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AN EXTENSION OF A RESULT OF ANDJEL

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We prove, using results for hydrodynamic limits, that an exclusion process starting from an ergodic initial distribution converges to product measure in one dimension. Our only assumption is the existence of a nonzero mean for the underlying random walk.

1. Introduction. In this note we consider a class of interacting particle systems closely related to systems of queues in tandem.

Classically interacting particle systems are Markov processes η_t on state space $\{0,1\}^{\mathbb{Z}^d}$. Then $\eta_t(x) = 1$ is interpreted to mean that there is a particle at site x at time t. Necessarily, there can be at most one particle per site at any given time t. The particles move or are created or destroyed at rates depending on the state of the configuration η_t locally. In this paper we are interested in the exclusion process; this is a process where particles are neither created nor destroyed. Each particle tries to move after waiting exponential amounts of time. After such times a new site is selected according to some fixed (for the process) random walk kernel. If at this time the selected site is already occupied by a particle, the move is suppressed and the first particle stays where it originally was; otherwise it does move to the new site.

We restrict attention to the one-dimensional exclusion process. This is an interacting particle system on $\{0,1\}^{\mathbb{Z}}$ with generator

$$\Omega f(\xi) = \sum_{x,\,y} (f(\xi^{xy}) - f(\xi))p(y-x)\xi(x)(1-\xi(y)),$$

where

$$\xi^{xy}(z) = \xi(z) \quad \text{for } z \neq x \text{ or } y,$$

$$\xi^{xy}(x) = \xi(y), \qquad \xi^{xy}(y) = \xi(x)$$

and where $p(\ )$ is a probability distribution on the integers.

The exclusion process and tandem systems of queues are intimately related. The connection has long been recognized (see [6]). There is an explicit mapping of the totally asymmetric nearest neighbor exclusion process [where $p(1) = 1$] to the queuing system consisting of a sequence of independent identically distributed memoryless servers and customers passing through from one queue to the succeeding queue: one identifies particles with customers and vacancies with servers. Given a vacancy, if its left neighbor is also vacant then it

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is empty; otherwise all 1's between the vacancy and the next vacancy to the left are regarded as customers queuing up for service at the first vacancy. The particle immediately to the left of the vacancy will be currently receiving service. One can therefore simply transport results from one set-up to the other. Another connection (at a heursitic level) is that both the exclusion process and queueing models possess a conservation property: particles or customers are neither created nor destroyed. This common property and similar coupling arguments enable one to prove similar results for stationary translation invariant processes or arrival processes for queues. One has the heuristic that what holds for one system holds for the other. This led to an argument of [2] for convergence in a translation invariant conservative particle system being adapted in [10] to give a convergence result for a stationary, ergodic arrival process that passed through a sequence of memoryless queues. It is our hope that the approach detailed in this paper can give results for queuing systems in tandem.

Though it is a bit artificial, one could think of our exclusion process as representing a queuing system where particles are customers and sites represent servers. When customers are served (after an exponential amount of time), they select a new server according to kernel $p(\ )$. If the server chosen is currently working, then customers engage in a new service with their original servers.

For technical reasons we will assume in this paper that

$$\text{(I)} \qquad \sum_{n\geq 0} p^n(z) + p^n(-z) > 0 \qquad \forall\, z \in \mathbb{Z},$$

$$\text{(II)} \qquad \sum_x |x| p(x) < \infty,$$

$$\text{(III)} \qquad \sum xp(x) \neq 0.$$

In fact, w.l.o.g., we suppose

$$\text{(III}'\text{)} \qquad \sum xp(x) = \ m\ > 0.$$

For a full account see [8], for an account of recent developments see [9] and for a rich account of "hydrodynamic limits" see [7].

For each $\alpha \in [0,1]$, ρ_α (product α-Bernoulli measure) is invariant for the process and (see [8]) it can be shown that there are no other extremal invariant distributions for the process that are stationary under translations (which is also the case for arrival processes to a $\cdot/M/1$ queue).

It is natural to conjecture that if ξ_0 has a distribution that is stationary under translations and of nonconstant density α, then $\xi_t \longrightarrow^D \rho_\alpha$ as t tends to infinity.

In generality all that is known is that

$$\xi_t \xrightarrow{D} \int_0^1 \lambda(d\alpha)\, \rho_\alpha,$$

a mixture of product measures. If p is symmetric (and irreducible) then we have indeed that $\xi_t \longrightarrow^D \rho_\alpha$ by self-duality (see [8]).

The first result for nonstationary p is [1] where it is proved that in the nearest neighbor case with $d = 1$, we have the appropriate convergence. Recently [2] showed that for finite range p (again in $d = 1$) we have convergence to product measure.

Our result is the following theorem.

THEOREM. *If ξ_0 is stationary under translations with*

$$\frac{1}{2n+1}\sum_{|x|\le n}\xi_0(x)\xrightarrow{a.s.}\alpha\in[0,1]$$

and p satisfies (I), (II), (III) *then*

$$\xi_t \xrightarrow{D} \rho_\alpha.$$

It should be noted that since we assume that $\Sigma x p(x) \neq 0$, our result is not an extension of that of [2].

