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# REPRODUCING KERNEL HILBERT SPACE METHODS FOR WIDE-SENSE SELF-SIMILAR PROCESSES[1]

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It has recently been observed that wide-sense self-similar processes have a rich linear structure analogous to that of wide-sense stationary processes. In this paper, a reproducing kernel Hilbert space (RKHS) approach is used to characterize this structure. The RKHS associated with a self-similar process on a variety of simple index sets has a straightforward description, provided that the scale-spectrum of the process can be factored. This RKHS description makes use of the Mellin transform and linear self-similar systems in much the same way that Laplace transforms and linear time-invariant systems are used to study stationary processes.

The RKHS results are applied to solve linear problems including projection, polynomial signal detection and polynomial amplitude estimation, for general wide-sense self-similar processes. These solutions are applied specifically to fractional Brownian motion (fBm). Minimum variance unbiased estimators are given for the amplitudes of polynomial trends in fBm, and two new innovations representations for fBm are presented.

**1. Introduction.** This paper is concerned with the linear problems associated with wide-sense self-similar processes, that is, with processes whose first and second moments are essentially scale-invariant. In the linear problems we are referring to, the solution is constrained to be a linear functional of an observed random process, and the criterion of optimality is such that the solution is determined by the first and second moments of the observed process. Examples include linear projection with the mean-square error metric, maximum signal-to-noise ratio signal detection and minimum-variance unbiased linear estimation. For Gaussian processes, these linear solutions are still optimal when the linearity constraint is removed.

The main finding of this paper is that wide-sense self-similar processes have essentially the same structure as wide-sense stationary processes, and that concepts such as autocorrelation and spectral density, used to study stationary processes, have simple analogs useful in the study of self-similar processes. The connection between these two classes was first made by Lamperti (1962), who pointed out a simple invertible transformation that connects any self-similar process with a stationary counterpart, and which is a central idea in this paper. Other recent work making use of Lamperti's transformation

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includes Albin (1998), Burnecki, Maejima and Weron (1997), Wornell (1991), Yazici and Kashyap (1997), Nuzman and Poor (2000) and Nuzman (2000).

Self-similar processes are used to model phenomena in a variety of disciplines. Hurst's investigations of variations of river water levels pioneered the use of such models; other areas of application include burst noise [Mandelbrot (1965)], heart-rate variability [Goldberger and West (1987)], synthetic aperture radar [Stewart, Moghaddam, Hintz and Novak (1993)], packet network traffic [Leland, Taqqu, Willinger and Wilson (1994)] and financial data [Willinger, Taqqu and Teverovsky (1999)]. The most commonly used model is the fractional Brownian motion (fBm). This paper includes general results for wide-sense self-similar processes as well as specific applications to the fBm.

In the next section, the class of wide-sense self-similar processes is defined, and the concept of a reproducing kernel Hilbert space (RKHS) is reviewed. In Section 3, the RKHS formalism is used to describe the special structure of wide-sense self-similar processes. Self-similar systems and scale spectral factorization are seen to be essential elements of this structure. These results are applied in Section 4, which gives general solutions for linear problems associated with self-similar processes. Finally, some of these problems are solved explicitly for fBm in Section 5.

## 2. Background.

2.1. *Wide-sense self-similarity.* A random process $\{Y(t), t > 0\}$ is considered to be *wide-sense self-similar with parameter* $H$ if there is a real $H$ such that the following properties are satisfied:

(i) $\mathbf{E}\{Y(t)^2\} < \infty, \forall\, t > 0$;
(ii) $\mathbf{E}\{Y(t)\} = a^H \mathbf{E}\{Y(t/a)\}, \forall\, a, t > 0$;
(iii) $\mathbf{E}\{Y(t_1)Y(t_2)\} = a^{2H}\mathbf{E}\{Y(t_1/a)Y(t_2/a)\}, \forall\, a, t_1, t_2 > 0$.

The parameter $H$ is often referred to as the *Hurst* parameter. This definition may be contrasted with the definition of (strict) $H$-self-similarity, which is that the processes $\{Y(t)\}$ and $\{a^H Y(t/a)\}$ have the same finite-dimensional distribution for every $a > 0$. The wide-sense definition is more general, in the sense that only the second moments are considered. However, it excludes self-similar processes with infinite second moments, such as non-Gaussian stable processes. Unless otherwise stated, $H$-self-similarity (denoted $H$-ss) will be taken to hold in the wide sense.

Given a wide-sense (resp. strict-sense) $H$-ss process $Y$, it is possible to form a wide-sense (resp. strict-sense) stationary process $X$ via the transformation

$$X(t) = e^{-Ht} Y(e^t).$$

The process $Y$ is premultiplied by $t^{-H}$ to form a 0-ss process, after which a change of variable turns the operation of scaling in time into time shifting. The linear, invertible mapping that takes $\{Y(t), t \in I\}$ to $\{X(t), t \in \ln I\}$

is referred to as *Lamperti's transformation with parameter $H$* and is denoted in the following by $L_H$. The stationary process $L_H Y$ is the *stationary generator* of $Y$.

Lamperti's transformation suggests that techniques developed for stationary processes might be useful in the study of self-similar processes. For example, to predict one self-similar process from another, one could first apply Lamperti's transformation to both processes, hence obtaining the well-studied problem of predicting one stationary process from another. In the next section, we use the reproducing kernel Hilbert space formalism to show how this approach works for general linear problems. Before doing so, we briefly review the RKHS concept.

2.2. *Reproducing kernel Hilbert spaces.* A Hilbert space $S$ of functions on an index set $I$ is called a reproducing kernel Hilbert space if there exists a doubly indexed function $R(t, v)$ on $I \times I$ which satisfies the following conditions:

1. $R(t, \cdot) \in S$ for each $t \in I$;
2. $\langle f, R(t, \cdot)\rangle_S = f(t)$ for each $f \in S$ and $t \in I$.

The function $R$ is called the reproducing kernel of $S$.

The linear space $L^2(Y, I)$ of a finite-variance random process $\{Y(t), t \in I\}$ is the closure under the mean-square norm of the set of all finite linear combinations

$$\sum_{k=1}^{N} a_k Y(t_k), \qquad t_k \in I.$$

This linear space is a Hilbert space, where the inner product between any two elements $Z_1, Z_2 \in L^2(Y, I)$ is given by $\mathrm{Cov}(Z_1, Z_2)$. Strictly speaking, the elements of this Hilbert space are equivalence classes of random variables such that $\mathrm{E}\{(Z_1 - Z_2)^2\} = 0$ for random variables in the same class.

The utility of RKHS's in the study of random processes stems from the fact that the linear space $L^2(Y, I)$ of the process is isomorphic to a deterministic RKHS, denoted $S(Y, I)$, for which the reproducing kernel $R$ is simply the covariance function

$$R(t, v) = \mathbf{E}\{Y(t)Y(v)\} - \mathbf{E}\{Y(t)\}\mathbf{E}\{Y(v)\}.$$

The isomorphism $J$ from $L^2(Y, I)$ to functions in $S(Y, I)$ is given by

$$J(Z) = \mathrm{Cov}(Y(\cdot), Z).$$

The RKHS associated with a random process provides a natural and elegant way to describe and prove results about the linear space of a random process. The answers to various linear problems can be expressed in RKHS terms, so that fully characterizing the RKHS of a process is equivalent to solving all of the associated linear problems. For example, for detection of a deterministic signal $m$ in Gaussian noise $Y$, the problem is nonsingular if and only if $m \in S(Y, I)$, in which case the likelihood ratio is given by $\exp(J^{-1}(m) - \langle m, m\rangle_S)$.

