D}(Y_+)$. □

PROOF. Consider the alternating renewal process, defined by the integer intervals of consecutive moves of the DF chain to the left, or of consecutive moves to the right. This process may be encoded by a sequence $(\varepsilon_n)_{n\geq 1}$ of zeroes and ones, such that $\varepsilon_n = 1$ if the nth move is to the right, and $\varepsilon_n = 0$ else. Since the chances of a move to the left are p, the DF chain spends in the long run a proportion p of its time making moves to the left. This implies (i).

As regards (ii), to condition, for instance, by a last move to the left is the same as to cancel the times n such that $\varepsilon_n = 1$, that is, such that the nth move is to the right. Hence, one considers an ordinary renewal process, defined by the concatenation of the intervals of integer lengths which represented consecutive moves to the left in the alternated renewal process defined above.

These lengths are i.i.d. and follow the distribution $\mathscr{D}(M)$ of (3). The analogue of Proposition 5 for discrete time renewal processes is as follows: the distribution of the overlapping interval is $\mathscr{D}(\widehat{M})$, and the past part $\widetilde{M}$ of this interval is uniform on $\{1, 2, \ldots, \widehat{M}\}$. [Caution: $\mathscr{D}(\widetilde{M})$ is not $\mathscr{D}(M_*)$.] Since $\mathscr{D}(M)$ is geometric, one line of computation shows that

$$\mathscr{D}(\widetilde{M}) = \mathscr{D}(M).$$

This implies that Y_0 and Y_- are the results of the same number of moves to the left from a local maximum; that is the first part of (ii). The proof of the second part is similar. □

We give a second proof of Theorem B, using the operators K defined in Section 3.1.

DEFINITION 16. Abbreviate $K_-(f, \cdot)$ and $K_+(g, \cdot)$ in k_- and k_+, and $K_-(F, \cdot)$ and $K_+(G, \cdot)$ in K_- and K_+.

The definition of the dynamics of the DF chain and the unicity result of [3] imply that π is the unique distribution solution of

$$\pi = p\,k_-(\pi) + q\,k_+(\pi). \tag{12}$$

OPERATOR PROOF OF THEOREM B. Using the random variables M and N of the randomization of Section 2, one gets

$$K_-(Q) = E(k_-^M(Q)), \qquad K_+(Q) = E(k_+^N(Q)),$$

where k_i^n is the composition of n operators k_i. Conditioned by $\{M \geq 2\}$, M follows the distribution of $1+M$. Furthermore, $M = 1$ with probability q. The same applies to N with p instead of q, hence

$$K_- = q\,k_- + p\,k_- \circ K_- \quad \text{and} \quad K_+ = p\,k_+ + q\,k_+ \circ K_+.$$

This yields

$$p\,k_-(P_0) = P_0 - q\,k_-(P_1) \quad \text{and} \quad q\,k_+(P_1) = P_1 - p\,k_+(P_0).$$

We apply these relations to $k(Q)$, where $Q := p\,P_0 + q\,P_1$ and $k := p\,k_- + q\,k_+$. The terms $k_-(P_1)$ and $k_+(P_0)$ cancel and there remains

$$k(Q) = p\,k_-(Q) + q\,k_+(Q) = p\,P_0 + q\,P_1 = Q,$$

a fact which proves that $\pi = p\,P_0 + q\,P_1$. $\square$

4. Algorithms. Recall that the stationary measure π of a DF chain is the only distribution solution of (12); that is,

$$\pi(y) = p \int \pi(y\,e^s)\,e^s\,f(s)\,ds + q \int \pi(1-(1-y)\,e^t)\,e^t\,g(t)\,dt.$$

The two algorithms to compute π that we now propose may be viewed as decompositions of this fixed point problem into two related parts. They summarize our results so far.

ALGORITHM I (The direct way to π).

1. Starting from $u =: e^{-\sigma}$ and $v =: 1 - e^{-\tau}$, compute the distribution F or G of $S := \sigma_1 + \cdots + \sigma_M$ or $T := \tau_1 + \cdots + \tau_N$, where M and N are geometric of means q^{-1} and p^{-1}.
2. Solve the system

$$P_0 = K_-(F, P_1), \quad P_1 = K_+(G, P_0).$$

3. Compute $\pi = p\,P_0 + q\,P_1$. $\square$

From Lemma 3, Step 2 has a unique solution. Steps 1 and 3 are trivial. Algorithm I bypasses the computation of P. However, it is sometimes simpler to compute P, even if one is only interested in the value of π (see the examples in the next sections). One then uses the following.

ALGORITHM II (Computing π through P).

1. Compute $R(f)$ and $R(g)$.
2. Compute P as a solution of (11), that is,

$$\frac{d}{dy}(y\,(1-y)\,P(y)) = q\,K_-(R(f), P_R)(y) - p\,K_+(R(g), P_L)(y).$$

Recall that

$$K_-(R(f), P_R)(y) = \int_y^1 R(f)\left(\log\left(\frac{x}{y}\right)\right) P(x)\,\frac{x}{y}\,dx$$

and

$$K_+(R(g), P_L)(y) = \int_0^y R(g)\left(\log\left(\frac{1-x}{1-y}\right)\right) P(x)\,\frac{1-x}{1-y}\,dx.$$

3. Deduce P_0 and P_1 from P through

$$P_0 = (\mu_0 + \mu_1)\,q\,K_-(R(f), P_R), \tag{13}$$

$$P_1 = (\mu_0 + \mu_1)\,p\,K_+(R(g), P_L). \tag{14}$$

4. Recover π through $\pi = p\,P_0 + q\,P_1$, or skip Step 3 and write π as

$$\pi = (pq)\,(\mu_0 + \mu_1)\,(K_-(R(f), P_R) + K_+(R(g), P_L)). \qquad \square$$

Algorithm II bypasses the computation of F and G, but makes use of $R(f)$ and $R(g)$. This suits best DF chains with distributions f and g such that $R(f)$ and $R(g)$ are simple. We now apply these two algorithms to specific examples.