The key ideas are the results of [11] and various coupling ideas. In Section 2 we simply note some "irreducibility" properties. In Section 3 we consider Rezakhanlou's work [11] and consequences for particle systems with "first" and "second" class systems. In Section 4 we give the overall argument and the theorem's proof is completed in Section 5.

In the remainder of the introduction we detail simple consequences of the natural coupling.

We assume that we are given a Harris system generating all processes on $\{0,1\}^{\mathbb{Z}}$. This system consists of an independent Poisson process $N^{x,y}$, $x, y \in \mathbb{Z}$ of rate $p(y-x)$. An exclusion process is constructed from the (N)'s by stipulating that at $t \in N^{x,y}$ "a particle tries to move from site x to site y." That is, for $t \in N^{x,y}$, if immediately before time t,

$$\xi(x)=1,\qquad \xi(y)=0\quad\Longrightarrow\quad \xi_t(x)=0,\qquad \xi_t(y)=1.$$

Otherwise there is no change in process ξ at time t.

A natural way to show that ξ_t converges to ρ_α in distribution as $t \longrightarrow \infty$ is via coupling. We introduce process ξ_t^α with $\xi_0^\alpha \sim \rho_\alpha$ (and so $\xi_t^\alpha \sim \rho_\alpha \,\forall\, t$) and generated by the same system of Poisson processes as ξ. If we could show that ξ and ξ^α couple in the sense that $\forall\, x P(\xi_t(x) = \xi_t^\alpha(x)) \longrightarrow 1$ as $t \longrightarrow \infty$ then we would have established the desired convergence.

A useful idea extensively exploited in, for example, [5] is to use first and second class particles. We consider a process ξ_t on $\{0,1,2\}^{\mathbb{Z}}$ [again generated by $(N^{x,y})_{x,y\in\mathbb{Z}}$] so that at time $t \in N^{x,y}$. If immediately before t,

$$\text{(I)}\qquad \xi(x)=1,\ \xi(y)=j\quad\text{then}\quad \xi_t(x)=j,\ \xi_t(y)=1,$$

$$\text{(II)}\qquad \xi(x)=2,\ \xi(y)=j\neq 1\quad\text{then}\quad \xi_t(x)=j,\ \xi_t(y)=2.$$

Otherwise ξ is unchanged at time t.

This is easily seen if we consider either

$$\text{(A)} \qquad \eta_t(x) = 1 \iff \xi_t(x) = 1$$

or

$$\text{(B)} \qquad \eta_t(x) = 1 \iff \xi_t(x) \neq 0.$$

Then we have the usual exclusion processes corresponding to p. One can basically have as many classes of particles as desired. We (as a limit) introduce a process η_t on $[0, 1]^{\mathbb{Z}}$ with generator

$$\Omega f(\eta) = \sum_{x, y} (f(\eta^{x, y}) - f(\eta)) p(y - x) I_{\eta(x) \le \eta(y)},$$

where (as before)

$$\eta^{x, y}(z) = \eta(z) \quad \text{for} \quad z \neq x, y,$$

$$\eta^{x, y}(x) = \eta(y), \quad \eta^{x, y}(y) = \eta(x)$$

and we fix $(\eta_0(x))_{x \in \mathbb{Z}}$ to be iid $U[0, 1]$ r.v's. Then $\forall \beta \in [0, 1]$ if we consider

$$\xi_t^\beta(x) = 1 \iff \eta_t(x) \le \beta.$$

Then ξ^β is an exclusion process starting in ρ_β.

Furthermore, if for $\beta < \gamma$,

$$\xi_t^{\beta, \gamma}(x) = 1 \iff \eta_t(x) \le \beta,$$

$$\xi_t^{\beta, \gamma}(x) = 2 \iff \eta_t(x) \in (\beta, \gamma),$$

$$\xi_t^{\beta, \gamma}(x) = 0 \quad \text{otherwise}.$$

Then $\xi^{\beta, \gamma}$ is an exclusion process of first and second class particles.

If η is generated by the same Harris system as our general exclusion process ξ_t, then, as already noted,

$$\xi_t \longrightarrow \rho_\alpha \quad \text{if} \quad \forall\, x P(\xi_t(x) = \xi_t^\alpha(x)) \longrightarrow 1.$$

In our coupling of (ξ_t, η_t) we suppose η_0 is independent of ξ_0 and (wlog) that ξ_0 is ergodic so that $\forall\, t (\xi_t, \eta_t)$ is ergodic. We define

$$\mu_t[0, c] = \lim_{n \to \infty} \frac{1}{2\alpha n} \sum_{|x| \le n} I_{\xi_t(x) = 1} I_{\eta_t(x) \le c}.$$