In this and other problems, the essential problems are finding representations for $J^{-1}$ and for the RKHS inner product. A recent review of this approach can be found in Kailath and Poor (1998), and other useful sources include Parzen (1963) and Kailath (1971), and other papers collected in Weinert (1982). In Barton and Poor (1988), an RKHS characterization was used to study signal detection in fractional Brownian motion.

**3. Reproducing kernel Hilbert spaces associated with self-similar processes.** The relationship between the RKHS of a self-similar process and that of its stationary generator is given in the following theorem.

THEOREM 3.1. *Suppose that* $\{Y(t)\}$ *is a second-order process on* $I \subset \mathbb{R}^+$ *and that* $\{X(\tau)\}$ *is the process on* $\ln I$ *defined by* $X = L_H Y$. *Denote by* $J_Y$: $L^2(Y, I) \to S(Y, I)$ *and* $J_X$: $L^2(X, \ln I) \to S(X, \ln I)$ *the RKHS isomorphisms associated with each process. Then* $L^2(Y, I) = L^2(X, \ln I)$ *and, for each* $Z \in L^2(Y, I)$, $J_Y(Z) = L_H^{-1} J_X(Z)$.

As illustrated in Figure 1, the processes $\{Y(t), t \in I\}$ and $\{X(\tau), \tau \in \ln I\}$ lie in a common linear space. If the Hilbert space isomorphism $J_X$ is known, then the corresponding isomorphism for $Y$ is simply $J_Y = L_H^{-1} J_X$. Thus the RKHS of a self-similar process is trivially related to that of its stationary generator.

A proof of this theorem is provided in Appendix A. Note that we need not necessarily restrict ourselves to $I \subset \mathbb{R}^+$. The theorem also holds, with essentially the same proof, for more general index sets, such as $\{1, 2, \ldots, N\} \times \mathbb{R}^+$, that arise in the study of jointly self-similar processes.

The following corollary shows that the RKHS associated with a self-similar process is itself self-similar in a sense.

COROLLARY 3.2. *Suppose that* $Y$ *is an* $H$*-ss process on* $\mathbb{R}^+$ *and that* $g \in S(Y, I)$ *for some* $I \subset \mathbb{R}^+$. *Then, for each* $\nu \in \mathbb{R}^+$, *the rescaled function*

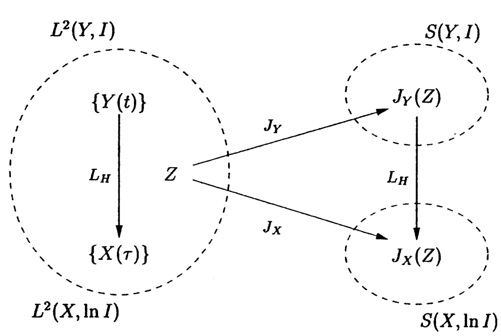


FIG. 1. *Relationships between the spaces in Theorem* 3.1.

*$g_\nu(t) = \nu^H g(t/\nu)$ is in $S(Y, \nu I)$ and*

$$\|g_\nu\|_{Y,\nu I} = \|g\|_{Y,I}.$$

PROOF. This result follows from a shift-invariance property for the RKHS associated with a stationary process. The scaled function $g_\nu$ is chosen so that

$$L_H g_\nu|_t = L_H g|_{t-\ln\nu}$$

is just a shifted version of $L_H g$. Then

$$\|g_\nu\|_{Y,\nu I} = \|L_H g_\nu\|_{X,\ln I+\ln\nu} = \|L_H g\|_{X,\ln I} = \|g\|_{Y,I}. \qquad \square$$

3.1. *Definitions.* To develop the consequences of Theorem 3.1 for the structure of self-similar processes, it is helpful to define analogs of some of the usual stationary tools. Several of these definitions are due to Yazici and Kashyap (1997). For an $H$-self-similar process $Y$ on $\mathbb{R}^+$ with covariance function $R$, the *scale-autocorrelation* is

$$\rho(t) \overset{\Delta}{=} t^{-H} R(t, 1) = (tv^2)^{-H} R(tv, v), \qquad t > 0.$$

For self-similar processes, the Mellin transform plays a role analogous to the role of the Laplace transform for stationary processes. The Mellin transform of a function $f$ will be denoted $\tilde{f}$ and is defined by

$$\tilde{f}(s) = \int_0^\infty f(t) t^{-s-1}\, dt.$$

Although the sign of $s$ in the integrand is positive in the usual definition of the Mellin transform, we use the opposite convention to simplify the connection with the Laplace transform, namely, that the Mellin transform of $f$ is the Laplace transform of $L_0 f$. The *scale-spectrum* of $Y$ is the Mellin transform $\tilde{\rho}$ of the scale-autocorrelation, that is, the Laplace spectrum of the stationary generator of $Y$.

We say that an $H_1$-ss process $Y_1$ and $H_2$-ss process $Y_2$ are *jointly* self-similar if $R_{12}(t, v) = \operatorname{Cov}(Y_1(t), Y_2(v))$ satisfies

$$R_{12}(t, v) = a^{H_1+H_2} R_{12}(t/a, v/a), \qquad t, v, a > 0,$$

in which case the scale-cross-correlation is

$$\rho_{12}(t) = t^{-H_1} R_{12}(t, 1) = t^{H_2} R_{12}(1, 1/t) = \rho_{21}(1/t).$$

A concept of spectral factorization can be defined for self-similar processes, analogous to the usual stationary concept [see, e.g., Wong (1971)]. If the scale-spectrum satisfies the Paley–Wiener condition

$$\int_{-\infty}^{\infty} \frac{|\log \tilde{\rho}(i\omega)|}{1+\omega^2}\, d\omega < \infty, \tag{1}$$

then it admits a factorization $\tilde{\rho}(s) = \tilde{\rho}^+(s)\tilde{\rho}^+(-s)$, where the inverse transform $\rho^+(t)$ is supported on $[1, \infty)$. Functions supported on $[1, \infty)$ will be referred to as *Mellin-causal*, because *scale-convolution* with such a function is a causal operation. The scale-convolution operator $\odot$ is defined by

$$(f \odot g)(t) = \int_0^\infty f(t/v) g(v) v^{-1}\, dv,$$

and the Mellin transform of $f \odot g$ is simply $\tilde{f}\tilde{g}$.

