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SELF-ATTRACTIVE RANDOM POLYMERS

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We consider a repulsion–attraction model for a random polymer of finite length in $\mathbb{Z}^d$. Its law is that of a finite simple random walk path in $\mathbb{Z}^d$ receiving a penalty $e^{-2\beta}$ for every self-intersection, and a reward $e^{\gamma/d}$ for every pair of neighboring monomers. The nonnegative parameters β and γ measure the strength of repellence and attraction, respectively.

We show that for $\gamma > \beta$ the attraction dominates the repulsion; that is, with high probability the polymer is contained in a finite box whose size is independent of the length of the polymer. For $\gamma < \beta$ the behavior is different. We give a lower bound for the rate at which the polymer extends in space. Indeed, we show that the probability for the polymer consisting of n monomers to be contained in a cube of side length $\varepsilon n^{1/d}$ tends to zero as n tends to infinity.

In dimension $d = 1$ we can carry out a finer analysis. Our main result is that for $0 < \gamma \le \beta - \frac{1}{2}\log 2$ the end-to-end distance of the polymer grows linearly and a central limit theorem holds.

It remains open to determine the behavior for $\gamma \in (\beta - \frac{1}{2}\log 2, \beta]$.

0. Introduction and main results.

0.1. *Model and motivation.* A polymer is a long chain of molecules (monomers) with two characteristic phenomenological properties: an irregular shape and a certain stiffness. The chemical motivation is that the monomers are lined up and are connected by "bonds" of the same length. For example, carbon-based polymers like polyethylene or polystyrene have a bond length of $1.54 \cdot 10^{-10}$ meters. The stereometric angles of neighboring bonds, however, are subject to randomness. Irregularity and stiffness are a result of entropy and repulsive and attractive forces between the monomers (and possibly a medium).

In material sciences an important question is to determine the end-to-end distance of the polymer and the average distance of monomers ("coil radius" or "radius of gyration"). We address this question in the present paper for a mathematical model of a random polymer.

In the simplest mathematical model for a random polymer it is assumed that the monomers are located at sites $S_0, S_1, \dots, S_n \in \mathbb{Z}^d$ and that $|S_i - S_{i-1}| = 1$, $i = 1, \dots, n$. $S = (S_i)_{i=0}^n$ is assumed to be a random variable. Its distribution is derived from that of a simple random walk (starting at $S_0 = 0$), denoted by P, by introducing interactions between monomers. More precisely,

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we define a Hamiltonian H_n that models repulsive and attractive forces. The distribution of the polymer is obtained by taking the Boltzmann distribution with respect to the simple random walk law.

For the model we consider in this paper we fix two parameters $\beta, \gamma \geq 0$ and define the Hamiltonian by

$$H_n(S) = H_n^{\beta,\gamma}(S) = \beta \sum_{i,j=0}^{n} \mathbf{1}_{\{|S_i - S_j|=0\}} - \frac{\gamma}{2d} \sum_{i,j=0}^{n} \mathbf{1}_{\{|S_i - S_j|=1\}}. \tag{0.1}$$

For $n \in \mathbb{N}$ the new path law $Q_n^{\beta,\gamma}$ is

$$\frac{dQ_n^{\beta,\gamma}}{dP}\big((S_i)_{i=0}^n\big) = \frac{1}{Z_n^{\beta,\gamma}} e^{-H_n(S)}, \tag{0.2}$$

where (with E denoting the expectation with respect to P) $Z_n^{\beta,\gamma}$ is the normalizing constant or partition function

$$Z_n^{\beta,\gamma} = E\big(e^{-H_n(S)}\big). \tag{0.3}$$

The law $Q_n^{\beta,\gamma}$ is called the *n-polymer measure with strength of repellence β and strength of attraction γ*. $Q_n^{\beta,\gamma}$ gives a penalty $e^{-2\beta}$ to every pair of monomers at the same site and a reward $e^{\gamma/d}$ for every pair of neighboring monomers. The penalty models polarization of the monomers, or the so-called *excluded volume effect*, which means that around each monomer there is a certain space in which it is energetically unfavorable to have another monomer. This space is called the excluded volume. For an explanation of the excluded volume effect and other properties of polymers from a physicist's point of view, see Vanderzande (1998).

The reward models attractive forces between monomers that are of short range, the so-called van der Waals forces. For an expository paper on mathematical polymer models, see den Hollander (1996). For a survey of one-dimensional polymer models, see van der Hofstad and König (2001).

The above model has received a lot of attention in the case where $\gamma = 0$, in which case it is called the Domb–Joyce model or the weakly self-avoiding walk. This is the case of a good solvent. In real situations, this corresponds to high temperatures. At lower temperatures the quality of the solvent deteriorates. Therefore, the excluded volume effect plays a less profound role, and the attractive forces between the monomers become more important. The temperature at which this phase transition occurs is called the θ-point. See Vanderzande [(1998), Chapter 8].

It is a folklore conjecture that the following scaling for the end-to-end distance $|S_n|$ holds for the weakly self-avoiding walk ($\gamma = 0$).

CONJECTURE 0.1. *For every $\beta \in \mathbb{R}^+$,*

$$E_{Q_n^{\beta,0}}\big[|S_n|^2\big] \sim D n^{2\nu_{\mathrm{SA}}} \quad (n \to \infty), \tag{0.4}$$

where $D = D(\beta, d) > 0$ *is some amplitude and* $\nu_{\mathrm{SA}} = \nu_{\mathrm{SA}}(d)$ *some* critical exponent. *The latter is believed to be* independent *of* β *and to assume the values*

$$\nu_{\mathrm{SA}} = \begin{cases} 1, & d = 1, \\ \frac{3}{4}, & d = 2, \\ 0.588\ldots, & d = 3, \\ \frac{1}{2}, & d > 4. \end{cases} \tag{0.5}$$

For $d = 4$ *it is believed that there are logarithmic corrections to the above behavior*; *that is*,

$$E_{Q_n^{\beta,0}}\big[|S_n|^2\big] \sim Dn(\log n)^{1/4} \qquad (n \to \infty). \tag{0.6}$$

See Madras and Slade [(1993), Section 2] or Vanderzande [(1998), Section 2] for a heuristic argument due to Flory (1949) that produces the right exponents, except in dimension 3, where the heuristic argument gives the slightly larger value $\frac{3}{5}$. In Vanderzande [(1998), Sections 3 and 4] there are also other heuristic explanations for the values of ν in dimensions 2 and 3.

For $d > 4$, two independent simple random walk paths typically intersect only finitely often. Conjecture 0.1 states that in this case the interaction is typically of short range and on a macroscopic scale the entropy is the decisive quantity. Therefore, we observe ordinary diffusive behavior. Here dimension 4 is the critical dimension, where the behavior is thought to be Gaussian with logarithmic corrections.

In lower dimensions the range of the interaction is larger and no Gaussian limit is expected. Finally, in dimension $d = 1$ the end-to-end distance behaves ballistically, that is, grows linearly in the number of monomers.

For $d \geq 5$, the lace expansion was used to prove the above conjecture [see, e.g., Brydges and Spencer (1985), Hara and Slade (1992a, b) or Madras and Slade (1993)]. In dimension $d = 1$, Greven and den Hollander (1993) showed the ballistic behavior of the polymer (law of large numbers). Later König (1996) was able to show a central limit theorem which we cite here as a basic theorem.

Denote by $\mathscr{N}(0,1)$ the standard normal distribution.

THEOREM 0. *For every* $\beta \in \mathbb{R}^+$ *there exist* $\theta^* = \theta^*(\beta) \in (0,1)$, $\sigma^* = \sigma^*(\beta) \in (0, \infty)$, *such that*

$$\lim_{n\to\infty} Q_n^{\beta,0}\left(\frac{|S_n| - \theta^* n}{\sigma^* \sqrt{n}} \in \cdot\right) = \mathscr{N}(0,1). \tag{0.7}$$

It is reasonable to expect that $\beta \mapsto \theta^*(\beta)$ is increasing. However this is still an open problem. It *is* known that $\beta \mapsto \theta^*(\beta)$ is analytic as a map from $(0,1)$ to $(0,1)$ [see Greven and den Hollander (1993)].

One main goal of this paper is to show a CLT for $Q_n^{\beta,\gamma}$ when γ is smaller than β (Theorem 3 below). Our strategy is to adapt the methods of Greven and den Hollander (1993), König (1996), and van der Hofstad, den Hollander and König (1997) to our model. The approach turns out to work for $\gamma \le \beta - \frac{1}{2}\log 2$. We expect the CLT to hold for all $\gamma < \beta$ but we could not show it with this method.

The fundamental theorem. The fundamental difference between $\gamma < \beta$ and $\gamma > \beta$ is that the polymer localizes if $\gamma > \beta$ while it does not if $\gamma < \beta$ (Theorem 1). For the behavior at $\gamma = \beta$ we only have a conjecture (Conjecture 0.3).

In order to state our first theorem we have to fix some terminology.

DEFINITION 0.2. The polymer is called *localized*, if for some $L \in \mathbb{N}$,

$$\lim_{n\to\infty} Q_n^{\beta,\gamma}(S_i \in C_L \ \forall\, i \le n) = 1, \tag{0.8}$$

where $C_L = \{-L, \dots, L\}^d$.

The polymer is trapped in a finite box in the localized regime. In fact, we will show that the probability to leave a certain large cube is exponentially small as $n \to \infty$.

THEOREM 1 (Fundamental theorem). *Fix $d \ge 1$.*

(i) *If $\gamma < \beta$, then the polymer is not localized. Furthermore, for $\varepsilon > 0$ small enough there exists a constant $c > 0$ such that for all $n \in \mathbb{N}$,*

$$Q_n^{\beta,\gamma}(S_i \in C_{[\varepsilon n^{1/d}]} \ \forall\, i \le n) \le e^{-cn}. \tag{0.9}$$

(ii) *If $\gamma > \beta$, then the polymer is localized. Moreover, there exists a constant $c > 0$ such that for L large enough and all $n \in \mathbb{N}$,*

$$Q_n^{\beta,\gamma}(\exists\, i \le n\colon S_i \notin C_L) \le e^{-cLn}. \tag{0.10}$$

Theorem 1 states that the transition from localization to nonlocalization takes place exactly at $\gamma = \beta$. The key ingredients for the proof of Theorem 1 are that for $\gamma > \beta$ we have

$$a := \lim_{n\to\infty} \frac{1}{n^2} \log Z_n^{\beta,\gamma} > 0, \tag{0.11}$$

while for $\gamma < \beta$,

$$-\infty < \limsup_{n\to\infty} \frac{1}{n} \log Z_n^{\beta,\gamma} < 0. \tag{0.12}$$

Indeed, for the behavior in (0.11) to occur, the Hamiltonian has to be of order an^2. This is only possible when the local times are of order n and is a clear indication that the polymer localizes. In this case, we will see that $a = \lim_{n\to\infty} n^{-2} \max H_n(S)$, where the maximum is taken over all n-step simple random walk paths (see Section 1). If, on the other hand, we have that $\gamma < \beta$, then the polymer pays a super-exponential price for large local times (see Section 1.1). Hence, if the bounds in (0.12) hold, then none of the local times are of order n, so that the polymer cannot localize.

We can think of $\beta - \gamma$ as the "effective parameter" of self-intersections. If this effective parameter is negative, there is an overall reward for self-intersections so that the polymer behaves like a self-attractive random walk ($\beta < 0$, $\gamma = 0$), which localizes in all dimensions even when $\beta = \beta_n = -\frac{\alpha}{n}$ with $\alpha > 0$ large enough [see Bolthausen and Schmock (1997)]. (However, for $\alpha > 0$ small enough, the behavior is diffusive in $d = 2$ [Brydges and Slade (1995)]. This shows that in $d = 2$ there is an interesting phase transition.)

If $\beta - \gamma$ is positive, then the polymer does not want to localize in the sense of Definition 0.2. However, it is unclear what the precise scaling behavior will be in this case. We will go deeper into conjectures and comparisons to other models in Section 0.2 below.

Shape theorem and the transition point. The next aim is to investigate the two regimes $\gamma > \beta$ and $\gamma < \beta$ in more detail. We start with the regime of localization ($\gamma > \beta$). How does the polymer localize? Does it reveal a particular profile? More precisely, if we assign to each monomer a mass of $\frac{1}{n}$, does the concentration of mass converge (weakly) to a random distribution and can we characterize this distribution?

In order to formulate our result we need to introduce the local times for simple random walk,

$$\ell_n(x) = \#\{0 \le i \le n \colon S_i = x\} \qquad \big(n \in \mathbb{N}_0, x \in \mathbb{Z}^d\big). \tag{0.13}$$

We want to show that up to translations, $n^{-1}\ell_n$ converges to some function $f^{\beta,\gamma}$, the "shape" or "profile" of the polymer. We are able to do so if $d = 1$. In our Corollary 1.5 we can even determine the "shape" $f^{\beta,\gamma}$.

THEOREM 2 (Shape theorem). *Assume $d = 1$ and and $\gamma > \beta$. There exist a finitely supported function $f^{\beta,\gamma}\colon \mathbb{Z} \to [0,\infty)$, $\|f^{\beta,\gamma}\|_1 = 1$ and constants $c', C' > 0$ such that for ξ large enough and all $n \in \mathbb{N}$,*

$$Q_n^{\beta,\gamma}\Big(\min_{x\in\mathbb{Z}} \|n^{-1}\ell_n - f^{\beta,\gamma}(x+\cdot)\|_1 > \xi n^{-1/2}\Big) \le C'e^{-c'n}. \tag{0.14}$$

Moreover, $f^{\beta,\gamma}$ is the function that minimizes the Hamiltonian among all functions that can occur as limits of rescaled local times $n^{-1}\ell_n$ ($n \to \infty$).

It is not too hard to show the *existence* of such a minimizer for any $d \ge 1$ (see Lemma 1.3). However, uniqueness requires more work. We are only able to show uniqueness in $d = 1$ where we can give an explicit formula in Corollary 1.5. A similar statement as in (0.14) holds for $d \ge 2$ if we replace the minimum over $x \in \mathbb{Z}$ by the minimum over the set $\mathscr{F}^*$ of functions that minimize the Hamiltonian. Since we do not know whether $\mathscr{F}^*$ is generated by one function (as in $d = 1$) or a finite number of functions or if it is more complicated, it does not seem worthwhile to state this as a result. However, we formulate the proof in Section 1.3 for $\mathbb{Z}^d$, under the assumption that there are finitely many maximizers.

Another interesting question is what happens at the transition point $\gamma = \beta$. We have the following conjecture.

CONJECTURE 0.3. *If $\gamma = \beta$, then the scale of the polymer is given by*

$$E_{Q_n^{\beta,\beta}}\big[|S_n|^2\big] \sim Dn^{2/(d+1)}(\log n)^{-1/(d+1)} \qquad (n \to \infty). \tag{0.15}$$

The rigorous proof in $d = 1$ will be presented in a forthcoming paper [see van der Hofstad, Klenke and König (2001)], in which the authors show that the properly normalized range of the random walk (i.e., the number of distinct sites visited by the walk) converges.

Central limit theorem. In dimension $d = 1$ there is a simple connection between the local times of simple random walk and a critical Galton–Watson branching process (Knight's theorem). Since it was first used in this context by Greven and den Hollander (1993) it has proved to be the most powerful tool for the investigation of one-dimensional random walks with interactions. Therefore, it is natural that we get the most precise result in $d = 1$. We are able to show ballistic behavior, which means $Q_n^{\beta,\gamma}(||S_n|/n - \theta^*(\beta,\gamma)| > \varepsilon) \to 0$, $n \to \infty$, for some $\theta^*(\beta,\gamma) > 0$ and all $\varepsilon > 0$. In addition we can show the central limit theorem for the fluctuations of $|S_n|$ around $n\theta^*(\beta,\gamma)$. Clearly the CLT is the stronger statement. Since our proof does not need the LLN as an intermediate step but is a direct approach via large deviation techniques we only state the CLT. Due to technical difficulties, we can only show this for γ such that $\gamma \leq \beta - \frac{1}{2}\log 2$.

THEOREM 3 (Central limit theorem). *For every $\beta, \gamma \in (0,\infty)$ such that $\gamma \leq \beta - \frac{1}{2}\log 2$, there exist $\theta^* = \theta^*(\beta,\gamma) \in (0,1]$, and $\sigma^* = \sigma^*(\beta,\gamma) \in (0,\infty)$ such that*

$$\lim_{n\to\infty} Q_n^{\beta,\gamma}\left(\frac{|S_n| - \theta^* n}{\sigma^*\sqrt{n}} \in \cdot\right) = \mathscr{N}(0,1). \tag{0.16}$$

Theorem 3 shows that for $\gamma \leq \beta - \frac{1}{2}\log 2$, the polymer is in the same universality class as the weakly self-avoiding walk for which $\gamma = 0$ (see Conjecture 0.1 and Theorem 1).