5. Exponential DF chains.

THEOREM C. *Assume that* $\mathscr{D}(u_n) = \mathrm{Beta}(a, 1)$ *and* $\mathscr{D}(v_n) = \mathrm{Beta}(1, b)$; *that is*,

$$a\,u^{a-1}\,du \quad \textit{and} \quad b\,(1-v)^{b-1}\,dv.$$

Then, $P_0 = \mathrm{Beta}(\lambda_0, \lambda_1 + 1)$, $P_1 = \mathrm{Beta}(\lambda_0 + 1, \lambda_1)$ *and* $P = \mathrm{Beta}(\lambda_0, \lambda_1)$, *with* $\lambda_0 := q\,a$ *and* $\lambda_1 := p\,b$.

Furthermore, π *is* Beta *if and only if* $\pi = P$ *if and only if* $a = b$. *If this is the case*, $\pi = P = \mathrm{Beta}(q\,a, p\,a)$. *In the general case*,

$$\pi(y) = \left(\frac{p}{a} + \frac{q}{b}\right)(a\,(1-y) + b\,y)\,\mathrm{Beta}(\lambda_0, \lambda_1)(y).$$

PROOF. Since $\mathscr{D}(\sigma)$ and $\mathscr{D}(\tau)$ are exponential of parameters a and b, $\mathscr{D}(S)$ and $\mathscr{D}(T)$ are exponential of parameters

$$\lambda_0 := q\,a \quad \text{and} \quad \lambda_1 := p\,b.$$

Using the density of the exponential distribution, the differentiation of (6) and (7) yields

$$y\,P_0'(y) + (1-\lambda_0)\,P_0(y) + \lambda_0\,P_1(y) = 0, \tag{15}$$

$$(1-y)\,P_1'(y) - (1-\lambda_1)\,P_1(y) - \lambda_1\,P_0(y) = 0. \tag{16}$$

In Lemma 11, $\mu_0 = \lambda_0^{-1}$ and $\mu_1 = \lambda_1^{-1}$. This yields the result. $\square$

REMARK 17. The case $a = b = 1$ of Theorem C, that is, $\mathscr{D}(u_n) = \mathscr{D}(v_n)$ uniform on $(0, 1)$, is Theorem 9.21 of [4]. If, furthermore, $p = \frac{1}{2}$, $\pi = \mathrm{Beta}(\frac{1}{2}, \frac{1}{2})$ has the arcsine density function, a fact noted in [3].

REMARK 18. Plugging (16) into (15) and vice versa, one gets equations which involve P_1 alone and P_0 alone:

$$y\,(1-y)\,P_1''(y) + (1-\lambda_0 + (\lambda_2 - 2)\,y)\,P_1'(y) + \lambda_2\,P_1(y) = 0, \tag{17}$$

$$y\,(1-y)\,P_0''(y) + (2-\lambda_0 + (\lambda_2 - 2)\,y)\,P_0'(y) + \lambda_2\,P_0(y) = 0, \tag{18}$$

where $\lambda_2 := \lambda_0 + \lambda_1 - 1$. Hence, P_0 and P_1 both solve Gauss's hypergeometric equations. By symmetry, one can check that P_1 of the form

$$P_1(y) =: Q(y, \lambda_0, \lambda_1)$$

solves (17) if and only if $P_0(y) := Q(1-y, \lambda_1, \lambda_0)$ solves (18). Hence, the system (17) and (18) is really one equation and its "symmetric" version.

The general solutions of (17) and (18) are hypergeometric functions of type ${}_2\mathrm{F}_1$. Hence, the Beta functions of Theorem C should be viewed as special cases of ${}_2\mathrm{F}_1$ functions.

6. Technical tools.

We state notations and basic facts about hypergeometric functions (see [1]), and we solve an integro-differential equation which appears in various guises below.

6.1. *Hypergeometric functions.* For $y \in [0, 1]$, $W(y) := 4\,y\,(1-y)$. Thus, $W(y) \in [0, 1]$ and

$$W'(y) = 4\,(1-2y), \qquad (W')^2 = 16\,(1-W), \qquad W'' = -8.$$

We sometimes write w for $W(y)$. For suitable a and b,

$$B(a, b) := \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}.$$

Gauss's hypergeometric differential operator $G_{a,b,c}$ is defined by

$$G_{a,b,c}(Q)(w) := w(1-w)Q''(w) + (c - (a+b+1)w)Q'(w) - abQ(w).$$

The hypergeometric function $F(a, b; c; w)$ is defined by the following series, which converges at least on the disc $|w| < 1$ (see 15.1.1 of [1]):

$$F(a, b; c; w) := \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{w^n}{n!}$$
$$= 1 + \frac{ab}{c} w + \frac{a(a+1)b(b+1)}{c(c+1)} \frac{w^2}{2!} + \cdots .$$

More generally, one defines $F(a_1, a_2, \ldots; b_1, b_2, \ldots; w)$ by the series

$$F(a_1, a_2, \ldots; b_1, b_2, \ldots; w) := \sum_{n \geq 0} \frac{(a_1)_n (a_2)_n \cdots}{(b_1)_n (b_2)_n \cdots} \frac{w^n}{n!}.$$

The symbol ${}_pF_q$ may be used instead of F to emphasize that p numbers a_i and q numbers b_j appear. Recall that $F(a, b; c; \cdot)$ solves

$$G_{a,b,c}(Q) = 0$$

on $(0, 1)$, and that the following function is also a solution:

$$Q(w) := w^{1-c} F(a - c + 1, b - c + 1; 2 - c; w).$$

As regards integrals of hypergeometric functions, if the real parts of r, s and $s + c - a - b$ are positive,

$$\int_0^1 w^{r-1}(1-w)^{s-1} F(a, b; c; w)\, dw = B(r, s)\, {}_3F_2(a, b, r; c, r + s; 1). \tag{19}$$

A similar formula applies to the integral of any ${}_pF_q$.