Finally, we introduce the concept of an $(\alpha, \beta)$-ss system. For a *scale-invariant system* $\Phi$, the input–output relationship $y(t) = \Phi[x(t)]$ implies that $y(at) = \Phi[x(at)]$ for all $a > 0$. Note that the scale-convolution implements such a system. Complex powers of the form $t^s$ are eigenfunctions of such systems, and the corresponding eigenvalues form the *frequency response* $\tilde{f}(s) = t^{-s}\Phi[t^s]$. An $(\alpha, \beta)$-ss system is then defined by the relationship $y(t) = t^\beta \Phi[t^{-\alpha} x(t)]$. The frequency response of an $(\alpha, \beta)$-ss system is defined to be that of the underlying scale-invariant system. In the Mellin domain, the input–output relationship is

$$\tilde{y}(s+\beta) = \tilde{f}(s)\tilde{x}(s+\alpha).$$

The motivation for this construction is that feeding an $\alpha$-ss process $X$ into such a system produces a $\beta$-ss process $Y$ with scale-spectrum

$$\tilde{\rho}_Y(s) = \tilde{f}(s)\tilde{f}(-s)\tilde{\rho}_X(s).$$

The $H$-ss systems defined in Wornell (1991) and Yazici and Kashyap (1997) correspond in our notation to $(-H, 0)$-ss and $(0, H)$-ss systems, respectively.

3.2. *Reproducing kernel Hilbert space structure on* $(0, T]$. The RKHS of any stationary process on a semiinfinite interval such as $(-\infty, \ln T]$ has a simple description, provided that the spectrum satisfies the Paley–Wiener condition. In view of Theorem 3.1, the RKHS of an $H$-ss process on $(0, T]$ or $[T, \infty)$ can also be simply described. For specificity, we focus on the former index set.

THEOREM 3.3. *Suppose that $Y$ is an $H$-ss process on $\mathbb{R}^+$ and that its scale-spectrum has a Mellin-causal factorization $\tilde{\rho}(s) = \tilde{\rho}^+(s)\tilde{\rho}^+(-s)$. Then the RKHS $S(Y, (0, T])$ consists of all functions of the form*

$$g(t) = t^H \int_0^t \rho^+(t/v) v^{1/2} w_g(v)\, dv/v, \tag{2}$$

*where $w_g$ is square integrable on $(0, T]$. For any $f, g \in S(Y, (0, T])$ the inner product is given by*

$$\langle f, g\rangle_{Y,(0,T]} = \int_0^T w_f(t) w_g(t)\, dt. \tag{3}$$

*There exists a process $W$ on $\mathbb{R}^+$ with the second-order structure of Brownian motion such that*

$$Y(t) = t^H \int_0^t \rho^+(t/v) v^{-1/2}\, dW(v) \tag{4}$$

*and*

$$J^{-1}(g) = \int_0^T w_g(v)\, dW(v). \tag{5}$$

PROOF. This theorem can be derived by applying Theorem 3.1 to an appropriate RKHS representation for stationary processes. Once the theorem is stated, however, it can be proved directly from the RKHS integral representation theorem given, for example, in Kailath (1971) and Parzen (1963). The covariance function of $R$ can be decomposed as

$$\begin{aligned} R(t,u) &= t^H u^H \rho(t/u) \\ &= t^H u^H \int_0^\infty \rho^+(t/v)\rho^+(u/v)\, dv/v \\ &= \int_0^\infty f_t(v) f_u(v)\, dv, \end{aligned}$$

where $f_t(v) = t^H \rho^+(t/v) v^{-1/2}$. Having noted this integral representation, the theorem follows directly. □

As Figure 2 illustrates, Theorem 3.3 has a simple interpretation in terms of $(\alpha, \beta)$-ss systems. The functions in $S(Y, (0, T])$ are obtained by passing square-integrable functions through a $(-1/2, H)$-ss system with impulse response $\tilde{\rho}^+(s)$. The process itself can be produced by passing white noise (the increments of $W$) through the same system, or equivalently, by feeding a Brownian motion $W$ through a $(1/2, H)$-ss system with response $(s + 1/2)\tilde{\rho}^+(s)$.

In the above theorem, the random variable $J^{-1}(g)$ is expressed in terms of $w_g$ and $W$, although linear problems are typically framed in terms of $g$ and $Y$. It is easy to see that the inverse of an $(\alpha, \beta)$-ss system with frequency response $\tilde{f}(s)$ is a $(\beta, \alpha)$-ss system with response $1/\tilde{f}(s)$, so that $w_g$ and $W$ can be obtained from $g$ and $Y$ using self-similar systems. The following theorem uses this idea to define a self-similar system for generating $J^{-1}(g)$ directly

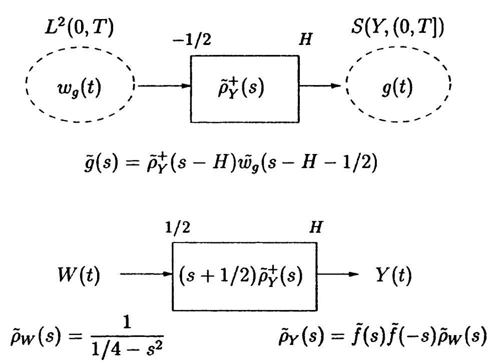


FIG. 2. *Self-similar system interpretation for Theorem* 3.3.

from $Y$. In the following, the notation $[\tilde{f}(s)]_{a+}$ indicates the Mellin transform of the truncation $f(t)1_{t>a}(t)$ of a function $f$.

THEOREM 3.4. *Suppose that $Y$ is $H$-ss on $(0, T]$ and that the scale-spectrum $\tilde{\rho}$ has no spectral nulls and admits a Mellin-causal spectral factorization. Suppose that a given function $g \in S(Y, (0, T])$ has Mellin transform $\tilde{g}$, and let $Z_g$ be the 0-ss process obtained by passing $Y$ through an $(H, 0)$-ss system with frequency response*

$$\frac{1}{\tilde{\rho}^+(s)}\left[T^{-s}\frac{\tilde{g}(H-s)}{\tilde{\rho}^+(-s)}\right]_{1+}.$$

*Then the inverse mapping $J^{-1}$ can be expressed as*

$$J^{-1}(g) = Z_g(T).$$

PROOF. Since the system mapping $w_g$ to $g(t)$ is causal, we can set $w_g(t) = 0$ for $t > T$ without loss of generality. We define $m_{T,g}(t) = (T/t)^{1/2} w_g(T/t)$ and define $Z_g$ as the result of passing $W$ through a $(1/2, 0)$-ss system

$$\begin{aligned} Z_g(t) &= \int_0^\infty m_{T,g}(t/\nu)\nu^{-1/2}\,dW(\nu) \\ &= \int_0^t (T/t)^{1/2} w_g(T\nu/t)\,dW(\nu) \end{aligned}$$

with impulse response $m_{T,g}(t)$. At time $T$, we get

$$Z_g(T) = \int_0^T w_g(\nu)\,dW(\nu) = J^{-1}(g).$$

To represent $Z_g$ in terms of $Y$, we use the $(H, 1/2)$-ss whitening filter to produce $W$ from $Y$, then the $(1/2, 0)$-ss system to produce $Z_g$ from $W$. The

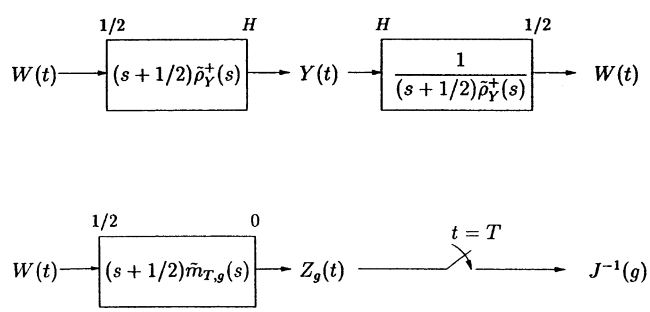


FIG. 3. *Self-similar systems in Theorem* 3.4.

