DOI: 10.1214/EJP.v9-186

Electronic Journal of Probability

Vol. 9 (2004), Paper no. 3, pages 37–56.

Journal URL
http://www.math.washington.edu/~ejpecp/

BROWNIAN BRIDGE ASYMPTOTICS FOR RANDOM p-MAPPINGS [1]

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Abstract: The Joyal bijection between doubly-rooted trees and mappings can be lifted to a transformation on function space which takes tree-walks to mapping-walks. Applying known results on weak convergence of random tree walks to Brownian excursion, we give a conceptually simpler rederivation of the Aldous-Pitman (1994) result on convergence of uniform random mapping walks to reflecting Brownian bridge, and extend this result to random **p**-mappings.

Keywords and phrases: Brownian bridge, Brownian excursion, Joyal map, random mapping, random tree, weak convergence.

AMS subject classification (2000): Primary 60C05; Secondary 60F17, 60J65.

Submitted to EJP on November 6, 2003. Final version accepted on December 10, 2003.

[1]Research supported in part by N.S.F. Grants DMS-0203062 and DMS-0071448.

1 Introduction

Write $[n] := \{1, 2, \ldots, n\}$. A *mapping* $m : [n] \to [n]$ is just a function, identified with its digraph $\mathcal{D}(m) = \{(i, m(i)),\ i \in [n]\}$. Exact and asymptotic properties of random mappings have been studied extensively in the combinatorial literature since the 1960s [12, 14]. Aldous and Pitman [4] introduced the method of associating a *mapping-walk* with a mapping, and showed that (for a uniform random mapping) rescaled mapping-walks converge in law to reflecting Brownian bridge. The underlying idea – that to rooted trees one can associate tree-walks in such a way that random tree-walks have tractable stochastic structure – has been developed by many authors in many directions over the last 15 years, and this paper, together with a companion paper [6], takes another look at invariance principles for random mappings with better tools.

As is well known, the digraph $\mathcal{D}(m)$ decomposes into trees attached to cycles. The argument of [4] was that for a uniform random mapping the walk-segments corresponding to different cycles, considered separately, converge to Brownian excursions, and that the process of combining these walk-segments into the mapping-walk turned out (by calculation) to be the same as the way that excursions of reflecting Brownian bridge are combined. That proof (and its reinterpretation by Biane [9]) made the result seem rather coincidental. In this paper we give a conceptually straightforward argument which both proves convergence and more directly identifies the limit. The argument is based on the *Joyal bijection* J between doubly-rooted trees and mappings. Being a bijection it takes uniform law to uniform law; less obviously, it takes the natural $\mathbf{p}$-*tree* model of random trees to the natural $\mathbf{p}$-*mapping* model of random mappings. Theorem 1 will show that under a natural hypothesis, mapping-walks associated with random $\mathbf{p}$-mappings converge weakly to reflecting Brownian bridge. We can outline the proof in four sentences.

- It is known that rescaled walks associated with random $\mathbf{p}$-trees converge in law to Brownian excursion, under the natural hypothesis (4) on $(\mathbf{p}^n)$ (section 2.5).
- There is a transformation $\mathbf{J} : D[0, 1] \to D[0, 1]$ which "lifts" the Joyal bijection trees$\to$mappings to the associated walks (section 3.3).
- $\mathbf{J}$ has appropriate continuity properties (section 3.2).
- $\mathbf{J}$ takes Brownian excursion to reflecting Brownian bridge (section 3.4).

Filling in the details is not difficult, and indeed it takes longer in section 2 to describe the background material (tree walks, mapping walks, the Joyal bijection in its probabilistic form, its interpretation for walks) than to describe the new arguments in section 3. One unusual aspect is that to handle the natural class (4) of $\mathbf{p}$-mappings, we need to use a certain $*$-topology on $D[0,1]$ which is weaker than the usual Skorokhod topology (in brief, it permits upward spikes of vanishing width but non-vanishing height).

A companion paper [6] used a quite different approach to studying a range of models for random trees or mappings, based on spanning subgraphs of random vertices. We will quote from there the general result (Theorem 4(b)) that rescaled random **p**-tree walks converge in the $*$-topology to Brownian excursion, but our treatment of random mappings will be essentially self-contained. We were motivated in part by a recent paper of O'Cinneide and Pokrovskii [16], who gave a more classically-framed study of random **p**-trees (under the same hypothesis (4)) from the viewpoint of limit distributions for a few explicit statistics. See [4, 6, 1, 5] for various explicit limit distributions derived from the Brownian bridge.

When (4) fails the asymptotics of **p**-trees and **p**-mappings are quite different: Brownian excursion and reflecting Brownian bridge are replaced by certain jump processes with infinite-dimensional parametrization. Technicalities become much more intricate in this case, but the general method of using the operator $\mathbf{J}$ will still work. We will treat this in a sequel [3].

The recent lecture notes of Pitman [18] provide a broad general survey of this field of probabilistic combinatorics and stochastic processes.

2 Background

2.1 Mappings

Let S be a finite set. For any mapping $m : S \to S$, write $\mathcal{D}(m)$ for the mapping digraph whose edges are $s \to m(s)$, and write $\mathcal{C}(m)$ for the set of cyclic points of m (i.e. the points that are mapped to themselves by some iterate of m).

