COUNTING INTERVALS IN THE PACKING PROCESS

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We consider a sequential interval packing process similar to Rényi's "car parking problem" but with a generator of random intervals which allows for arbitrarily small lengths. Embedding the process in continuous time, we view it as a self-similar interval splitting process. We determine the asymptotical behavior of the quantities such as the number of intervals packed to some instant and obtain convergence results in the context of the more general splitting model.

1. Introduction. Sequential interval packing problems extending the celebrated Rényi "car-parking problem" [13] have a wide range of applications which include models of liquid structure in physics, absorption models in chemistry and construction of error-correcting codes in information theory, to mention a few. Many references to the early literature are found in the survey [14]. Recently, new interest in this class of problems arose in connection with modeling of communication networks (cf. [3], [4], [5]). We continue this line of research by extending the analysis of the counting problem initiated in [5].

Let $I_1, I_2, \ldots$ be independent copies of a random interval $I \subset [0,1]$. The sequential packing process is defined as follows. The first interval I_1 is always *packed*, that is, fixed at its position. The interval I_2 is packed if it does not overlap I_1. Similarly, each subsequent I_j, $j > 2$, is packed if it does not overlap any of the intervals packed so far.

The process comes eventually to an absorbing state, if the size of I cannot be arbitrarily small, as in Rényi's problem, and much of the literature on the subject is devoted to characterization of the terminal state. In sharp contrast to this, the process can proceed unlimitedly if I fits in any fixed gap with positive probability, in which case it is natural to ask about the number of packed intervals after n packing attempts. Coffman, Mallows and Poonen [5] raised this question under the assumption that I has the uniform distribution (that is to say, I is the span between two points drawn uniformly at random from $[0,1]$). They proved that for large n,

(1) $$\text{the expected number of packed intervals} \sim cn^{(\sqrt{17}-3)/4}$$

and gave an exact formula for the constant $c = 1.84 \ldots$. In the same paper they asked about the asymptotic behavior of the variance of the number of packed intervals and about the distribution of the sample size needed to pack a given number of intervals.

A generalization of the uniform distribution for random intervals is the family of distributions given by

$$(2) \qquad P(I \subset [x, 1 - y]) = (1 - x - y)^{\alpha}, \qquad (x, y) \in \Delta,$$

where $\alpha > 1$ and $\Delta = \{(x, y): x \geq 0, y \geq 0, x + y \leq 1\}$. For $\alpha = 2, 3, \ldots, I$ can be seen as the span of α random points chosen from the uniform distribution on the unit interval. In the present paper we find asymptotics for all moments of the number of packed intervals and related quantities for distributions (2). We adopt here the approach which was mentioned in [5] but not exploited in that paper: embed the packing process in continuous time and convert it, via Poissonization, into a version of the generalized Brennan–Durrett interval splitting process [1]. The key benefit of this device is that it brings exact independence into the problem where otherwise only a kind of asymptotic independence is present. Technically, the analysis is considerably simplified because each quantity like the mean number of packed intervals satisfies a single integral-differential equation rather than a system of difference-integral equations for each n. We show that the basic quantities are functions of hypergeometric type with powerlike growth at infinity.

The generalized interval-splitting process considered here differs radically from the original Brennan–Durrett model [1, 2] in that the sum of the pieces which undergo further subdivision is less than the whole. A distinguished feature appearing in the new model is that the number of parts normalized by its mean has a nontrivial weak limit which we identify with a fixed point of the "smoothing transformation" [6].

The packing–splitting model can be viewed as a continuous time version of a recursive construction of random fractals, as studied in [10]. From this viewpoint, the exponent in (1) should be interpreted as β^*/α where $\beta^* = (\sqrt{17}-3)/2$ is the Hausdorff dimension of the random Cantor set obtained by cutting out all packed intervals, while $\alpha = 2$ is the rate parameter of the waiting time between splits.

2. Self-similar packing process. Suppose that random intervals $I_1, I_2, \ldots$ arrive by a unit rate Poisson stream which is independent of the I_j's. Let the I_j's be independent replicas of a random interval I with probability law (2). We are interested in the configuration of packed intervals after the time t is elapsed.

Tractability of distributions (2) stems from their scale invariance properties. First, the probability that I fits in a fixed gap $[x, 1 - y]$ depends solely on the size of the gap and not on its location. Second, for the affine order-preserving transformation $\phi_{x, y}$ which maps $[x, 1 - y]$ onto $[0, 1]$, the conditional distribution of $\phi_{x, y}(I)$ given that $I \subset [x, 1 - y]$ coincides with the unconditional distribution of I. The invariance taken together with the total independence property of the extended Poisson process on $[0, \infty) \times \Delta$ yields a self-similarity of the packing process, which is best understood via the evolution of gaps between packed intervals. The packing process within each gap is a scaled

stochastic copy of the whole process, with scaling factors depending only on the size of the gap.