We will use a trick due to Appell. The general solution of the third-order differential equation

$$U''' + 3AU'' + (2A^2 + A' + 4B)U' + (4AB + 2B')U = 0, \tag{20}$$

where A and B are arbitrary functions, is a linear combination of V_1^2, $V_1 V_2$ and V_2^2 (see, e.g., Exercise 10, Chapter XIV of Whittaker and Watson [10]), where V_1 and V_2 are two linearly independent solutions of the second-order differential equation

$$V'' + AV' + BV = 0. \tag{21}$$

Finally, when $c = a + b + \frac{1}{2}$, Clausen's formula reads

$${}_2F_1(a, b; c; w)^2 = {}_3F_2(2a, 2b, a + b; c, 2c - 1; w). \tag{22}$$

6.2. *A basic integration.*

PROPOSITION 19. (i) *Assume that* P *solves*

$$\begin{aligned}\frac{d}{dy}(y(1-y)P(y)) = cy^{2\lambda-1}\int_y^1 (1-x)x^{-2\lambda}P(x)\,dx \\ - c(1-y)^{2\lambda-1}\int_0^y x(1-x)^{-2\lambda}P(x)\,dx,\end{aligned} \tag{23}$$

for some given c *and* λ. *Then*, P *is smooth and solves*

$$\begin{aligned}y^2(1-y)^2P''' + \alpha y(1-y)(1-2y)P'' \\ + (\beta + \beta' y(1-y))P' + \gamma(1-2y)P = 0,\end{aligned} \tag{24}$$

for the following values of the parameters:

$$\begin{aligned}\alpha &:= 5 - 2\lambda, \\ \beta &:= c - 4\lambda + 4, \\ \beta' &:= -2c - 24 + 22\lambda - 4\lambda^2, \\ \gamma &:= (2\lambda - 3)(c + 2 - 2\lambda).\end{aligned}$$

(ii) *Assume that* $P =: Q \circ W$ *with* $W(y) := 4y(1-y)$ *and that* P *solves* (24). *Then*, Q *solves*

$$w^2(1-w)Q''' + w(\alpha - \alpha' w)Q'' + (\beta + \beta'' w)Q' + \gamma' Q = 0, \tag{25}$$

where α', β'' *and* γ' *are*

$$\alpha' := \alpha + \tfrac{3}{2}, \qquad \beta'' := \tfrac{1}{4}\beta' - \tfrac{1}{2}\alpha, \qquad \gamma' := \tfrac{1}{4}\gamma.$$

(iii) *Define* Q *by* $Q(w) := w^{-n}\,{}_3F_2(a_1, a_2, a_3; b_1, b_2; w)$. *Then*, Q *solves* (25) *if and only if* n *solves the indicial equation*

$$n(n^2 + (3-\alpha)n + \beta - \alpha + 2) = 0, \tag{26}$$

and $(a_i)_i$ *and* $(b_i)_i$ *solve, for every* k,

$$\prod_{i=1}^{3}(k + n + a_i) = k(k-1)(k-2) + \alpha' k(k-1) - \beta'' k - \gamma'$$

and

$$(k + n + 1)\prod_{i=1}^{2}(k + n + b_i) = (k+1)(k^2 + (\alpha - 1)k + \beta).$$

(iv) *Starting from* (23), *the indicial equation reads*

$$n = 0 \quad \textit{or} \quad n = 1 - \lambda \pm \sqrt{\lambda^2 - c},$$

and, introducing $b_3 := 1$, *the parameters* a_i *and* b_i *satisfy*

$$\{n + a_i\}_i = \left\{\tfrac{3}{2} - \lambda, 1 - \tfrac{1}{2}\lambda + \tfrac{1}{2}\sqrt{\lambda^2 - 2c}, 1 - \tfrac{1}{2}\lambda - \tfrac{1}{2}\sqrt{\lambda^2 - 2c}\right\},$$
$$\{n + b_i\}_i = \left\{1, 2 - \lambda + \sqrt{\lambda^2 - c}, 2 - \lambda - \sqrt{\lambda^2 - c}\right\}.$$

PROOF. Direct computations give (24). From it, one gets (25) through the values of W', W'' and the chain rule. One gets part (iii), by equating first the lowest degree terms of the expansions, which yields the indicial equation, then the coefficients of w^{k-n}, for $k \geq 0$. Part (iv) follows from the values of the parameters in (i) and (ii), and from the factorization of the polynomials of (iii). □

7. Gamma DF chains.

In this section, we assume that $\mathscr{D}(\sigma) = \mathscr{D}(\tau) = \mathrm{Gamma}(\lambda, 2)$, that is,

$$\lambda^2 t e^{-\lambda t}\, dt,$$

and we call this distribution simply a Gamma distribution. We show that π, P, P_0 and P_1 are related to hypergeometric functions.