We can label the ξ-particles so that a particle only moves from site x to site y at times $t \in N^{xy}$ or N^{yx} and so that the associated η-value for a ξ-particle never increases. Then the quantity

$$\frac{1}{2\alpha n} \sum_{|x| \le n} I_{\xi_t(x)=1} I_{\eta_t(x) \le c}$$

is equal to a nondecreasing process plus "edge effects" resulting from particles moving in and out of spatial interval $[-n, n]$. As n becomes large these "edge effects become negligible." Thus we have (simultaneously over all $c \in [0, 1]$) that $\mu_t[0, c]$ increases in time. It is easy to see that we have

$$\mu_t[0, c] \uparrow \mu_\infty[0, c] \quad \text{for } \mu_\infty \quad \text{a measure on} \quad [0, 1].$$

As noted if μ_∞ is $U[0, \alpha]$ (the uniform distribution on $[0, \alpha]$) then we have that asymptotically all ξ-particles are coalesced with η-particles of value in $[0, \alpha]$ and vice versa. That is, $\xi_t \longrightarrow^D \rho_\alpha$. So in Section 2 we will begin an argument by contradiction and suppose $\mu_\infty \neq U[0, \alpha]$.

2. Coupling foundations. We begin this section with a basic result concerning two finite exclusion processes generated by the same Harris system.

An exclusion process on $\{0, 1\}^I$ for finite interval I is the Markov process with generator

$$\Omega f(\xi) = \sum_{x,\, y \in I} (f(\xi^{xy}) - f(\xi)) p(y - x) \xi(x)(1 - \xi(y)).$$

We denote exclusion processes on interval $[0, m]$ by ξ^m, η^m. We say two such processes η^m and ξ^m are coupled if they are generated by the same Harris sytem.

LEMMA 2.1. *Under the assumptions for $p(\cdot)$, there exist an m_0 so that for $m \ge m_0$, if ξ^m, η^m are coupled exclusion processes on $\{0, 1\}^{[0, m]}$,*

$$\xi_0^m(0) = 1 = 1 - \eta_0^m(0),$$
$$\xi_0^m(m) = 0 = 1 - \eta_0^m(m),$$

then

$$P\left(\sum_{i=0}^m |\xi_0^m(i) - \eta_1^m(i)| \le \sum_{i=0}^m |\xi_0^m(i) - \eta_0^m(i)| - 2 \right) > 0.$$

REMARK. In other words, the lemma asserts that there is a strictly positive chance that an uncoupled ξ^m-particle will couple with an uncoupled η^m-particle if there exist two such particles sufficiently far apart.

PROOF OF LEMMA 2.1. It is easy to see that for m sufficiently large there will exist a random walk path $0 = x_0, x_1, x_2, \dots, x_R = m$ contained in $[0, m]$. Here a random walk path satisfies $p(x_i - x_{i-1}) > 0 \,\forall\, i$. If $R = 1$, then trivally there is a strictly positive chance that the ξ^m-particle at 0 jumps to site m

(thereby coupling with the η^m-particle at m) before any other jump. So we suppose $R > 1$, use induction on R and suppose that whenever (for whatever m) there exists a random walk path of length $R-1$ or less from an uncoupled ξ-particle to an uncoupled η-particle, a coupling of these two particles may occur.

First suppose that for some $0 < j < R$ we have $\xi_0^m(x_j) \neq \eta_0^m(x_j)$. If $\xi_0^m(x_j) = 1 = 1-\eta_0^m(x_j)$ then $x_j, x_{j+1} \cdots x_R$ is a path of length shorter than R from an uncoupled ξ^m-particle to an uncoupled η^m-particle and so the result follows by induction. Similarly, if $\xi_0^m(x_j) = 0 = 1-\eta_0^m(x_j)$, then $x_0, x_1, \dots, x_j$ is such a path.

So we assume $\xi_0^m(x_i) = \eta_0^m(x_i) \, \forall\, 0 < i < R$.

Let $j = \inf\{i\colon \xi_0^m(x_i) = 0\}$; in this case with positive probability (ξ^m, η^m) moves to $(\xi^{m,1}, \eta^{m,1})(\xi^{m,2}, \eta^{m,2}) \cdots (\xi^{m,j}, \eta^{m,j})$ where $(\xi^{m,k}, \eta^{m,k})(y) = (\xi_0^m, \eta_0^m)(y)$ for $y \notin \{x_0, x_1, \dots x_j\}$ and for $0 < k < j$,

$$(\xi^{m,k}, \eta^{m,k})(x_{j-k}) = (\xi_0^m, \eta_0^m)(x_j),$$

$$(\xi^{m,k}, \eta^{m,k})(x_r) = (1,1) \quad \text{for} \quad r > j-k,$$

$$(\xi^{m,k}, \eta^{m,k})(x_r) = (\xi_0^m, \eta_0^m)(x_r), \qquad r < j-k$$

and $(\xi^{m,j}, \eta^{m,j})$ is obtained from $(\xi^{m,j-1}, \eta^{m,j-1})$ by the ξ-particle at 0 moving to site x_i. If $x_j = x_R = m$ then this move delivers the coupling. On the other hand, if $x_j \neq x_R$ then for configuration $(\xi^{m,j}, \eta^{m,j}) x_1, x_2, \dots x_R$ is a path of length $R-1$, from an uncoupled ξ-particle to an uncoupled η-particle.