It is convenient to enumerate the gaps by finite dyadic sequences, denoting a generic sequence by $*$. Writing $I = [X, 1-Y]$ we identify a random interval with a point $(X, Y) \in \Delta$, and let (X_*, Y_*)'s be independent copies of (X, Y).

Initially there is only one gap $[0, 1]$ of size $S_0 = 1$. After a mean one exponential time an interval is packed and two new gaps of sizes $S_{00} = X_0 S_0$ and $S_{01} = Y_0 S_0$ are created. Now, those incoming intervals which do not fit in any of the two gaps cannot alter the configuration of packed intervals, while the substreams of intervals fitting in the gaps are Poisson and independent. The process iterates: after a gap of size S_* is created it waits a random mean $S_*^{-\alpha}$ exponential time and then splits into three pieces of sizes

$$S_{*0} = X_* S_*, \ (1 - X_* - Y_*)S_* \text{ and } S_{*1} = Y_* S_*,$$

where the first and the third pieces are the newly created gaps and the second piece is the packed interval. Given the gap size, the waiting time for a gap to split is independent of the past of the process, splitting proportions and the evolution of the other coexising gaps.

3. Generalized interval splitting model. Essential in the above representation of the packing process is only the description of the random splitting mechanism.

More generally, we consider the Markovian splitting of $[0, 1]$ with a rate parameter α governing the exponential waiting times and a probability distribution F on Δ determining the random proportions $X, 1 - X - Y$ and Y. We will still speak of "packed intervals" and "gaps" to make a clear difference between divisible and indivisible pieces, although for the general F the splitting process cannot be induced by a Poisson stream of i.i.d. intervals.

We are interested in the functionals which depend only on gap sizes and are insensible to the arrangement of gaps within $[0, 1]$; thus we do not loose generality by assuming that X and Y are exchangeable. In particular, X and Y have the same distribution which we denote G. We will assume throughout the paper that G is absolutely continuous.

We will view the moments of X as particular values of the function

$$g(s) = \int_0^1 x^s \, dG(x),$$

defined in the fundamental half-plane, to the right from the convergence abscissa σ, $-\infty \leq \sigma \leq 0$, where the integral converges absolutely. If $\sigma > -\infty$ and the integral converges for $s = \sigma$ then the fundamental half-plane includes the vertical line $\operatorname{Re} s = \sigma$ and is therefore closed. Otherwise the fundamental half-plane is open. In any case, g is analytical in the interior of the fundamental half-plane.

REMARK. In loose terms, σ is responsible for the "flatness" of G near 0. For example, for $G(x) = x^a L(x)$, $a > 0$ and L slowly varying we have $\sigma = -a$ and

the fundamental half-plane is open. On the other hand, for $G(x) = -(\log x)^{-1}$ we have $\sigma = 0$ and the fundamental half-plane is closed. The function g is rational with poles at x_α when G is a linear combination of terms x^α with coefficients being polynomials in $\log x$.

The asymptotic analysis to follow involves the roots of the equation

$$(3) \qquad\qquad 2g(s) = 1.$$

Because g is strictly decreasing on the intersection of the real axis and the fundamental half-plane, and because $g(0) = 1$ and $g(1) \leq 1/2$, there is a unique solution $\beta^* \in\,]0, 1]$ which we call *the main root*. The main root is the only real solution to the right from σ and it is simple because

$$(4) \qquad\qquad g'(\beta^*) = \int_0^1 x^{\beta^*} \log x \, dG(x) < 0.$$

The Brennan–Durrett interval splitting model appears as a special case when $X + Y = 1$ with probability 1, and a gap of size S_* breaks effectively into two pieces of sizes $X_* S_*$ and $Y_* S_*$. A characteristic feature of this case is that $\beta^* = 1$.

4. Basic recursion. We fix now the ingredients α and F and proceed with the generalized interval splitting model.

For β in the fundamental half-plane, define $L(t, \beta)$ as the sum of the βth powers of the sizes of all gaps present at instant t. Thus $L(t, 0)$ is the number of gaps and $L(t, 1)$ is the total gap length. Denote by T the waiting time for the first split. Because future splitting of any of the two gaps appearing at T reproduces the whole process we have

$$(5) \qquad L(t, \beta) \stackrel{d}{=} \begin{cases} 1, & \text{for } t < T, \\ X^\beta L_0(X^\alpha(t - T)) + Y^\beta L_1(Y^\alpha(t - T)), & \text{for } t \geq T, \end{cases}$$

where L_0 and L_1 have the same probability law as L, T is mean one exponential, (X, Y) follow the splitting law F and $L_0, L_1, T, (X, Y)$ are mutually independent.