7.1. *Equations for* P, P_0 *and* P_1. Since $\mu_0 = E(\sigma)/q = 2/(q\lambda)$, $\mu_1 = 2/(p\lambda)$ and

$$R(f)(t) = R(g)(t) = \lambda^2 e^{-2\lambda t},$$

(13) and (14) read

$$\begin{aligned} P_0(y) &= (\mu_0 + \mu_1) q \lambda^2 y^{2\lambda - 1} \int_y^1 P(x)(1-x)x^{-2\lambda}\, dx, \\ P_1(y) &= (\mu_0 + \mu_1) p \lambda^2 (1-y)^{2\lambda - 1} \int_0^y P(x) x (1-x)^{-2\lambda}\, dx, \end{aligned} \tag{27}$$

and (11) reads

$$\begin{aligned} &\frac{d}{dy}(y(1-y)P(y)) = q\lambda^2 y^{2\lambda-1} \int_y^1 (1-x)x^{-2\lambda} P(x)\, dx \\ &- p\lambda^2 (1-y)^{2\lambda-1} \int_0^y x(1-x)^{-2\lambda} P(x)\, dx. \end{aligned} \tag{28}$$

The value $\lambda = \frac{1}{2}$ is special in (28) since the monomials before the integrals vanish. Then, P solves a second-order differential equation, and we compute completely the solution in the symmetric case $p = \frac{1}{2}$. For $\lambda \neq \frac{1}{2}$, P solves a third-order differential equation, and we only give the form of the solution.

Convention. For the rest of this section, and unless we mention the contrary, we assume that $p = \frac{1}{2}$.

7.2. *Computing P and π when $\lambda = \frac{1}{2}$.*

PROPOSITION 20. *Assume that $p = \frac{1}{2}$ and $\lambda = \frac{1}{2}$. Then P is a linear combination of two $w^n {}_2F_1(w)$ functions, and π is an affine combination of two $w^n {}_3F_2(w)$ functions (see the explicit values of all the coefficients below).*

One computes P and π in four steps: (1) P solves a differential equation; (2) the general solution of this equation involves hypergeometric functions; (3) normalizing conditions give the values of the constants in P; (4) P yields π.

PROOF. When $p = \frac{1}{2}$, the symmetry of the problem yields $P_1(y) = P_0(1-y)$ and $P(1-y) = P(y)$. Since $\lambda = \frac{1}{2}$, (28) reduces to

$$8\frac{d}{dy}(y(1-y)P(y)) = \int_y^1 \frac{1-x}{x} P(x)\,dx - \int_0^y \frac{x}{1-x} P(x)\,dx. \tag{29}$$

This is (23) for $c := \frac{1}{8}$ and $\lambda = \frac{1}{2}$. One could use Proposition 19. We prefer to solve (29) directly, differentiating once, setting $P =: Q \circ W$ and applying the chain rule. After tedious computations, one gets

$$w(1-w)Q''(w) + (2 - \tfrac{5}{2}w)Q'(w) - \tfrac{9}{16}Q(w) = -\tfrac{1}{8}w^{-1}Q(w),$$

that is,

$$wG_{3/4,\,3/4,\,2}(Q)(w) = -\tfrac{1}{8}Q(w). \tag{30}$$

An Ansatz to deal with the right-hand member $-\frac{1}{8}Q$ of (30) is to look for solutions, proportional to

$$Q(w) =: w^{-n}R(w),$$

where R is a power series in w such that $R(0) = 1$. Equating the lowest degree terms of (30), one gets

$$n(n-1) = -\tfrac{1}{8},$$

that is, $n = n_\pm$ with

$$n_- := \tfrac{1}{2}\left(1 - \tfrac{1}{\sqrt{2}}\right) = .146+, \qquad n_+ := \tfrac{1}{2}\left(1 + \tfrac{1}{\sqrt{2}}\right) = 1 - n_- = 0.854-.$$

If $n = n_i$ with $i \in \{-,+\}$, (30) is equivalent to $G(R) = 0$ for the Gauss's operator G of parameters (a_i, a'_i, b_i) such that

$$b_i := 2 - 2n_i,$$

and such that $a_i + a'_i = \frac{3}{2} - 2n_i$ and $a_i a'_i = \frac{7}{16} - \frac{1}{2}n_i$. Hence,

$$a_i := \tfrac{3}{4} - n_i =: a'_i,$$

which gives $a_- = 0.603+$ and $a_+ = -0.207-$. The general solution of (30) on $(0,1)$ is a linear combination $Q = A_+Q_+ + A_-Q_-$, with

$$Q_i(w) := w^{-n_i}F_i(w) \quad \text{where} \quad F_i := F(a_i, a_i; b_i; \cdot).$$

Since $n_+ \neq n_-$ and $F_i(0) = 1$, the asymptotic behavior of $Q_i(w)$ near $w = 0$ proves that Q_- and Q_+ are linearly independent. $\square$

The exact computation of P and π involves two technical results. We omit the proof of Lemma 21.

LEMMA 21. *For any nonnegative B,*

$$\int_0^1 P(y)B(w(y))\,dy = \tfrac{1}{2}\int_0^1 Q(w)B(w)(1-w)^{-1/2}\,dw.$$

LEMMA 22. *Assume that $\mathscr{D}(\sigma) = \mathscr{D}(\tau)$ and $p = \frac{1}{2}$. Then,*

$$E(Y_0) = \tfrac{1}{2}E(e^{-\sigma}).$$

PROOF. $E(Y_0)$ is a rational fraction in $E(e^{-S})$, because

$$\mathscr{D}(Y_0) = \mathscr{D}(e^{-S}e^{-T}Y_0 + e^{-S}(1-e^{-T})),$$

with independent S, T and Y_0. Lemma 12 shows that $E(e^{-S})$ is a rational fraction in $E(e^{-\sigma})$. When $p = \frac{1}{2}$, one gets the lemma. $\square$