concatenation of the two systems, illustrated in Figure 3, is an $(H, 0)$-ss system with causal frequency response

$$\begin{aligned}\frac{\tilde{m}_{T,g}(s)}{\tilde{\rho}^+(s)} &= \frac{T^{-s}\tilde{w}_g(-1/2-s)}{\tilde{\rho}^+(s)}\\ &= \frac{T^{-s}}{\tilde{\rho}^+(s)}\left[\frac{\tilde{g}(H-s)}{\tilde{\rho}^+(-s)}\right]_{1/T+}\\ &= \frac{1}{\tilde{\rho}^+(s)}\left[T^{-s}\frac{\tilde{g}(H-s)}{\tilde{\rho}^+(-s)}\right]_{1+},\end{aligned}$$

which completes the proof. □

A simpler expression for $J^{-1}$ can be obtained in the limit $T \to \infty$. To see this, let $X_g(t) = Z_g(Tt)$ so that $J^{-1}(g) = X_g(1)$. Then $J^{-1}(g)$ is obtained by feeding $Y$ into an $(H, 0)$-ss system with response

$$\frac{1}{\tilde{\rho}^+(s)}\left[\frac{\tilde{g}(H-s)}{\tilde{\rho}^+(-s)}\right]_{1/T+},$$

and sampling the output at time $t = 1$. Taking the limit as $T \to \infty$ of the frequency response, the truncation disappears, and it follows that, for $g \in S(Y, \mathbb{R}^+)$, the inverse mapping is $J^{-1}(g) = X_g(1)$, where $X$ is produced using the simple frequency response $\tilde{g}(H-s)/\tilde{\rho}(s)$.

3.3. *Reproducing kernel Hilbert space structure of jointly self-similar processes.* In this section, we consider sets of $N$ jointly $H$-ss processes. Here, the concept of matrix spectral factorization can be used to describe the RKHS in terms of a matrix of self-similar systems. In matrix notation, the results below are very similar to those of the previous section. However, a matrix spectral factorization is generally much harder to perform than a one-dimensional factorization. Useful techniques for $2 \times 2$ spectral factorization are discussed later in this section.

If $\{\mathbf{Y}(t), t > 0\}$ is an $N$-vector of jointly self-similar processes, its covariance matrix function satisfies

$$\begin{aligned}\mathbf{R}(t, v) &\stackrel{\Delta}{=} \mathbf{E}\{\mathbf{Y}(t)\mathbf{Y}(v)^\mathsf{T}\}\\ &= a^{2H}\mathbf{R}(t/a, v/a)\\ &= v^H t^H \boldsymbol{\rho}(t/v),\end{aligned}$$

where $\boldsymbol{\rho}$ is the scale-autocorrelation matrix of $\mathbf{Y}$. Taking the Mellin transform of each element of this matrix yields the spectral matrix $\tilde{\boldsymbol{\rho}}$. By a causal matrix spectral factorization, we mean a factorization of the form

$$\tilde{\boldsymbol{\rho}}(s) = \tilde{\boldsymbol{\rho}}^+(s)\tilde{\boldsymbol{\rho}}^+(-s)^\mathsf{T}$$

in which every element of $\boldsymbol{\rho}^+(t)$ is Mellin causal.

THEOREM 3.5. *Suppose that* $\{\mathbf{Y}(t), t > 0\}$ *is a vector of* $N$ *jointly* $H$*-ss processes with scale-spectral matrix* $\tilde{\boldsymbol{\rho}}$*. If the spectrum has a completely causal factorization*

$$\tilde{\boldsymbol{\rho}}(s) = \tilde{\boldsymbol{\rho}}^{+}(s)\tilde{\boldsymbol{\rho}}^{+}(-s)^{\mathsf{T}},$$

*then the RKHS* $S(\mathbf{Y}, (0, T])$ *consists of all vector functions of the form*

$$\mathbf{g}(t) = t^H \int_0^t \boldsymbol{\rho}^{+}(t/v)\mathbf{w}_{\mathbf{g}}(v)v^{-1/2}\,dv,$$

*where* $\mathbf{w}_{\mathbf{g}}$ *is an* $N$*-vector of square-integrable functions on* $(0, T]$*. Further, for any* $\mathbf{f}, \mathbf{g} \in S(\mathbf{Y}, \mathbb{R}^{+})$*, the inner product is given by*

$$\langle \mathbf{f}, \mathbf{g} \rangle_S = \int_0^T \mathbf{w}_{\mathbf{f}}(t)^{\mathsf{T}}\mathbf{w}_{\mathbf{g}}(t)\,dt.$$

*There exists a vector* $\mathbf{W}$ *of* $N$ *mutually uncorrelated processes with the second-order structure of Brownian motion such that*

$$\mathbf{Y}(t) = t^H \int_0^t \boldsymbol{\rho}^{+}(t/v)v^{-1/2}\,d\mathbf{W}(v)$$

*and such that*

$$J^{-1}(\mathbf{g}) = \int_0^T \mathbf{w}_{\mathbf{g}}(v)^{\mathsf{T}}\,d\mathbf{W}(v).$$

PROOF. By virtue of the matrix spectral factorization, the covariance matrix has the integral representation

$$\begin{aligned}\mathbf{R}(t, u) &= t^H u^H \boldsymbol{\rho}(t/u)\\ &= t^H u^H \int_0^\infty \boldsymbol{\rho}^{+}(t/v)\boldsymbol{\rho}^{+}(u/v)^{\mathsf{T}}\,dv/v.\end{aligned}$$

The result then follows directly from a matrix version of the integral representation theorem, given in Nuzman (2000). The matrix version is closely related to a characterization of jointly stationary processes in Kailath (1971). □

If the elements of the spectral matrix are not all causal, we use the notation $\tilde{\boldsymbol{\rho}}^{o}$ instead of $\tilde{\boldsymbol{\rho}}^{+}$. In this case, the above theorem still holds, except that the index set of interest becomes $\mathbb{R}^{+}$ rather than $(0, T]$, and the upper limit of each of the integrals is infinite. If the factorization is partially causal, as in the second example below, it may be possible to restrict some components of $\mathbf{Y}$ to $(0, T]$.