By our choice of β the expectation $l(t, \beta) = EL(t, \beta)$ is finite and, averaging in (5), we derive an integral equation

$$(6) \qquad l(t, \beta) = e^{-t} + 2 \int_0^t e^{-s} \int_0^1 l((t - s)x^\alpha, \beta) x^\beta \, dG(x) \, ds,$$

where the symmetry between X and Y reflects in the factor 2. Differentiating in t we get

$$(7) \qquad\qquad l'(t, \beta) = -l(t, \beta) + 2 \int_0^1 l(x^\alpha t, \beta) x^\beta \, dG(x),$$

$$(8) \qquad\qquad l(0, \beta) = 1.$$

Repeated differentiation shows that $l(t, \beta)$ is infinitely smooth.

THEOREM 1. *The initial value problem* (7), (8) *has a unique* C^∞ *solution. The solution is given by the power series*

$$(9) \qquad l(t, \beta) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \prod_{j=0}^{k-1} (1 - 2g(\alpha j + \beta)),$$

which converges absolutely for all t.

PROOF. The convergence claim follows by noting that the product in (9) varies slowly with k (see Lemma 3 below); thus the series defines an entire function. Substituting into (7) we see that (9) is indeed a solution. The uniqueness is justified by the contraction principle; see [8] for details. □

The uniqueness implies that the derivatives satisfy

$$l^{(k)}(t, \beta) = l^{(k)}(0, \beta)l(t, k\beta + \alpha).$$

It follows that for real β the $l(t, \beta)$'s are strictly monotone in t, except for $l(t, \beta^*) \equiv 1$.

Call β *singular* if $\alpha K + \beta$ is a solution to (3) for some nonnegative integer K. For such a β, $l(t, \beta)$ is a polynomial of degree K and we have for $t \to \infty$,

$$(10) \qquad l(t, \beta) = (K!)^{-1} \left(\prod_{j=0}^{K-1} (1 - 2g(\alpha j + \beta)) \right) t^K + O(t^{K-1}).$$

In particular, the series terminates if $(\beta^* - \beta)/\alpha$ is a nonnegative integer; in which case the nonzero coefficients of the series are positive. For other real values of β the series starts alternating from some term and the situation is more complicated. If $c(\beta)t^{(\beta^* - \beta)/\alpha}$ is plugged into (7) the leading terms cancel, leading to the guess that this is the correct asymptotics, although this heuristics itself gives no hint of the value of the coefficient. Deriving the asymptotics requires a more sophisticated treatment, which we give in the next section.

5. Asymptotics of the expectations. Assume first that β is not singular. Write

$$\gamma(k, \beta) = \prod_{j=0}^{k-1} (1 - 2g(\alpha j + \beta))$$

for the coefficient of the series at $(-t)^k/k!$. Our plan is to extend γ to a function of the complex variable s and to derive a Mellin–Barnes integral representation for the series. This technique is classical and can be found in many books (see, e.g., [11]).

Obviously, g has a ridge at the real axis,

$$(11) \qquad |g(s)| \le g(\text{Re } s)$$

and therefore goes to zero as $\text{Re } s \to \infty$.

Mimicking the proof of the Riemann–Lebesgue theorem (see [15], page 11), we obtain a more delicate property.

LEMMA 2. *For G absolutely continuous, $g(s)$ tends to 0 as s varies in any closed sub-half-plane of the fundamental half-plane so that $|\operatorname{Im} s| \to \infty$. The convergence is uniform in $\operatorname{Re} s$.*

A consequence of the lemma is that (3) has finitely many roots in any closed sub-half-plane of the fundamental half-plane. By (11) all roots lie to the left from β^*. Also, there are no further roots on the line $\operatorname{Re} s = \beta^*$ because otherwise G would have a purely atomic component.

For β in the fundamental half-plane, set

$$\mathscr{P}_\beta = \{(s - \beta)/\alpha - j \colon s > \sigma, \ 2g(s) = 1, \ j = 0, 1, \ldots\}$$

and for s such that $\operatorname{Re} s > (\sigma - \beta)/\alpha$, $s \notin \mathscr{P}_\beta$, consider the product

$$(12) \qquad \gamma(s, \beta) = \prod_{j=0}^{\infty} \frac{1 - 2g(\alpha j + \beta)}{1 - 2g(\alpha j + \beta + \alpha s)}.$$

Passing to logarithms we see that the product converges, hence $\gamma(s, \beta)$ is a meromorphic function with poles in $\mathscr{P}_\beta$. The rightmost pole is $(\beta^* - \beta)/\alpha$ and it is simple.

The identity

$$\gamma(s, \beta) = \frac{\gamma(s + \beta/\alpha, 0)}{\gamma(\beta/\alpha, 0)}$$

shows that all the γ's are essentially versions of a single function.