The quantity θ^* is called the *speed of the polymer*, while σ^* is called the *spread of the polymer*. In Section 2 we give a characterization of these quantities in terms of a largest eigenvalue problem. It is reasonable to believe that $(\beta,\gamma) \mapsto \theta^*(\beta,\gamma)$ is increasing in β and decreasing in γ and that $\theta^*(\beta,\gamma) \to 0$ as $\gamma \uparrow \beta$. However, we are only able to show analyticity for $0 < \gamma < \beta$.

The gap of $\frac{1}{2}\log 2$ is due to a technical difficulty that we could not overcome here. We know from Theorem 1(i) that $\max_{0\leq i\leq n}|S_i| > \varepsilon n$ with high probability. This suggests that we would also have ballistic behavior here, which presumably goes along with the central limit theorem behavior for all $0 < \gamma < \beta$.

0.2. *Discussion and conjectures.* The model considered in this paper is related to the attractive (strictly) self-avoiding walk studied in Brak, Owczarek and Prellberg (1993), obtained by taking the limit of $Q_n^{\beta,\gamma}$ as β tends to infinity. Evidently self-avoiding walk cannot intersect itself and thus cannot localize in the sense of Definition 0.2. Brak, Owczarek and Prellberg (1993) conjecture that there exists a $\gamma^*(d) \in (0,\infty)$ such that for $\gamma < \gamma^*(d)$ the attractive self-avoiding walk behaves like ordinary self-avoiding walk while it is contained in a ball of radius of a multiple of $n^{1/d}$ if $\gamma > \gamma^*(d)$. Note that for $\beta = \infty$ the phase transition observed in this paper at $\gamma = \beta$ does not occur. The transition point γ^* is expected to take a nontrivial value. It is believed that for $0 < \beta < \infty$, a similar picture holds. Indeed, it is conjectured that there is a second critical curve $\beta \mapsto \gamma^*(\beta, d)$ such that for $\gamma \in [0, \gamma^*(\beta, d))$ the scale of the polymer is n^ν, with $\nu = \nu_{\mathrm{SA}}$ as in Conjecture 0.1, while for $\gamma \in (\gamma^*(\beta, d), \beta)$ the scale is $n^{1/d}$. (Note that in dimension 1, $\nu = 1/d = 1$, so there cannot be a second phase transition.) In Theorem 1(i) we show that if ν exists, then $\nu \geq \frac{1}{d}$.

The value $\gamma^*(\beta)$ is called the θ-point. Thus, two phase transitions are expected, one at $\gamma = \beta$ and one at the θ-point. We mention that $\nu_\theta = \nu(\gamma^*(\beta), \beta)$ is expected to be $\frac{4}{7}$ in $d = 2$ and $\frac{1}{2}$ for $d \geq 3$. See Vanderzande (1998) for all these conjectures for the attractive self-avoiding walk ($\beta = \infty$, $\gamma \in [0,\infty)$). For the critical case $\gamma = \beta$ there exist no conjectures in the literature, to our best knowledge. We think that $\nu(\beta, \beta) = \frac{1}{d+1}$ in all dimensions (recall Conjecture 0.3), and that there are logarithmic corrections. However, the heuristic argument for $d > 1$ is very weak. In Figure 1, we give a summary of the expected critical exponents ν. A general proof of the existence of the critical exponent ν does not exist, so all values are conjectured unless a rigorous proof and identification exists.

For $d \geq 5$, the lace expansion has been used to prove that weakly self-avoiding walk can be rescaled to Brownian motion, that is, weakly self-avoiding walk is diffusive. However, the lace expansion technique depends sensitively on the strict self-repellence property of that model. Even for $\gamma \ll \beta$ the attractive random polymer is self-repellent only on a macroscopic scale. This is not sufficient to use the lace expansion. Still we believe that the attractive weakly

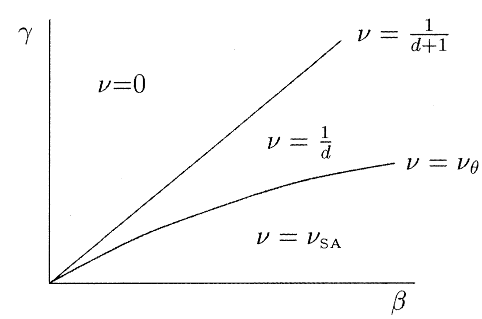

FIG. 1. *Sketch of the conjectured values for $\nu(\gamma, \beta)$.*

self-avoiding walk behaves diffusively for $d \geq 5$ and γ small enough. Possibly the lace expansion can be adapted to handle the case where $0 \leq \gamma \ll \beta \ll 1$.

Oono (1975, 1976) investigates the above model for $\gamma = 0$ and $\beta < 0$. He shows that the path jumps back and forth between two points with high probability. The above problem is easier than the one we consider for $\gamma > \beta$, since one can explicitly compute the maximizer of the Hamiltonian, whereas in the case where $\gamma > \beta$ we cannot.

0.3. *Outline.* In Section 1 we prove the fundamental theorem (Theorem 1) and the shape theorem (Theorem 2). The main step is to compute the exact scaling of the partition function as $n \to \infty$. This is done by solving a variational problem for the Hamiltonian. Furthermore, we give a heuristic argument that explains the formulas in the special case $\gamma = 0$.

In Section 2 we identify $\theta^*(\beta, \gamma)$ and $\sigma^*(\beta, \gamma)$ in terms of derivatives of the largest eigenvalue of an $\mathbb{N}^2 \times \mathbb{N}^2$ matrix, acting as a compact operator on $\ell^2(\mathbb{N}^2)$. Existence and analyticity of this eigenvalue (as a function of β and γ) are proved by employing standard functional analytic methods.

In Section 3 and 4 we prove the central limit theorem (Theorem 3) using a variation of a method introduced by van der Hofstad, den Hollander and König (1997). We recall Knight's theorem in Section 3. This is a Markov chain description of the local times of one-dimensional simple random walk. We use this description to write the moment enerating function of S_n under $Q_n^{\beta,\gamma}(\cdot|S_n > 0)$ as the expectation of an exponential functional of three Markov chains. These Markov chains correspond to the local times in the intervals $(-\infty, 0)$, $[0, S_n]$ and (S_n, ∞).

In Section 4 we absorb the exponential functional $e^{-H_n^{\beta,\gamma}(S)}$ into the transition kernels of the Markov chains and rewrite the moment generating function as a correlation function involving three Markov processes. We show that, in the limit as $n \to \infty$, the correlation function factors into a product of three parts. The part corresponding to $[0, S_n]$ gives the CLT in Theorem 3, the parts corresponding to $(-\infty, 0)$ and (S_n, ∞) result into constants that drop out in the normalization. This proves the CLT (Theorem 3).

1. Proof of the fundamental and shape theorem. The proof of Theorem 1(i) is fairly simple and is the content of the next short subsection. The proof of localization for $\gamma > \beta$ requires a firm grip on the asymptotics of the partition function $Z_n^{\beta,\gamma}$. This is Proposition 1.1 in Section 1.2. The proof is rather involved and includes solving a variational problem. At the end of that subsection we prove Theorem 1(ii). The various steps in the study of the variational problem serve to prove the shape theorem in the final subsection.

1.1. *No localization for $\gamma < \beta$.* First we introduce some notation. We consider the local time ℓ_n of a random walk path [see (0.13)] as an element of $l^2(\mathbb{Z}^d)$, the space of square-summable sequences with scalar product $\langle \cdot, \cdot \rangle$. We first reformulate the Hamiltonian [recall (0.1)] in terms of the local times ℓ_n

[see (0.13)],

$$H_n(S) = \beta \sum_{x\in\mathbb{Z}^d} \ell_n^2(x) - \frac{\gamma}{2d} \sum_{x,\, e\in\mathbb{Z}^d:\, |e|=1} \ell_n(x)\ell_n(x+e). \tag{1.1}$$

We see that the Hamiltonian $H_n^{\beta,\gamma}$ is a quadratic functional on $l^2(\mathbb{Z}^d)$ that is given by a symmetric bilinear form, that is, by a matrix $F = F_{\beta,\gamma}$ indexed by $\mathbb{Z}^d$ which is given by

$$F(x, y) = F_{\beta,\gamma}(x, y) = \begin{cases} -\beta, & x = y, \\ \dfrac{\gamma}{2d}, & |x-y| = 1, \\ 0, & \text{else}. \end{cases} \tag{1.2}$$

Thus $H_n(S_n) = -\langle \ell_n, F\ell_n\rangle$. We will frequently use the inequality

$$\begin{aligned} \langle \ell_n, F_{\beta,\gamma}\ell_n\rangle &= -(\beta-\gamma)\sum_{x\in\mathbb{Z}^d} \ell_n^2(x) - \frac{\gamma}{4d}\sum_{x,\, e\in\mathbb{Z}^d:\, |e|=1} (\ell_n(x) - \ell_n(x+e))^2 \\ &\le -(\beta-\gamma)\sum_{x\in\mathbb{Z}^d} \ell_n^2(x) \\ &= -(\beta-\gamma)\langle \ell_n, \ell_n\rangle = \langle \ell_n, F_{\beta-\gamma,0}\ell_n\rangle. \end{aligned} \tag{1.3}$$

Fix $\gamma < \beta$ and $L \in \mathbb{N}$. We give a lower bound for the energy of a path that stays in C_L at all times. To do so, let $(\ell_n(x))_{x\in\mathbb{Z}^d}$ be the local time of such a path and use (1.3) to get the estimate

$$H_n(S) \ge (\beta-\gamma)\langle \ell_n, \ell_n\rangle \ge (\beta-\gamma)n^2(2L+1)^{-d}.$$

Combine this with the trivial estimate (using a straight path) $Z_n^{\beta,\gamma} \ge e^{-\beta}\times (e^{-\beta+\gamma/d}/2d)^n$ to conclude that for some $c > 0$,

$$\begin{aligned} Q_n^{\beta,\gamma}(S_j \in C_L \forall j \le n) &\le e^{-(\beta-\gamma)n^2(2L+1)^{-d}}/Z_n^{\beta,\gamma} \\ &\le e^{-cn} \to 0, \qquad n\to\infty. \end{aligned} \tag{1.4}$$

Note that (1.4) still holds if $L = L_n = \varepsilon n^{1/d}$ and $\varepsilon > 0$ small enough. This proves (0.9).

In addition to (1.4) we will give bounds for the normalizing constant $Z_n^{\beta,\gamma}$ for $\gamma < \beta$. Use (1.3) to get

$$Z_n^{\beta,\gamma} = E\big[e^{\langle \ell_n, F_{\beta,\gamma}\ell_n\rangle}\big] \le E\big[e^{\langle \ell_n, F_{\beta-\gamma,0}\ell_n\rangle}\big] = Z_n^{\beta-\gamma,0}. \tag{1.5}$$

Note that $Z_n^{\beta-\gamma,0}$ is the normalizing constant for the weakly self-avoiding walk with interaction parameter $\beta-\gamma > 0$. By submultiplicativity (i.e., $Z_{n+m}^{\beta-\gamma,0} \le Z_n^{\beta-\gamma,0} Z_m^{\beta-\gamma,0}$), it is clear that

$$\lim_{n\to\infty} \frac{1}{n}\log Z_n^{\beta-\gamma,0} = \inf_{n\in\mathbb{N}} \frac{1}{n}\log Z_n^{\beta-\gamma,0} \le \frac{1}{2}\log Z_2^{\beta-\gamma,0} < 0, \tag{1.6}$$

where we used that $Z_2^{\beta-\gamma,0} = (1-\frac{1}{2d}) + \frac{1}{2d}e^{-(\beta-\gamma)} < 1$. This proves (0.12).

1.2. *Localization for* $\gamma > \beta$. Fix $\gamma > \beta$. We will start by bounding the normalizing constant $Z_n^{\beta,\gamma}$ from above and below in Proposition 1.1.

First we need some notation. Define

$$M_n = M_n(\beta,\gamma) = \max_{S=(S_i)_{i=0}^n} \langle \ell_n, F_{\beta,\gamma}\ell_n\rangle \tag{1.7}$$

and note that

$$\left(\frac{1}{2d}\right)^n e^{M_n} \le Z_n^{\beta,\gamma} \le e^{M_n}. \tag{1.8}$$

The main ingredients for the proof of Theorem 1(ii) are good upper and lower bounds for M_n. This is the content of the next proposition which will be proved in the pages following. The proof of Theorem 1(ii) follows at the end of this section.

PROPOSITION 1.1. *There exist constants* $a, C > 0$ *such that for all* $n \in \mathbb{N}$,

$$an^2 - Cn \le M_n \le an^2 + \frac{\gamma}{d}n. \tag{1.9}$$

The proof of (0.11) is a simple consequence of (1.9) and (1.8).

For the proof of (1.9) we scrutinize a variational problem for the Hamiltonian (Lemma 1.3). A major point is that due to periodicity of the random walk not all functions are admissible as possible limits of the rescaled local times. Rather than considering the local times directly, we reformulate the problem in terms of the numbers of *bond crossings*. Here no restrictions apply (apart from nonnegativity), at least in the limit $n \to \infty$, as will follow from the proof.

In order to formulate the problem we have to introduce some notation.

For a random walk path $(S_i)_{i=0}^n$, not necessarily starting in 0, define the *averaged local time*

$$\tilde{\ell}_n(x) = \frac{1}{2}\sum_{i=0}^{n-1}(\mathbf{1}_{S_i=x} + \mathbf{1}_{S_{i+1}=x}). \tag{1.10}$$

Clearly, $\tilde{\ell}_n = \ell_n - \frac{1}{2}\mathbf{1}_{\{S_0\}} - \frac{1}{2}\mathbf{1}_{\{S_n\}}$.

Furthermore, let

$$\mathscr{G} = \{g = (g^1,\dots,g^d) | g^i\colon \mathbb{Z}^d \to [0,\infty); i = 1,\dots,d, \|g\| = 1\}, \tag{1.11}$$

where

$$\|g\| = \sum_{i=1}^d \sum_{x\in\mathbb{Z}^d} |g^i(x)|. \tag{1.12}$$

Let e_i be the ith unit vector in $\mathbb{Z}^d$ and define the linear map α by

$$(\alpha g)(x) = \frac{1}{2}\sum_{i=1}^d \bigl[g^i(x-e_i) + g^i(x)\bigr]. \tag{1.13}$$

Here $g^i(x)$ measures the number of crossings of the bond between x and $x+e_i$, in either direction and $\alpha g(x)$ is the corresponding local time.

Finally, let $\mathscr{F} = \alpha(\mathscr{G})$. It is clear that for every random walk path $(S_i)_{i=0}^n$,

$$\frac{1}{n}\tilde{\ell}_n \in \mathscr{F}. \tag{1.14}$$

Now, define for $L \in \mathbb{N}$,

$$\mathscr{G}_L = \{g \in \mathscr{G}\colon\ \operatorname{supp}(g) \subset \{0, \dots, L-1\}^d\} \tag{1.15}$$

and let $\mathscr{F}_L = \alpha(\mathscr{G}_L)$.

LEMMA 1.2. *For every $f \in \mathscr{F}_L$ and every $n \in \mathbb{N}$ there exists a random walk path $(S_i)_{i=0}^n$ such that* [*recall* (1.10)]

$$\|\tilde{\ell}_n - nf\|_1 \le (2d+2)L^d. \tag{1.16}$$

PROOF. Choose $g \in \mathscr{G}_L$ with $f = \alpha g$. Choose $\bar{g} \in \mathscr{G}_L$ with $\bar{g}^i(x) \in \frac{2}{n}\mathbb{N}_0$, $x \in \mathbb{Z}^d$, $i \in \{1, \dots, d\}$ and $|\bar{g}^i(x) - g^i(x)| \le 2/n$. Hence $\|\alpha\bar{g} - f\| \le 2dL^d/n$. Then $\bar{g}$ corresponds in an obvious way to a family of (at most) L^d random walk paths starting and ending in the same position x and visiting only their neighbors $x + e_i$, $i = 1, \dots, d$ [exactly $\frac{n}{2}\bar{g}^i(x)$ times].

Now connect these paths by a random walk path that visits every point in $\{0, \dots, L-1\}^d$ exactly once. This gives a random walk path of length $n + L^d$. Cutting off the last L^d steps gives the desired path S for which (1.16) holds. □

The previous lemma is the connection of (1.9) to the following variational problem:

$$\sup\{\langle f, Ff\rangle, f \in \mathscr{F}\}. \tag{1.17}$$

LEMMA 1.3. *The supremum in* (1.17) *is attained and there exists an $L_0 \in \mathbb{N}$ such that* (*up to translations*) *$\operatorname{supp}(f^*) \subset \{0, \dots, L_0\}^d$ for any maximizer f^* of* (1.17). *Furthermore, $a := \langle f^*, Ff^*\rangle > 0$.*

PROOF. First we rewrite our problem in terms of functions in $\mathscr{G}$. Define the matrix $G = \alpha^* F \alpha$, where α^* is the adjoint of α. Hence we have to show that for the variational problem

$$\sup\{\langle g, Gg\rangle, g \in \mathscr{G}\} \tag{1.18}$$

the supremum is assumed, that there exists an $L_0 \in \mathbb{N}$ such that any maximizer g^* has (up to translations) its support in $\{0, \dots, L_0\}^d$, and that $a = \langle g^*, Gg^*\rangle > 0$.