EXAMPLE 3.1 (Self-similar process on $[-T, T]$). An $H$-ss process $Y$ defined on the entire real line can be thought of as a vector of two jointly $H$-ss processes on $\mathbb{R}^{+}$, with components $\{Y_1(t)\} = \{Y(t)\}$ and $\{Y_2(t)\} = \{Y(-t)\}$. If $Y$

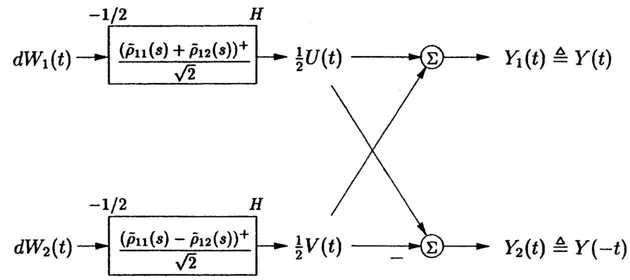


FIG. 4. *Symmetric self-similar process representation.*

is symmetric, in the sense that $R(t,u) = R(-t,-u)$ for all $t, u \in \mathbb{R}$, then the spectral matrix $\tilde{\boldsymbol{\rho}}$ can be written $\tilde{\boldsymbol{\rho}}(s) = \mathbf{A}\widetilde{\mathbf{D}}(s)\mathbf{A}^{\mathsf{T}}$, where

$$\mathbf{A} = 2^{-1/2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \widetilde{\mathbf{D}}(s) = \begin{bmatrix} \tilde{\boldsymbol{\rho}}_{11}(s) + \tilde{\rho}_{12}(s) & 0 \\ 0 & \tilde{\rho}_{11}(s) - \tilde{\rho}_{12}(s) \end{bmatrix}.$$

If the diagonal elements of $\widetilde{\mathbf{D}}$ can be individually causally factored to form $\widetilde{\mathbf{D}}^{+}$, then one has

$$\tilde{\boldsymbol{\rho}}^{+}(s) = \mathbf{A}\widetilde{\mathbf{D}}^{+}(s).$$

In this case, the characterization of $\mathbf{Y}$ on $(0, T]$ given by Theorem 3.5 is really a characterization of $Y$ on $[-T, T]$. The corresponding innovations representation for $Y$ is illustrated in Figure 4. There, independent white noise processes are fed into self-similar systems to produce $U(t) = Y(t) + Y(-t)$ and $V(t) = Y(t) - Y(-t)$, which are then combined to form $Y$ itself.

EXAMPLE 3.2 (Self-similar processes on $(-\infty, T]$). Another $2 \times 2$ spectral factorization is based on Cholesky matrix factorization. Suppose that $Y_2$ has a (possibly noncausal) spectral factorization $\tilde{\rho}_{22}^{o}$ and that the function

$$\tilde{f}(s) = \tilde{\rho}_{11}(s) - \frac{\tilde{\rho}_{12}(s)\tilde{\rho}_{12}(-s)}{\tilde{\rho}_{22}(s)}$$

has a causal spectral factorization $\tilde{f}(s) = \tilde{f}^{+}(s)\tilde{f}^{+}(-s)$. Then $\tilde{\rho}$ has a partially causal factorization

$$\tilde{\rho}^{o}(s) = \begin{bmatrix} \tilde{f}^{+}(s) & \dfrac{\tilde{\rho}_{12}(s)}{\tilde{\rho}_{22}^{o}(-s)} \\ 0 & \tilde{\rho}_{22}^{o}(s) \end{bmatrix}$$

which can be used to characterize $\mathbf{Y}$ on $\mathbb{R}^{+}$ and $Y$ on $\mathbb{R}$. The resulting innovation representation for $Y$ is depicted in Figure 5. There are three self-similar systems, corresponding to the three nonzero elements of $\tilde{\boldsymbol{\rho}}^{o}$. Likewise, the whitening filter for $Y$ is composed of three self-similar systems whose frequency responses are the nonzero elements of $(\tilde{\rho}^{o})^{-1}$. Because the systems

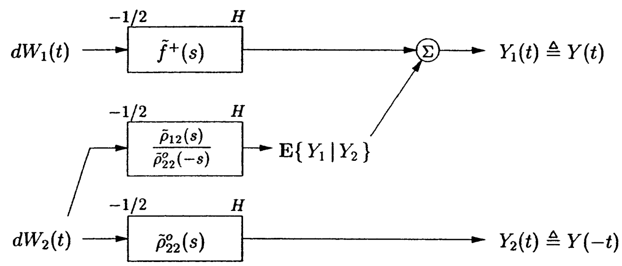


FIG. 5. *Asymmetric (Cholesky) self-similar process representation.*

fed by $W_1$ and $Y_1$ are causal, the index set for those two processes can be restricted to $(0, T]$. Hence, the above structure can characterize the RKHS of $Y$ on $(-\infty, T]$. Likewise, it is straightforward to obtain representations for index sets such as $\mathbb{R}\backslash(0, T)$ via triangular factorizations.

**4. Solutions to linear problems.** For many linear problems, the general solutions have been formulated in RKHS terms. The RKHS characterizations of the previous section then provide explicit solutions for self-similar processes. In this section, we examine a few specific applications of this approach in which special structure is apparent.

4.1. *Projection.* In the first application, the goal is to project each element of an $H'$-ss process $Z$ onto the linear space of an $H$-ss process $Y$. We assume that $Z$ and $Y$ are jointly self-similar. In general, the projection can be expressed as

$$\widehat{Z}(a) = J^{-1}(\mathbf{E}\{Z(a)Y(t)\}).$$

If observations of $Y$ are available on the entire real line, then the discussion in the last paragraph of Section 3.2 shows that $\widehat{Z}(a)$ is the output at time $t = 1$ of an $(H, 0)$-ss system with frequency response $\tilde{g}(H - s)/\tilde{\rho}_Y(s)$, where

$$g(t) = a^{H'} t^H \rho_{ZY}(t/a).$$

Equivalently, $\widehat{Z}(a)$ is the output at time $t = a$ of an $(H, H')$-ss system with frequency response $\tilde{\rho}_{ZY}(s)/\tilde{\rho}_Y(s)$. Hence $\{\widehat{Z}(a), a > 0\}$ and the error process $\{Z(a) - \widehat{Z}(a), a > 0\}$ are themselves $H'$-ss. The variance of either process can be obtained from its scale-spectrum, using the identity

$$\operatorname{Var}(Z(t)) = t^{2H}\rho_Z(1) = \frac{t^{2H}}{2\pi}\int_{\infty}^{\infty} \tilde{\rho}_Z(i\omega)\, d\omega.$$

In a second scenario, suppose that the projection of $Z(cT)$ onto $L^2(Y, (0, T])$ is desired for some fixed $c$ as $T$ varies. Theorem 3.4 can be applied to show that $\widehat{Z}(cT)$ is the output at time $T$ of a causal $(H, H')$-ss system which is fed by $Y$ and which has frequency response

$$\frac{c^{H'}}{\tilde{\rho}_Y^+(s)}\left[c^s \frac{\tilde{\rho}_{ZY}(s)}{\tilde{\rho}_Y^+(-s)}\right]_{1+}.$$

4.2. *Signal detection and amplitude estimation.* The detectability of a deterministic signal $m(t)$ embedded in a Gaussian process $\{Y(t)\}$ is an increasing function of the RKHS norm of the signal, and the detection problem is singular if $m$ is not an element of $S(Y, I)$. The likelihood ratio for detection is given by

$$\mathscr{L} = \exp\Big(J^{-1}(m) - 1/2\|m\|_S^2\Big).$$

Whether or not $Y$ is Gaussian, $J^{-1}(m)$ maximizes the signal-to-noise ratio

$$\frac{(\mathrm{E}_1\{f(Y)\} - \mathrm{E}_0\{f(Y)\})^2}{\mathrm{Var}(f(Y))}$$

over all linear functionals $f(Y)$ of the observed process, where $\mathrm{E}_1\{\cdot\}$ and $\mathrm{E}_0\{\cdot\}$, respectively, denote expectation with and without the signal present. The maximum signal-to-noise ratio is $\|m\|_S^2$. If $Y$ is $H$-ss and if the index set is $(0, T]$, problems of Gaussian or maximum signal-to-noise ratio signal detection are solved by Theorems 3.3 and 3.4.