Let $\mathcal{T}_c(m)$ be the tree component of the mapping digraph with root $c \in \mathcal{C}(m)$. The tree components are bundled by the disjoint cycles $\mathcal{C}_j(m) \subseteq \mathcal{C}(m)$ to form the basins of attraction $\mathcal{B}_j(m)$ of the mapping, say

$$\mathcal{B}_j(m) := \bigcup_{c \in \mathcal{C}_j(m)} \mathcal{T}_c(m) \quad \supseteq \mathcal{C}_j(m) \text{ with } \bigcup_j \mathcal{B}_j(m) = S \text{ and } \bigcup_j \mathcal{C}_j(m) = \mathcal{C}(m) \qquad (1)$$

where all three unions are disjoint unions, and the $\mathcal{B}_j(m)$ and $\mathcal{C}_j(m)$ are indexed by $j = 1, 2, \ldots$ in such a way that these sets are non-empty iff $j \leq k$, the number of cycles of the digraph, which is also the number of basins of the digraph. The choice of ordering will be important later, but first we define the random mappings we will consider.

From now on, suppose that $S = \{1, 2, \ldots, n\} =: [n]$. Consider a probability law $\mathbf{p}$ on $[n]$, and assume that $p_i > 0$ for each i. A random mapping M is called a $\mathbf{p}$-mapping if for every $m \in [n]^{[n]}$,

$$P(M = m) = \prod_{x \in [n]} p_{m(x)}. \tag{2}$$

In other words, each point of $[n]$ is mapped independently of the others to a point of $[n]$ chosen according to the probability law $\mathbf{p}$.

We now define an order on the basins of attraction and cycles of a $\mathbf{p}$-mapping which will be relevant to our study. Consider a random sample $(X_2, X_3, \ldots)$ of i.i.d. points of $[n]$ with common law $\mathbf{p}$, independent of M (the reason for our unusual choice of index set $\{2, 3, \ldots\}$ will become clear in section 2.4). Then order the basins of M in their order of appearance in the $\mathbf{p}$-sample. More precisely, since $p_i > 0$ for every $i \in [n]$, we have that $\{X_2, X_3, \ldots\} = [n]$ a.s., so the following procedure a.s. terminates:

- Let $\mathcal{B}_1(M)$ be the basin of M containing X_2 and $\mathcal{C}_1(M)$ be the cycle included in $\mathcal{B}_1(M)$. Define $\tau_1 = 2$.
- Given $(\tau_i)_{1 \le i \le j}$ and the non-empty $(\mathcal{B}_i(M))_{1 \le i \le j}$ and $(\mathcal{C}_i(M))_{1 \le i \le j}$, as long as $\cup_{1 \le i \le j} \mathcal{B}_i(M) \neq [n]$, let $\tau_{j+1} = \inf\{k : X_k \notin \cup_{1 \le i \le j} \mathcal{B}_i(M)\}$ and let $\mathcal{B}_{j+1}(M)$ be the basin containing $X_{\tau_{j+1}}$; also let $\mathcal{C}_{j+1}(M)$ be the cycle included in $\mathcal{B}_{j+1}(M)$.

For the purpose of defining a useful marked random walk in the next section, we shall also introduce an order on all the cyclic points, as follows. With the above notations, let $c_j \in \mathcal{C}_j(M)$ be the cyclic point which is the root of the subtree of the digraph of M that contains X_{τ_j}. Then within $\mathcal{C}_j(M)$ the vertices are ordered as follows:

$$M(c_j), M^2(c_j), \ldots, M^{|\mathcal{C}_j(M)|-1}(c_j), c_j.$$

Together with the order on basins, this induces an order on all cyclic points.

Call this order (on basins, cycles, or cyclic points) the **p-biased random order**.

2.2 Coding trees and mappings by marked walks

Let $\mathbf{T}_n^o$ be the set of ordered rooted trees on n vertices. By ordered, we mean that the children of each vertex of the tree, if any, are ordered (i.e. we are given a map from the set of children into $\{1, 2, 3, \ldots\}$). Consider some tree T in $\mathbf{T}_n^o$. Denote by $H_i(T)$ the height of vertex i in this tree (*height* = number of edges between i and the root). Suppose that each vertex i has a weight $w_i > 0$, to be interpreted as the duration of time that the walk spends at each vertex. Then one can define the *height process* of the tree as follows. First put the vertices in *depth-first order* (the root is first, and coming after a certain vertex is either its first child, or (if it has no children) its next brother, or (if he has no brother either), the next brother of its parent, and so on). This order can be written as a permutation σ: we say that $\sigma(i)$ is the label of the i-th vertex. For $s \leq \sum_{i=1}^{n} w_{\sigma(i)}$ set

$$H_s^T = H_{\sigma(i)}(T) \qquad \text{if } \sum_{j=1}^{i-1} w_{\sigma(j)} \leq s < \sum_{j=1}^{i} w_{\sigma(j)},$$

and $H^T_{\sum_{i=1}^{n} w_{\sigma(i)}} = H_{\sigma(n)}(T)$ (so the process is right-continuous). This also induces a map $s \mapsto s^T$ from $[0, \sum_i w_i]$ to $[n]$, where $s^T = \sigma(i)$ whenever $\sum_{j=1}^{i-1} w_{\sigma(j)} \leq s < \sum_{j=1}^{i} w_{\sigma(j)}$. With this notation, $H_s^T = H_{s^T}(T)$. We say that s is a time at which the vertex s^T is *visited* by the height process of T.