It is easily seen from the definition (12) that the function γ satisfies a functional equation analogous to the well-known equation for the gamma function,

$$(13) \qquad \gamma(s + 1, \beta) = (1 - 2g(\alpha s + \beta))\gamma(s, \beta).$$

But the behavior of γ at infinity is very different from the asymptotics of the gamma function.

LEMMA 3. *The function $\gamma(s, \beta)$ is bounded in any closed sub-half-plane of the fundamental half-plane, outside a sufficiently large circle enclosing $\mathscr{P}_\beta$. Besides that, as $|s| \to \infty$,*

$$|\gamma(s, \beta)| > \operatorname{const} e^{-\varepsilon|s|}$$

for arbitrary $\varepsilon > 0$.

PROOF. Let N be a positive integer larger than $(\beta^* - \operatorname{Re} \beta)/\alpha$. For $\operatorname{Re} s > N$ we have

$$|2g(\alpha s + \beta + \alpha j)| \leq 2g(\alpha \operatorname{Re} s + \operatorname{Re} \beta + \alpha j) < 2g(\alpha N + \operatorname{Re} \beta + \alpha j),$$

and this is less than 1, hence $|\gamma(s, \beta)| < \gamma(N, \beta)$. To bound the function for s on the left from N use (13) and Lemma 2.

To estimate γ in the vertical direction we decompose γ into two products with indices $j < J$ and $j \geq J$. Away from the poles, the absolute value of the

first product is bounded from below by Lemma 2. To estimate the absolute value of the second product we use first the uniform estimate (11). Then

$$\left| \log\left(\prod_{j=J}^{\infty} \frac{1 - 2g(\alpha j + \beta)}{1 - 2g(\alpha j + \beta + \alpha s)} \right) \right| \le C \sum_{j=J}^{\infty} |g(\alpha j + \beta) - g(\alpha j + \beta + \alpha s)|$$

for some constant C. Using, further, the obvious estimate $|1 - x^{\alpha s}| \le \alpha |s|(1 - x) \le \alpha |s|(1 - x^{\alpha})$ (valid for $0 < x \le 1$) and telescoping the resulting sum, we arrive at

$$\alpha |s| C g(\beta + J)$$

as an upper bound for the absolute value of the second product. Taking J large enough we see that the integral grows slower than $\varepsilon |s|$. $\square$

Following the well-known arguments (see [11], pages 300 and 301) we obtain an integral representation of the power series

$$l(t, \beta) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma(-s)\gamma(s, \beta)t^s \, ds,$$

where the integration contour goes along the imaginary axis and is indented so that the integers $0, 1, 2, \ldots$ [which are the poles of $\Gamma(-s)$] lie on one side and the poles $\mathscr{P}_\beta$ on the other side from the contour. Such a contour exists because $\mathscr{P}_\beta$ contains no nonnegative integers. The integral converges absolutely for all complex t in the sector $|\mathrm{Arg}\, t| < \pi/2$ as it follows from the asymptotics of the gamma function and the first part of Lemma 3. The equality follows from the residue theorem by taking a large segment of the imaginary axis and closing it by a rectangular contour lying on the right.

Pulling the rectangular contour to the left across the poles of γ gives an asymptotic expansion for large t,

$$l(t, \beta) \asymp \sum_{s_\alpha \in \mathscr{P}_\beta} \Gamma(-s_\alpha)t^{s_\alpha} \, \mathrm{Re}\, s_\alpha \, \gamma(\cdot, \beta).$$

Evaluating the leading term yields the conjectured result.

THEOREM 4.   *For all β in the fundamental half-plane and $t \to \infty$,*

$$(14) \qquad l(t, \beta) = c(\beta)t^{(\beta^* - \beta)/\alpha} + O\big(t^{(\beta^* - \beta)/\alpha - \min(1, \theta/\alpha)}\big),$$

where θ is the width of the strip left from β^* which is free from the roots of (3) *and lies within the fundamental half-plane, and*

$$(15) \qquad c(\beta) = \Gamma\left(\frac{\beta - \beta^*}{\alpha} \right) \frac{1 - 2g(\beta)}{-2\alpha g'(\beta^*)} \prod_{j=1}^{\infty} \frac{1 - 2g(\alpha j + \beta)}{1 - 2g(\alpha j + \beta^*)}.$$

Note that there is no restriction on β in the formulation of the theorem. For β singular a possible pole of the gamma function is always annihilated by a zero factor in (15), so that the formula becomes (10).

We emphasize that the most important value, $\beta = 0$, is always covered by the theorem, including the boundary case $\sigma = 0$.