For two coupled processes η, ξ on an interval I, the quantity $\sum_I |\xi_t(x) - \eta_t(x)|$ cannot increase at any time t. If this quantity strictly decreases at time t, we say that a ξ-particle has *joined* an η-particle.

We return to the coupled process $(\xi_t, \eta_t)_{t\geq 0}$. Here ξ_t is our exclusion process starting from translation invariant distribution of (nonrandom) density α and η_t is our process on $[0,1]^{\mathbb{Z}}$.

If the associated measures $(\mu_t = \text{disn}\, \eta_t(0) \mid \xi_t(0) = 1)$ satisfy $\mu_t \longrightarrow^D U[0,\alpha]$, then we have $\xi_t \longrightarrow^D \rho_\alpha$. We therefore suppose that $\mu_t \not\to U[0,\alpha]$. In this case we must have the existence of $\varepsilon, \delta > 0$ and $0 < a < \alpha - 4\delta < \alpha + 4\delta < b < b + \delta < 1$, so that

$$\mu_\infty([a, a+\delta]) \leq \frac{\delta}{\alpha}(1-\varepsilon),$$

$$\mu_\infty([b, b+\delta]) \geq \frac{\delta\varepsilon}{\alpha}.$$

Fix such ε, δ, a and b. These quantities are given to us by our assumption of nonconvergence. The quantities ε, δ can be thought of as small but need not be so in principle. We will now introduce ε' (and later γ) to be numbers which are very small in comparison to $\varepsilon\delta$. The large number N will be chosen so that various laws of large numbers hold (to within some multiple of this ε' and γ').

Choose $\varepsilon' \ll (\varepsilon\delta)^3 \wedge \frac{\varepsilon\delta}{1800(1+\sum|x|p(x))}$ (we will introduce other upper bounds later). There exist $T(= T(\varepsilon'))$ so that $\forall\, c \in \{a, a+\delta, b, b+\delta\}$; the density of ξ particles having η-label $\leq c$ does not increase by more than $\varepsilon'/4$ for $t \geq T$.

Now we collapse η_t onto a system of particles with five classes η_t^i, $i = 1, 2, \dots, 5$, where

$$\begin{aligned}
\eta_t^1(x) = 1 \quad &\Longleftrightarrow \quad \eta_t(x) \leq a,\\
\eta_t^2(x) = 1 \quad &\Longleftrightarrow \quad \eta_t(x) \in (a, a+\delta],\\
\eta_t^3(x) = 1 \quad &\Longleftrightarrow \quad \eta_t(x) \in (a+\delta, b],\\
\eta_t^4(x) = 1 \quad &\Longleftrightarrow \quad \eta_t(x) \in (b, b+\delta],\\
\eta_t^5(x) = 1 \quad &\Longleftrightarrow \quad \eta_t(x) > b+\delta.
\end{aligned}$$

Thus each ξ_t-particle (or vacancy) has an associated η-class in $\{1, 2, 3, 4, 5\}$. Our choice of $a, b, \varepsilon, \delta$ and T ensure that $\forall\, t \geq T$, the density of ξ_t vacancies of η-class 2 is at least $\varepsilon\delta - \varepsilon'/4$, and that the density of ξ_t-particles of η-class 4 is at least $\delta\varepsilon - \varepsilon'/4$.

Thus we have the following lemma.

LEMMA 2.2. *For N sufficiently large,*

$$\text{(I)} \qquad \sum_{x=0}^{N} \big(1 - \xi_T(x)\big)\eta_T^2(x) \geq N(\varepsilon\delta - \varepsilon') \equiv M,$$

$$\text{(II)} \qquad \sum_{x=N+m_0}^{2N} \xi_T(x)\eta_T^4(x) \geq N(\varepsilon\delta - \varepsilon') \equiv M,$$

outside of probability $\varepsilon'^2/10^{10}$.

In other words, Lemma 2.2 is asserting the existence of many η-type 2 ξ-vacancies and many η-type 4 ξ-particles outside of very small probability.

Labeling. In the last two sections the arguments will turn on the behavior of ξ-particles and ξ-vacancies. We introduce a labeling of these particles and vacancies. We first stipulate that the labels should move so that the η-class of a ξ-vacancy does not decrease; the η-class of a ξ-particle does not increase.

Thus if u and v are ξ-particles at sites x, y, respectively, immediately before t and immediately before t, $\eta^i(x) = 1$, $\eta^j(y) = 1$ for $i < j$, then if $t \in N^{x,y}$, at time t label u moves from site x to site y while label v moves from site y to site x. Thus labels can move from site x to site y even though the ξ process is unchanged at these sites.