Exact computation of P and π. One can deduce the values of A_- and A_+ from the fact that P is the density of a distribution [see (31) below] and from the determination of $E(Y_0)$ [see (32) below]. Lemma 21 with $B = 1$ yields

$$\int_0^1 Q(w)(1-w)^{-1/2}\,dw = 2. \tag{31}$$

Since $\mu_0 = \mu_1 = E(\sigma)/p = 2\,E(\sigma)$, the first equation of (27) reads

$$P_0(y) = \frac{1}{2}E(\sigma)\int_y^1 P(x)\frac{1-x}{x}\,dx.$$

After the interversion of the order of integration, Lemma 22 and Lemma 21 with $B(w) = w$ yield

$$\int_0^1 Q(w)w(1-w)^{-1/2}\,dw = 16E(e^{-\sigma})/E(\sigma). \tag{32}$$

Since $E(e^{-\sigma}) = \frac{1}{9}$ and $E(\sigma) = 4$, (19) implies that (A_-, A_+) solves the following linear system

$$\sum_i B(1-n_i, \tfrac{1}{2})F(a_i, a_i, 1-n_i; b_i, \tfrac{3}{2}-n_i; 1)A_i = 2,$$

$$\sum_i B(2-n_i, \tfrac{1}{2})F(a_i, a_i, 2-n_i; b_i, \tfrac{5}{2}-n_i; 1)A_i = \tfrac{4}{9},$$

where the sums are over $i \in \{-, +\}$. Numerically, one finds

$$(A_-, A_+) = (-0.0172-, 0.249+).$$

The plot of P over $[0,1]$ is arcsinelike but a little bit flatter in its middle part than the plot of the arcsine density. The density P yields π through

$$\pi(y) = \frac{1}{2}\int_y^1 P(x)\frac{1-x}{x}\,dx + \frac{1}{2}\int_0^y P(x)\frac{x}{1-x}\,dx.$$

The plot of π is even more arcsinelike than the plot of P. One gets a more explicit form of π by differentiation. Then,

$$\pi'(y) = 2(2y-1)W(y)^{-1}P(y),$$

that is, π' involves a linear combination of ${}_2F_1$ functions. Integrating π' term by term as a series in w, one gets

$$\pi(y) = \pi\left(\frac{1}{2}\right) + \frac{1}{2}\sum_i \frac{A_i}{n_i}[{}_3F_2(a_i, a_i, -n_i; b_i, 1-n_i; w)w^{-n_i} \\ - {}_3F_2(a_i, a_i, -n_i; b_i, 1-n_i; 1)],$$

The fact that the integral of π should be 1 yields the value of $\pi(\frac{1}{2})$. One gets

$$\pi(y) = 1 + \frac{1}{2}\sum_i \frac{A_i}{n_i}\Big[{}_3F_2(a_i, a_i, -n_i; b_i, 1-n_i; w)w^{-n_i} \\ -\frac{1}{2}\,{}_3F_2\Big(a_i, a_i, -n_i; b_i, \frac{3}{2}-n_i; 1\Big)B\left(1-n_i, \frac{1}{2}\right)\Big].$$

7.3. *Computing P.*

THEOREM D. *Assume that $p=\frac{1}{2}$ and $\lambda>0$. Then, if $\lambda \neq 2\pm\sqrt{2}$, P is a linear combination of three $w^n\,{}_3F_2(w)$ functions.*

PROOF. (28) is equivalent to (23) for $c=\frac{1}{2}\lambda^2$. For generic values of λ (see below for the exceptions), Proposition 19 gives three different values of n, that are $n_0 := 0$, and

$$n_\pm := 1 - \lambda(1\pm 1/\sqrt{2}).$$

If $n = n_0$,

$$a_1 = a_2 = 1-\tfrac{1}{2}\lambda,\ a_3 = \tfrac{3}{2}-\lambda \quad \text{and} \quad b_1,\ b_2 = 2-\lambda(1\pm 1/\sqrt{2}).$$

If $n = n_\pm$, one gets

$$a_1 = a_2 = \lambda(\tfrac{1}{2}\pm\tfrac{1}{\sqrt{2}}),\ a_3 = \tfrac{1}{2}\pm\lambda\tfrac{1}{\sqrt{2}},\ b_1 = \lambda(1\pm\tfrac{1}{\sqrt{2}}) \quad \text{and} \quad b_2 = 1\pm\lambda\sqrt{2}.$$

This defines three functions Q_i solutions of (25) with $i\in\{0,+,-\}$, where

$$Q_i(w) := w^{-n_i}F_i(w),$$

and each F_i is an hypergeometric function of type ${}_3F_2$. Finally, $P(y)$ is a linear combination of the functions $Q_i\circ W$. The degenerate cases are $n_\pm = n_0$, that is,

$$\lambda = 2\pm\sqrt{2}. \qquad \square$$

REMARK 23. Theorem D is the first result in this paper based on Proposition 19, that is on (23), twice differentiated to get (25). Thus, solutions of (25) appear, which are not solutions of (23).

For instance, we suspect that P is a linear combination of Q_+ and Q_-. Besides, Theorem D probably still holds in the degenerate cases, thanks to a continuity principle for P, as a function of λ. We do not pursue this idea here.

7.4. *Computing P_0 and P_1.* From (27), P_0 and P_1 are indefinite integrals of P. These formulas are not very explicit and we now try to compute P_0 and P_1 directly. The cases of complete success are only sporadic.

PROPOSITION 24. *Assume that* $p = \frac{1}{2}$.

(i) *P_0 and P_1 solve third-order differential equations.*

(ii) *If $\lambda = \frac{1}{2}$, P_0' is determined by a linear combination of two Whittaker's functions* (*see a more precise statement in the proof*).