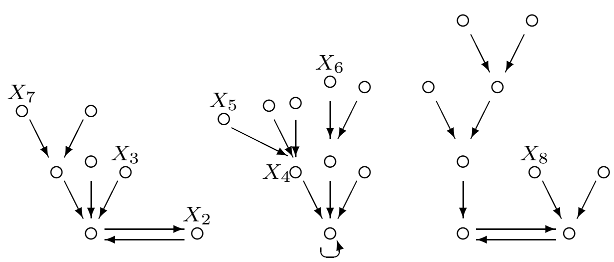

Fig. 1: A mapping pattern digraph and a **p** sample run until it has visited the three basins.

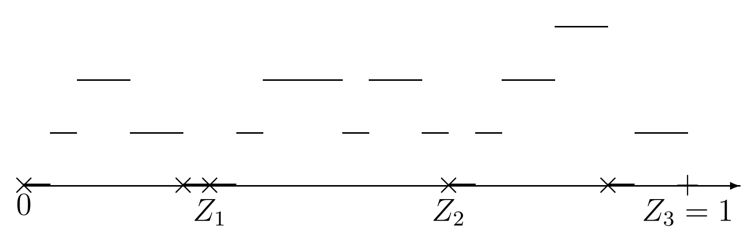

Fig. 2: The corresponding marked walk. Crosses indicate visits to cyclic points.

Now consider a **p**-mapping M on $[n]$ with the assumptions above on **p**. Given the choice of a particular order on the cyclic points, say $(c_1, \ldots, c_K)$, one can construct the "height processes" associated with the **p**-mapping, as follows: in the digraph of M, delete the edges between cyclic points and consider the tree components $\mathcal{T}_{c_1}, \mathcal{T}_{c_2}, \ldots, \mathcal{T}_{c_K}$ of the resulting random forest, with respective roots $c_1, c_2, \ldots, c_K$. The tree components are unordered trees, but we can make them into ordered trees by putting each set of children of the vertices of the $\mathcal{T}_{c_i}$'s into uniform random order. This induces a depth-first order on each $\mathcal{T}_{c_i}$. Let $H^{\mathcal{T}_{c_i}}$ be the height process of $\mathcal{T}_{c_i}$ (where the weight w_x of a point x is its $\mathbf{p}$-value p_x). Now define the *mapping walk* $(H_s^M, 0 \leq s \leq 1)$ to be the concatenation of these tree-walks, in the order dictated by the order on the cyclic points. That is, for $0 \leq s \leq 1$ set

$$H_s^M = H^{\mathcal{T}_{c_i}}_{s - \sum_{j<i} p(\mathcal{T}_{c_j})} \qquad \text{if } \sum_{j<i} p(\mathcal{T}_{c_j}) \leq s < \sum_{j \leq i} p(\mathcal{T}_{c_j}), \tag{3}$$

and $H_1^M = H_{1-}^M$. Here $p(\mathcal{T}_{c_j})$ denotes the $\mathbf{p}$-measure of the vertex-set of $\mathcal{T}_{c_j}$. As with the trees, we denote by s^M the vertex that is visited by H^M at time s. Also, for $x \in [n]$ let $[g^M(x), d^M(x))$ be the interval where x is visited by the walk associated with M. Several features of the mapping M are coded within this walk, such as the number of cyclic points (which is the number of points x such that $H^M_{g^M(x)} = 0$), and the shapes of the trees planted on the cyclic points, which can be deduced from the excursions of the walk away from 0.

Now suppose that $c_1, \ldots, c_K$, and the basins $\mathcal{B}_1(M), \mathcal{B}_2(M), \ldots$ are in the $\mathbf{p}$-biased random order. Put a mark Z_i at the time when the i-th non-empty basin of M has been entirely visited. This must be a time when H^M is 0 (this is the time when the walk visits the first cyclic point of the next basin), unless Z_i is the time when the last basin has been visited, and then one has $Z_i = 1$. The marks $0 = Z_0, Z_1, Z_2, \ldots$ determine the visits of each basin, i.e. the portion of H^M between Z_{j-1} and Z_j is the mapping walk corresponding to the j-th basin of the mapping. In particular $p(\mathcal{B}_j) = Z_j - Z_{j-1}$.

Last, we denote by ℓ_s^M the number of cyclic points that are before s^M in depth-first order. Precisely,

$$\ell_s^M = \sum_{j \leq i} 1\{H^M_{\sum_{k=1}^{j} p_{\sigma(k)}} = 0\} \quad \text{if } \sum_{j<i} p_{\sigma(j)} \leq s < \sum_{j \leq i} p_{\sigma(j)}$$

where σ is the ordering of vertices implicit in the construction of the mapping-walk.

Remark. (a) Because the walk can visit two cyclic points consecutively, some information about the mapping pattern (i.e. the digraph with unlabeled vertices) is lost in $(H^M, (Z_1, Z_2, \ldots))$. But when we are also given $((g^M(i), d^M(i)))_{i \in [n]}$, which is a partition of $[0, 1]$, we can recover the mapping pattern.

(b) The height process of a tree is a particular instance of a "tree walk", i.e. a walk associated with a tree. The fact that the walk spends time p_i at vertex i is important; but other walks with this property might also be usable.