If $m$ is of the special form $t^a$ for some $a \in \mathbb{R}$, these computations become straightforward. In general, the function $w_m$ is obtained by passing $m$ through an $(H, -1/2)$-ss system with response $1/\tilde{\rho}^+(s)$. Because the signal $t^a$ is an eigenfunction of the underlying linear scale-invariant system, we have

$$w_m(t) = \frac{t^{a-H-1/2}}{\tilde{\rho}^+(a-H)} 1_{0,T}(t)$$

so that, if $a > H$,

$$\|t^a\|_{S(Y,(0,T])}^2 = \|w_m\|_{L^2}^2 = \frac{T^{2a-2H}}{(2a-2H)\tilde{\rho}^+(a-H)^2}.$$

In general, a causal spectral factor $\tilde{\rho}^+(s)$ has no poles or zeros for $\mathrm{Re}(s) > 0$ [see, e.g., Wong (1971)], so that detection of a signal $t^a$ is non-singular if and only if $a > H$.

In the proof of Theorem 3.4, it is shown that the frequency response for the $(H, 0)$-ss system mapping $Y$ to $Z_m$ can be expressed

$$\frac{T^{-s}\bar{w}_m(-1/2-s)}{\tilde{\rho}^+(s)}.$$

Inserting the explicit expression for $w_m$ in the case of $m(t) = t^a$, the frequency response becomes

$$\frac{T^{a-H}}{\tilde{\rho}^+(a-H)(a-H+s)\tilde{\rho}^+(s)}. \tag{6}$$

Equivalently, $J_T^{-1}(t^a)$ is the output at time $T$ of an $(H, a-H)$-ss system whose frequency response is obtained by omitting the numerator term from (6). Then $J_T^{-1}(t^a)$ is an $(a-H)$-ss process in $T$, with scale-spectrum

$$\frac{1}{\tilde{\rho}^+(a-H)^2((a-H)^2-s^2)}.$$

Given that the signal is of the form $t^a$, the spectrum of $Y$ affects only the amplitude of the spectrum of $J_T^{-1}(t^a)$.

Closely related to signal detection is the problem of estimating the amplitude of a signal $\alpha m(t)$ which is observed in additive noise. In general, the minimum-variance unbiased linear estimate $\hat{\alpha}$ is $J^{-1}(m)/\langle m, m\rangle_S$, which has variance $\langle m, m\rangle_S^{-1}$. In the case of a polynomial signal in $H$-ss noise, with observations restricted to $(0, T]$, the estimate $\hat{\alpha}_T$ is obtained by passing $Y$ through an $(H, H-a)$-ss system with frequency response

$$\frac{(2a-2H)\tilde{\rho}^+(a-H)}{(a-H+s)\,\tilde{\rho}^+(s)},$$

essentially the same response used for detection. Because $\hat{\alpha}_T$ is $(H-a)$-ss, its variance decreases as $T^{2(H-a)}$.

If the sum of $N$ signals $m_i$ are observed in noise, then the minimum-variance unbiased linear estimate of the vector of amplitudes can be expressed in terms of RKHS quantities. In particular, suppose that $\mathbf{B}$ is a matrix with $B_{ij} = \langle m_i, m_j\rangle_S$, that $\boldsymbol{\nu}$ is a vector with $\nu_i = J^{-1}m_i$ and that the unknown amplitude of the $i$th signal is $\alpha_i$. Hájek (1962) states that the optimal estimate is

$$\hat{\boldsymbol{\alpha}} = \mathbf{B}^{-1}\boldsymbol{\nu},$$

and the covariance matrix of $\hat{\boldsymbol{\alpha}}$ is $\mathbf{B}^{-1}$. For the sum of two power-law signals $\alpha_1 t^{a_1} + \alpha_2 t^{a_2}$, the covariance matrix is

$$\mathbf{B}^{-1} = \frac{(a_1+a_2-2H)^2}{(a_1-a_2)^2}\boldsymbol{\Lambda}\begin{bmatrix} 2a_1-2H & -\dfrac{(2a_1-2H)(2a_2-2H)}{a_1+a_2-2H} \\ -\dfrac{(2a_1-2H)(2a_2-2H)}{a_1+a_2-2H} & 2a_2-2H \end{bmatrix}\boldsymbol{\Lambda},$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix with entries $\boldsymbol{\Lambda}_i = T^{H-a_i}\tilde{\rho}^+(a_i-H)$. Comparing with the previous results for $N=1$, we see that making the estimator for $\alpha_1$ insensitive to $\alpha_2$ always increases the variance of the estimator by the multiplicative factor $(a_1+a_2-2H)^2/(a_1-a_2)^2$.

**5. Applications to fractional Brownian motion.** One of the most widely used self-similar processes is the fractional Brownian motion, the class of covariance-continuous $H$-ss Gaussian processes with stationary increments. For any given $0 < H < 1$, the fBm has covariance

$$R(t,u) = \sigma^2(|t|^{2H} + |u|^{2H} - |t-u|^{2H})/2,$$

and the case $H = 1/2$ reduces to ordinary Brownian motion. In this section, we apply the general results of previous sections to problems involving the fBm. An up-to-date introduction to fBm may be found in Samorodnitsky and Taqqu [(1994), Chapter 7].

To solve problems associated with the fBm, it is sufficient to be able to compute the spectral factorization or matrix spectral factorization, and to be able to take the resulting inverse Mellin transforms. Some representative results obtained by this approach are given in the following sections. Some of the Mellin transforms used in these computations are collected in Appendix B.

5.1. *Fractional Brownian motion on* $(0, T]$. The RKHS of fractional Brownian motion on index sets such as $\mathbb{R}^+$ and $(0, T]$, along with associated innovations and whitening filters, was described in Barton and Poor (1988), Decreusefond and Üstünel (1999) and Molčan and Golosov (1969). In Nuzman and Poor (2000), scale-spectral factorization was used to confirm these results and to simplify some of the filter representations. Prediction and interpolation of fBm were also studied in this latter work.

A problem that seems not to have been studied previously is polynomial amplitude estimation on $(0, T]$. One useful example is detrending, in which the amplitude of a linear trend embedded in fBm is to be estimated. For an fBm for which the variance at unit time is normalized to unity, the spectral factorization is

$$\tilde{\rho}^+(s) = \sqrt{\sin(\pi H)\Gamma(2H+1)}\frac{\Gamma(1-H+s)}{\Gamma(1/2+s)(H+s)}.$$

In view of Section 4.2, the minimum variance of an unbiased estimate of the amplitude of a linear trend, based on observations on $(0, T]$, is

$$\begin{aligned}\operatorname{Var}(\hat{\alpha}_T) &= T^{2H-2}(2-2H)\tilde{\rho}^+(1-H)^2 \\ &= T^{2H-2}\frac{(2-2H)\sin(\pi H)\Gamma(2H+1)\Gamma(2-2H)^2}{\Gamma(3/2-H)^2} \\ &= T^{2H-2}\,2^{3-4H}\frac{\Gamma(2H+1)\Gamma(2-H)}{\Gamma(H)}.\end{aligned}$$

The solid curve in Figure 6 plots the normalized variance $T^{2-2H}\operatorname{Var}(\hat{\alpha}_T)$ as a function of $H$. It is interesting to note that the normalized variance is very close to unity for $1/2 \le H \le 1$ and is equal to 1 at the endpoints of that interval. Hence the simple unbiased estimate

$$\hat{\alpha}_T = \frac{Y(T)}{T},$$

with variance $T^{2H-2}$, is nearly optimal for Hurst parameters in this range.