REMARK. The series $l(t, \beta)$ generalize certain generalized hypergeometric functions of the type ${}_pF_p$. These are indeed generalized hypergeometric functions in the case when g is extendible to a rational function on the whole plane, with p being the degree of the denominator. In the latter case the infinite products can be evaluated in terms of the gamma function or we can adopt the classical hypergeometric asymptotics. We refer the reader to [12] for a transparent exposition of the hypergeometrical asymptotics and integral representations).

6. Higher moments and the limiting distribution of the number of gaps.
In this section we use the notation

$$l_k(t) = EL^k(t, 0)$$

for the kth moment of the number of gaps [thus $l(t, 0) = l_1(t)$]. Setting $\beta = 0$ and averaging the kth powers on both sides of (5) we derive

$$(16) \qquad l_k'(t) = -l_k(t) + 2\int_0^1 l_k(x^\alpha t)\, dG(x) + h_k(t),$$
$$l_k(0) = 1,$$

where the inhomogeneous term involves the joint distribution of (X, Y),

$$h_k(t) = \sum_{j=1}^{k-1} \binom{k}{j} \int_\Delta l_j(tx^\alpha) l_{k-j}(ty^\alpha)\, dF(x, y).$$

We use the notation

$$f(\lambda, \mu) = \int_\Delta x^\lambda y^\mu\, dF(x, y)$$

for joint moments of (X, Y).

THEOREM 5. *For $k \geq 1$ and $t \to \infty$,*

$$l_k(t) \sim c_k t^{k\beta^*/\alpha},$$

where $c_1 = c(0)$ is as in (15) and, recursively,

$$(17) \qquad c_k = \sum_{j=1}^{k-1} c_j c_{k-j} \binom{k}{j} \frac{f(j\beta^*, (k-j)\beta^*)}{1 - 2g(k\beta^*)}, \qquad k = 2, 3, \ldots.$$

PROOF. The proof is by induction on k and exploits efficiently the observation that $t^{\beta^*/\alpha}$ satisfies the integral equation

$$-l(t) + 2\int_0^1 l(x^\alpha t)\,dG(x) = 0.$$

Theorem 4 states that the asymptotics holds for $k = 1$. Suppose the statement is valid for $j < k$, then the inhomogeneous term satisfies

(18) $$h_k(t) \sim \left(\sum_{j=1}^{k-1} c_j c_{k-j}\binom{k}{j} f(j\beta^*, (k-j)\beta^*)\right) t^{k\beta^*/\alpha},$$

and let b denote the coefficient in this formula. Introduce a comparison function

$$u(t) = (c_k + \delta)t^{k\beta^*/\alpha} + at^{\beta^*/\alpha} + 2,$$

where δ and a are some positive constants. We claim that for a sufficiently large, $l_k(t) \le u(t)$ for all $t \ge 0$. Suppose not, then for any fixed a from $u(0) > l_k(0)$ follows that there is a minimal τ where l_k becomes larger than u. Observe that $\tau \to \infty$ as $a \to \infty$. By the definition of τ we have $u(\tau) = l_k(\tau)$ and $u(t) > l_k(t)$ for $t < \tau$; thus using (16) and (18) we get for $a \to \infty$,

$$l_k'(\tau) = -l_k(\tau) + 2\int_0^1 l_k(\tau x^\alpha)\,dG(x) + h_k(\tau)$$

$$< -u(\tau) + 2\int_0^1 u(\tau x^\alpha)\,dG(x) + h_k(\tau)$$

$$= -(c_k + \delta)\tau^{k\beta^*/\alpha} - a\tau^{\beta^*/\alpha}(1 - 2g(\beta^*))$$
$$\quad + 2(c_k + \delta)\tau^{k\beta^*/\alpha}g(k\beta^*) + 2 + h_k(t)$$

$$\sim \tau^{k\beta^*/\alpha}\big(-(c_k + \delta)(1 - 2g(k\beta^*)) + b\big)$$

$$= -\tau^{k\beta^*/\alpha}\delta(1 - 2g(k\beta^*)) \to -\infty,$$

because $1 - 2g(k\beta^*) > 0$ for $k > 1$. But this is a contradiction because $u'(\tau) > 0$ and l_k cannot become larger than u with negative derivative (or simply because $l_k(t)$ is increasing). From $u(t) > l_k(t)$ by selecting δ arbitrarily small we obtain for $t \to \infty$,

$$\limsup \frac{l_k(t)}{t^{k\beta^*/\alpha}} \le c_k.$$

To get the lower bound we take another comparison function,

$$v(t) = (c_k - \delta)t^{k\beta^*/\alpha} - at^{\beta^*/\alpha}$$

and show that $l_k(t) > v(t)$ provided a is sufficiently large. Indeed, assuming the contrary and considering the first intersection point τ we have again $\tau \to \infty$ as $a \to \infty$. Arguing as in the proof of the upper bound we have

$$l_k'(\tau) > \text{const } \tau^{k\beta^*/\alpha}.$$

On the other hand,

$$v'(\tau) < \text{const } \tau^{k\beta^*/\alpha - 1}$$

which is definitely smaller than the derivative of l_k for τ sufficiently large: a contradiction. Letting $\delta \to 0$ we see that

$$\liminf \frac{l_k(t)}{t^{k\beta^*/\alpha}} \geq c_k.$$

Putting the two sides together proves the induction step, whence the result. □

We are now in a position to show that the normalized number of gaps has a weak limit.