2.3 The convergence theorem

At this point we can state precisely the result of this paper, Theorem 1. For a probability law $\mathbf{p}$ on $[n]$ write

$$c(\mathbf{p}) := \sqrt{\sum_i p_i^2}.$$

For a sequence $(\mathbf{p}^{(n)})$ of probability laws on $[n]$, introduce the *uniform asymptotic negligibility* condition

$$\frac{\max_i p_i^{(n)}}{c(\mathbf{p}^{(n)})} \to 0 \text{ as } n \to \infty. \tag{4}$$

This turns out to be natural because of the *birthday tree* construction of $\mathbf{p}$-trees [10], or the direct study of iterates of random mappings [6]. It is easy to check that (4) implies $c(\mathbf{p}^{(n)}) \to 0$.

Let M_n be a $\mathbf{p}^{(n)}$-mapping. Consider the associated marked random walks $(H^{M_n}, (Z_1^n, Z_2^n, \ldots))$. Let $B^{|\mathrm{br}|}$ be standard reflected Brownian bridge on $[0,1]$, let $(L_s, 0 \le s \le 1)$ be **half** its local time at 0, which is normalized to be the density of the occupation measure at 0 of the reflecting Brownian bridge. Define the random points $(D_1, D_2, \ldots)$ as follows: take U_1 uniform on $[0,1]$ independent of $B^{|\mathrm{br}|}$, and let $D_1 = \inf\{s \ge U_1 : B_s^{|\mathrm{br}|} = 0\}$. Then conditionally on D_1 take U_2 uniform on $[D_1, 1]$ independent of $(B_s^{|\mathrm{br}|}, D_1 \le s \le 1)$, and let $D_2 = \inf\{s \ge U_2 : B_s^{|\mathrm{br}|} = 0\}$, and so on.

Theorem 1 *Suppose $(\mathbf{p}^{(n)})$ satisfies (4).*
(i) There is convergence in law

$$c(\mathbf{p}^{(n)}) H^{M_n} \to 2B^{|\mathrm{br}|} \tag{5}$$

with respect to the $*$-topology on $D[0,1]$ defined in section 2.5. If $\mathbf{p}^{(n)}$ is uniform on $[n]$ then we can use the usual Skorokhod topology on $D[0,1]$.
(ii) Jointly with the convergence in (i), the marks $(Z_1^n, Z_2^n, \ldots)$ converge in law to the sequence $(D_1, D_2, \ldots)$.
(iii) Jointly with the above convergences we have the limit in law (for the uniform topology)

$$(c(\mathbf{p}^{(n)}) \ell_s^{M_n}, 0 \le s \le 1) \to (L_s, 0 \le s \le 1). \tag{6}$$

This immediately yields

Corollary 2 *The following convergence in law holds jointly with (6) in Theorem 1 :*

$$\left(p^{(n)}(\mathcal{B}_j(M_n)), c(\mathbf{p}^{(n)})|\mathcal{C}_j(M_n)|\right)_{j\ge 1} \underset{n\to\infty}{\overset{(d)}{\longrightarrow}} (D_j - D_{j-1}, L_{D_j} - L_{D_{j-1}})_{j \ge 1}. \tag{7}$$

For uniform $\mathbf{p}^{(n)}$ we have $c(\mathbf{p}^{(n)}) = n^{-1/2}$ and these results rederive the results of [4]. For $(\mathbf{p}^{(n)})$ satisfying (4), these results imply results proved by other methods in [6] while adding assertion (iii) which cannot be proved by those methods.

2.4 p-trees, p-mappings and the Joyal bijection

Let $\mathbf{T}_n$ be the set of unordered rooted labeled trees on $[n]$. We define a random object, the **p-tree**, as a random rooted unordered labeled tree whose law is given by

$$P(\mathcal{T} = T) = \prod_{x \in [n]} p_x^{|T_x|}, \tag{8}$$

where T_x is the set of children of vertex x. It is not obvious that the normalizing factor on the right hand side of (8) is 1, that is, that this formula indeed defines a probability law. This known fact [17] can be seen as a consequence of our following discussion.

As shown by Joyal [13] and reviewed in Pitman [17] one can define a bijection J between $\mathbf{T}_n \times [n]$ and $[n]^{[n]}$ which pushes forward the law of the $\mathbf{p}$-tree, together with an independent $\mathbf{p}$-vertex X_1, to the law of the $\mathbf{p}$-mapping. This bijection maps the *spine* of the tree, that is, the vertices of the path from the root X_0 to the distinguished vertex X_1, to the cyclic points of the mapping. As a deterministic bijection it would involve an arbitrary matching of two sets of some cardinality $K!$, but for our probabilistic uses it is more convenient to have the matching implemented by an explicit rule based on external randomization, as follows.