(iii) *If* $\lambda = 1/\sqrt{2}$,

$$\begin{aligned} P_0 &= \mathrm{Beta}(\lambda - \tfrac{1}{2}, \lambda + \tfrac{3}{2}), \\ P &= \mathrm{Beta}(\lambda - \tfrac{1}{2}, \lambda - \tfrac{1}{2}), \\ \pi &= \tfrac{1}{2}\,\mathrm{Beta}(\lambda - \tfrac{1}{2}, \lambda + \tfrac{3}{2}) + \tfrac{1}{2}\,\mathrm{Beta}(\lambda + \tfrac{3}{2}, \lambda - \tfrac{1}{2}). \end{aligned}$$

REMARK 25. Part (iii) follows from McFadden's results since the computation of a Laplace transform yields that, if $\mathscr{D}(\sigma) = \mathrm{Gamma}(\frac{1}{\sqrt{2}}, 2)$,

$$\mathscr{D}(e^{-S}) = \mathrm{Beta}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}, 2\right),$$

and [5] shows that, if $\mathscr{D}(e^{-S}) = \mathscr{D}(e^{-T}) = \mathrm{Beta}(a, b-a)$, then $P_0 = \mathrm{Beta}(a, b)$ and $P_1 = \mathrm{Beta}(a, b)$. If furthermore $b = a + 2$, then $P = \mathrm{Beta}(a, a)$.

Part (iii) follows also from the application of our algorithms, and we omit its proof.

PROOF OF (i). From the computation of its Laplace transform, F is a linear combination of two exponentials. Differentiating twice $P_0 = K_-(F, P_1)$, one gets

$$y^2 P_0''(y) + (3 - 2\lambda) y P_0'(y) + (1 - 2\lambda + \tfrac{1}{2}\lambda^2) P_0(y) = \tfrac{1}{2}\lambda^2 P_1(y). \tag{33}$$

Setting $P_2(y) := y^2\, P_0'(y) + (1 - 2\lambda)\, y\, P_0(y)$, and noticing that, by symmetry, $P_1(y) = P_0(1 - y)$, one gets the fact that

$$P_2'(y) = \tfrac{1}{2}\lambda^2 (P_1(y) - P_0(y))$$

assumes opposite values at y and $1 - y$, hence that P_2 assumes the same value at y and $1 - y$. Considering that P_2 is also

$$P_2(y) = -(1 - y)^{1+2\lambda} \frac{d}{dy}((1 - y)^{1-2\lambda} P_1(y)),$$

and using (33) as an expression of $P_1(y)$ in terms of $P_0(y)$ and its derivatives, one gets after some cumbersome computations that P_0 solves the following third-order equation:

$$y^2(1-y)^2P_0''' + y(1-y)A(y)P_0'' + B(y)P_0' + (1-2\lambda)C(y)P_0 = 0, \tag{34}$$

where $A(y)$ and $C(y)$ are polynomials in y of degree at most 1, and $B(y)$ is a polynomial in y of degree at most 2. More precisely,

$$\begin{aligned} A(y) &= 5 - 2\lambda - (6-4\lambda)y, \\ B(y) &= (5\lambda^2 - 12\lambda + 7)y^2 - (7\lambda^2 - 13\lambda + 9)y + \tfrac{1}{2}\lambda^2 - 4\lambda + 4, \\ C(y) &= \tfrac{1}{2}\lambda^2 - (1-\lambda)^2(1-y). \end{aligned} \qquad \square$$

PROOF OF (ii) (The case $\lambda = \frac{1}{2}$). The P_0 term of (34) cancels out. Then,

$$A(y) = 4(1-y), \qquad B(y) = \tfrac{17}{8} - \tfrac{17}{4}y + \tfrac{9}{4}y^2.$$

Setting $Q(y) := (1-y)^2\, P_0'(1-y)$, one gets

$$y^2Q''(y) + (\tfrac{1}{4}y^2 - \tfrac{1}{4}y + \tfrac{1}{8})Q(y) = 0. \tag{35}$$

We claim that this is a rewriting of Whittaker's equation

$$U''(z) + (-\tfrac{1}{4} + \kappa z^{-1} + (\tfrac{1}{4} - m^2)z^{-2})U(z) = 0.$$

For $m \neq 0$, two independent solutions are the confluent hypergeometric solutions $M_{\kappa,m}$ and $M_{\kappa,-m}$, also called Whittaker's functions (see [1]), defined by

$$M_{\kappa,m}(z) := e^{-z/2}\, z^{m+1/2}\,{}_1F_1(m - \kappa + 1/2, 1 + 2m, z).$$

Assuming that U solves Whittaker's equation, $F(z) := U(iz)$ solves

$$F''(z) + (\tfrac{1}{4} + i\kappa z^{-1} + (\tfrac{1}{4} - m^2)z^{-2})F(z) = 0.$$

This is equivalent to (35) when $\kappa := i/4$ and $m := 1/(2\sqrt{2})$. Finally, two independent solutions of (35) are Q_+ and Q_-, with

$$Q_\pm(y) := \mathrm{Re}(M_{i/4,\pm m}(iy)),$$

and P_0' is a linear combination of

$$Q_+(1-y)/(1-y)^2 \quad \text{and} \quad Q_-(1-y)/(1-y)^2. \qquad \square$$

REMARK 26. Despite the mysterious form of P_0 above, recall that, from Proposition 20, π is easy to write. One sees that, depending on the cases, some of the distributions π or P or P_0 may assume a tractable form, while the others do not.