The optimal estimate is obtained by passing $Y$ through an $(H, H-1)$-ss system with response

$$(H+s)\frac{\Gamma(3-2H)\Gamma(1/2+s)}{\Gamma(3/2-H)\Gamma(2-H+s)},$$

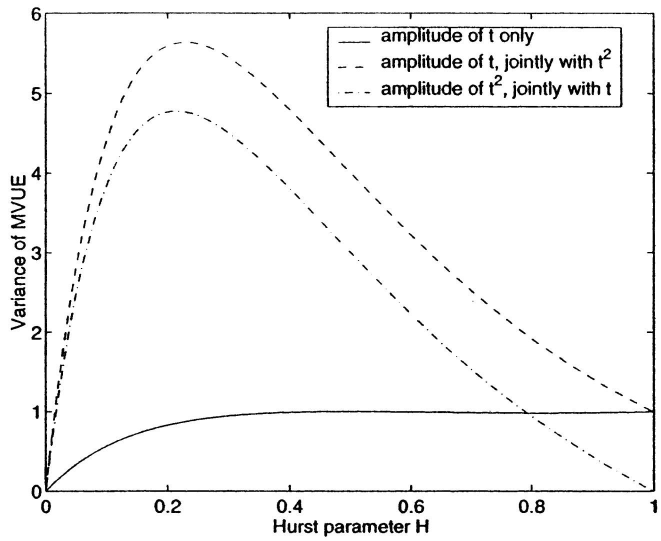


FIG. 6. *Minimum variance obtainable by unbiased estimates of linear and quadratic signals in fBm, as a function of Hurst parameter $H$. The fBm is observed on* $(0, 1]$ *and normalized to have unit variance at time* 1.

which can be expressed

$$\hat{\alpha}_1 = \frac{\int_0^T (T-\nu)^{1/2-H}\nu^{1/2-H}\,dY(\nu)}{\int_0^T (T-\nu)^{1/2-H}\nu^{1/2-H}\,d\nu}$$

for $1/2 < H < 1$.

If we seek to jointly estimate the coefficients of a trend of the form $\alpha_1 t + \alpha_2 t^2$, the minimum variance of the linear amplitude is multiplied by the factor $(3-2H)^2$, and the minimum variance of the quadratic term is

$$\mathrm{Var}(\hat{\alpha}_2) = T^{2H-4}2^{5-4H}\frac{(1-H)\Gamma(3-H)\Gamma(2H+1)}{\Gamma(H)}.$$

The normalized variances $T^{2i-2H}\,\mathrm{Var}(\hat{\alpha}_i)$ for $i = (1, 2)$ are plotted as functions of $H$ in Figure 6.

5.2. *Fractional Brownian motion on extended index sets.* Matrix spectral factorization can be used to characterize the RKHS of fBm on extended index sets such as $(-\infty, T]$ and $[-T, T]$. In this section we illustrate this technique by showing how the standard representation given in Mandelbrot and Van Ness (1968) fits into this framework, and by giving two new innovations representations. The matrix factorizations given below can also be used to solve other problems such as prediction and amplitude estimation.

Because of the symmetry of fBm about the time origin, its spectral matrix is of the form

$$\tilde{\rho}(s) = \begin{bmatrix} \tilde{\rho}_{11}(s) & \tilde{\rho}_{12}(s) \\ \tilde{\rho}_{12}(s) & \tilde{\rho}_{11}(s) \end{bmatrix}.$$

The matrix elements, derived in Nuzman and Poor (2000), are

$$\tilde{\rho}_{11}(s) = \frac{\Gamma(1-H+s)\Gamma(1-H-s)}{\Gamma(1/2+s)\Gamma(1/2-s)(H^2-s^2)}$$

and

$$\tilde{\rho}_{12}(s) = \frac{\cos(\pi H)\Gamma(1-H+s)\Gamma(1-H-s)}{\pi(H^2-s^2)}.$$

One possible factorization of this matrix is

$$\tilde{\rho}^o(s) = \begin{bmatrix} \dfrac{\Gamma(1/2+s)}{\Gamma(1+H+s)} & \dfrac{\cos(\pi H)\Gamma(1-H-s)\Gamma(1/2+s)}{\pi(H+s)} \\ 0 & \dfrac{\Gamma(1-H-s)}{\Gamma(1/2-s)(H+s)} \end{bmatrix}.$$

Using Theorem 3.5 in conjunction with Example 3.2, the fBm on $(-\infty, T]$ can be expressed

$$Y(t) = \int_{-\infty}^{0} \frac{|t-\tau|^{H-1/2} - |\tau|^{H-1/2}}{\Gamma(H+1/2)}\, dW(\tau) + \int_0^t \frac{|t-\tau|^{H-1/2}}{\Gamma(H+1/2)}\, dW(\tau),$$

as given in Mandelbrot and Van Ness (1968). Mellin transforms useful for this computation are given in Appendix B. The corresponding whitening filter for $Y$ can be obtained from the inverse matrix of $\tilde{\rho}^o$.

In the above factorization, the 2, 2 element of $\tilde{\rho}^o$ is a noncausal factorization of the scale-spectrum of $Y_2$. If the *causal* spectral factorization is used, we obtain instead

$$\tilde{\rho}^o(s) = \begin{bmatrix} \dfrac{\Gamma(1/2+s)}{\Gamma(1+H+s)} & \dfrac{\cos(\pi H)\Gamma(1-H+s)\Gamma(1/2-s)}{\pi(H+s)} \\ 0 & \dfrac{\Gamma(1-H+s)}{\Gamma(1/2+s)(H+s)} \end{bmatrix}.$$

In the case $H > 1/2$, this can be expressed in the time domain as

$$Y(t) = -\frac{1}{\Gamma(H-1/2)} \int_t^0 B_{1-v/t}(H-1/2, 1-2H) v^{H-1/2}\, dW(v)$$

for $t < 0$ and as

$$\begin{aligned} Y(t) = {} & \frac{1}{\Gamma(H+1/2)} \int_0^t (t-v)^{H-1/2}\, dW(v) - \Gamma\left(\frac{3}{2} - H\right) \\ & \times \frac{\cos(\pi H)}{\pi} \int_{-\infty}^{0} B_{t/(t-v)}\left(H + \frac{1}{2}, 1-2H\right)(-v)^{H-1/2}\, dW(-v) \end{aligned}$$

for $t > 0$, where the incomplete beta function is defined as

$$B_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1}\, dt, \qquad a > 0, 0 \le x < 1.$$

A similar expression can be obtained for $H < 1/2$. Although less elegant than the Mandelbrot representation, this representation has the property that, for each $t < 0$, $Y(t)$ depends on $\{W(u)\colon t < u < 0\}$ rather than $\{W(u)\colon u < 0\}$.