Let $(\mathcal{T}, X_1)$ denote a $\mathbf{p}$-tree $\mathcal{T}$, rooted at some vertex X_0, together with an independent $\mathbf{p}$-point X_1. Let $X_0 = c_1, c_2, \ldots, c_K = X_1$ be the path from the root to X_1, which we call the *spine* of the tree. Delete the edges between the vertices of the spine, obtaining K trees $\mathcal{T}_{c_1}, \ldots, \mathcal{T}_{c_K}$. Recall that $(X_2, X_3, \ldots)$ is an independent random $\mathbf{p}$-sample. As before the following construction a.s. terminates:

• Let $\tau_1 = 2$ and $\mathcal{T}_{c_{k_1}}$ be the tree containing X_2. Then bind the trees $\mathcal{T}_{c_1}, \ldots, \mathcal{T}_{c_{k_1}}$ together by putting edges $c_1 \to c_2 \to \ldots \to c_{k_1} \to c_1$. Let $\mathcal{C}_1 = \{c_1, \ldots, c_{k_1}\}$ and $\mathcal{B}_1 = \cup_{1 \le i \le k_1} \mathcal{T}_{c_i}$.

• Knowing $(\tau_i)_{1 \le i \le j}$, $(k_i)_{1 \le i \le j}$, $(\mathcal{C}_i)_{1 \le i \le j}$ and $(\mathcal{B}_i)_{1 \le i \le j}$ whose union is not $[n]$, let $\tau_{j+1} = \inf\{k : X_k \notin \cup_{1 \le i \le j} \mathcal{B}_i\}$. Then let $\mathcal{T}_{c_{k_{j+1}}}$ be the tree containing $X_{\tau_{j+1}}$, bind the trees $\mathcal{T}_{c_{k_j+1}}, \ldots, \mathcal{T}_{c_{k_{j+1}}}$ by putting edges $c_{k_j+1} \to c_{k_j+2} \to \ldots \to c_{k_{j+1}} \to c_{k_j+1}$. Let $\mathcal{C}_{j+1} = \{c_{k_j+1}, \ldots, c_{k_{j+1}}\}$ and $\mathcal{B}_{j+1} = \cup_{k_j+1 \le i \le k_{j+1}} \mathcal{T}_{c_i}$.

• When this ends (i.e. all the tree is examined), write $J(\mathcal{T}, X_1)$ for the mapping whose basins are $\mathcal{B}_1, \mathcal{B}_2, \ldots$, and whose digraph is given by the following edges within each basin: within each tree $\mathcal{T}_c$ for $c \in \mathcal{C} = \cup \mathcal{C}_i$, the edges are pointing towards the root c, and the cyclic points are pointing to each other according to the binding of trees described above.

Proposition 3 *The mapping $J(\mathcal{T}, X_1)$ is a* $\mathbf{p}$*-mapping, and its basins and cyclic points are in* $\mathbf{p}$*-biased order.*

Proof. Fix m, a particular mapping on $[n]$. We condition on the $\mathbf{p}$-sample $(X_2, X_3, \ldots)$. It is then not difficult to see that there exists a unique (T, y) such that $J(T, y) = m$. This tree is obtained as follows: take the first cyclic point c of m to which X_2 is mapped by some iterate of m. If it is not the unique cyclic point of the basin of m in which X_2 has fallen, we delete the edge between the previous cyclic point (i.e. the cyclic point c' such that $m(c') = c$) and c. We then write $c_1 = c, c_2 = m(c), c_3 = m^2(c), \ldots, c_{k_1} = c'$. We reverse the edges between these cyclic points, i.e. we put directed edges $c_{k_1} \to \ldots \to c_2 \to c_1$. Then we do the same with the next basin discovered by $(X_2, X_3, \ldots)$, and, with obvious notations, we put an edge $c_{k_1+1} \to c_{k_1}$. We then call y the *top* of the spine of the tree T thus built, so that y is the root of the tree in which the point of the $\mathbf{p}$-sample $(X_2, X_3, \ldots)$ that has "discovered" the last basin of m has fallen. In fact, what we have done here is the way to invert the map J.

Now, the probability that $(\mathcal{T}, X_1)$, the $\mathbf{p}$-tree with an independent $\mathbf{p}$-vertex, is equal to (T, y), is easily seen to be equal to $\prod_{x \in [n]} p_x^{|m^{-1}(x)|}$. Indeed, for each vertex x of T except y, the number of edges pointing to x is the same as in the mapping digraph, and for y there is one ingoing edge missing, but this is compensated by our choice of $X_1 = Y$ which has probability p_y.

Moreover, the probability does not depend on the values of $X_2, X_3, \ldots$. So we can uncondition on $(X_2, X_3, \ldots)$ and then the fact that the basins of $J(\mathcal{T}, X_1)$ are in $\mathbf{p}$-biased random order is obvious. $\square$

Remark. As hinted before, there are different ways of implementing the Joyal bijection in a probabilistic context. In the Brownian bridge limit setting of Theorem 1, these lead to different recursive decompositions of Brownian bridge, discussed in detail in [1].

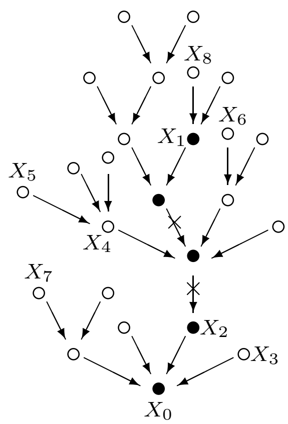

Fig. 3: A tree and a **p**-sample giving
the mapping of Fig. 1 by the Joyal map.

In Figure 3, we draw a tree with a spine (• vertices) and we run a **p** sample on it. The crosses indicate the edges that must be removed to form the mapping digraph, which is the same as in Fig. 1.