7.5. *Computing π directly.* Last but not least, one can try to compute π directly, starting from (12). For Gamma intervals, k_- and k_+ are defined by

$$k_-(Q)(y) = \lambda^2 y^{\lambda-1} \int_y^1 Q(x) \log\left(\frac{x}{y}\right) \frac{dx}{x^\lambda},$$
$$k_+(Q)(y) = \lambda^2 (1-y)^{\lambda-1} \int_0^y Q(x) \log\left(\frac{1-x}{1-y}\right) \frac{dx}{(1-x)^\lambda}.$$

Hence, π is the only density of a distribution such that

$$\pi(y) = p\lambda^2 y^{\lambda-1} \int_y^1 \pi(x) \log\left(\frac{x}{y}\right) \frac{dx}{x^\lambda}$$
$$+ q\lambda^2 (1-y)^{\lambda-1} \int_0^y \pi(x) \log\left(\frac{1-x}{1-y}\right) \frac{dx}{(1-x)^\lambda}.$$

PROPOSITION 27. *Assume that $p = \frac{1}{2}$ and $\lambda = 1$. Then, π is a linear combination of three $w^n\,{}_3F_2(w)$ functions.*

PROOF. The prefactors $y^{\lambda-1}$ and $(1-y)^{\lambda-1}$ cancel out and π solves

$$\pi(y) = \frac{1}{2} \int_y^1 \pi(x) \log\left(\frac{x}{y}\right) \frac{dx}{x} + \frac{1}{2} \int_0^y \pi(x) \log\left(\frac{1-x}{1-y}\right) \frac{dx}{1-x}.$$

From here, we proceed as in the proof of Proposition 19. Differentiating yields

$$\pi'(y) = -\frac{1}{2} y^{-1} \int_y^1 \pi(x) \frac{dx}{x} + \frac{1}{2}(1-y)^{-1} \int_0^y \pi(x) \frac{dx}{1-x}.$$

Differentiating $(1-y)\pi'(y)$ cancels the second integral term, leaving a multiple of the integral from y to 1. Therefore, differentiating a multiple of the result yields an equation in π''', π'', π' and π, with no integral term. One sets $\pi =: \rho \circ W$, getting

$$2w^3(1-w)\rho''' + w^2(4-7w)\rho'' - w(1+\tfrac{5}{2}w)\rho' + (1+\tfrac{1}{4}w)\rho = 0. \tag{36}$$

One looks for solutions ρ such that $w^{-n}\rho(w)$ is a series. Three solutions arise: for $n = 1$,

$$\rho_1(w) := {}_3F_2\left(\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}; 2 + \frac{1}{\sqrt{2}}, 2 - \frac{1}{\sqrt{2}}; w\right) w,$$

and, for $n = \pm\frac{1}{\sqrt{2}}$,

$$\rho_\pm(w) := {}_3F_2(n - \tfrac{1}{2}, n + \tfrac{1}{2}, n + \tfrac{1}{2}; n, 1 + 2n; w)\, w^n.$$

Finally, π is a linear combination of $\rho_i \circ W$, for $i \in \{1, +, -\}$. $\square$

8. Gamma RC filters. Pawula and Rice [7] (see also [8]) consider the case

$$\mathscr{D}(S) = \mathscr{D}(T) = \mathrm{Gamma}(\lambda, 2).$$

This is different from the case $\mathscr{D}(\sigma) = \mathscr{D}(\tau) = \mathrm{Gamma}(\lambda, 2)$ studied in Section 7. In fact, such S and T are never the randomizations of some σ and τ in the sense of Section 2 (we omit the proof). In [7], (11) is solved when $\lambda = \frac{1}{2}$. Then, P is proportional to

$$W(x)^{-1/2}\, F(a, b; \tfrac{1}{2}; 1 - W(x)), \tag{37}$$

where $a + b = \frac{1}{2}$ and $ab = \frac{1}{8}$, that is,

$$a := \tfrac{1}{4}(1 - i), \qquad b := \tfrac{1}{4}(1 + i). \tag{38}$$

When $\lambda \neq \frac{1}{2}$, [7] does not solve (11), but provides intricate recurrence formulas for the coefficients of the series expansion of P.

Since (11) is (23) for $c := \lambda^2$, Proposition 19 yields solutions $w^{-n}\,{}_3F_2(w)$ for the exponents

$$n_0 = 0 \quad \text{and} \quad n_- = n_+ = 1 - \lambda.$$

Hence, the indicial equation is degenerate. We now solve the case $\lambda = 1$, when the three exponents coïncide.

PROPOSITION 28. *If $\lambda = 1$, P is a linear combination of F_1 and F_2, with*

$$F_1(y) := F^2(y) + F^2(1 - y) \quad \textit{and} \quad F_2(y) := F(y)F(1 - y),$$

where $F := {}_2F_1(a, b; 1; \cdot)$. Thus, $F^2 = {}_3F_2(2a, 2b, \frac{1}{2}; 1, 1; \cdot)$.

PROOF. From Proposition 19, P solves

$$\begin{aligned} &y^2(1 - y)^2 P''' + 3(1 - 2y)y(1 - y)P'' \\ &\qquad + (8y^2 - 8y + 1)P' - (1 - 2y)P = 0. \end{aligned} \tag{39}$$

This is (20) of Appell's trick in Section 6.1, with

$$A(y) := \frac{1 - 2y}{y(1 - y)} \quad \text{and} \quad B(y) := \frac{-1}{2y(1 - y)}.$$

Then, (21) becomes

$$y(1 - y)V'' + (1 - 2y)V' - \tfrac{1}{2}V = 0, \tag{40}$$

that is, $G_{a,b,1}(V) = 0$. Hence, one can choose

$$F(y) := F(a, b; 1; y),$$

and one line of computation shows that $F(1 - y)$ is also solution of (40), and is independent of $F(y)$ (for instance, because F is convex and increasing). Hence, using the symmetry of P, $P(y)$ is a linear combination of

$$F(y)^2 + F(1 - y)^2 \quad \text{and} \quad F(y)\, F(1 - y). \qquad \square$$

9. Partial results on Beta DF chains. McFadden [5] studies the case where $\mathscr{D}(S) = \mathscr{D}(T) = \mathrm{McFa}(a, b)$ has density

$$e^{-at}(1-e^{-t})^{b-a-1}/B(a, b-a),$$

with $b > a$. Then, $\mathscr{D}(e^{-S}) = \mathrm{Beta}(a, b-a)$. From [5],

$$P_0 = \mathrm{Beta}(a, b) \quad \text{and} \quad P_1 = \mathrm{Beta}(b, a),$$

and P involves incomplete Beta functions. This last result shows that P is a linear combination involving ${}_2F_1$ functions.