The symmetric spectral factorization of Example 3.1 leads to an innovations representation for fBm on $[-T, T]$. The spectrum of $U(t) = Y(t) + Y(-t)$ is

$$\begin{aligned}\tilde{\rho}_U(s) &= (\cos \pi s + \cos \pi H)\frac{\Gamma(1-H+s)\Gamma(1-H-s)}{\pi(H^2 - s^2)} \\ &= \frac{2^{1-2H}\Gamma((2-H+s)/2)\Gamma((2-H-s)/2)}{(H^2-s^2)\Gamma((1+H+s)/2)\Gamma((1+H-s)/2)}\end{aligned}$$

and that of $V(t) = Y(t) - Y(-t)$ is

$$\tilde{\rho}_V(s) = \frac{2^{-1-2H}\Gamma((1-H+s)/2)\Gamma((1-H-s)/2)}{\Gamma((2+H+s)/2)\Gamma((2+H-s)/2)}.$$

The causal spectral factorizations $\tilde{\rho}_U^+$ and $\tilde{\rho}_V^+$ are obtained by collecting terms for which the sign of $s$ is positive. When these factorizations are expressed in the time domain, they lead to the following representations for $U$ and $V$, for $0 < H < 1$:

$$\begin{aligned}U(t) &= \frac{2^{-1/2-H}}{\Gamma(H+1/2)}\int_0^t \nu^{H-1/2} B_{1-\nu^2/t^2}(H+1/2, 1-H)\, dW_1(\nu) \\ &\quad + \frac{2^{1/2-H}t^{-1}}{\Gamma(H+1/2)}\int_0^t \nu^{3/2-H}(t^2-\nu^2)^{H-1/2}\, dW_1(\nu), \\ V(t) &= \frac{2^{1/2-H}}{\Gamma(H+1/2)}\int_0^t \nu^{1/2-H}(t^2-\nu^2)^{H-1/2}\, dW_2(\nu).\end{aligned}$$

When $H > 1/2$, the expression for $U$ simplifies to

$$U(t) = \frac{2^{1/2-H}}{\Gamma(H-1/2)}\int_0^t \nu^{H-1/2} B_{1-\nu^2/t^2}(H-1/2, 1-H)\, dW_1(\nu).$$

The corresponding whitening filters can be derived from $1/\tilde{\rho}_U^+$ and $1/\tilde{\rho}_V^+$.

**6. Conclusions.** The scale-invariance of wide-sense self-similar processes can greatly simplify the analysis of their linear structure. Once the connection with shift-invariance has been established, the utility of concepts such as self-similar systems, scale-spectra and scale-spectral factorization become apparent.

The structure of self-similar processes has been described in this paper using reproducing kernel Hilbert spaces. For various index sets of interest, the

RKHS inverse isomorphism $J^{-1}(m)$ can be computed by passing an observed self-similar process through a self-similar system whose frequency response depends on $m$. Likewise, the RKHS norm $\|m\|_S$ can be computed as the $L^2$-norm of a function $w_m$, where $w_m$ is obtained by passing $m$ through a self-similar system defined by the scale-spectrum of the observed process. Once these RKHS quantities are known, it is straightforward to write down the solutions to various linear problems.

Using this approach, we have shown that the signal detection and amplitude estimation problems are especially tractable when the signals are of the form $t^a$. We have studied such problems explicitly for fractional Brownian motion; we also have given some new representations for fBm.

## APPENDIX A

PROOF OF THEOREM 3.1. The linear space of $Y$ is composed of finite sums of the form $\sum a_i Y(t_i)$ along with their mean-square limits; since each such finite sum of $Y(t_i)$ can also be expressed as a finite sum of $X(\tau_i)$, and vice versa, the linear spaces are identical.

We will show that $J_X J_Y^{-1} = L_H$. Given that $L^2(X, \ln I) = L^2(Y, I)$, the map $Q$: $S(Y, I) \to S(X, \ln I)$ given by $Q(g) = J_X(J_Y^{-1}(g))$ is a well-defined Hilbert space isomorphism. First suppose that $g(v)$ is of the form $g(v) = \sum a_i R_Y(v, t_i)$. Then

$$\begin{aligned} J_X J_Y^{-1}(g) &= J_X J_Y^{-1}\left(\sum_{i=1}^N a_i R_Y(\cdot, t_i)\right) = J_X\left(\sum_{i=1}^N a_i Y(t_i)\right) \\ &= J_X\left(\sum_{i=1}^N a_i t_i^H X(\ln t_i)\right) = \sum_{i=1}^N a_i t_i^H r_X(\cdot - \ln t_i) \\ &= \sum_{i=1}^N a_i t_i^H \rho_Y(e^{\cdot}/t_i) = e^{-H\cdot} \sum_{i=1}^N a_i R_Y(e^{\cdot}, t_i) = L_H g. \end{aligned}$$

An arbitrary $g(v) \in H(Y, I)$ is the limit in the RKHS norm of a sequence of functions $g_n(v)$, where each $g_n(v)$ is a finite sum as above. Since $Q$ is an isomorphism of Hilbert spaces, we get

$$Q(g) = Q(\lim g_n) = \lim Q(g_n) = \lim L_H g_n,$$

where the limits are limits in RKHS norm. A sequence of functions that is convergent in norm must also be pointwise convergent [see, e.g., Kailath (1971, p. 540)]. On a pointwise basis, it is easy to see that limits pass through $L_H$, so that $\lim L_H g_n = L_H \lim g_n = L_H g$. Hence $Q(g) = L_H g$ for arbitrary $g$. □

TABLE 1
*Inverse Mellin transforms*

| **Mellin transform** $\tilde{f}(s) = \int_0^\infty f(t)t^{-s-1}\,dt$ | **Function** $f(t)$ |
|---|---|
| $\frac{\Gamma(a+s)}{\Gamma(c+s)}$ | $\frac{1}{\Gamma(c-a)} t^{-a}(1-1/t)^{c-a-1}u(\ln t)$ |
| $-a < \mathrm{Re}(s)$ | $c > a$ |
| $\frac{\Gamma(a+s)}{\Gamma(c+s)(b+s)}$ | $\frac{t^{-b}}{\Gamma(c-a)} B_{1-1/t}(c-a, a-b)u(\ln t)$ |
| $\max(-a, -b) < \mathrm{Re}(s)$ | $c > a$ |
| $\Gamma(a+s)\Gamma(b-s)$ | $\Gamma(a+b)t^{-a}(1+1/t)^{-a-b}$ |
| $-a < \mathrm{Re}(s) < b$ | $a+b > 0$ |
| $\frac{\Gamma(a+s)\Gamma(b-s)}{c+s}$ | $\Gamma(a+b)t^{-c}B_{t/(1+t)}(c+b, a-c)$ |
| $\max(-a, -c) < \mathrm{Re}(s) < b$ | $a+b > 0,\ b+c > 0$ |

# APPENDIX B

**B.1. Inverse Mellin transforms.** See Table 1.

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