2.5 Weak convergence of random tree walks

Let $\mathcal{T}_n$ be a random $\mathbf{p}^{(n)}$-tree and let $H^{(n)} = H^{\mathcal{T}_n}$ be the associated height process from section 2.2. Let B^{exc} be standard Brownian excursion. We quote the following theorem: part (a) is from [2] (see [15] for recent variations) and part (b) is [6] Theorem 4.

Theorem 4 *(a) If $\mathbf{p}^{(n)}$ is uniform on $[n]$ then*

$$n^{-1/2}H^{(n)} \to 2B^{\text{exc}} \textit{ in law}$$

with respect to the usual Skorokhod topology on $D[0,1]$.
(b) If the sequence $(\mathbf{p}^{(n)})$ satisfies the uniform asymptotic negligibility condition (4) then

$$c(\mathbf{p}^{(n)})H^{(n)} \to 2B^{\text{exc}} \textit{ in law}$$

with respect to the $*$-topology on $D[0,1]$ described below.

Examples show [6] that Skorokhod convergence does not hold in the complete generality of (4). In unpublished work we have sufficient conditions on $(\mathbf{p}^{(n)})$ for Skorokhod convergence, but we do not have a conjecture for the precise necessary and sufficient conditions.

Here are the properties of the $*$-topology that we need (stated slightly differently than in [6]). Write $\stackrel{\text{unif}}{\to}$ for uniform convergence on $[0,1]$.

Lemma 5 *Let $f^n \in D[0,1]$ and $f^\infty \in C[0,1]$. Then $f^n \to^* f^\infty$ if and only if there exist functions $g^n, h^n \in D[0,1]$ such that*

$$\begin{aligned} f^n &= g^n + h^n \\ g^n &\stackrel{\mathit{unif}}{\to} f^\infty \\ h^n &\geq 0 \\ \mathrm{Leb}\{x : h^n(x) > 0\} &\to 0. \end{aligned}$$

3 Proof of Theorem 1

3.1 Representation of the mapping walk with p-trees

As in section 2.4, let $(\mathcal{T}, X_1)$ denote a **p**-tree $\mathcal{T}$, rooted at some vertex X_0, together with an independent **p**-point X_1. Recall the definition of the mapping $J(\mathcal{T}, X_1)$ defined in terms of $(\mathcal{T}, X_1)$ and a **p**-sample $(X_2, X_3, \ldots)$. We are now going to use Proposition 3 to construct the **p**-mapping walk H^M, for $M = J(\mathcal{T}, X_1)$, from $(\mathcal{T}, X_1)$. Recall that $\mathcal{T}_{c_1}, \ldots, \mathcal{T}_{c_K}$ are the subtrees of $\mathcal{T}$ obtained when the edges between the vertices of the spine are deleted, and rooted at these vertices. To each of these we can associate the height processes $H^{\mathcal{T}_{c_i}}$ (with weights on vertices being the **p**-values). If we now concatenate these walks together, just as in (3), it should be clear from Proposition 3 that the resulting process is the walk H^M associated with the **p**-mapping $M = J(\mathcal{T}, X_1)$, with the order on basins induced by the Joyal map. With this interpretation, the mapping walk is thus what we call the height process of the **p**-tree *above the spine.*

Next, we need to incorporate the time-marks of the mapping walk. Recall that these time-marks give the successive intervals $[Z_j, Z_{j+1})$ of exploration of the j-th basin. By Proposition 3, the order on basins is determined by the visits of a **p**-sample of components of the **p**-tree. So it should be clear that we may obtain the marks as follows (this has to be understood as a conditional form of the recursive constructions above). Let $Z_0 = 0$. Recall the notation $[g^M(i), d^M(i))$ for the interval during which the walk H^M visits the point i.

• Take U_2 uniform $(0,1)$ independent of the **p**-tree. Then let $Z_1 = \inf\{g^M(i) : g^M(i) > U_2, H^M_{g^M(i)} = 0\} \wedge 1$. If $Z_1 = 1$ we are done.

• Given $(Z_i)_{0\le i\le j}$ with $Z_j < 1$, let U_{j+2} be uniform on $(Z_j, 1)$ independent of the tree, and $Z_{j+1} = \inf\{g^M(i) : g^M(i) > U_{j+2}, H^M_{g^M(i)} = 0\} \wedge 1$. If this is 1 we are done.

Thus we can study mapping-walks directly in terms of trees, as summarized in

Proposition 6 *Let $\mathcal{T}$ be a $\mathbf{p}$-tree and X_1 a $\mathbf{p}$-sample. The marked height process above the spine, $(H, (Z_1, \ldots))$, has the law of the marked walk of the $\mathbf{p}$-mapping $J(\mathcal{T}, X_1)$, with basins in $\mathbf{p}$-biased random order.*

3.2 A transformation on paths

Motivated by the discrete transformation (height process → height process above spine) above, we introduce a transformation $\mathbf{J}^u : D[0,1] \to D[0,1]$. Fix $0 \le u \le 1$. Consider $f = (f_t) \in D[0,1]$. Define the *pre- and post- infimum process of f before and after u*, written $\underline{f}(u)$, as follows:

$$\underline{f}_s(u) = \begin{cases} \inf_{t\in[s,u]} f_t & \text{for } s < u \\ \inf_{t\in[u,s]} f_t & \text{for } s \ge u. \end{cases}$$