While the methods of [5] seem difficult to apply to DF chains, our methods (integro-differential equations) seem difficult to apply to Beta distributions in general. Hence, we consider cases where $R(F)$, defined in Section 3.1, has a simple form. For example, if $b = a+1$, $R(F) = a\,\delta_0$ (this is the case studied in Section 5). If $b = a+2$,

$$R(F)(t) = a(a+1)e^{-(2a+1)t} \tag{41}$$

and, if $b = a+3$,

$$R(F)(t) = 2a(a+1)(a+2)A^{-1}e^{-3(a+1)t/2}\ \sin(At/2),$$

with $A := \sqrt{3a^2+6a-1}$. We focus on the case $b = a+2$.

THEOREM E. *Assume that* $\mathscr{D}(u) = \mathscr{D}(1-v) = \mathrm{Beta}(a, 2)$ *and* $p = \frac{1}{2}$. *Then, if* $a \neq 3$, P *is a linear combination of three functions* $w^n{}_3F_2(w)$.

PROOF. Since $R(f)$ is given by (41) and $R(F) = q\,R(f)$, and since the same applies to G and g with p instead of q, P solves

$$\begin{aligned}\frac{d}{dy}(y(1-y)P(y)) &= qa(a+1)y^{2a}\int_y^1 (1-x)x^{-(2a+1)}P(x)\,dx \\ &\quad - pa(a+1)(1-y)^{2a}\int_0^y x(1-x)^{-(2a+1)}P(x)\,dx.\end{aligned}$$

When $p = \frac{1}{2}$, this is (23), with $c := \frac{1}{2}\,a\,(a+1)$ and $\lambda := a+\frac{1}{2}$. Thus, $c = \frac{1}{2}\,\lambda^2 - \frac{1}{8}$, and the solutions of the indicial equation of Proposition 19 are

$$n = 0, \qquad n = \tfrac{1}{2} - a \pm \tfrac{1}{2}\sqrt{2a^2+2a+1}.$$

When $a \neq 3$, these numbers are all different, and P is a linear combination of the three corresponding functions. □

REMARK 29. *The methods of our paper yield a proof of Theorem* 6.1 *of* [3], *which we sketch below.*

THEOREM 6.1 OF [3]. *Assume that* $p = \frac{1}{2}$, $\mathscr{D}(u) = \mathscr{D}(v) = \mathrm{Beta}(a, a)$ *and* $\pi = \mathrm{Beta}(b, b)$. *Then,* $a = 1$ *and* $b = \frac{1}{2}$, *that is,* $\mathscr{D}(u) = \mathscr{D}(v)$ *is uniform.*

SKETCH OF THE PROOF. Assume that p, u, v and π are as prescribed above, start from (12) and use along the way the fact that

$$\int_0^1 (1-x)^{a-1} x^{b-1} (1-yx)^{-c}\, dx = B(a,b)\,{}_2F_1(b,c;a+b;y).$$

Then, one gets that $G(x) := x^a\, F(x) + (1-x)^a\, F(1-x)$ must be constant, with

$$F(x) := {}_2F_1(2b-a, a; a+b; x).$$

Now, use the fact that F solves Gauss's hypergeometric differential equation to write a second-order differential equation that G solves. Use $x^a F(x) = x^a + O(x^{a+1})$ when $x \to 0$ to compute the x^{a-1} term of this equation.

This term must be zero, that is, $a(a-2b) = 0$. Since $a > 0$, $a = 2b$ and $F = 1$. Finally, the only case where $x^a + (1-x)^a$ is constant is $a = 1$.

REFERENCES

[1] ABRAMOWITZ, M. and STEGUN, I. A. (eds.) (1992). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York. (Reprint of 1972 edition.)

[2] CHAMAYOU, J.-F. and LETAC, G. (1991). Explicit stationary distributions for compositions of random functions and products of random matrices. *J. Theoret. Probab.* **4** 3–36.

[3] DIACONIS, P. and FREEDMAN, D. (1999). Iterated random functions. *SIAM Rev.*, **41** 45–76.

[4] DUBINS, L. E. and FREEDMAN, D. A. (1967). Random distribution functions. *Proc. Fifth Berkeley Symp. Math. Statist. Probab.* **2** 183–214. Univ. California Press, Berkeley.

[5] MCFADDEN, J. A. (1959). The probability density of the output of an RC filter when the input is a binary random process. *IRE Trans.* IT-**5** 174–178.

[6] MUNFORD, A. G. (1986). Moments of a filtered binary process. *IEEE Trans. Inform. Theory* **32** 824–826.

[7] PAWULA, R. F. and RICE, S. O. (1986). On filtered binary processes. *IEEE Trans. Inform. Theory* **32** 63–72.

[8] PAWULA, R. F. and RICE, S. O. (1987). A differential equation related to a random telegraph wave problem. Computer calculation of series solution. *IEEE Trans. Inform. Theory* **33** 882–888.

[9] THORISSON, H. (2000). *Coupling, Stationarity, and Regeneration*. Springer, New York.

[10] WHITTAKER, E. T. and WATSON. G. N. (1996). An introduction to the general theory of infinite processes and of analytic functions; with an account of the principal transcendental functions. *A Course of Modern Analysis*, 4th ed. (Reprint of the 1927 editon.) Cambridge Univ. Press.

LAPCS
UNIVERSITÉ CLAUDE BERNARD LYON-I
DOMAINE DE GERLAND
50, AVENUE TONY-GARNIER
69366 LYON CEDEX 07
FRANCE
E-MAIL: Christian.Mazza@univ-lyon1.fr
Didier.Piau@univ-lyon1.fr