Clearly the set $\mathscr{G}_L$ is compact for every $L \in \mathbb{N}$ and $\mathscr{G}_L \to \mathbb{R}$, $g \mapsto \langle g, Gg\rangle$ is continuous. Hence there exists a solution $g_L^* \in \mathscr{G}_L$ of

$$\langle g_L^*, Gg_L^*\rangle = \sup\{\langle g, Gg\rangle, g \in \mathscr{G}_L\}. \tag{1.19}$$

Denote by $\mathscr{G}_L^*$ the set of such g_L^* and let $a_L = \langle g_L^*, Gg_L^*\rangle$. The sequence $(a_L)_{L\in\mathbb{N}}$ is nondecreasing and $a = \lim_{L\to\infty} a_L$. We have to show that there exists an $L_0 \in \mathbb{N}$ such that for any $K \geq L_0$ and $g \in \mathscr{G}_K^*$ we have (up to translations) $g \in \mathscr{G}_{L_0}^*$. In this case clearly $a = a_{L_0}$.

Let $\varphi_L = (L^{-d}\mathbf{1}_{\{0,\dots,L-1\}^d}, 0, \dots, 0) \in \mathscr{G}_L$. Clearly, $\alpha(\varphi_L)(x) = L^{-d}$ if $x \in \{0, \dots, L-2\} \times \{0, \dots, L-1\}^{d-1}$ and $= \frac{1}{2}L^{-d}$ if $x \in \{-1, L-1\} \times \{0, \dots, L-1\}^{d-1}$. Hence for $L \geq \frac{3\gamma}{\gamma-\beta}$,

$$
\begin{aligned}
a_L &\geq \langle \varphi_L, G\varphi_L\rangle = \langle \alpha(\varphi_L), F\alpha(\varphi_L)\rangle \\
&\geq (\gamma-\beta)L^{-d} - \frac{\gamma-\beta}{2}L^{-(d+1)} - \gamma L^{-(d+1)} \\
&\geq (\gamma-\beta)L^{-d} - \frac{3}{2}\gamma L^{-(d+1)} \geq \frac{\gamma-\beta}{2L^d}.
\end{aligned}
\tag{1.20}
$$

Choosing $L_0 = \lfloor \frac{4\gamma}{\gamma-\beta} \rfloor$ we get for $L \geq L_0$ that $a_L \geq \frac{(\gamma-\beta)^{d+1}}{2(4\gamma)^d} > 0$.

Let $L \in \mathbb{N}$ and fix $g_L^* \in \mathscr{G}_L^*$. For $x, y \in \{0, \dots, L-1\}^d$ and $i, j \in \{1, \dots, d\}$ such that $(g_L^*)^i(x) > 0$ and for $\varepsilon \in [0, 1]$ define $h_\varepsilon = (h_\varepsilon^1, \dots, h_\varepsilon^d)$ by

$$
h_\varepsilon^k(z) = \begin{cases} (g_L^*)^j(y) + \varepsilon, & k = j,\ z = y, \\ (g_L^*)^i(x) - \varepsilon, & k = i,\ z = x, \\ (g_L^*)^k(z), & \text{else}. \end{cases}
\tag{1.21}
$$

Clearly $h_\varepsilon \in \mathscr{G}_L$ for $\varepsilon \in [0, (g_L^*)^i]$ and $h_0 = g_L^*$. Thus

$$
0 \geq \frac{d}{d\varepsilon}\langle h_\varepsilon, Gh_\varepsilon\rangle|_{\varepsilon=0} = 2\big[(Gg_L^*)^j(y) - (Gg_L^*)^i(x)\big].
\tag{1.22}
$$

If also $(g_L^*)^j(y) > 0$, then the reverse inequality holds. Hence there exists $b_L \in \mathbb{R}$ such that

$$
(Gg_L^*)^i(x) = b_L, \qquad x \in \operatorname{supp}(g_L^*)^i, \qquad i = 1, \dots, d.
\tag{1.23}
$$

It follows that $a_L = \langle g_L^*, Gg_L^*\rangle = \langle g_L^*, b_L\mathbf{1}\rangle = b_L$. G is a continuous operator on $l^1(\mathbb{Z}^d \times \{1, \dots, d\})$, $\|G\| < \infty$, with entries of absolute value not exceeding $d\gamma$. Furthermore, G is translation invariant and symmetric and $G((x,i),(y,j)) = 0$, $|x-y| \geq 3$. Thus $\#\{(y,j): G((x,i),(y,j)) > 0\} \leq d5^d$. This implies

$$
\|(Gg_L^*)^i\|_1 \leq \gamma d^2 5^d \|g_L^*\|_1 = \gamma d^2 5^d.
$$

Thus $|\operatorname{supp}(g_L^*)^i| \leq \gamma d^2 5^d / a_L$. Using the estimate for a_L we get

$$
|\operatorname{supp}(g_L^*)| \leq \frac{d^3}{10}\left(\frac{20\gamma}{\gamma-\beta}\right)^{d+1} < \infty.
\tag{1.24}
$$

We must exclude the possibility that $\mathrm{supp}(g_L^*)$ has large gaps. For $y \in \mathbb{Z}$ and $i \in \{1, \dots, d\}$ define

$$H^{i,-}(y) = \{x \in \mathbb{Z}^d \colon x^i \le y\},\qquad H^{i,+}(y) = \{x \in \mathbb{Z}^d \colon x^i \ge y\} \tag{1.25}$$

and

$$H^i(y) = H^{i,-}(y) \cap H^{i,+}(y).$$

Assume that the support of $g \in \mathscr{G}_L$ has a gap of three hyperplanes,

$$g(x) = 0, \qquad x \in H^i(y-1) \cup H^i(y) \cup H^i(y+1),$$

for some $y \in \mathbb{Z}$. Then define g^+ and g^- by $g^-(x) = g(x)\mathbf{1}_{H^{i,-}(y)}(x)$ and $g^+(x) = g(x)\mathbf{1}_{H^{i,+}(y)}(x)$. Since $G((x, j), (x', j')) = 0$ if $|x - x'| \ge 3$ we get

$$\begin{aligned} \langle g, Gg\rangle &= \langle g^- + g^+, G(g^- + g^+)\rangle = \langle g^-, Gg^-\rangle + \langle g^+, Gg^+\rangle \\ &\le [\|g^-\|^2 + \|g^+\|^2]\langle g_L^*, Gg_L^*\rangle. \end{aligned} \tag{1.26}$$

[Recall the norm $\|g\|$ from (1.12).] If $g^- \ne 0$ and $g^+ \ne 0$ then $\langle g, Gg\rangle < \langle g_L^*, Gg_L^*\rangle$. Hence we can rule out the possibility that g_L^* has gaps of more than two hyperplanes in the support. This implies that (up to translations)

$$\mathrm{supp}(g_L^*) \subset \{0, \dots, 3|\mathrm{supp}(g_L^*)|\}^d \subset \{0, \dots, L_0\}^d, \tag{1.27}$$

where $L_0 = \lfloor (\frac{20\gamma}{\gamma-\beta})^{d+1} d^3 \rfloor$ is independent of L. Now, if $L \ge L_0$, then $g_L^* \in \mathscr{G}_{L_0}^*$ which completes the proof. $\square$

Proof of Proposition 1.1. Let $a = \langle f^*, Ff^*\rangle$ as in the proof of the previous lemma and let L_0 be such that $\mathrm{supp}(f^*) \subset \{0, \dots, L_0 - 1\}^d$. The upper bound of (1.9) is immediate from the fact that $\tilde{\ell}_n = \ell_n - \frac{1}{2}\mathbf{1}_{\{S_0\}} - \frac{1}{2}\mathbf{1}_{\{S_n\}} \in \mathscr{F}$, the fact that absolute values of the entries of $F_{\beta,\gamma}$ are bounded from above by γ and that $\sum_{x\sim S_i} \tilde{\ell}_n(x) \le n$ for $i = 0, n$. Let $C = (2d+2)\gamma L_0^d$. Similarly, the lower bound in (1.9) follows from Lemma 1.2, the fact that the entries of the matrix F have absolute values not exceeding γ and that $\tilde{\ell}_n(x) \le n$ for all $x \in \mathbb{Z}^d$. $\square$

In dimension $d = 1$ we can show that the maximizers g^* and f^* are unique and we determine their shapes. Unfortunately, we are not able to give the exact maximizer but we can give a class of functions in which this maximizer lies. This class is indexed by the size L of the support of these functions. We have a conjecture for the size of the optimal L, but rigorously we can only use the previous lemmas to get bounds on the optimal L that are rather poor.

Lemma 1.4. *Assume $d = 1$. Let $\omega = \arccos(\beta/\gamma)$ and define for $L \in \mathbb{N}$ the function g_L with support in $\{0, \dots, L-1\}$ and $\|g_L\| = 1$ for $x \in \{0, \dots, L-1\}$*

by

$$g_L(x) = \begin{cases} b\Big[\alpha_0 + \alpha_1\Big(1 - \dfrac{2}{L-1}x\Big)(-1)^x \\ \qquad + \cos\Big(\omega\Big(x - \dfrac{L-1}{2}\Big)\Big)\Big], & L \text{ even}, \\ b\Big[\alpha_0 + \alpha_1(-1)^x + \cos\Big(\omega\Big(x - \dfrac{L-1}{2}\Big)\Big)\Big], & L \text{ odd}, \end{cases} \tag{1.28}$$

where $b > 0$ is a normalizing constant and

$$\alpha_0 = \begin{cases} \dfrac{1}{2(L+2)}\Big((L+1)\cos\Big(\omega\dfrac{L+3}{2}\Big) \\ \qquad + (L+3)\cos\Big(\omega\dfrac{L+1}{2}\Big)\Big), & L \text{ even}, \\ -\cos\Big(\dfrac{\omega}{2}\Big)\cos\Big(\dfrac{\omega}{2}(L+2)\Big), & L \text{ odd}, \end{cases} \tag{1.29}$$

$$\alpha_1 = \begin{cases} \dfrac{1}{L+2}\sin\Big(\dfrac{\omega}{2}\Big)\sin\Big(\dfrac{\omega}{2}(L+2)\Big), & L \text{ even}, \\ \sin\Big(\dfrac{\omega}{2}\Big)\sin\Big(\dfrac{\omega}{2}(L+2)\Big), & L \text{ odd}. \end{cases} \tag{1.30}$$

Then $g^* = g_L$ for some $L \in \mathbb{N}$.

PROOF. Refining the argument of (1.26) we see that in $d = 1$ the support has no gaps at all, hence for any L there exists $K \le L$ such that $\operatorname{supp}(g_L^*) = \{0, \dots, K-1\}$. If $L > K$ then $g_L^* = g_K^* = g^*$.

We will next solve the equation

$$(Gg_L)(x) = c \quad \text{for all } x \in \{0, \dots, L-1\},$$

for h_L with $\operatorname{supp}(h_L) = \{0, \dots, L-1\}$. Clearly, $(G\mathbf{1}_L)(x) = \gamma - \beta$ for all $x \in \{0, \dots, L-1\}$, where we define $\mathbf{1}_L(x) = \mathbf{1}_{\{-2,\dots,L+1\}}(x)$. Hence,

$$g_L(x) = b[\mathbf{1}_L(x) + f(x)],$$

where $(Gf)(x) = 0$ for all $x \in \{0, \dots, L-1\}$, whereas $f(-2) = f(-1) = f(L) = f(L+1) = -1$. However, $(Gf)(x) = 0$ for all $x \in \{0, \dots, L-1\}$ if and only if

$$f \in \big\{x \mapsto \big(\xi_1(-1)^x + \xi_2 x(-1)^x + \xi_3\cos(\omega x) + \xi_4\sin(\omega x)\big), \xi_1, \dots, \xi_4 \in \mathbb{R}\big\},$$

where the right-hand side is considered to be a set of functions on $\{-2, \dots, L+1\}$. Indeed, the above can easily be checked by using that

$$G(x, y) = \begin{cases} \dfrac{\gamma}{4} - \dfrac{\beta}{2}, & \text{for } x = y, \\ \dfrac{\gamma}{4} - \dfrac{\beta}{4}, & \text{for } |x - y| = 1, \\ \dfrac{\gamma}{8}, & \text{for } |x - y| = 2, \\ 0, & \text{else.} \end{cases}$$

The solution for $\xi_1, \dots, \xi_4$ is given in the statement of the lemma. Next, $b = b_L$ and $c = c_L$ are determined by

$$\|g_L\| = 1 \quad \text{and} \quad c = b(\gamma - \beta).$$

Consequently, we find the optimal L by maximizing c_L over all L such that $g_L(x) > 0$ for all $x \in \{0, \dots, L-1\}$, and we see that $g^* = g_L$ for this value of L. □

REMARK. Numerical computations suggest that the optimal choice is $L = [\frac{2\pi}{\omega}] - 2 \sim \pi\sqrt{2}\sqrt{(\gamma/\gamma - \beta)}$, $\gamma - \beta \downarrow 0$. This is consistent with the corresponding optimization problem in continuous space where the optimizer is $(1 - \cos(\omega x))\mathbf{1}_{[0, 2\pi/\omega]}(x)$. However, we have not been able to prove this. Note that for the correct choice of L, automatically $g_L = g^* \geq 0$. However, it need not be that the maximal L with this property is the correct choice. This makes it difficult to determine L analytically.

We give the following corollary of Lemma 1.4 that determines the shape of the maximizer f^* in dimension 1.

COROLLARY 1.5. *Assume $d = 1$. The maximizer f^* has the form $f_L \in \mathscr{F}_L$ for some $L \in \mathbb{N}$, where*

$$f_L(x) = \begin{cases} b\left[\alpha_0 + \dfrac{\alpha_1}{L-1}(-1)^x + \cos\left(\dfrac{\omega}{2}\right)\cos\left(\omega\left(x - \dfrac{L-2}{2}\right)\right)\right], & L \text{ even}, \\ b\left[\alpha_0 + \cos\left(\dfrac{\omega}{2}\right)\cos\left(\omega\left(x - \dfrac{L-2}{2}\right)\right)\right], & L \text{ odd}, \end{cases} \tag{1.31}$$

where b, α_0 and α_1 are defined as in Lemma 1.4.

With Proposition 1.1 at hand it is not difficult to prove localization of the polymer. In fact, we can show immediately the stronger statement of (0.10).

PROOF OF THEOREM 1(ii). We may assume for convenience that $\frac{1}{3}L \in \mathbb{N}$. Recall that $\tilde{\ell}_n$ is the averaged local time of the path defined in (1.10).

For any $x \in \mathbb{Z}^d$, let x^i be the ith component, $i = 1, \dots, d$. Let $(S_i)_{i=0}^n$ be a random walk path from the event in (0.10). Without loss of generality we may assume that $S_i^1 > L$ for some $i \in \{0, \dots, n\}$. For $y \in \mathbb{Z}$, define the hyperplane [recall the notation of (1.25)]

$$H^1(y) = \{x \in \mathbb{Z}^d \colon x^1 = y\}. \tag{1.32}$$

Since $\sum_{y=L/3}^{2L/3} \sum_{x \in H^1(y)} \tilde{\ell}_n(x) \le n$, there exists a $y_0 \in \{\frac{L}{3} + 1, \dots, \frac{2L}{3} - 1\}$ with

$$\sum_{x \in H^1(y_0-1) \cup H^1(y_0) \cup H^1(y_0+1)} \tilde{\ell}_n(x) \le \frac{9n}{L-3}. \tag{1.33}$$

The next step is to decompose the path into the pieces that are "left" of $H^1(y_0)$ and "right" of $H^1(y_0)$. Of course, a path may re-enter a half-space at a different place than where it left it. Thus, rather than one path we get a collection of paths in the left and right half-space.