In time interval $[T, kN]$ (k to be fixed later) we will introduce a labeling scheme for η particles that will respect the priorities. We will subdivide priority (or class 2) so that η-class 2 particles in spatial interval $[0, N]$ at time T

will be assigned a new class 2^*; η-class η-4 particles in $[N+m_0, 2N]$ at time T will be assigned a new class 4^*. The priority scheme will be

$$1 \text{ over } 2 \text{ over } 2^* \text{ over } 3 \text{ over } 4^* \text{ over } 4 \text{ over } 5.$$

It should be noted that while the labeling of particles will thus differ from previous labelings, the overall coupled processes are the same. Of course as far as our original process is concerned, η-class 2^* particles are simply η-class 2 particles, similarly for η-class 4^* particles.

Assuming the conclusions of Lemma 2.2, let us label M ξ-vacancies of η-class 2^* in $[0, N]$ at time T, $i_1, i_2, \dots, i_M$. Let us label M ξ-particles of η-class 4^* $j_1, \dots, j_M$. We now describe the motion of labels i_h, j_l.

Let $x_t(i_h)$ denote the position of ξ-vacancy i_h at time t. If η-particle u (u of whatever class) at time t is at site $w = x_t(i_h)$, we say that u is associated with i_h at time t (and vice versa).

First we detail the motion of i_h if $x_{t-}(i_h) = w$ and i_h is associated with a class ℓ η-particle at $t-$ and $t \in N^{w,y}$.

If at $t-$ the η-class of site y is $\le \ell$ then nothing happens to label i_h (it remains at w). If the η-class at y greater than ℓ, then i_h moves to y at time t if and only if $\xi_{t-}(y) = 0$. Otherwise i_h stays at w and its η-class increases.

Next, if $x_{t-}(i_h) = w$ and i_h is associated with a class ℓ η-particle ($\ell \in \{1, 2, 2^*, 3, 4^*, 4, 5\}$) and $t \in N^{y,w}$ then if (at $t-$) the η-class of site y is less than ℓ, i_h moves to site y (and remains associated to the same η-particle). If the η-class of site y is greater than or equal to ℓ then i_h moves to y if and only if $\xi_{t-}(y) = 1$. In this case the η-class of i_h may increase.

The motion of ξ_t-particles j_l is similarly determined. The important fact is that:

(*) For ξ-vacancies i_h the class of the associated η-particle never decreases.
(**) For ξ-particles j_l the class of associated η-particle never increases.

3. A result of Rezakhanlou. The key tool of this paper is a result of Rezakhanlou [11]; for our purposes (though much more is proved in his paper) his theorem is as follows.

THEOREM. *Let f be a positive function that is bounded by 1 and continuous except at finitely many points. Let η_t^N be a sequence of exclusion processes on $\mathbb{Z}/N$ with $(\eta_0^N(x))$ independent Bernoulli, $P(\eta_0^N(x) = 1) = f(x)$ for $x \in \mathbb{Z}/N$. Then for interal I we have $\forall\, \gamma > 0$,*

$$P\left(\left|\frac{1}{N}\sum_{x\in(\mathbb{Z}/N)\cap I} \eta_{Nt}^N(x) - \int_I u(t,x)\,dx\right| > \gamma\right) \longrightarrow 0 \quad \text{as } N \longrightarrow \infty,$$

where u is the unique entropy solution to

$$\frac{\partial u}{\partial t} + m\frac{\partial(u(1-u))}{\partial x} = 0, \qquad u(0,x) = f(x).$$

[Note that η^N has particles that try to jump from x to y at rate $p(N(y-x))$.]

REMARK. [11] actually only explicitly treats the case where p is finite range (but otherwise assumptions I–III hold). However, in $d = 1$, the arguments carry over to our case without much trouble.

We consider this theorem first for f_1 and f_2 with $f_1 \equiv a + \delta$ and $f_2 \equiv a + \delta$ on $[0,1]^c \equiv a$ on $[0,1]$. If u_1, u_2 are the corresponding solutions to Burger's equation with initial f_1 and f_2, then

$$u_1 \geq u_2 \ \forall\, t, x \quad \text{and} \quad \int (u_1 - u_2)\, dx$$

is constant in t (conservation property). However (see, e.g., [7]), $u_1 = u_2$ on $[m(1 - 2a - 2\delta)t, 1 + m(1 - 2a - \delta)t]^c$.

Now by a very elementary argument we have that the number of particles in space interval $[0, N]$ at time 0 but not in $[0, N]$ at time T or not in $[0, N]$ at time 0 but in $[0, N]$ at time T is very small compared to N with very large probability. Thus we have the following.

PROPOSITION 3.1. *As $N \longrightarrow \infty$ we have $\forall\, \gamma > 0$, P [number of 2^* particles outside $[mN(1 - 2a - 2\delta)t,\ N(1 + m(1 - 2a - \delta))t]$ at time $Nt \geq N\gamma] \longrightarrow 0$.*

Similarly we have $u_2 \equiv b$, taking Burger's equation solutions u_i with u_1 initially equal to b on $[1,2]^c$ and equal to $b + \delta$ on $[1, 2]$.