An "excursion" of f above $\underline{f}(u)$ is a portion of path of $f - \underline{f}(u)$ on a constancy interval of $\underline{f}(u)$. Each of these excursions has a starting time g which is at some height $h = f_g = \underline{f}_g(u)$, and if two or more of these excursions have the same starting height, we stick them together in the order induced by $(0,1)$, so that each height specifies at most one "generalized" excursion of f above $\underline{f}$. Write $\varepsilon_1(\cdot), \varepsilon_2(\cdot), \ldots$ for these generalized excursions, ranked for example in decreasing order of lifetimes $l_1, l_2, \ldots$, and let h_i be the height of the starting point of excursion ε_i. We now concatenate these excursions in increasing order of starting height. That is, for $s \in [0,1)$, let $h = h_i$ be the unique height such that $\sum_{j:h_j<h_i} l_j \le s < \sum_{j:h_j\le h_i} l_j$ and define

$$(\mathbf{J}^u(f))_s = \varepsilon_i\left(s - \sum_{j:h_j<h_i} l_j\right).$$

If the sum s_0 of lengths of constancy intervals of $\underline{f}$, that is $\sum_j l_j$, equals 1, then $\mathbf{J}^u(f)$ is defined for all $0 \le s \le 1$; otherwise we just define $\mathbf{J}^u(f)$ to equal 0 on $s_0 < s \le 1$. We call $\mathbf{J}^u(f)$ the *process f reflected above $\underline{f}(u)$*.

Lemma 7 *Let $f^n \in D[0,1]$ and $f^\infty \in C[0,1]$. Suppose that, for each $0 \le u \le 1$, the lengths of intervals of constancy of $\underline{f}^\infty(u)$ sum to 1, and suppose that the different excursions of f^∞ above $\underline{f}^\infty(u)$ start at different heights.*

(a) If $f^n \stackrel{unif}{\to} f^\infty$ *then* $\mathbf{J}^u(f^n) \stackrel{unif}{\to} \mathbf{J}^u(f^\infty)$.
(b) If $f^n \to^* f^\infty$ *and* U *has uniform*$(0,1)$ *law then* $\mathbf{J}^U(f^n) \to^* \mathbf{J}^U(f^\infty)$ *in probability.*

Proof. We outline the argument, omitting some details. Fix u. Consider an interval of constancy of $\underline{f}^\infty(u)$, say $[a_k, b_k]$. From the hypotheses on f^∞ we have $f(s) > f(a_k)$ on $a_k < s < b_k$. Consider the case $f^n \stackrel{\text{unif}}{\to} f^\infty$. Then for large n there must be intervals of constancy of $\underline{f}^n(u)$, say $[a_k^n, b_k^n]$, such that $a_k^n \to a_k, b_k^n \to b_k,\ f^n(a_k^n) \to f^\infty(a_k)$. This implies

$$(f^n(a_k^n + s) - f^n(a_k^n), 0 \le s \le b_k^n - a_k^n) \stackrel{\text{unif}}{\to} (f^\infty(a_k + s) - f^\infty(a_k), 0 \le s \le b_k - a_k).$$

Since $\sum_k (b_k^n - a_k^n) \to \sum_k (b_k - a_k) = 1$, we easily see that in the case $u = 1$ we have $\mathbf{J}^u(f^n) \stackrel{\text{unif}}{\to} \mathbf{J}^u(f^\infty)$. For general u, apply the argument above separately to $[0, u]$ and $[u, 1]$, and check that the operation of "concatenation of excursions in order of starting height" is continuous; again we deduce $\mathbf{J}^u(f^n) \stackrel{\text{unif}}{\to} \mathbf{J}^u(f^\infty)$.

Now consider the case $f^n \to^* f^\infty$. Recall the Lemma 5 decomposition $f^n = g^n + h^n$. By passing to a subsequence we may assume that for almost all $0 \le u \le 1$

$$h^n(u) = 0 \text{ ultimately }. \tag{9}$$

Fix such a u; it is enough to show $\mathbf{J}^u(f^n) \to^* \mathbf{J}^u(f^\infty)$. The previous case implies that $\mathbf{J}^u(g^n) \stackrel{\text{unif}}{\to} \mathbf{J}^u(f^\infty)$. Consider, as in the previous argument, an interval of constancy of $[a_k^n, b_k^n]$ of $\underline{g^n}$ converging to an interval of constancy of $[a_k, b_k]$ of $\underline{f^\infty}$. Since $f^n = g^n + h^n$ with $h^n \ge 0$, there is a corresponding interval of constancy of $\underline{f^n}$ which contains the interval $[\tilde{a}_k^n, \tilde{b}_k^n]$ defined by

$$\tilde{a}_k^n = \inf\{a \ge a_k^n : h^n(a) = 0\}, \quad \tilde{b}_k^n = \sup\{b \le b_k^n : h^n(b) = 0\}.$$

Use (9) to see that $\tilde{b}_k^n - \tilde{a}_k^n \to b_k - a_k$. We now see that the analog of $\mathbf{J}^u(g^n)$ using only excursions over $\cup_k [\tilde{a}_k^n, \tilde{b}_k^n]$ will converge uniformly to $\mathbf{J}^u(f^\infty)$. After adding the contribution of h^n over these intervals, we will still have $*$-convergence; and the contribution to $\mathbf{J}^u(f^n)$ from the complement of $\cup_k [\tilde{a}_k^n, \tilde{b}_k^n]$ is asymptotically negligible for $*$-convergence.