Here are the precise definitions. Define the random times τ_k^- and τ_k^+, $k \in \mathbb{N}$ inductively by $\tau_1^- = 0$ and

$$\begin{aligned} \tau_k^+ &= \inf\{m \ge \tau_k^-;\ S_m^1 > y_0\} - 1, \\ \tau_{k+1}^- &= \inf\{m \ge \tau_k^+;\ S_m^1 < y_0\} - 1. \end{aligned} \tag{1.34}$$

Consider now the families of random walk paths

$$\begin{aligned} &\big\{ (S_i^{k,-})_{i=0,\dots,(\tau_k^+ \wedge n) - \tau_k^-},\ k \in \mathbb{N},\ \tau_k^- < n \big\}, \\ &\big\{ (S_i^{k,+})_{i=0,\dots,(\tau_{k+1}^- \wedge n) - \tau_k^+},\ k \in \mathbb{N},\ \tau_k^+ < n \big\}, \end{aligned} \tag{1.35}$$

where $S_i^{k,-} = S_{i+\tau_k^-}$, and $S_i^{k,+} = S_{i+\tau_k^+}$. Define the associated averaged local times $\tilde{\ell}^-$ and $\tilde{\ell}^+$, so that $\tilde{\ell}_n = \tilde{\ell}^- + \tilde{\ell}^+$. Note that $\tilde{\ell}^-$ and $\tilde{\ell}^+$ are supported by $H^{1,-}(y_0)$ and $H^{1,+}(y_0)$. Let $m^\pm = \|\tilde{\ell}^\pm\|_1$, hence $n = m^- + m^+$. Furthermore, clearly $\tilde{\ell}^\pm / m^\pm \in \mathscr{F}$, so that by Lemma 1.4 we have

$$\langle \tilde{\ell}^\pm, F\tilde{\ell}^\pm \rangle \le a[m^\pm]^2. \tag{1.36}$$

Furthermore, use (1.33) and assume $L \ge 30$ [which implies $\sum_{x \in H^1(y_0 \pm 1)} \tilde{\ell}^\pm(x) \le \frac{9n}{L-3} \le \frac{10n}{L}$] to get

$$\begin{aligned} \langle \tilde{\ell}^-, F\tilde{\ell}^+ \rangle &= \langle \tilde{\ell}^+, F\tilde{\ell}^- \rangle \\ &= \frac{\gamma}{2d} \sum_{x \in H^1(y_0)} \big[\tilde{\ell}^-(x - e_1)\tilde{\ell}^+(x) + \tilde{\ell}^-(x)\tilde{\ell}^+(x + e_1) \big] \\ &\le \frac{\gamma}{2d} \bigg(\sum_{x \in H^1(y_0-1)} \tilde{\ell}^-(x) \sum_{x \in H^1(y_0)} \tilde{\ell}^+(x) \\ &\qquad + \sum_{x \in H^1(y_0)} \tilde{\ell}^-(x) \sum_{x \in H^1(y_0+1)} \tilde{\ell}^+(x) \bigg) \\ &\le \frac{10\gamma}{d} \frac{n}{L} [m^- \wedge m^+]. \end{aligned} \tag{1.37}$$

Note that by construction $m^- \wedge m^+ \geq \frac{L}{3}$ and thus $m^- m^+ \geq [m^- \wedge m^+] \cdot \frac{n}{2} \geq \frac{nL}{6}$. Hence,

$$\begin{aligned}
\langle \ell_n, F\ell_n\rangle &\leq \langle \tilde{\ell}_n, F\tilde{\ell}_n\rangle + \frac{\gamma}{2d}n \\
&= \langle \tilde{\ell}^-, F\tilde{\ell}^-\rangle + \langle \tilde{\ell}^+, F\tilde{\ell}^+\rangle + 2\langle \tilde{\ell}^-, F\tilde{\ell}^+\rangle + \frac{\gamma}{2d}n \\
&\leq a[(m^-)^2 + (m^+)^2] + \frac{20\gamma n}{dL}[m^- \wedge m^+] + \frac{\gamma}{2d}n \\
&= an^2 - am^-m^+ - Cn + \left(C + \frac{\gamma}{2d} - \frac{am^-m^+}{2n}\right)n \\
&\quad + \left(\frac{40\gamma}{dL}\frac{n}{2}\frac{m^- \wedge m^+}{m^-m^+} - \frac{a}{2}\right)m^-m^+ \\
&\leq an^2 - am^-m^+ - Cn + \left(C + \frac{\gamma}{2d} - \frac{La}{12}\right)n \\
&\quad + \left(\frac{40\gamma}{dL} - \frac{a}{2}\right)m^-m^+.
\end{aligned} \tag{1.38}$$

Now for $L \geq a^{-1}(12C + 80\gamma/d)$ the last two terms on the r.h.s. of (1.38) are negative. Letting $c = a/12$ we get

$$\begin{aligned}
\langle \ell_n, F\ell_n\rangle &\leq an^2 - Cn - 2cLn \\
&= an^2 - Cn - cLn - \frac{aL}{12}n \\
&\leq M_n - cLn - \frac{aL}{12}n.
\end{aligned} \tag{1.39}$$

Finally, assume in addition $L \geq a^{-1}12\log(2d)$ and use the fact that $Z_n^{\beta,\gamma} \geq (2d)^{-n}e^{M_n}$ to conclude

$$\begin{aligned}
Q_n^{\beta,\gamma}(\exists\, i \leq n\colon S_i \notin C_L) &\leq \left(Z_n^{\beta,\gamma}\right)^{-1}\exp(an^2 - cLn - (aL/12)n) \\
&\leq (2d)^n \exp(-cLn - (aL/12)n) \leq e^{-cLn}.
\end{aligned} \tag{1.40}$$

□

1.3. *Proof of the shape theorem.* With the estimates at hand about the partition function and the maximizers from the previous subsection it is not too hard to prove the shape theorem in the case where the number of maximizers is finite. This proves the shape theorem for dimension $d = 1$ by Lemma 1.4. For $d \geq 2$ we do not have an analogue for Lemma 1.4 that proves uniqueness of the maximizer.

Here is the quick argument that works whenever we have a finite number of maximizers of the quadratic functional $\mathscr{F} \to \mathbb{R}$, $f \mapsto \langle f, Ff\rangle$. Let f^* be a maximizer of the quadratic functional $f \mapsto \langle f, Ff\rangle$ from $\mathscr{F} \to \mathbb{R}$; f^* has finite support uniformly for all maximizers f^*. Let L and c be as in Theorem 1(ii).

We may assume that L is large enough such that $\operatorname{supp}(f^*) \subset C_L$ for all maximizers f^* for which the origin is in the support. By Theorem 1(ii) it suffices to consider paths $(S_i)_{i=0}^n$ which are contained in C_L for some sufficiently large but fixed L.

Define $V = \{f \in \mathscr{F}\colon \operatorname{supp}(f) \subset C_L\}$, $V^* = \{f^*; f^* \text{ maximizes } \langle f, Ff\rangle\} \cap V$, and $V_1 = \{f \in V\colon \|f\|_1 = 1\}$. Note that $V^* \subset V_1$ and that (with a from Proposition 1.1)

$$a = \sup_{f \in V_1} \langle f, Ff\rangle = \langle f^*, Ff^*\rangle \quad \text{for all } f^* \in V^*.$$

By definition we have $\langle f, Ff\rangle < a$ if $f \in V_1 \setminus V^*$. By assumption, V^* is a finite set and $f \mapsto \langle f, Ff\rangle$ is quadratic. Thus, there exists an open neighborhood N of V^* and a $c' > 0$ such that for $f \in N$,

$$\langle f, Ff\rangle - a \le -c' \min_{f^* \in V^*} \|f - f^*\|_1^2. \tag{1.41}$$

On the other hand, $V_1 \setminus N$ is compact and hence

$$\sup_{f \in V_1 \setminus N} \langle f, Ff\rangle < a.$$

Thus (maybe by making c' a little smaller), (1.41) holds for all $f \in V_1$.

Recall C from Proposition 1.1, assume $\xi > ((C + \log(2d))/c')^{1/2}$, and let $c'' = \min(cL, c'\xi^2 - (C + \log(2d))) > 0$. Hence by Theorem 1(ii),

$$\begin{aligned}
&Q_n^{\beta,\gamma}\Big(\inf_{f^* \in V^*} \|\ell_n - nf^*\|_1 > \xi n^{1/2}\Big)\\
&\quad \le e^{-cLn} + Q_n^{\beta,\gamma}\Big(\inf_{f^* \in V^*} \|\ell_n - nf^*\|_1 > \xi n^{1/2};\ \operatorname{supp}(\ell_n) \subset C_L\Big)\\
&\quad \le e^{-cLn} + (2d)^n e^{Cn} e^{-c'\xi^2 n}\\
&\quad \le 2e^{-c''n}.
\end{aligned}$$

□

2. Speed and variance in dimension 1. In the rest of the paper we only consider $d = 1$ and $\gamma < \beta$. As explained earlier, in this case we get the best results due to the availability of a particularly powerful method. Before we start with the details we give an outline of the method and some heuristics.

Greven and den Hollander (1993) identified the speed of a polymer in the case $\gamma = 0$. A similar method was used in König (1996) to prove the central limit theorem (Theorem 0) for $\gamma = 0$. We give a nonrigorous sketch of the underlying ideas of their work in order to motivate this and the next three sections.

Let us assume $\gamma = 0$ and that the end-to-end distance grows like $\theta^* n$ as $n \to \infty$, for some $\theta^* \in (0, 1]$. We want to identify the speed θ^* and the exponential rate of the normalizing constant $r^* = r^*(\beta)$. With equal probability the polymer extends to the left or right of the origin. Without loss of generality we assume that it extends to the right. Assume that n is very large and that S_n is precisely $\lfloor \theta^* n \rfloor$. In the subsequent heuristic argument we neglect all boundary effects coming from local times left of 0 and right of $\lfloor \theta^* n \rfloor$. Hence the local times $(\ell_n(x))_{x=0}^{\lfloor \theta^* n \rfloor}$ form a stationary (non-Markov) sequence and we should have $1/\theta^* = E_{Q_n^{\beta,0}}(\ell_n(x))$. Let $m(x)$ be the number of up-crossings from x to $x+1$. Hence $\ell_n(x) = m(x-1) + m(x) - 1$. Note that the stationary sequence $(m(x))_{0 \le x \le \lfloor \theta^* n \rfloor}$ *is* Markov. In order to determine θ^* we have to obtain information on the stationary distribution of m under $Q_n^{\beta,0}$.

Knight's theorem (see Section 3) relates the up-crossings of simple random walk to a critical Galton–Watson branching process with geometric offspring distribution and one immigrant per generation. This process has the transition matrix

$$P(i, j) = \binom{i+j-2}{i-1} \left(\frac{1}{2}\right)^{i+j-1}. \tag{2.1}$$

Our polymer is a random walk with interaction. There is a penalty of $e^{-\beta \sum_x \ell_n(x)^2} = \prod_x e^{-\beta(m(x)+m(x+1)-1)^2}$. The normalizing constant $Z_n^{\beta,0} = E \times (e^{-\beta \sum_{x \in \mathbb{Z}} \ell_n(x)^2})$ behaves like $e^{-r^*(n+1)} = e^{-r^* \sum_{x \in \mathbb{Z}} \ell_n(x)} = \prod_x e^{-r^*(m(x)+m(x+1)-1)}$. We want to identify r^*. Again neglecting boundary effects, we can write

$$e^{r(n+1)} Z_n^{\beta,0} \approx \sum \prod_x A_{r,\beta}(m(x), m(x+1)),$$

where we sum over all sequences $(m(x))$ such that $\sum_{x=0}^{\lfloor \theta^* n \rfloor}(m(x)+m(x+1)-1) = \sum_x \ell_n(x) = n$, and we define

$$A_{r,\beta}(i, j) = e^{r(i+j-1) - \beta(i+j-1)^2} P(i, j), \qquad i, j \in \mathbb{N} \text{ and } r \in \mathbb{R}.$$

Then we can write

$$\begin{aligned} e^{r^*(n+1)} Z_n^{\beta,0} &\approx \sum \prod_x A_{r^*,\beta}(m(x), m(x+1)) \\ &= \sum \frac{\tau_{r^*}(m(0))}{\tau_{r^*}(m(\lfloor \theta^* n \rfloor))} \prod_x P_\beta(m(x), m(x+1)). \end{aligned} \tag{2.2}$$

Here P_β is defined by

$$P_\beta(i, j) = A_{r^*,\beta}(i, j) \frac{\tau_{r^*}(j)}{\tau_{r^*}(i)},$$

where $r^* \in \mathbb{R}$ has to be chosen appropriately, and $\tau_r \in l^2(\mathbb{N})$ is the unique positive and normalized eigenvector of $A_{r,\beta}$ corresponding to the largest eigenvalue $\lambda(r, \beta)$. The ratio of τ_{r^*}'s in (2.2) is of order 1. If r^* is the exponential rate of $Z_n^{\beta,0}$, then $e^{r^*(n+1)} Z_n^{\beta,0}$ has exponential rate 0. This is the case if and only if P_β is a stochastic matrix. Therefore its largest eigenvalue $\lambda(r^*, \beta) = \sum_j P_\beta(i, j)$ must equal 1. Thus we have to pick r^* such that $\lambda(r^*, \beta) = 1$. Furthermore, it is easy to check that $\tau_{r^*}^2$ is the invariant distribution of $(m(x))$

under P_β. Therefore,

$$
\begin{aligned}
\frac{1}{\theta^*} &= E_{\tau^2_{r^*}}(m(x)+m(x+1)-1) = \sum_{i,\,j}(i+j-1)\tau^2_{r^*}(i)P_\beta(i,\,j)\\
&= \frac{d}{dr}\sum_{i,\,j}\tau_r(i)A_{r,\,\beta}(i,\,j)\tau_r(j)\bigg|_{r=r^*} \qquad (2.3)\\
&= \frac{d}{dr}\lambda(r,\,\beta)\bigg|_{r=r^*}.
\end{aligned}
$$

Computing the second derivatives yields an expression for the variance σ^{*2}.

The situation in the present paper is somewhat more involved. There are interactions between monomers on neighboring sites. Hence, the $m(x)$-chain under the transformed measure does not form a Markov process. This forces us to consider a bivariate process of the type $((m(x), m(x+1))_{x\in\mathbb{N}}$. The analogue of the matrix $A_{r,\,\beta}$ becomes a matrix $A_{r,\,\beta,\,\gamma}(\mathbf{i},\mathbf{j})$, $\mathbf{i},\mathbf{j}\in\mathbb{N}^2$.

The program for the rest of this paper is as follows. In this section we define $A_{r,\,\beta,\,\gamma}$, we show analyticity of the largest eigenvalue $\lambda(r,\,\beta,\,\gamma)$ and define the quantities r^*, θ^* and σ^*. The methods employed are adapted from Greven and den Hollander (1993) and Baillon, Clément, Greven and den Hollander (1994).

In Section 3 we quote Knight's theorem, introduce the bivariate branching chain $\mathfrak{M}$ [in (3.4)], and formulate the connection of the end-to-end distance of the polymer to exponential functionals of $\mathfrak{M}$ (Lemma 3.1).

In Section 4 we construct for every $r\in\mathbb{R}$ a positive recurrent bivariate chain with the equilibrium distribution corresponding to $\tau^{(R)}_{r,\,\beta,\,\gamma}\tau^{(L)}_{r,\,\beta,\,\gamma}$ of $A_{r,\,\beta,\,\gamma}$ (Lemma 4.1). We write the Laplace transform of the end-to-end distance in terms of this chain (Lemma 4.2 and 4.3). Recall that in the heuristics we used that the sequence of local times was stationary (no boundary effects). Lemma 4.3 would lead directly to the proof of the CLT if we really had stationarity. It is the content of Proposition 4.4 and 4.5 to show that the boundary terms are negligible, hence showing the asymptotic stationarity of the local times as $n\to\infty$. In Proposition 4.4 we state pointwise convergence to the equilibrium while Proposition 4.5 states summability of the boundary terms needed for a dominated convergence argument. To prove the summability we have to make the assumption $\gamma\le\beta-\frac{1}{2}\log 2$. At the end of Section 4, dominated convergence and the ideas leading to (2.3) are combined to a proof of Theorem 3.

2.1. *Defining speed and variance.* Now we come to the technical details. For $\mathbf{i}=(\mathbf{i}_1,\mathbf{i}_2)\in\mathbb{N}^2$ let

$$s(\mathbf{i})=\mathbf{i}_1+\mathbf{i}_2-1 \quad\text{and}\quad s^*(\mathbf{i})=\mathbf{i}_1+\mathbf{i}_2. \qquad (2.4)$$

For $r \in \mathbb{R}$ and $\beta, \gamma \in \mathbb{R}^+$ we define the matrix $A_{r,\beta,\gamma}$ by [recall (2.1)]

$$A_{r,\beta,\gamma}(\mathbf{i},\mathbf{j}) = e^{(r/2)[s(\mathbf{i})+s(\mathbf{j})]-(\beta/2)[s(\mathbf{i})^2+s(\mathbf{j})^2]+\gamma s(\mathbf{i})s(\mathbf{j})} \times \mathbf{1}_{\{\mathbf{i}_2=\mathbf{j}_1\}}\sqrt{P(\mathbf{i}_1,\mathbf{i}_2)P(\mathbf{j}_1,\mathbf{j}_2)} \qquad (\mathbf{i},\mathbf{j} \in \mathbb{N}^2). \tag{2.5}$$

Define $\lambda(r,\beta,\gamma)$ to be the unique largest eigenvalue of $A_{r,\beta,\gamma}$ in $l^2(\mathbb{N}^2)$. The analytic heart of this section are the following propositions. They are needed to define r^*, θ^* and σ^*.