PROPOSITION 3.2. *As $N \longrightarrow \infty$ we have $\forall\, \gamma > 0$, P [number of 4^* particles outside $[(1 + mt(1 - 2b - \delta))N, (2 + mt(1 - 2b))N]$ at time $Nt \geq N\gamma] \longrightarrow 0$.*

Taken together we have the following proposition.

PROPOSITION 3.3. *Fix $k > \frac{1}{m(b-a-\delta)}$, then $\forall\, \gamma > 0$ fixed we have P[all but $\gamma N 2^*$ particles are to the right of all but $\gamma N 4^*$ particles at time $kN] > 1 - \gamma$ for N sufficiently large.*

In addition to previous demands on N we assume N is sufficiently large that Proposition 3.3 holds with $\gamma = (\varepsilon')^2/10^{10}$.

4. Statement of Proof. Consider a ξ-vacancy that is at time T associated with a η-type 2^* particle i_h (necessarily at a site on $[0, N]$ at time T).

If this ξ-vacancy remains associated to a η-type 2^* particle (not necessarily the same one) throughout $[T, kN]$ then (with overwhelming probability) it should (by Proposition 3.3) have had to pass over many ξ-particles associated with a η-type 4^* particle. From our coupling, the rate at which a ξ-vacancy moves from site x to site y is bounded by $p(y-x) + p(x-y)$. [What it actually is depends on the (ξ, η) configuration.]

Thus we have that N can be chosen so large that $\forall\, 1 \leq h \leq M$,

$$\text{(D)} \qquad \begin{aligned} &P[i_h \text{ leaves interval } [-2CN, 2CN] \text{ in time } kN] \\ &\quad \leq (\varepsilon')^3 \text{ for } C = 3k\big(1 + \textstyle\sum |x|\, p(x)\big), \end{aligned}$$

where the bound is independent of $(\xi, \eta)_T$, similarly for ξ-particles j_l, $1 \le l \le M$.

Let m_0 be the constant of Lemma 2.1. We can choose m_1 so that $\Sigma_{|x|\ge m_1}(|x|+m_0)p(x) \le (\varepsilon')^2/(k+1)$.

Now for ξ-vacancy i_h we say an m_1 jump occurs at time t if $x_t(i_h)-x_{t-}(i_h) \ge m_1$. Let

$$X(i_h) = \sum_{m_1 \text{ jumps } t\le kN} (x_t(i_h) - x_{t-}(i_h) + m_0).$$

We similarly say for ξ-particle j_l that an m_1 jump occurs if $x_t(j_l)-x_{t-}(j_l) \le -m_1$.

Let $X(j_l) = \Sigma_{m_1 \text{ jump at } t<kN}(m_0 + x_{t-}(j_l) - x_t(j_l))$. By the law of large numbers we can choose N so large that

$$\text{(E)} \qquad \begin{gathered} \forall\ 1 \le h, l \le M \\ P(X(i_h) \ge (\varepsilon')^{3/2}N) \le (\varepsilon')^2, \\ P(X(j_l) \ge (\varepsilon')^{3/2}N) \le (\varepsilon')^2. \end{gathered}$$

The idea of our proof is that, with large probability by time kN, either a reasonable number of the ξ-vacancies of η-type 2^* are now associated with higher η-types or a reasonable number of ξ-particles of η-type 4^* are now associated with lower η-types [and by assumption (D) this will have taken place inside $[-2CN, 2CN]$], or (by Proposition 3.3) a large number of ξ-vacancies of η-type 2^* have passed by a large number of ξ-particles of η-type 4^*.

However, the former case is forbidden by our choice of T. For the latter, stipulation (E) on the size of N means that, outside of small probability, most ξ-vacancies will pass close to most ξ-particles while the former travel to the right of the latter, outside of very small probability.

5. Completion of proof. An ξ-vacancy i_h can pass ξ-particle j_l in three ways:

1. For some $t \subset [T, kN]$, $x_t(j_\ell) \in [x_t(i_h) + m_0,\ x_t(i_h) + m_1 + m_0]$;
2. For some $t \in [T, kN]$, $x_t(i_h) + m_0 > x_t(j_\ell) > x_{t-}(i_h) + m_1 + m_0$;
3. For some $t \in [T, kN]$, $x_t(j_\ell) - m_0 < x_t(i_h) < x_{t-}(j_\ell) - (m_1 + m_0)$.

Assumptions (E) on the size of N imply that the expected number of j_ℓ passed in manner (2) or (3) is likely to be small. So it remains to consider the possibility of many j_ℓ passing by i_h after coming "within range." By Lemma 2.1 if j_ℓ is in $[x_t(i_h)+m_0,\ x_t(i_h)+m_1+m_0]$ then (if both have their original η-class) there is a reasonable chance of some coupling taking place. A problem (to be addressed via "windows") is that the coupling between η-2^*-particles and ξ-particles in $[x_t(i_h) + m_0, x_t(i_h) + m_1 + m_0]$ might not involve j_ℓ or i_h.