THEOREM 6. *There exists a weak limit*

$$\frac{L(t,0)}{l(t,0)} \xrightarrow{d} Z, \qquad t \to \infty,$$

where the random variable $Z \geq 0$ is a unique solution of the distributional equation,

(19) $$Z \stackrel{d}{=} X^{\beta^*} Z_0 + Y^{\beta^*} Z_1,$$

with $EZ = 1$. Here, Z_0 and Z_1 have the same distribution as Z, (X,Y) has distribution F and $Z_0, Z_1, (X,Y)$ are independent.

PROOF. Equation (19) defines Z as a fixed point of Durrett–Liggett's "smoothing transformation." Because the derivative (4) is strictly negative, Theorem (5.1) from [6] implies the existence and uniqueness of the solution, while finiteness of all moments of Z is a consequence of Theorem (5.3) from that paper. Setting $z_k = EZ^k$ we have $z_1 = 1$ and manipulating with (19) we conclude that the z_k's satisfy the recursion (17); therefore $z_k = c_k/c_1^k$. It is not hard to check that $z_k < k!(1 - 2g(2\beta^*))^{-k}$, and applying Carleman's condition we see that the distribution of Z is uniquely determined by the moments z_k. Finally, by Theorem 4 we have

$$\frac{l^k(t)}{l_1^k(t)} \to \frac{c_k}{c_1^k} = z_k$$

and the weak convergence follows from the convergence of moments. □

REMARK. Similar arguments show the weak convergence $L(t,\beta)/l(t,\beta) \to_d Z$ for any real β between the convergence abscissa σ and β^*. In the critical case, $L(t,\beta^*)$ is a martingale and converges with probability 1 to the total mass of the Hausdorff measure of the Cantor set mentioned in the Introduction; see [10] for details.

In the case $X + Y \equiv 1$ studied by Brennan and Durrett we have $\beta^* = 1$, $c_k = c_1^k$, $z_k = 1$; thus $\text{Var } L(t,0) = o(t^{2\beta^*/\alpha})$ and the limit of $L(t,0)/l(t,0)$ is the degenerate variable $Z \equiv 1$. Brennan and Durrett [1] derived a better

estimate for the variance and concluded that the convergence to 1 holds with probability 1.

7. Limiting empirical law for gap lengths. We have seen that the mean total gap size is of the order $t^{(\beta^*-1)/\alpha}$ while the mean number of gaps grows like $t^{\beta/\alpha}$. This suggests that the size of a typical gap must be of the order $t^{-1/\alpha}$. We will justify the guess in the form of a limiting empirical distribution for the normalized gap sizes.

Following the suggestion, define for $x \geq 0$,

$$H_t(x) = \frac{\text{expected number of gaps with lengths} \leq xt^{-1/\alpha}}{\text{expected total number of gaps}}.$$

Clearly, H_t is a distribution function with the moments

$$\int_0^\infty x^\beta H_t(x) = \frac{t^{\beta/\alpha} l(t, \beta)}{l(t, 0)}.$$

Multiplying the right-hand side by $t^{-\beta^*/\alpha}$ and letting $t \to \infty$ we derive from Theorem 4 that these integrals converge, whence H_t has a limit.

THEOREM 7. *There exists a distribution function H such that for $x \geq 0$,*

$$H(x) = \lim_{t \to \infty} H_t(x).$$

The limit is uniquely characterized by the formula

$$(20) \qquad \int_0^\infty x^\beta \, dH(x) = \frac{\Gamma\big((\beta - \beta^*)/\alpha\big)}{\Gamma\big(-\beta^*/\alpha\big)} \frac{1}{\gamma(\beta/\alpha, 0)}$$

which holds for all β in the fundamental half-plane.

For $\beta = k\alpha$; $k = 0, 1, \ldots$, the infinite product (12) telescopes and we get

$$\int_0^\infty x^{k\alpha} \, dH(x) = \frac{\Gamma(k - \beta^*/\alpha)}{\Gamma(-\beta^*/\alpha)} \prod_{j=1}^{k-1} \frac{1}{1 - 2g(\alpha j)}.$$

These moments grow slower than $k!2^k$, thus the Carleman's criterion guarantees that they determine H unambiguously.