PROPOSITION 2.1 (Unique maximal eigenvector). *Fix $0 \le \gamma < \beta$. Then*:

(i) *The operator $A_{r,\beta,\gamma}$: $l^2(\mathbb{N}^2) \to l^2(\mathbb{N}^2)$ is compact and has nonnegative matrix elements for all $r \in \mathbb{R}$. Its square $A^2_{r,\beta,\gamma}$ has strictly positive matrix entries*.

(ii) *For every $r \in \mathbb{R}$, there exist unique left and right eigenvectors $\tau^{(L)}_{r,\beta,\gamma}$, $\tau^{(R)}_{r,\beta,\gamma} \in l^2(\mathbb{N}^2)$ corresponding to $\lambda(r,\beta,\gamma)$ and normalized in $l^2(\mathbb{N}^2)$. Moreover, $\tau^{(L)}_{r,\beta,\gamma}(i_1,i_2) = \tau^{(R)}_{r,\beta,\gamma}(i_2,i_1) > 0$ for all $i_1, i_2 \in \mathbb{N}$. In particular*,

$$p_{r,\beta,\gamma} := \left\langle \tau^{(L)}_{r,\beta,\gamma}, \tau^{(R)}_{r,\beta,\gamma} \right\rangle > 0. \tag{2.6}$$

We will need the following properties of the dependence of the maximal eigenvalue on its parameters.

PROPOSITION 2.2 (Analyticity of the maximal eigenvalue).

(i) *The map $(r,\beta,\gamma) \mapsto \lambda(r,\beta,\gamma)$ is analytic on $\{(r,\beta,\gamma) \in \mathbb{R} \times [0,\infty)^2;\ \beta > \gamma\}$.*

(ii) *The map $r \mapsto \lambda(r,\beta,\gamma)$ is strictly increasing and strictly log-convex, $\lambda(0,\beta,\gamma) \le e^{-(\beta-\gamma)}$ and $\lim_{r\to\infty} \lambda(r,\beta,\gamma) = \infty$.*

(iii) *The map $r \mapsto p_{r,\beta,\gamma}$ is continuous.*

For fixed $0 \le \gamma < \beta$ let $r^* = r^*(\beta,\gamma)$ be the unique solution of

$$\lambda(r^*,\beta,\gamma) = 1. \tag{2.7}$$

Now we are in the position to define θ^* and σ^*.

DEFINITION 2.3. We define the speed $\theta^* = \theta^*(\beta,\gamma)$ and the spread $\sigma^* = \sigma^*(\beta,\gamma)$ of the random polymer by

$$\theta^* = \left[\frac{\partial}{\partial r}\lambda(r,\beta,\gamma)\right]^{-1}_{r=r^*}, \qquad \sigma^{*2} = \theta^{*3}\left[\frac{\partial^2}{\partial r^2}\lambda(r,\beta,\gamma) - \frac{1}{\theta^{*2}}\right]_{r=r^*}. \tag{2.8}$$

2.2. *Proof of Proposition* 2.1. We will prove the different parts of Proposition 2.1 one by one. For notational convenience we suppress the (β,γ)-dependence in the notation where no ambiguities may occur, and write $A_r = A_{r,\beta,\gamma}$, $\lambda_r = \lambda(r,\beta,\gamma)$, $\tau^{(L)}_r = \tau^{(L)}_{r,\beta,\gamma}$ and $\tau^{(R)}_r = \tau^{(R)}_{r,\beta,\gamma}$.

PART (I). If $0 \le \gamma < \beta$, or $0 \le \gamma \le \beta$ and $r < 0$, then A_r is a Hilbert–Schmidt matrix. To see this, we estimate the Hilbert–Schmidt norm $||A_r||_{\mathrm{HS}}$ as

$$\begin{aligned}
||A_r||^2_{\mathrm{HS}} &= \sum_{\mathbf{i},\mathbf{j}\in\mathbb{N}^2} A_r(\mathbf{i},\mathbf{j})^2 \\
&= \sum_{\mathbf{i},\mathbf{j}\in\mathbb{N}^2} e^{r[s(\mathbf{i})+s(\mathbf{j})]-\beta[s(\mathbf{i})^2+s(\mathbf{j})^2]+2\gamma s(\mathbf{i})s(\mathbf{j})} \\
&\qquad\qquad \times \mathbf{1}\{\mathbf{i}_2=\mathbf{j}_1\}P(\mathbf{i}_1,\mathbf{i}_2)P(\mathbf{j}_1,\mathbf{j}_2) \\
&\le \sum_{\mathbf{i},\mathbf{j}\in\mathbb{N}^2} e^{r[s(\mathbf{i})+s(\mathbf{j})]-(\beta-\gamma)[s(\mathbf{i})^2+s(\mathbf{j})^2]-\gamma(s(\mathbf{i})-s(\mathbf{j}))^2} P(\mathbf{i}_1,\mathbf{i}_2)P(\mathbf{j}_1,\mathbf{j}_2) \qquad (2.9)\\
&\le \sum_{\mathbf{i},\mathbf{j}\in\mathbb{N}^2} e^{r[s(\mathbf{i})+s(\mathbf{j})]-(\beta-\gamma)[s(\mathbf{i})^2+s(\mathbf{j})^2]} P(\mathbf{i}_1,\mathbf{i}_2)P(\mathbf{j}_1,\mathbf{j}_2) \\
&= \left(\sum_{\mathbf{i}\in\mathbb{N}^2} e^{rs(\mathbf{i})-(\beta-\gamma)s(\mathbf{i})^2} P(\mathbf{i}_1,\mathbf{i}_2)\right)^2 < \infty.
\end{aligned}$$

This implies that A_r: $l^2(\mathbb{N}^2) \mapsto l^2(\mathbb{N}^2)$ is a compact operator [see, e.g., Yosida (1980), Chapter X.2, Example 2].

The fact that $A^2_{r,\beta,\gamma}(\mathbf{i},\mathbf{j}) > 0$ for all $\mathbf{i},\mathbf{j} \in \mathbb{N}^2$ is easiest to see by writing down the explicit formula for $A^2_{r,\beta,\gamma}(\mathbf{i},\mathbf{j})$. This is left to the reader.

PART (II). Since $A = A_{r,\beta,\gamma}$ is compact on $l^2(\mathbb{N}^2)$, it has unique positive left and right eigenvectors $\tau^{(L)} = \tau^{(L)}_{r,\beta,\gamma}$ and $\tau^{(R)} = \tau^{(R)}_{r,\beta,\gamma}$ corresponding to $\lambda = \lambda(r,\beta,\gamma)$, normalized such that $\langle\tau^{(L)},\tau^{(L)}\rangle = \langle\tau^{(R)},\tau^{(R)}\rangle = 1$.

Define $\bar{\mathbf{i}} = (\mathbf{i}_2,\mathbf{i}_1)$ for $\mathbf{i} \in \mathbb{N}^2$. To see $\tau^{(L)}(\mathbf{i}) = \tau^{(R)}(\bar{\mathbf{i}})$, we note that the transposed matrix A^* fulfills the relation $A^*(\mathbf{i},\mathbf{j}) = A(\bar{\mathbf{i}},\bar{\mathbf{j}})$ for all $\mathbf{i},\mathbf{j} \in \mathbb{N}^2$. Thus $\mathbf{i} \mapsto \tau^{(R)}(\bar{\mathbf{i}})$ is the left eigenvector with eigenvalue λ. Since A^2 is strictly positive the same is true for the eigenvectors which proves the final claim in Part (ii). □

2.3. *Proof of Proposition* 2.2.

PART (I). Let

$$A^{(N)}_r = (A_r(\mathbf{i},\mathbf{j})\mathbf{1}_{\{1,\dots,N\}^4}(\mathbf{i},\mathbf{j}))_{\mathbf{i},\mathbf{j}\in\mathbb{N}^2}$$

be the restriction of A to $l^2(\{1,\dots,N\}^2)$ and denote by $\lambda^{(N)}_r$ its largest eigenvalue. Clearly $(r,\beta,\gamma) \mapsto \lambda^{(N)}(r,\beta,\gamma)$ is analytic for each N. Furthermore,

$$|\lambda(r,\beta,\gamma) - \lambda^{(N)}(r,\beta,\gamma)| \le \left\|A_{r,\beta,\gamma} - A^{(N)}_{r,\beta,\gamma}\right\|_{\mathrm{HS}}.$$

A calculation similar to (2.9) shows that the latter quantity converges to 0 as $N \to \infty$ uniformly in (r, β, γ) on compact subsets of $\mathbb{R} \times \{(\beta, \gamma),\ \beta > 0, \gamma > \beta\}$. Hence, as a uniform limit of analytic functions $(r, \beta, \gamma) \mapsto \lambda(r, \beta, \gamma)$ is analytic.

PART (II). By the Perron–Frobenius theorem, the largest eigenvalue is a strictly increasing function of the entries of the (nonnegative) matrix. Hence $r \mapsto \lambda_r$ is strictly increasing. Furthermore, the largest eigenvalue of the matrix $(A(\mathbf{i}, \mathbf{j})\mathbf{1}_{\{((1,1),(1,1))\}}(\mathbf{i}, \mathbf{j}))_{\mathbf{ij}}$ is simply $A_r((1,1),(1,1))$. Thus we get the following inequality, which we need below:

$$\lambda_r > A_r((1,1),(1,1)) = e^r A_0((1,1),(1,1)). \tag{2.10}$$

For each $\mathbf{i}, \mathbf{j} \in \mathbb{N}^2$ the map $r \mapsto A_r(\mathbf{i}, \mathbf{j})$ is log-linear. Moreover, $\lambda(r, \beta, \gamma) = \lim_{N\to\infty}(A^N(\mathbf{i}, \mathbf{j}))^{1/N}$. Since log-convexity is preserved under positive combinations and under taking pointwise limits [see Kingman (1961) and Kato (1982)], $r \mapsto \lambda(r, \beta, \gamma)$ is log-convex in r.

We will show that it is strictly log-convex by contradiction. Assume that $r \mapsto \log \lambda_r$ is not strictly convex. Since it is convex, analytic and increasing there exist $a, b \geq 0$ such that $\lambda_r = ae^{br}$. For $r < 0$ we get (2.9) that

$$\begin{aligned}
\lambda_r &\leq \|A_r\|_{\mathrm{HS}} \\
&\leq e^r A_0((1,1),(1,1)) + \sum_{\mathbf{i},\, s(\mathbf{i})\geq 2} e^{rs(\mathbf{i})} P(\mathbf{i}_1, \mathbf{i}_2) \\
&= e^r A_0((1,1),(1,1)) + \tfrac{1}{2} e^{2r}(1 - e^r)^{-1}.
\end{aligned}$$

Letting $r \to -\infty$ yields $b \geq 1$. Together with (2.10) this implies $b = 1$ and $a = A_0((1,1),(1,1))$. However, this is a contradiction to (2.10). Thus we have proved strict log-convexity of $r \mapsto \lambda_r$.

Recall that $0 \leq \gamma < \beta$. Then we use an estimate as in (2.9) and Cauchy–Schwarz to get

$$\begin{aligned}
\lambda_0 &\leq \sup_{\|x\|=1} \langle x, A_0 x\rangle \\
&\leq \sup_{\|x\|=1} e^{-(\beta-\gamma)} \sum_{\mathbf{i},\mathbf{j}} x(\mathbf{i}) \mathbf{1}\{\mathbf{i}_2 = \mathbf{j}_1\} \sqrt{P(\mathbf{i}_1, \mathbf{i}_2) P(\mathbf{j}_1, \mathbf{j}_2)}\, x(\mathbf{j}) \\
&\leq \sup_{\|x\|=1} e^{-(\beta-\gamma)} \biggl(\sum_{\mathbf{i},\mathbf{j}} x^2(\mathbf{i}) \mathbf{1}_{\{\mathbf{i}_2=\mathbf{j}_1\}} P(\mathbf{j}_1, \mathbf{j}_2) \biggr)^{1/2} \qquad (2.11) \\
&\quad \times \biggl(\sum_{\mathbf{i},\mathbf{j}} x^2(\mathbf{j}) \mathbf{1}_{\{\mathbf{i}_2=\mathbf{j}_1\}} P(\mathbf{i}_1, \mathbf{i}_2) \biggr)^{1/2} \\
&= e^{-(\beta-\gamma)} < 1,
\end{aligned}$$

where the last equality follows from the facts that P is a doubly stochastic matrix and that x is normalized.

Finally, $\lim_{r\to\infty}\lambda_r=\infty$ since $r\mapsto\log\lambda_r$ is (strictly) increasing and convex.

PART (III). This works quite similarly to Part (i). We omit the details. □

3. Branching process and local times. In this section we quote Knight's theorem, a representation of random walk local times in terms of a branching process. We write the exponential in the definition of $Q_n^{\beta,\gamma}$ [recall (0.1) and (0.2)] in terms of this branching process (Lemma 3.1).

3.1. *Knight's theorem.* This subsection provides an important tool for the proof of Theorem 3, namely, a family of Markov chains that describes the local times of simple random walk on $\mathbb{Z}$ [recall (0.13)] at certain stopping times, viewed as a process in the spatial parameter. The following material is based upon the work of Knight (1963). It is the discrete space–time analogue of the Ray–Knight theorem for local times of Brownian motion. The present form is taken from van der Hofstad, den Hollander and König (1997), to which we refer for some of the proofs.

Recall that $(S_i)_{i=0}^n$ is a path of simple random walk in $\mathbb{Z}$. Fix $s\in\mathbb{N}_0$. Define the successive times at which the walker makes steps $s\to s+1$ and $s+1\to s$, by putting $T^{\uparrow}_{0,s}=T^{\downarrow}_{0,s}=0$ and for $k\in\mathbb{N}$,

$$\begin{aligned} T^{\uparrow}_{k,s}&=\inf\{i>T^{\uparrow}_{k-1,s}\colon S_{i-1}=s,S_i=s+1\},\\ T^{\downarrow}_{k,s}&=\inf\{i>T^{\downarrow}_{k-1,s}\colon S_{i-1}=s+1,S_i=s\}.\end{aligned}\tag{3.1}$$

By discarding null sets we can assume that all these stopping times are finite (one-dimensional simple random walk is recurrent!). Note that $T^{\uparrow}_{k,s}<T^{\downarrow}_{k,s}<T^{\uparrow}_{k+1,s}$ for $s\in\mathbb{N}_0$. Recall the definition of the stochastic $\mathbb{N}\times\mathbb{N}$ matrix P in (2.1), and introduce a stochastic $\mathbb{N}_0\times\mathbb{N}_0$ matrix $P^\star$ by putting

$$P^\star(i,j)=\mathbf{1}_{\mathbb{N}}(i)P(i,j+1)+\mathbf{1}_{\{(0,0)\}}(i,j)\qquad(i,j\in\mathbb{N}_0).\tag{3.2}$$

Let

$$(m(x))_{x\in\mathbb{N}_0}\quad\text{and}\quad(m^\star(x))_{x\in\mathbb{N}_0}\tag{3.3}$$

be the Markov chains with transition kernels P and $P^\star$, respectively.

We introduce the bivariate chains

$$\begin{aligned}\mathfrak{M}(x)&=(m(x),m(x+1))\quad\text{and}\\ \mathfrak{M}^\star(x)&=(m^\star(x),m^\star(x+1)),\qquad x\in\mathbb{N}_0.\end{aligned}\tag{3.4}$$

Recall that $s^\star(\mathbf{i})=\mathbf{i}_1+\mathbf{i}_2$ and $s(\mathbf{i})=\mathbf{i}_1+\mathbf{i}_2-1$.

In terms of these Markov chains, we can describe the distribution of the local times of simple random walk at the stopping times $T^{\uparrow}_{k,s}$, respectively $T^{\downarrow}_{k,s}$ as follows. (We write $\overset{d}{=}$ for equality in distribution.)