Initially, we have no windows. If at time t, i_h is not "in a window" then a window $[x_t(i_h),\ x_t(i_h)+m_0+m_1]$ is created at time t. If (i) i_h has an associated

η-value 2^* at time t; (ii) in interval $[x_t(i_h), x_t(i_h)+m_0+m_1]$ is a ξ-particle j_ℓ associated with a η-particle of type 4^*.

A window $[x_{t_0}(i_h), x_{t_0}(i_h)+m_0+m_1]$ will survive until (a) there is a joining (inside the interval) of a ξ-particle with a type 2^* (or lower) η-particle, or the coupled exclusion process (ξ, η) evolves so that such a joining is impossible on $[x_{t_0}(i_h), x_{t_0}(i_h)+m_0+m_1]$ without particles entering or leaving the window; (b) A ξ- or η-particle on $(x_{t_0}(i_h)+m_1+m_0, \infty)$ tries to jump to a site in $(-\infty, x_t(i_h)+m_1+m_0]$ or a ξ- or η-particle in $(-\infty, x_{t_0}(i_h))$ tries to jump to a site in $[x_{t_0}(i_h), \infty)$.

The various windows in existence at a given time may overlap. Thus when there is a joining of a ξ-particle with a η-type 2^* (or lower) particle, it may be simultaneously occurring within (at most m_0+m_1) many windows. We wish to credit this coupling to a marked ξ-vacancy i_h. To do this we randomly choose among the windows having a claim, and then, given our window, we randomly choose an i_h to credit among the (again as most m_0+m_1) eligible i_h's in that window.

We define $W(i_h)$ (a lower bound on the number of windows that i_h has been in) as follows.

Let

$$S_0^h = 0,$$

$$T_1^h = \inf\{t \geq S_0^h \colon (i_h \text{ is in a window})\}.$$

For $v \geq 1$,

$$S_v^h = \inf\{t > T_v^h \colon \text{the window } [x_{T_v^h},\ x_{T_v^h}+m_0+m_1] \text{ dies}\},$$

$$T_v^h = \inf\{t > S_{v-1}^h \colon (i_h \text{ is in a window})\}.$$

Of course, often $T_v^h = S_{v-1}^h$.

DEFINITION. $W(i_h) = \sup\{v \colon S_v^h \leq kN\}$.

The idea is that a typical i_h cannot pass through or past many j_ℓ without W becoming large and that W cannot become large without i_h becoming credited.

LEMMA 5.1. *For some c (depending on m_0, m_1 but not on N) we have [uniformly over all $(\xi, \eta)_T$] and i_h, $P[W(i_h) > R,\ i_h$ is not credited$] \leq (1-c)^R$.*

PROOF. This simply follows from the Markov property and the fact that there are only finitely possible (ξ, η) configurations on $(\{0,1\}^{[0,\, m_0+m_1]})^2$.

PROOF OF THEOREM. Let B be the event that in time $[T, kN]$ on interval $[-(2CN+m_1+m_0),\ 2CN+m_1+m_0]$ at least $\frac{\varepsilon\delta}{20}N\xi$-particles change their η-class to 3 or lower or at least $\frac{\varepsilon\delta}{20}N\,\xi$-vacancies change their class to 3 or higher.

Obviously, if $P(B) > 1/3$ (say) then we have that [by ergodicity of $(\xi, \eta)_T$] the density of ξ-particles taking η-value in $[0, a+\delta]$ or in $[0, b]$ will increase by at least $1/6 \frac{\varepsilon\delta}{20(4C+\frac{2(m_1+m_0)}{N})} \gg \varepsilon'$ which will contradict our definition of T, and thereby the assumption that $M_t \not\to U[0, \alpha]$.

We consider the following "bad" events:

1. The number of 2^* η-particles that are ξ-vacant is less than $M = N(\varepsilon\delta - \varepsilon')$ or the number of 4^* η-particles that are ξ-occupied is less than M.
2. The number of i_h, $1 \le h \le M$ so that $X(i_h) \ge \varepsilon'^{3/2} N$ is $\ge \varepsilon' N$ or the number of j_l, $1 \le l \le M$ so that $X(j_l) \ge \varepsilon'^{3/2} N$ is greater than or equal to $\varepsilon' N$.
3. The number of i_h or j_ℓ that leave $[-2CN, 2CN]$ in time interval $[T, kN]$ is greater than or equal to $\varepsilon' N$.
4. The conclusion of Proposition 3.3 does not hold with $\gamma = (\varepsilon')^2/10^{10}$.
5. At least one i_h has the property that i_h is associated with a η- 2^* particle as time kN, i_h is not credited and $W(i_h) \ge \frac{\varepsilon\delta N}{5(m_1+2m_0)^2}$.

By lower bounds for N, D, and E and Proposition 3.3, we have that the probability of (1) $\to$ (4) occurring is bounded by 1/10. By Lemma 5.1 the probability of (5) is bounded by $N(1-c)^{\varepsilon\delta N/5(m_1+2m_0)^2} < 1/10$ if N is sufficiently large (as we will take N to be).