The limiting distribution is absolutely continuous for $x > 0$ and its density can be recovered as the inverse Mellin integral,

$$H'(x) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} \frac{\Gamma\big((s - \beta^*)/\alpha\big)}{\Gamma\big(-\beta^*/\alpha\big)} \frac{x^{-s-1}}{\gamma(s/\alpha, 0)} \, ds,$$

where ν is an arbitrary positive constant. The Stirling formula for the gamma function and the second part of Lemma 3 imply that the integral converges absolutely for $x > 0$; therefore H indeed has a density.

REMARK. The function $H(x)$ behaves near zero like $G(x)$ and is of the order of $x^{-\sigma}$. If g admits a meromorphic continuation to the left from σ, an asymptotic expansion of H is determined from the spacing of poles of g (see [7], Theorem 7.8). Explicitly,

$$\frac{1}{\gamma(s/\alpha, 0)} = \prod_{j=0}^{\infty} \frac{1 - 2g(\alpha j + s)}{1 - 2g(\alpha j)},$$

which shows that singularities in (20) come with those of $1 - 2g(s)$.

8. Packed intervals and gaps. The reader might have been surprised to learn things about gaps while the title of the paper suggests that the packed intervals must be our primary topic. To remedy the situation we shall reduce the "interval counting" to the "gap counting."

The number of packed intervals is equal to the number of gaps minus 1. The total area covered by the packed intervals is 1 minus the total area covered by the gaps. Taking the squared sums breaks the direct connection, but there is still a simple relation between the expectations. Surprisingly enough, this applies to almost all β.

Let $M(t, \beta)$ be the sum of the βth powers of the lengths of *packed intervals* present at instant t, $m(t, \beta) = EM(t, \beta)$. Clearly,

$$M(t, 0) = L(t, 0) - 1, \qquad M(t, 1) = 1 - L(t, 1).$$

THEOREM 8. *If β is not a root of* (3) *and* $\operatorname{Re} \beta > \sigma$, *then*

$$m(t, \beta) = a(\beta)(l(t, \beta) - 1),$$

where

$$a(\beta) = \frac{\int_{\Delta}(1 - x - y)^{\beta}\, dF(x, y)}{-1 + 2g(\beta)}.$$

PROOF. An argument similar to that which led us to (7) shows that $m(t, \beta)$ is a solution to

$$m'(t, \beta) = -m(t, \beta) + 2\int_0^1 m(tx^{\alpha})x^{\beta}\, dG(x) + \int_{\Delta}(1 - x - y)^{\beta}\, dF(x, y),$$

$$m(0, \beta) = 0.$$

Substitution $m(t, \beta) = a(\beta)(l(t, \beta) - 1)$ converts this equation into (6) and the statement follows by uniqueness of the solutions. □

9. Packing time. What is the limiting distribution of the time needed to pack a large number N of intervals?; that is, how long should we wait until N splits occur? This question was posed in the last section of [5]. We can prove here that a limiting distribution distribution does exist.

Let T_N be the time of the Nth split. Thus T_1 is mean one exponential, but already the law of T_2 must involve the joint distribution of splitting proportions.

Since each split adds one gap and one interval we have

$$P(T_N \le t) = P(L(t, 0) > N + 1).$$

Inverting this relation and applying Theorem 6 yields the following.

THEOREM 9.   *There exists a weak limit, as $N \to \infty$,*

$$\frac{T_N}{(N/c(0))^{\alpha/\beta^*}} \xrightarrow{d} Z^{-\alpha/\beta^*}.$$

REMARK.   In the Brennan–Durrett case the limit is degenerate and the convergence holds with probability 1.

10. Packing problem with distributions (2).   We return to the original problem and consider the packing problem with $\alpha > 1$ and

$$dF(x, y) = \alpha(\alpha - 1)(1 - x - y)^{\alpha-2} \, dx \, dy.$$

As a splitting model, this problem has a special feature that α enters both the waiting time distribution and the splitting distribution. The marginal distribution of a splitting proportion is

$$dG(x) = \alpha(1 - x)^{\alpha-1} \, dx$$

and the integral

$$g(s) = \alpha B(s + 1, \alpha) = \frac{\Gamma(\alpha + 1)\Gamma(s + 1)}{\Gamma(\alpha + s + 1)}$$

has the convergence abscissa at $\sigma = -1$. The function g is meromorphic on the whole plane, with finitely or infinitely many poles at negative integers. The main root is a single positive solution to

(21) $$1 - 2\alpha B(\beta^* + 1, \alpha) = 0$$

An easy computation yields the joint moments

$$f(\lambda, \mu) = \frac{\Gamma(\alpha + 1)\Gamma(\lambda + 1)\Gamma(\mu + 1)}{\Gamma(\alpha + \lambda + \mu + 1)}.$$

REMARK.   Upon a change of variable the integral term in (7) becomes

$$\int_0^1 l(ty, \beta) y^{(\beta+1-\alpha)/\alpha} (1 - y^{1/\alpha})^{\alpha-1} \, dy,$$

where we recognize an integral of Erdélyi–Kober type appearing in the fractional calculus [9]. The solutions are generalized hypergeometric functions of type ${}_\alpha F_\alpha$ for $\alpha = 2, 3, \ldots$. For noninteger values of α the $l(t, \beta)$'s are entire functions which seem not to have been considered before.