THEOREM 4 (Knight's theorem). *Fix* $k, s \in \mathbb{N}$. *Let* $(m(x))_{x\in\mathbb{N}_0}$ *start at* $m(0) = k$. *Let* $(m_1^\star(x))_{x\in\mathbb{N}_0}$ *and* $(m_2^\star(x))_{x\in\mathbb{N}_0}$ *be two independent copies of* $(m^\star(x))_{x\in\mathbb{N}_0}$ *starting at* $m_1^\star(0) = m(0)$, *respectively*, $m_2^\star(0) = m(s)$. *Assume that* m, $m_1^\star$ *and* $m_2^\star$ *are independent given* $m(0)$ *and* $m(s)$. *Then*

$$\begin{aligned}&\big[\big(\ell_{T^{\uparrow}_{k,s}}(s+1-x)\big)_{x=1,\dots,s}, \big(\ell_{T^{\uparrow}_{k,s}}(s+x)\big)_{x\in\mathbb{N}_0}, \big(\ell_{T^{\uparrow}_{k,s}}(1-x)\big)_{x\in\mathbb{N}_0}\big]\\ &\quad\stackrel{d}{=}\big[(s(\mathfrak{M}(x-1)))_{x=1,\dots,s}, (s^\star(\mathfrak{M}_1^\star(x-1)))_{x\in\mathbb{N}_0}, (s^\star(\mathfrak{M}_2^\star(x-1)))_{x\in\mathbb{N}_0}\big].\end{aligned}\tag{3.5}$$

Furthermore,

$$\ell_{T^{\downarrow}_{k,s}}(x) = \begin{cases}\ell_{T^{\uparrow}_{k,s}}(x) + \mathbf{1}_{\{s\}}(x), & \textit{if } x \le s,\\ \ell_{T^{\uparrow}_{k+1,s}}(x) - \mathbf{1}_{\{s+1\}}(x), & \textit{otherwise}.\end{cases}\tag{3.6}$$

For the proof, see van der Hofstad, den Hollander and König (1997).

In the sequel $\mathbb{P}_{\mathbf{i}}$ and $\mathbb{P}_k^\star$ will denote the laws of the two Markov chains in (3.3) starting in $\mathfrak{M}(0) = \mathbf{i} \in \mathbb{N}^2$ respectively $m^\star(0) = k \in \mathbb{N}_0$. We write $\mathbb{E}_{\mathbf{i}}$ and $\mathbb{E}_k^\star$ for expectation with respect to $\mathbb{P}_{\mathbf{i}}$, respectively, $\mathbb{P}_k^\star$.

3.2. *The distribution of the local times.* The description of the local times given in Knight's theorem has the disadvantage that the local times are observed at certain stopping times. For the description of the polymer we need to go back to the *fixed* time n. One of the problems we consequently have to deal with is the global restriction $\sum_{x\in\mathbb{Z}} \ell_n(x) = n+1$.

Fix $s, n \in \mathbb{N}$. In this subsection we derive a representation for the expression $E(e^{-H_n(S_n)}\mathbf{1}_{\{S_n=s\}})$ in terms of the Markov chains introduced in the preceding subsection. The idea is to sum over the number of steps $0 \to 1$, $s \to s+1$ (respectively, $s+1 \to s$), and over the amount of time the walker spends in the three intervals $-\mathbb{N}_0$, $\{1,\dots,s\}$ and $\{s+1, s+2, \dots\}$ until time n.

Define the functionals

$$\begin{aligned}U(s) &= \sum_{x=0}^{s-1} s(\mathfrak{M}(x)), & U^\star &= \sum_{x=0}^{\infty} s^\star(\mathfrak{M}^\star(x)),\\ V(s) &= \sum_{x=0}^{s-1} s(\mathfrak{M}(x))^2, & V^\star &= \sum_{x=0}^{\infty} s^\star(\mathfrak{M}^\star(x))^2,\\ W(s) &= \sum_{x=0}^{s-1} s(\mathfrak{M}(x))s(\mathfrak{M}(x+1)), & W^\star &= \sum_{x=0}^{\infty} s^\star(\mathfrak{M}^\star(x))s^\star(\mathfrak{M}^\star(x+1)).\end{aligned}\tag{3.7}$$

We will need the notation $\bar{\mathbf{i}} = (\mathbf{i}_2, \mathbf{i}_1)$ for $\mathbf{i} \in \mathbb{N}^2$. In terms of these new objects we may write the following.

LEMMA 3.1. *For all* n, $s \in \mathbb{N}$,

$$E\big(e^{-H_n(S)}\mathbf{1}_{\{S_n=s+1,\, S_{n-1}=s\}}\big)$$

$$= \sum_{n_1,\, n_2\in\mathbb{N}} \sum_{\mathbf{j}^1,\, \mathbf{j}^2\in\mathbb{N}^2} \prod_{i=1}^{2} \mathbb{E}^{\star}_{\mathbf{j}^i_2}\big[e^{-\beta V^\star+\gamma W^\star+\gamma s(\mathbf{j}^i)s^\star(\mathfrak{M}^\star(0))}\mathbf{1}_{\{n_i\}}(U^\star)\big] \times \mathbb{E}_{\mathbf{j}^1}\big[e^{-\beta V(s)+\gamma W(s)}\mathbf{1}_{\{n-n_1-n_2+1\}}(U(s))\mathbf{1}_{\{\overline{\mathbf{j}^2}\}}(\mathfrak{M}(s-1))\big] \tag{3.8}$$

and

$$E\big[e^{-H_n(S)}\mathbf{1}_{\{S_n=s,\, S_{n-1}=s+1\}}\big]$$

$$= \sum_{k_1\in\mathbb{N}\setminus\{1\},\, n_1\in\mathbb{N}_0} \sum_{k_2,\, n_2\in\mathbb{N}} \prod_{i=1}^{2} \mathbb{E}^{\star}_{k_i}\big[e^{-\beta V^\star+\gamma W^\star-\delta_i}\mathbf{1}_{\{n_i\}}(U^\star)\big] \times \mathbb{E}_{k_1-1}\big[e^{-\beta[V(s)+\delta_3]+\gamma[W(s)+\delta_4]}\mathbf{1}_{\{n-n_1-n_2+1\}}(U(s))\mathbf{1}_{\{k_2\}}(m(s))\big], \tag{3.9}$$

with

$$\begin{aligned} \delta_1 &= 2\beta m^\star(1) - \gamma(m^\star(1)+m^\star(2)), \quad \delta_2 = 0,\\ \delta_3 &= 2m(1), \quad \delta_4 = k_1 + m(1) - 1. \end{aligned} \tag{3.10}$$

For the proof, see van der Hofstad, den Hollander and König (1997), proof of Lemma 3.

In the proof of Theorem 3 we shall focus on the contribution coming from the right-hand side of (3.8). It will be argued at the end of Section 4.4 that (3.9) behaves in the same manner as (3.8) as $n \to \infty$; that is, the small perturbations $\delta_1, \dots, \delta_4$ are harmless.

In Lemma 3.1 we have rewritten $Q_n^{\beta,\gamma}$ in terms of exponential functionals of the two Markov chains defined in (3.3). We can henceforth forget about the underlying random walk. Note that in Lemma 3.1 we have *products* of expectations.

4. Proof of the CLT. In this section we perform the main steps of the proof of Theorem 3. Our approach is a variation of the method used in van der Hofstad, den Hollander and König (1997). In Section 4.1 (Lemma 4.2) we reformulate Lemma 3.1 in terms of the equilibrium distribution of a transformed Markov chain. Then in Section 4.2 we give the final reformulation in terms of a Markov renewal chain (Lemma 4.3). The representation in Lemma 4.3 allows to give in Section 4.3 the key propositions (Propositions 4.4 and 4.5) that are the technical core of the argument. In Section 4.4 we complete the proof of Theorem 3.

4.1. *A transformed Markov chain.* In this subsection we define a transformation of the Markov chain $(\mathfrak{M}(x))_{x\in\mathbb{N}_0}$ introduced in Section 3.1. The goal of this transformation is to absorb the random variable $e^{-\beta V(s)+\gamma W(s)}$ (see (3.7)) into the new transition probabilities.

Recall the definition of $A_{r,\beta,\gamma}$ and r^* from Section 2.1 [(2.5) and (2.7)] and recall that we usually suppress the (β,γ)-dependence in the notation. Fix $r \in \mathbb{R}$ and $\beta,\gamma \in \mathbb{R}^+$ such that $\gamma < \beta$. As was pointed out in Proposition 2.1, the matrix A_r has a unique largest eigenvalue λ_r. Consequently, similarly as an h-transform we can define a stochastic matrix P_r by

$$P_r(\mathbf{i},\mathbf{j}) = \frac{A_r(\mathbf{i},\mathbf{j})}{\lambda_r}\frac{\tau_r^{(R)}(\mathbf{j})}{\tau_r^{(R)}(\mathbf{i})} \qquad (\mathbf{i},\mathbf{j} \in \mathbb{N}^2). \tag{4.1}$$

We shall write $\mathbb{P}_{\mathbf{k}}^r$ to denote the law of the Markov chain $(\mathfrak{M}(x))_{x\in\mathbb{N}_0}$ [recall (3.4)], starting at $\mathbf{k} \in \mathbb{N}^2$ and having P_r as its transition kernel. We write $\mathbb{E}_{\mathbf{k}}^r$ for the corresponding expectation.

LEMMA 4.1. *$(\mathfrak{M}(x))_{x\in\mathbb{N}_0}$ is positive recurrent and ergodic with invariant distribution* [*recall* (2.6)] $(p_r^{-1}\tau_r^{(L)}(\mathbf{i})\tau_r^{(R)}(\mathbf{i}))_{\mathbf{i}\in\mathbb{N}^2}$.

PROOF. Since A_r^2 is strictly positive [Proposition 2.1(i)], the same is true for the eigenvector $\tau_r^{(R)}$ and for P_r^2. Hence P_r has a unique invariant measure. However, it is immediate from (4.1) that $p_r^{-1}\tau_r^{(L)}\tau_r^{(R)}P_r = p_r^{-1}\tau_r^{(L)}\tau_r^{(R)}$; hence $p_r^{-1}\tau_r^{(L)}\tau_r^{(R)}$ is an invariant measure for P_r. Since it is a probability measure by the definition of p_r, $(\mathfrak{M}(x))_{x\in\mathbb{N}_0}$ is positive recurrent and ergodic. □

We write $\mathbb{P}^r, \mathbb{E}^r$ when the chain starts in its invariant distribution.

We will reformulate the right-hand side of (3.8) in terms of $(\mathfrak{M}(x))_{x\in\mathbb{N}_0}$, since this is the natural object for our analysis. First we need some more notation. For $r \in \mathbb{R}$ and $\beta,\gamma \in \mathbb{R}^+$, define the functions $w_r^{(R)}$, $w_r^{(L)}$: $\mathbb{N}_0^2 \times \mathbb{N}_0 \to \mathbb{R}_0^+$ by [see (3.7)]

$$\begin{aligned} w_r^{(R)}(\mathbf{k},l) = {}& \tau_r^{(R)}(\mathbf{k})^{-1} \\ &\times \mathbb{E}_{\mathbf{k}_2}^{\star}\big[e^{rU^\star - \beta V^\star + \gamma W^\star + (r/2)s(\mathbf{k}) - (\beta/2)s(\mathbf{k})^2 + \gamma s(\mathbf{k})(\mathbf{k}_2 + m^\star(1))}\mathbf{1}_{\{l\}}(U^\star)\big] \\ &\times \sqrt{P(\mathbf{k}_1,\mathbf{k}_2)}, \end{aligned} \tag{4.2}$$

and $w_r^{(L)}(\mathbf{k},l) = w_r^{(R)}(\bar{\mathbf{k}},l)$. Furthermore, define

$$T_l = \min\{s\colon U(s) \geq l\} \tag{4.3}$$

to be the exceeding time of l and let for $n \in \mathbb{N}$,

$$I(n) = \bigcup_{s\in\mathbb{N}_0}\{U(s) = n\} = \{U(T_n) = n\}, \tag{4.4}$$

the event that U hits n exactly.

Our aim is to obtain a convenient description of the Laplace transforms of $Q_n^{\beta,\gamma}(\frac{|S_n|-\theta^* n}{\sigma^* n^{1/2}} \in \cdot)$. A first step is the following lemma [recall p_r from (2.6)].

LEMMA 4.2. *For $\mu \in \mathbb{R}$ and $n \in \mathbb{N}$,*

$$\begin{aligned} &e^{-2\mu/\sigma^*\sqrt{n}} e^{(n+1)r_n^*} E\big[e^{-H_n(S)} e^{\mu S_n/\sigma^*\sqrt{n}} \mathbf{1}_{\{0\le S_{n-1}<S_n\}}\big] \\ &\qquad = p_{r_n^*} \sum_{n_1, n_2\in\mathbb{N}} \mathbb{E}^{r_n^*}\Big[w_{r_n^*}^{(L)}(\mathfrak{M}(0), n_1) w_{r_n^*}^{(R)}(\mathfrak{M}(T_{n-n_1-n_2+1}-1), n_2); \\ &\qquad\qquad I(n-n_1-n_2+1)\Big], \end{aligned} \tag{4.5}$$

where $r_n^* = r_n^*(\mu)$ is given by

$$\lambda_{r_n^*} = e^{-\mu/\sigma^*\sqrt{n}}. \tag{4.6}$$

PROOF. Note that by definition of the transformed Markov chain in Section 4.1 we can write for $\mathbf{i}, \mathbf{j} \in \mathbb{N}^2$ and $N, s \in \mathbb{N}$,

$$\begin{aligned} &\lambda_r^{1-s} \mathbb{P}_{\mathbf{i}}^r[U(s) = N, \mathfrak{M}(s-1) = \mathbf{j}] \sqrt{P(\mathbf{i}_1, \mathbf{i}_2)} \sqrt{P(\mathbf{j}_1, \mathbf{j}_2)} \\ &\qquad = \frac{\tau_r^{(R)}(\mathbf{j})}{\tau_r^{(R)}(\mathbf{i})} \sum_{\substack{\mathbf{k}^1,\dots,\mathbf{k}^{s-2}\in\mathbb{N}^2 \\ s(\mathbf{i})+s(\mathbf{k}^1)+\cdots+s(\mathbf{k}^{s-2})+s(\mathbf{j})=N}} A_r(\mathbf{i}, \mathbf{k}^1) \cdots A_r(\mathbf{k}^{s-2}, \mathbf{j}) \\ &\qquad\qquad \times \sqrt{P(\mathbf{i}_1, \mathbf{i}_2)} \sqrt{P(\mathbf{j}_1, \mathbf{j}_2)}. \end{aligned} \tag{4.7}$$

Letting $k_0 = \mathbf{i}_1$, $k_1 = \mathbf{i}_2$, $k_{s-1} = \mathbf{j}_1$, $k_s = \mathbf{j}_2$ and

$$u(s) = \sum_{l=0}^{s-1} (k_l + k_{l+1} - 1), \qquad v(s) = \sum_{l=0}^{s-1} (k_l + k_{l+1} - 1)^2,$$

$$w(s) = \sum_{l=0}^{s-2} (k_l + k_{l+1} - 1)(k_{l+1} + k_{l+2} - 1),$$

this equals

$$\begin{aligned} &\frac{\tau_r^{(R)}(\mathbf{j})}{\tau_r^{(R)}(\mathbf{i})} \sum_{\substack{k_2,\dots,k_{s-2}\in\mathbb{N} \\ u(s)=N}} P(k_0, k_1) \cdots P(k_{s-1}, k_s) e^{r(u(s)-(s(\mathbf{i})+s(\mathbf{j}))/2)} \\ &\qquad \times e^{-\beta(v(s)-(s(\mathbf{i})^2+s(\mathbf{j})^2)/2)} e^{\gamma w(s)} \\ &\quad = \frac{\tau_r^{(R)}(\mathbf{j})}{\tau_r^{(R)}(\mathbf{i})} \mathbb{E}_{\mathbf{i}}\big[e^{-\beta V(s)+\gamma W(s)};\ U(s) = N,\ \mathfrak{M}(s-1) = \mathbf{j}\big] \\ &\qquad \times e^{-r(s(\mathbf{i})+s(\mathbf{j}))/2} e^{\beta(s(\mathbf{i})^2+s(\mathbf{j})^2)/2} e^{rN}. \end{aligned} \tag{4.8}$$

Hence, if we let $N = n - n_1 - n_2 + 1$ and $r = r_n^*$ and use the representation in (3.8), then we can rewrite the left-hand side of (4.5) as $p_{r_n^*}$ times

$$\begin{aligned}
&\sum_{n_1, n_2 \in \mathbb{N}} \sum_{\mathbf{i}, \mathbf{j} \in \mathbb{N}^2} \sum_{s \in \mathbb{N}_0} w_{r_n^*}^{(L)}(\mathbf{i}, n_1) w_{r_n^*}^{(R)}(\mathbf{j}, n_2) \tau_{r_n^*}^{(L)}(\mathbf{i}) \tau_{r_n^*}^{(R)}(\mathbf{i}) \\
&\qquad \times \mathbb{P}_{\mathbf{i}}^{r_n^*}[U(s) = n - n_1 - n_2 + 1, \mathfrak{M}(s-1) = \mathbf{j}] \\
&= \sum_{n_1, n_2 \in \mathbb{N}} \sum_{\mathbf{i}, \mathbf{j} \in \mathbb{N}^2} w_{r_n^*}^{(L)}(\mathbf{i}, n_1) w_{r_n^*}^{(R)}(\mathbf{j}, n_2) \tau_{r_n^*}^{(L)}(\mathbf{i}) \tau_{r_n^*}^{(R)}(\mathbf{i}) \\
&\qquad \times \mathbb{P}_{\mathbf{i}}^{r_n^*}[\mathfrak{M}(T_{n-n_1-n_2+1} - 1) = \mathbf{j}; I(n - n_1 - n_2 + 1)].
\end{aligned}$$

This completes the proof. □

In the right-hand side of (4.5) appears a *correlation function*. In the sequel we shall prove that the first and the last factor in this correlation function are asymptotically independent as $n \to \infty$.