So with probability $1 - 1/10 - 1/10 = 8/10$ (at least) we have that (1) $\to$ (5) do not occur. We claim that in this case event B must occur. If not, then by $(1)^c$ there are M labelled η- 2^* ξ-vacancies i_h in $[0, N]$ and M labelled η-4^* ξ-particles j_l in $[N+m_0, 2N]$ at time T.

By $(3)^c$ and B^c occurring we have that for at least $(M - \varepsilon' N - \frac{N\varepsilon\delta}{20})h$ in $\{1, \dots, M\}$ and $(M - \varepsilon' N - \frac{N\varepsilon\delta}{20})\ell$ in $\{1, \dots, M\} i_h$, j_ℓ remains inside $[-2CN, +2CN\}$ in time interval $[T, kN]$ and i_h retains a η-2^* value, j_ℓ an η-4^* value throughout.

By $(4)^c$ of these at least $M - \varepsilon' N - \frac{N\varepsilon\delta}{20} - \varepsilon' N$ i_h's will be to the right of $M - \varepsilon' N - \frac{N\varepsilon\delta}{20} - \varepsilon' N$ of the j_ℓ's. Let the two indexes be

$$I \subseteq \{1, \dots, M\}, \qquad |I| \ge M - \varepsilon' N - \frac{N\varepsilon\delta}{20} - \varepsilon' N,$$

$$J = \{1, \dots, M\}, \qquad |J| \ge M - \varepsilon' N - \frac{N\varepsilon\delta}{20} - \varepsilon' N.$$

By $(2)^c$ we finally have subsets

$$I' \subseteq \{1, \dots, M\}, \qquad J' \le \{1, \dots, M\},$$

$$|I'|, |J'| \ge M - \varepsilon' N - \frac{N\varepsilon\delta}{20} - \varepsilon' N - \varepsilon' N \ge \frac{8N\varepsilon\delta}{10}.$$

Taking subsets if necessary we suppose $|I'|, |J'| = \frac{8N\varepsilon\delta}{10}$. For N large so that $h \in I'$, $\ell \in J'$:

(a) $x_t(i_h),\ x_t(j_\ell) \in [-2CN, 2CN]\,\forall\, t \in [T, kN]$.
(b) $x_{kN}(i_h) > x_{kN}(j_\ell)$.
(c) At time kN, i_h has η-value 2^* and j_ℓ has value 4^*.
(d) $X(i_h) \le \varepsilon'^{3/2}N$, $X(j_\ell) \le \varepsilon'^{3/2}N$.

For $\ell \in J'$, let $D_\ell = \{h\colon$ for some $t \in [T, kN]$, $x_t(j_\ell) - m_0 < x_t(i_h) < x_{t-}(j_\ell) - (m_1 + m_0)\}$.

By hypothesis, $\forall\, \ell \in J' |D_\ell| \le \varepsilon'^{3/2}N$ so $\Sigma_{\ell\in J'}|D_\ell| \le \varepsilon'^{3/2}N\frac{8N\varepsilon\delta}{10}$. So by Chebyshev number of $h \in I'$: $\Sigma_{\ell\in J'} I_{h\in D_\ell} \ge \varepsilon' N$ is less than or equal to $N\varepsilon\delta\varepsilon'^{1/2}$. So let $I'' = \{h \in I' : \Sigma_{\ell\in J'} I_{h\in D_\ell} \le \varepsilon' N\}$; then $|I''| \ge \frac{7}{10}N\varepsilon\delta$.

As noted, if (b) occurs for $h \in I''$, $\ell \in J'$ then either for some $t \in [T, kN]$,

(A) $x_t(i_h) + m_0 \le x_t(j_\ell) \le x_t(i_h) + m_0 + m_1$;
(B) $x_t(j_\ell) - m_0 < x_t(i_h) < x_{t-}(j_\ell) - (m_1 + m_0)$;
(C) $x_t(i_h) + m_0 + m_1 < x_t(j_\ell) < x_t(i_h) + m_0$.

By property (d) of I' and therefore of I'', the number of ℓ such that (C) holds is bounded by $\varepsilon'^{3/2}N$; by definition of I'' the number of ℓ so that (B) holds is bounded by $\varepsilon' N$, so for $h \in I''$ at least $|J'| - \varepsilon' N - \varepsilon'^{3/2}N \ge \frac{7N\varepsilon\delta}{10}\ell$ satisfy (A).

We denote this subset of J' by G_h. By definition of window, for every S_i^h at most m_1+2m_0 j_ℓ in G_h can pass from m_1+m_0 to the right of i_h to less than m_0 to the right. Thus it must be the case that for $h \in I'' W(i_h) \ge \frac{7N\varepsilon\delta}{10}/m_1 + 2m_0$.

Thus by (5) we must have i_h is credited. $\forall\, h \in I''$ but as $x_t(i_h) \in [-2CN, 2CN]\forall t \in [T, kN]$. This means that at least (I'') ξ-particles decreased their η-value to 2^* or lower in $[-2CN-(m_0+m_1)2CN+m_0+m_1]$; that is, B occurs. □

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