We specialize now in the case $\alpha = 2, 3, \dots$. Then (21) becomes a polynomial equation

$$(\beta + 1)(\beta + 2) \cdots (\beta + \alpha) - 2\alpha! = 0$$

which has α roots $\beta^*, \beta_2, \dots, \beta_\alpha$, of which β^* is the rightmost one. Since g is rational we can write $1 - 2g(s)$ as a ratio of two polynomials which we decompose into a product of monomials. The resulting series in the usual hypergeometric form is

$$l(t, \beta) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \prod_{j=0}^{k-1} \frac{\left(j + (\beta - \beta^*)/\alpha\right)\left(j + (\beta - \beta_2)/\alpha\right) \cdots \left(j + (\beta - \beta_\alpha)/\alpha\right)}{\left(j + (\beta + 1)/\alpha\right)\left(j + (\beta + 2)/\alpha\right) \cdots \left(j + (\beta + \alpha)/\alpha\right).}$$

Theorem 4 applies and evaluating the infinite product in terms of the gamma function (as in [16]) we get

$$(22) \qquad c(\beta) = \frac{\prod_{j=2}^{\alpha} \Gamma\left((\beta^* - \beta_j)/\alpha\right) \prod_{j=1}^{\alpha} \Gamma\left((\beta + j)/\alpha\right)}{\prod_{j=2}^{\alpha} \Gamma\left((\beta - \beta_j)/\alpha\right) \prod_{j=1}^{\alpha} \Gamma\left((\beta^* + j)/\alpha\right),}$$

a result which could be read off from the classical asymptotics (see [11, 12]).

The limiting empirical distribution of gap sizes has density

$$H'(x) = \frac{1}{2\pi i} \frac{\prod_{j=2}^{\alpha} \Gamma\left(-\beta_j/\alpha\right)}{\prod_{j=1}^{\alpha} \Gamma\left(j/\alpha\right)} \int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{\alpha} \Gamma\left((s + j)/\alpha\right)}{\prod_{j=2}^{\alpha} \Gamma\left((s - \beta_j)/\alpha\right)} x^{-s-1} \, ds.$$

The integral represents Meijer's G-function of type $G_{\alpha-1, \alpha}^{\alpha, 0}$ (see [12], page 32) and converges absolutely in the sector $|\mathrm{Arg}\, x| < \pi/(2\alpha)$. The density $H'(x)$, $x \geq 0$, has an exponential decay at infinity and a positive value at $x = 0$.

The variance of the number of gaps satisfies

$$\mathrm{Var}\, L(t, 0) \sim (c_2 - c_1^2) t^{2\beta^*/\alpha},$$

where $c_1 = c(0)$ is given by (22) and

$$(23) \qquad c_2 = 2c_1^2 \frac{\Gamma(\alpha + 1)\Gamma(\beta^* + 1)\Gamma(\beta^* + 1)}{\Gamma(\alpha + 2\beta^* + 1) - 2\Gamma(\alpha + 1)\Gamma(2\beta^* + 1)}.$$

EXAMPLE. We get an additional insight into the problem studied in [5]. In this case $\alpha = 2$ and the main root is $\beta^* = (\sqrt{17} - 3)/2$. The second root $\beta_2 = (-\sqrt{17} - 3)/2$ lies to the left from the convergence abscissa at $\sigma = -1$.

The (smooth version of) density H' has a positive value at $x = 0$. Coffman, Mallows and Poonen demonstrated an integral representation for H' different from that given here and expressed the density via a Whittaker function. [It should be noted that the $x^{-1/2}$ factor in their equation (4.2) gets absorbed by a singularity of the Whittaker function, thus confirming the behavior of H' at zero.]

The coefficient $c(0)$ appeared as formula (3.7) in [5], which is in full accord with our (22). [Note: their c_k's correspond to the $c(\beta)$'s in our notation and

our c_k's have a different meaning.] Now we can do the next step and apply Theorem 6 and (23) to compute the asymptotics of the variance,

$$\mathrm{Var}\, L(t, 0) \sim (c_2 - c_1^2) t^{(\sqrt{17}-3)/2},$$

where the numeric values are $c_2 \approx 3.849820$ and $c_2 - c_1^2 \approx 0.464725$.

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