4.2. *Markov renewal chain.* It turns out that a convenient way to show that the correlations vanish is to replace the chain $\mathfrak{M}(x)$ by a related renewal chain $(\Gamma(l))_{l \in \mathbb{N}_0}$. In this subsection we define $(\Gamma(l))$ and reformulate Lemma 3.1 in terms of $(\Gamma(l))$.

Define

$$X(l) = U(T_l) - l, \qquad Y(l) = (m(T_l - 1), m(T_l)). \tag{4.10}$$

Then $I(n) = \{X(n) = 0\}$. The pair

$$\Gamma(l) = (X(l), Y(l)) \tag{4.11}$$

is a random element of the set

$$\Sigma = \{(i, \mathbf{j}) \in \mathbb{N}_0 \times \mathbb{N}^2 \colon i \le s(\mathbf{j}) - 1\}.$$

For any $\mathbf{j} \in \mathbb{N}^2$, under the law $\mathbb{P}_{\mathbf{j}}^r$ the process $(\Gamma(l))_{l \in \mathbb{N}_0}$ is a Markov renewal process with transition kernel Q_r on Σ given by

$$Q_r((i, \mathbf{j}), (i', \mathbf{j}')) = \mathbf{1}_{\{i=0,\, i'=s(\mathbf{j}')-1\}} P_r(\mathbf{j}, \mathbf{j}') + \mathbf{1}_{\{i'=i-1,\, \mathbf{j}'=\mathbf{j}\}} \tag{4.12}$$

and starting at $\Gamma(0) = (0, \mathbf{j})$. It is easily checked that the probability distribution ν_r on Σ defined by

$$\nu_r(i, \mathbf{j}) = \theta_r p_r^{-1} \tau_r^{(L)}(\mathbf{j}) \tau_r^{(R)}(\mathbf{j}) \tag{4.13}$$

and

$$\theta_r = \frac{\lambda_r}{\lambda_r'}$$

is the associated invariant distribution on Σ. Indeed,

$$\begin{aligned}(\nu_r Q_r)(i',\mathbf{j}') &= \frac{\theta_r}{p_r}\biggl(\sum_{\mathbf{j}\in\mathbb{N}^2}\tau_r^{(L)}(\mathbf{j})\tau_r^{(R)}(\mathbf{j})P_r(\mathbf{j},\mathbf{j}')\mathbf{1}_{\{i'=s(\mathbf{j}')-1\}}\\ &\qquad\qquad + \tau_r^{(L)}(\mathbf{j}')\tau_r^{(R)}(\mathbf{j}')\mathbf{1}_{\{i'\le s(\mathbf{j}')-2\}}\biggr)\\ &= \frac{\theta_r}{p_r}\tau_r^{(L)}(\mathbf{j}')\tau_r^{(R)}(\mathbf{j}')\mathbf{1}_{\{i'\le s(\mathbf{j}')-1\}} = \nu_r(i',\mathbf{j}'),\end{aligned}$$

since $\tau_r^{(L)}\tau_r^{(R)}$ is invariant for P_r. To see that ν_r is normed, note that

$$\|\nu_r\| := \sum_{(i,\mathbf{j})\in\Sigma}\nu_r(i,\mathbf{j}) = \frac{\theta_r}{p_r}\langle\tau_r^{(L)}, S\tau_r^{(R)}\rangle, \tag{4.14}$$

where S is the diagonal matrix $S(\mathbf{i},\mathbf{j}) = s(\mathbf{i})\mathbf{1}_{\{\mathbf{i}=\mathbf{j}\}}$, $\mathbf{i},\mathbf{j}\in\mathbb{N}^2$. Clearly $A_r = \exp(\frac{r}{2}S)A_0\exp(\frac{r}{2}S)$. Hence $\partial_r A_r = \frac{1}{2}(SA_r + A_r S)$. Moreover, we let A^* be the adjoint operator of A. Then

$$\begin{aligned}\langle\tau_r^{(L)}, S\tau_r^{(R)}\rangle &= \frac{1}{\lambda_r}\langle A_r^*\tau_r^{(L)}, S\tau_r^{(R)}\rangle\\ &= \frac{1}{\lambda_r}\frac{1}{2}\bigl(\langle SA_r^*\tau_r^{(L)}, \tau_r^{(R)}\rangle + \langle\tau_r^{(L)}, A_r S\tau_r^{(R)}\rangle\bigr)\\ &= \frac{1}{\lambda_r}\Bigl\langle\tau_r^{(L)}, \frac{1}{2}(SA_r + A_r S)\tau_r^{(R)}\Bigr\rangle = \frac{1}{\lambda_r}\langle\tau_r^{(L)}, (\partial_r A_r)\tau_r^{(R)}\rangle.\end{aligned} \tag{4.15}$$

Thus we get $\|\nu_r\| = \frac{1}{\lambda_r'}\partial_r(p_r^{-1}\langle\tau_r^{(L)}, A_r\tau_r^{(R)}\rangle) = 1$.

We write $\widetilde{\mathbb{P}}^r$ and $\widetilde{\mathbb{E}}^r$ to denote probability and expectation w.r.t. the Markov chain $(\Gamma(l))_{l\in\mathbb{N}_0}$ starting in its invariant distribution ν_r.

Before we reformulate the right-hand side of (3.8) in terms of $(\Gamma(l))_{l\in\mathbb{N}_0}$ we need some more notation. Recall that for $\mathbf{i}\in\mathbb{N}^2$ we defined $\bar{\mathbf{i}} = (\mathbf{i}_2, \mathbf{i}_1)$.

For $r\in\mathbb{R}$ and $\beta,\gamma\in\mathbb{R}^+$, define the functions $f_r^{(R)}$ and $f_r^{(L)}\colon \Sigma\times\mathbb{N}_0\to\mathbb{R}^+$ by

$$f_r^{(R)}((i,\mathbf{j});l) = w_r^{(R)}(\mathbf{j},l)\mathbf{1}_{\{i=0\}}, \qquad f_r^{(L)}((i,\mathbf{j});l) = f_r^{(R)}((i,\bar{\mathbf{j}});l), \tag{4.16}$$

where $w_r^{(L)}$ and $w_r^{(R)}$ are defined in (4.2).

We can now reformulate the left-hand side in Lemma 4.2 as follows.

LEMMA 4.3. *For $\mu\in\mathbb{R}$ and $n\in\mathbb{N}$,*

$$\begin{aligned}&e^{-2\mu/\sigma^*\sqrt{n}}e^{(n+1)r_n^*}\,E\bigl[e^{-H_n(S_n)}e^{\mu S_n/\sigma^*\sqrt{n}}\mathbf{1}_{\{0\le S_{n-1}<S_n\}}\bigr]\\ &\qquad = \frac{p_{r_n^*}}{\theta_{r_n^*}}\sum_{n_1,n_2\in\mathbb{N}}\widetilde{\mathbb{E}}^{r_n^*}\Bigl[f_{r_n^*}^{(L)}(\Gamma(0);n_1)f_{r_n^*}^{(R)}(\Gamma(n-n_1-n_2);n_2)\Bigr],\end{aligned} \tag{4.17}$$

where r_n^* is given in (4.6).

PROOF. Using Lemma 4.2 we rewrite the left-hand side of (4.17) as

$$
\begin{aligned}
&p_{r_n^*} \sum_{n_1, n_2 \in \mathbb{Z}} \mathbb{E}^{r_n^*}\Big[w_{r_n^*}^{(L)}(\mathfrak{M}(0), n_1) w_{r_n^*}^{(R)}(\mathfrak{M}(T_{n-n_1-n_2+1} - 1), n_2); \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad I(n - n_1 - n_2 + 1)\Big] \\
&\quad = p_{r_n^*} \sum_{n_1, n_2 \in \mathbb{Z}} \mathbb{E}^{r_n^*}\Big[w_{r_n^*}^{(L)}(\mathfrak{M}(0), n_1)\, \mathbf{1}_{\{0\}}(X(n - n_1 - n_2 + 1)) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times w_{r_n^*}^{(R)}(Y(n - n_1 - n_2 + 1), n_2)\Big] \\
&\quad = p_{r_n^*} \sum_{n_1, n_2 \in \mathbb{Z}} \mathbb{E}^{r_n^*}\Big[w_{r_n^*}^{(L)}(\mathfrak{M}(0), n_1) f_{r_n^*}^{(R)}(\Gamma(n - n_1 - n_2 + 1); n_2)\Big] \\
&\quad = p_{r_n^*} \sum_{n_1, n_2 \in \mathbb{Z}} \mathbb{E}^{r_n^*}\Big[w_{r_n^*}^{(L)}(Y(1), n_1)\, f_{r_n^*}^{(R)}(\Gamma(n - n_1 - n_2 + 1); n_2)\Big].
\end{aligned}
$$

Note that $X(0) = 0$ if and only if $X(1) = s(Y(1)) - 1$ and use the fact that $\mathbb{P}^{r_n^*} = \widetilde{\mathbb{P}}^{r_n^*}[\cdot | X(0) = 0] = \frac{1}{\theta_{r_n^*}} \widetilde{\mathbb{P}}^{r_n^*} \cdot \mathbf{1}_{X(0)=0}$ to rewrite this equation as

$$
\begin{aligned}
&= \frac{p_{r_n^*}}{\theta_{r_n^*}} \sum_{n_1, n_2 \in \mathbb{Z}} \widetilde{\mathbb{E}}^{r_n^*}\Big[w_{r_n^*}^{(L)}(Y(1), n_1) \mathbf{1}_{\{0\}}(X(0)) f_{r_n^*}^{(R)}(\Gamma(n - n_1 - n_2 + 1); n_2)\Big] \\
&= \frac{p_{r_n^*}}{\theta_{r_n^*}} \sum_{n_1, n_2 \in \mathbb{Z}} \widetilde{\mathbb{E}}^{r_n^*}\Big[f_{r_n^*}^{(L)}(\Gamma(1); n_1) f_{r_n^*}^{(R)}(\Gamma(n - n_1 - n_2 + 1); n_2)\Big].
\end{aligned}
$$

Using the fact that Γ is stationary under $\widetilde{\mathbb{E}}^{r_n^*}$ yields the claim. □

4.3. *Convergence to the equilibrium.* It is clear from the ergodicity of the renewal chain that its distribution converges to its equilibrium ν_r for every fixed r. However, in Lemma 4.3 we need convergence where $r = r_n^*$ depends on n and converges to r^*. We state pointwise convergence in Proposition 4.4 and uniform integrability in Proposition 4.5. This sets the stage for the proof of Theorem 3.

PROPOSITION 4.4. *Fix $(i, \mathbf{k}), (i', \mathbf{k}') \in \Sigma$ and $n_1 \in \mathbb{N}$. For any sequence $r_n \to r^*$,*

$$
\lim_{n\to\infty} \mathbb{P}_{(i, \mathbf{k})}^{r_n}[X(n - n_1) = i', Y(n - n_1) = \mathbf{k}'] = \nu_{r^*}(i', \mathbf{k}'). \tag{4.18}
$$

PROOF. The statement is equivalent to $\|\delta_{(i, \mathbf{k})} Q_{r_n}^{n-n_1} - \nu_{r^*}\|_{\mathrm{TV}} \to 0$, $n \to \infty$. Note that $N \mapsto \|\delta_{(i, \mathbf{k})} Q_{r_n}^N - \nu_{r_n}\|_{\mathrm{TV}}$ is decreasing for all $n \in \mathbb{N}$ since $\nu_{r_n} Q_{r_n} = \nu_{r_n}$

and Q_{r_n} is a contraction. Thus for $n \ge n_1 + N$,

$$
\begin{aligned}
\|\delta_{(i,\mathbf{k})} Q_{r_n}^{n-n_1} - \nu_{r^*}\|_{\mathrm{TV}} &\le \|\delta_{(i,\mathbf{k})} Q_{r_n}^{n-n_1} - \nu_{r_n}\|_{\mathrm{TV}} + \|\nu_{r_n} - \nu_{r^*}\|_{\mathrm{TV}} \\
&\le \|\delta_{(i,\mathbf{k})} Q_{r_n}^{N} - \nu_{r_n}\|_{\mathrm{TV}} + \|\nu_{r_n} - \nu_{r^*}\|_{\mathrm{TV}} \\
&\le \|\delta_{(i,\mathbf{k})} Q_{r^*}^{N} - \delta_{(i,\mathbf{k})} Q_{r_n}^{N}\|_{\mathrm{TV}} + \|\delta_{(i,\mathbf{k})} Q_{r^*}^{N} - \nu_{r^*}\|_{\mathrm{TV}} \\
&\quad + 2\|\nu_{r_n} - \nu_{r^*}\|_{\mathrm{TV}}.
\end{aligned} \tag{4.19}
$$

By continuity of $r \mapsto \tau_r$, $r \mapsto \lambda'_r$ and $r \mapsto \lambda_r$, we get $Q_{r_n} \to Q_{r^*}$ and $\nu_{r_n} \to \nu_{r^*}$ as $n \to \infty$, and hence the first and third term on the r.h.s. of the above equation vanish, so that

$$
\limsup_{n\to\infty} \|\delta_{(i,\mathbf{k})} Q_{r_n}^{n-n_1} - \nu_{r^*}\|_{\mathrm{TV}} \le \|\delta_{(i,\mathbf{k})} Q_{r^*}^{N} - \nu_{r^*}\|_{\mathrm{TV}} \quad \text{for all } N \in \mathbb{N}. \tag{4.20}
$$

However, by ergodicity of Q: $\|\delta_{(i,\mathbf{k})} Q_{r^*}^{N} - \nu_{r^*}\| \to 0$, $N \to \infty$. This completes the proof. $\square$

The next proposition states summability of the correlation function. This will be needed to impose a dominated convergence argument in the proof of Theorem 3. Note that it is only here that we have to assume $\gamma \le \beta - \frac{1}{2}\log 2$.

PROPOSITION 4.5. *Assume that* $0 \le \gamma \le \beta - \frac{1}{2}\log 2$. *Then there exists an* $n_0 < \infty$ *such that*

$$
\begin{aligned}
&\sum_{\mathbf{k},\mathbf{j}\in\mathbb{N}^2} \sum_{n_1,n_2\in\mathbb{N}} \sup_{n\ge n_0} \frac{1}{\theta_{r_n^*}} \nu_{r_n^*}(0,\mathbf{k}) \mathbb{E}_{(0,\mathbf{k})}^{r_n^*} \\
&\qquad \times \Big[f_{r_n^*}^{(L)}(\Gamma(0); n_1) f_{r_n^*}^{(R)}(\Gamma(n-n_1-n_2); n_2) \mathbf{1}_{\{\mathbf{j}\}}(Y(n-n_1-n_2)) \Big] < \infty.
\end{aligned} \tag{4.21}
$$

We divide the proof of this proposition into three lemmas. The statement (4.21) will be immediate from Lemmas 4.6, 4.7 and 4.8.

LEMMA 4.6. *Assume that* $0 \le \gamma \le \beta - \frac{1}{2}\log 2$. *Then there exists an* $n_0 < \infty$ *such that for* $n \ge n_0$,

$$
\begin{aligned}
&\nu_{r_n^*}(0,\mathbf{k}) \mathbb{E}_{(0,\mathbf{k})}^{r_n^*} \Big[f_{r_n^*}^{(L)}(\Gamma(0); n_1) f_{r_n^*}^{(R)}(\Gamma(n-n_1-n_2); n_2) \\
&\qquad\qquad \times \mathbf{1}_{\{\mathbf{j}\}}(Y(n-n_1-n_2)) \Big] \\
&\quad \le \theta_{r_n^*} \lambda_r \sqrt{\tau_{r_n^*}^{(L)}(\mathbf{k}) \tau_{r_n^*}^{(R)}(\mathbf{k})}\, w_{r_n^*}^{(L)}(\mathbf{k}, n_1) \sqrt{\tau_{r_n^*}^{(L)}(\mathbf{j}) \tau_{r_n^*}^{(R)}(\mathbf{j})}\, w_{r_n^*}^{(R)}(\mathbf{j}, n_2).
\end{aligned} \tag{4.22}
$$

PROOF. The proof is easy. Write

$$
\begin{aligned}
&\nu_{r_n^*}(0,\mathbf{k})\mathbb{E}_{(0,\mathbf{k})}^{r_n^*}\Big[f_{r_n^*}^{(L)}(\Gamma(0);n_1)f_{r_n^*}^{(R)}(\Gamma(n-n_1-n_2);n_2)\\
&\qquad\qquad\qquad\qquad \times \mathbf{1}_{\{\mathbf{j}\}}(Y(n-n_1-n_2))\Big]\\
&\quad=\widetilde{\mathbb{E}}^{r_n^*}\Big[f_{r_n^*}^{(L)}(\Gamma(0);n_1)\mathbf{1}_{\{(0,\mathbf{k})\}}(\Gamma(0))f_{r_n^*}^{(R)}(\Gamma(n-n_1-n_2);n_2)\\
&\qquad\qquad\qquad\qquad \times \mathbf{1}_{\{(0,\mathbf{j})\}}(\Gamma(n-n_1-n_2))\Big],
\end{aligned}
\tag{4.23}
$$

and use Cauchy–Schwarz and the fact that $\nu_{r_n^*}$ is the stationary measure. This gives

$$
\begin{aligned}
&\nu_{r_n^*}(0,\mathbf{k})\mathbb{E}_{(0,\mathbf{k})}^{r_n^*}\big[f_{r_n^*}^{(L)}(\Gamma(0);n_1)f_{r_n^*}^{(R)}(\Gamma(n-n_1-n_2);n_2)\\
&\qquad\qquad\qquad\qquad \times \mathbf{1}_{\{\mathbf{j}\}}(Y(n-n_1-n_2))\big]\\
&\quad\le \big(\nu_{r_n^*}(0,\mathbf{k})\big(f_{r_n^*}^{(L)}\big)^2((0,\mathbf{k});n_1)\big)^{1/2}\\
&\quad\quad\times \big(\nu_{r_n^*}(0,\mathbf{j})\big(f_{r_n^*}^{(R)}\big)^2((0,\mathbf{j});n_2)\big)^{1/2}.
\end{aligned}
\tag{4.24}
$$

Finally, substitute $f_{r_n^*}^{(L)}$, $f_{r_n^*}^{(R)}$ [recall (4.16) and (4.2)] and $\nu_{r_n^*}$ [recall (4.13)]. $\square$

Recall $U^\star$ and $V^\star$ from (3.7) and r_n^* from (4.6). The crucial quantity for summability of $w_{r_n^*}^{(R)}$ is

$$
\bar{r} := \sup\{r>0\colon \alpha_r<\infty\},
\tag{4.25}
$$

where

$$
\alpha_r := \mathbb{E}_1^\star\big[e^{-(\beta-\gamma)V^\star+rU^\star}\big].
\tag{4.26}
$$

LEMMA 4.7. *If $\bar{r}>r^*$, then for sufficiently large $n_0\in\mathbb{N}$,*

$$
\sum_{\substack{\mathbf{k}\in\mathbb{N}^2\\ m\in\mathbb{N}}}\sup_{n\ge n_0}\Big(w_{r_n^*}^{(R)}(\mathbf{k},m)\sqrt{\tau_{r_n^*}^{(L)}(\mathbf{k})\tau_{r_n^*}^{(R)}(\mathbf{k})}\Big)<\infty.
$$

PROOF. By convexity of the map $r\mapsto\lambda_r$ we get that $r_n^*<r^*+|\mu|/\sigma^*\sqrt{n}$. In particular, there exists an $r'\in(r^*,\bar{r})$ and an $n_0\in\mathbb{N}$ such that $r_n^*<r'$ for all $n\ge n_0$.

Note that $\tau_{r_n^*}^{(R)}(\mathbf{k}) \ge \frac{1}{\lambda_{r_n^*}}A_{r_n^*,\beta,\gamma}(\mathbf{k},\bar{\mathbf{k}})\tau_{r_n^*}^{(R)}(\bar{\mathbf{k}})$. Since $A_{r_n^*,\beta,\gamma}(\mathbf{k},\bar{\mathbf{k}}) = e^{r_n^* s(\mathbf{k})+(\gamma-\beta)s(\mathbf{k})^2}P(\mathbf{k}_1,\mathbf{k}_2)$, and since $r\mapsto\lambda_r$ is increasing we get

$$
\sqrt{\tau_{r_n^*}^{(L)}(\mathbf{k})\tau_{r_n^*}^{(R)}(\mathbf{k})}\le\tau_{r_n^*}^{(R)}(\mathbf{k})\lambda_{r'}^{1/2}e^{-(r_n^*/2)s(\mathbf{k})+((\beta-\gamma)/2)s(\mathbf{k})^2}\Big/\sqrt{P(\mathbf{k}_1,\mathbf{k}_2)}.
$$

Now, sum out over m and note that $V^\star - W^\star = \frac{1}{2}s^\star(\mathfrak{M}^\star(0))^2 + \frac{1}{2}\sum_{x=0}^\infty (s^\star(\mathfrak{M}^\star(x+1) - s^\star(\mathfrak{M}^\star(x))))^2 \geq \frac{1}{2}s^\star(\mathfrak{M}^\star(0))^2 + \frac{1}{2}(s^\star(\mathfrak{M}^\star(1) - s^\star(\mathfrak{M}^\star(0))))^2$ to bound the exponential in the expectation in the definition of $w_{r_n^*}^{(R)}$ to get

$$\sum_{\mathbf{k}, m} \sup_{n \geq n_0} \Big(w_{r_n^*}^{(R)}(\mathbf{k}, m) \sqrt{\tau_{r_n^*}^{(L)}(\mathbf{k}) \tau_{r_n^*}^{(R)}(\mathbf{k})} \Big) \leq \sum_{\mathbf{k}} v_{r'}(\mathbf{k}), \tag{4.27}$$

where

$$v_r(\mathbf{k}) = \mathbb{E}_{\mathbf{k}_2}^\star \big[e^{-(\beta-\gamma)V^\star + r'U^\star - (\gamma/2)(\mathbf{k}_1 - m^*(1)+1)^2} \big]. \tag{4.28}$$

We show that $v_{r'}(\mathbf{k})$ is in $l^1(\mathbb{N}^2)$. We first define $c_\gamma = 2\sum_{j=0}^\infty e^{-\frac{\gamma}{2}j^2}$ to bound

$$\sum_{\mathbf{k} \in \mathbb{N}^2} v_{r'}(\mathbf{k}) \leq c_\gamma \sum_{k \in \mathbb{N}} \mathbb{E}_k^\star [e^{-(\beta V^\star + rU^\star)}] =: c_\gamma \sum_{k \in \mathbb{N}} z_{r'}(k).$$

We next use that z satisfies for $k \geq 1$,

$$z_{r'}(k) = \sum_{l \in \mathbb{N}} P(k, l+1) e^{r'(k+l) - ((\beta-\gamma)/2)(k+l)^2} z_{r'}(l),$$

to see that it is sufficient to have $z_{r'}(l) \leq \alpha^l$ for some $\alpha < \infty$. This is what we will prove now.

To this end let $\{(m_i^\star(x))_{x \in \mathbb{N}_0}; i \in \mathbb{N}\}$ be independent copies of $(m^\star(x))_{x \in \mathbb{N}_0}$ starting with $m_i^\star(0) = 1$, $i \in \mathbb{N}$. Here we make use of the fact that $m^\star$ has the branching property, that is, is the sum of independent branching chains. More precisely, if we fix $\mathbf{k}_2 \in \mathbb{N}$ and define

$$(m^\star(x))_{x \in \mathbb{N}_0} = \big(m_1^\star(x) + \cdots + m_{\mathbf{k}_2}^\star(x) \big)_{x \in \mathbb{N}_0}, \tag{4.29}$$

then $(m^\star(x))_{x \in \mathbb{N}_0}$ is a Markov chain with transition matrix $P^\star$ and $m^\star(0) = \mathbf{k}_2$. Define the functionals $U_i^\star$ and $V_i^\star$ as in (3.7), now for the chain $m_i^\star$, $i \in \mathbb{N}$. Hence

$$U^\star = U_1^\star + \cdots + U_{\mathbf{k}_2}^\star \quad \text{and} \quad V^\star \geq V_1^\star + \cdots + V_{\mathbf{k}_2}^\star. \tag{4.30}$$

From (4.30) it is clear that

$$z_{r'}(k) \leq \mathbb{E}_k^\star [e^{-(\beta-\gamma)\sum_{i=1}^k V^\star(i) + r'\sum_{i=1}^k U^\star(i)}] = \alpha_{r'}^k.$$

However, by assumption, $\alpha_{r'} < \infty$, which completes the proof. $\square$

The final step in the proof of Proposition 4.5 is to show that $\gamma \leq \beta - \frac{1}{2}\log 2$ implies $\bar{r} > r^*$.

LEMMA 4.8. *If* $\gamma \leq \beta - \frac{1}{2}\log 2$, *then* $\bar{r} > r^*$.

PROOF. Define $r' = \beta - \gamma + \log 2 > r^*$. Then $r' \leq 3(\beta - \gamma)$, hence $r's^\star(\mathbf{i}) - (\beta - \gamma)s^\star(\mathbf{i})^2 \leq 0$ unless $\mathbf{i} = (1, 0)$ or $\mathbf{i} = (1, 1)$, where it assumes the values $r' - (\beta - \gamma) = \log 2$, respectively, $2r' - 4(\beta - \gamma) = 2\log 2 - 2(\beta - \gamma) \leq \log 2$.

Let $E = \{(1,1),(1,0),(0,0)\}$. Thus

$$r'U^\star - (\beta-\gamma)V^\star \le \sum_{x=0}^{\infty} \mathbf{1}_E(\mathfrak{M}^\star(x))(r's^\star(\mathfrak{M}^\star(x)) - (\beta-\gamma)s^\star\mathfrak{M}^\star(x)^2) \tag{4.31}$$
$$\le \log 2(1+\#\{x\colon \mathfrak{M}^\star(x) = (1,1)\}).$$

Let $\mathfrak{M}'$ be the chain $\mathfrak{M}^\star$ observed only when it is in E. This definition makes sense since E is absorbing. Hence the right-hand side of (4.31) equals $\log 2(1+\tau)$, where $\tau = \sup\{x \in \mathbb{N}\colon \mathfrak{M}'(x) = (1,1)\}$. Note that we can compute the transition probabilities of $\mathfrak{M}'$: $P((0,0),(0,0)) = 1$, $P((1,0),(0,0)) = 1$, $P((1,1),(1,1)) = 1/(4(1-\rho))$, $P((1,1),(1,0)) = 1/(2(1-\rho))$ and $P((1,1),(0,0)) = (1-4\rho)/(4(1-\rho))$, where

$$\rho = P^1(m^\star(1) \ge 2, m^\star(x) = 1 \text{ for some } x \ge 2) < \frac{1}{4}.$$

Hence $1+\tau$ is geometrically distributed with parameter $\frac{1}{4(1-\rho)} < \frac{1}{3}$ and we get

$$\alpha_{r'} = \mathbb{E}_1^\star[e^{r'U^\star - (\beta-\gamma)V^\star}] \le E^{(1,1)}[2^{1+\tau}]$$
$$= 2\frac{1-1/(4(1-\rho))}{1-1/(2(1-\rho))} = \frac{3-4\rho}{1-2\rho} < \infty. \qquad \square$$

REMARK. Numerical computations show that it is possible that $\alpha_{r^*} = \infty$. More precisely, we can show analytically that if $\beta - \gamma \le 0.1$ and $r^* \ge 0.65$, then $\alpha_{r^*} = \infty$. The numerics yield, for example, that $r^*(8, 7.91) \approx 0.685 > 0.65$ and that hence this case in fact occurs. This means that with our estimates we have wasted too much. However, we have not found a way how we can substantially improve the bounds presented here.

4.4. *Completion of the proof.* We prove Theorem 3 by showing that there is an $L \in \mathbb{R}^+$ such that for every $\mu \in \mathbb{R}$,

$$\lim_{n\to\infty} e^{r^* n} E\big[e^{-H_n(S_n)} e^{\mu(S_n - \theta^* n)/\sigma^*\sqrt{n}} \mathbf{1}_{\{S_n > 0\}}\big] = Le^{\mu^2/2}. \tag{4.32}$$

Note that (4.32) implies that under the law $Q_n^{\beta,\gamma}(\cdot|S_n > 0)$ the moment generating function of $(S_n - \theta^* n)/\sigma^*\sqrt{n}$ converges pointwise to the one of the standard normal distribution as $n \to \infty$ (divide the left-hand side of (4.32) by the same expression for $\mu = 0$ and use (1.1)). Therefore (4.32) implies the central limit theorem as stated in Theorem 3.

Now we show (4.32). Fix $\mu \in \mathbb{R}$. First we analyze the asymptotics of the exponential on the right-hand side of (4.17). Recall that we abbreviate $\lambda_r = \lambda(r, \beta, \gamma)$. We write $s \mapsto \lambda^{-1}(s)$ for the inverse of $r \mapsto \lambda_r$ for fixed β and γ, and we write ∂_s for the derivative with respect to s. Expand $\lambda^{-1}(s)$ in a Taylor series around $s = 1$. Abbreviate $\mu_n = \mu/\sigma^*\sqrt{n}$. Then, from the definition of $r_n^*(\mu)$ in (4.6), we obtain the existence of some number ξ_n in between 1 and

$e^{-\mu_n}$ such that

$$
\begin{aligned}
r_n^*(\mu) &= \lambda^{-1}(e^{-\mu_n}) \\
&= \lambda^{-1}(1) + (e^{-\mu_n} - 1)\partial_s \lambda^{-1}(1) + \tfrac{1}{2}(e^{-\mu_n} - 1)^2 \partial_s^2 \lambda^{-1}(s)|_{s=\xi_n} \qquad (4.33)\\
&= r^* + (e^{-\mu_n} - 1)\theta^* + \tfrac{1}{2}(e^{-\mu_n} - 1)^2 \partial_s^2 \lambda^{-1}(s)|_{s=\xi_n}.
\end{aligned}
$$

Here the last equality follows from (2.7) and (2.8).

Next, we calculate

$$
\begin{aligned}
\partial_s^2 \lambda^{-1}(\xi_n) &= \big[\partial_s[\partial_r \lambda_r]^{-1}_{r=\lambda^{-1}(s)}\big]_{s=\xi_n} = -\left[\frac{\partial_r^2 \lambda_r}{(\partial_r \lambda_r)^3}\right]_{r=\lambda^{-1}(\xi_n)} \qquad (4.34)\\
&= -\sigma^{*2} - \theta^* + o(1).
\end{aligned}
$$

Note that the θ^* term cancels the second-order term of $e^{-\mu_n} - 1$ in (4.33). Thus

$$
\begin{aligned}
r^* - r_n^*(\mu) &= \mu_n \theta^* + \frac{1}{2}\mu_n^2 \sigma^* 2 + o(\mu_n^2) \qquad (4.35)\\
&= \frac{\mu \theta^*}{\sigma^* \sqrt{n}} + \frac{1}{2}\frac{\mu^2}{n} + o\left(\frac{1}{n}\right).
\end{aligned}
$$

Recall from Lemma 4.3 that

$$
\begin{aligned}
&e^{r^* n} E\big[e^{-H_n(S_n)} e^{\mu(S_n - \theta^* n)/\sigma^* \sqrt{n}} \mathbf{1}_{\{0 \le S_{n-1} < S_n\}}\big] \\
&\quad = e^{(r^* - r_n^*)n - \mu(\theta^*/\sigma^*)\sqrt{n}} \frac{p_{r_n^*} e^{-r_n^*} e^{2\mu/\sigma^* \sqrt{n}}}{\theta_{r_n^*}} \qquad (4.36)\\
&\qquad \times \sum_{n_1, n_2 \in \mathbb{N}} \widetilde{\mathbb{E}}^{r_n^*}\Big[f_{r_n^*}^{(L)}(\Gamma(0); n_1) f_{r_n^*}^{(R)}(\Gamma(n - n_1 - n_2); n_2)\Big].
\end{aligned}
$$

From (4.35) it is clear that the first term converges to $e^{\mu^2/2}$ as $n \to \infty$. Further, by continuity, the middle term converges to $p_{r^*} e^{-r^*}/\theta^* \in (0, \infty)$.

To finish the proof, use Lemmas 4.1 and 4.2, Propositions 4.4 and 4.5, together with dominated convergence to get that the sum converges to

$$
\left(\sum_{k \in \mathbb{N}} \widetilde{\mathbb{E}}^{r^*}\Big[f_{r^*}^{(L)}(\Gamma(0); k)\Big]\right)\left(\sum_{k \in \mathbb{N}} \widetilde{\mathbb{E}}^{r^*}\Big[f_{r^*}^{(R)}(\Gamma(0); k)\Big]\right).
$$

This yields (4.32) with the additional indicator on the event $\{0 \le S_{n-1} < S_n\}$ in the l.h.s. The limit assertion with the indicator on $\{0 \le S_n < S_{n-1}\}$ is similar. The constant L in (4.32) is the sum of both limits. □

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