3.3 Pushing forward tree walks to mapping walks

Let $\mathcal{T}$ be a $\mathbf{p}$-tree on $[n]$, and put the children of each vertex in uniform random order. Let U be uniform on $(0, 1)$, independent of $\mathcal{T}$, and let X_1 be the vertex visited by the height process $H^{\mathcal{T}}$ at time U. The fact that the height process spends time p_x at vertex x

implies that X_1 is a $\mathbf{p}$-sample. By Proposition 3, $M = J(\mathcal{T}, X_1)$ is a random $\mathbf{p}$-mapping with basins in $\mathbf{p}$-biased random order. Let H^M be the associated marked random walk, constructed as in section 3.1, which by Proposition 6 is the height process of $\mathcal{T}$ above the spine.

So to get H^M from $H^{\mathcal{T}}$ we have to extract from $H^{\mathcal{T}}$ the height processes of the subtrees rooted on the spine. This will be done by applying the transformation $\mathbf{J}$ to a slightly modified version of $H^{\mathcal{T}}$.

Write $c_1, c_2, \ldots, c_K = X_1$ for the vertices of the spine of $\mathcal{T}$ in order of height, and as before write $[g(c_i), d(c_i))$ for the interval in which the height process $H^{\mathcal{T}}$ "visits" c_i. Now we consider the process

$$K_s = \begin{cases} H^{\mathcal{T}}_{g(c_i)} + 1 & \text{if } s \in (g(c_i), d(c_i)), \text{ for some } i \\ H^{\mathcal{T}}_s & \text{else.} \end{cases} \tag{10}$$

In other words, we "lift" the heights of the spine vertices by 1, but we use a small artifact here: at the point $g(c_i)$, the process stays at the value $H^{\mathcal{T}}_{g(c_i)}$, and the process is not càdlàg in general. Now reflect this process K above $\underline{K}(U)$ to obtain the process $\mathbf{J}^U(K)$.

Lemma 8

$$\mathbf{J}^U(K)\big|_s = (H^M_s - 1)^+, \quad 0 \leq s \leq 1. \tag{11}$$

Proof. Suppose that the height process $H^{\mathcal{T}}$ of the tree is currently visiting a spine vertex, say c_i, which is not the top of the spine. Write h for its height ($h = H^{\mathcal{T}}(c_i) = K_{g(c_i)} = K_{g(c_i+)} - 1$). Then c_i has some children, one of them being c_{i+1}. Now we want to recover the height process of the subtree $\mathcal{T}_{c_i}$ rooted at c_i when we delete the edges between the vertices of the spine. First, during the time interval $(g(c_i), g(c_{i+1}))$, the height process of $\mathcal{T}_n$ visits c_i and the vertices of $\mathcal{T}_{c_i}$ that are located to the left of the spine (i.e. the descendants of the children of c_i located before c_{i+1}), if any. Then the process examines all the descendants of c_{i+1}, hence staying at heights greater than $h+1$, and after that visits the children of c_i that are to the right of the spine, if any, starting say at time $g'_i > U$.

Hence, $K_{g(c_i)} = h$, $K_s \geq h+1$ for $s \in (g(c_i), g(c_{i+1})]$ and $K_s > h+1$ for $s \in (g(c_{i+1}), U)$. So $(g(c_i), g(c_{i+1}))$ is an excursion interval of K above $\underline{K}$, for an excursion starting at height $h+1$. This excursion is easily seen as being $(H^{\mathcal{T}_{c_i}} - 1)^+$ restricted to the vertices that are to the left hand side of the spine, where $H^{\mathcal{T}_{c_i}}$ is the height process of $\mathcal{T}_{c_i}$.

Then $K_{g'_i} = h+1$, so g'_i is the starting time of an excursion of K above $\underline{K}(U)$, with starting height $h+1$, and this excursion is now $(H^{\mathcal{T}_{c_i}} - 1)^+$ restricted to the vertices that are to the right hand side of the spine. The analysis is easier if $c_i = X_1$ is the top of the spine, in which case there is no child of c_i at the left or right-hand side of the spine. This gives the result. $\square$

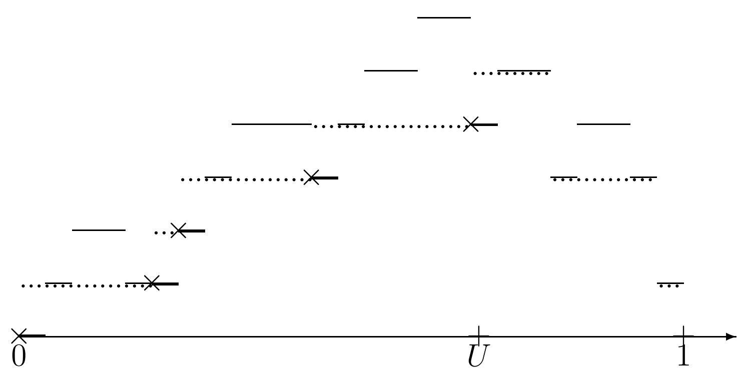

Fig. 4: The process $H^{\mathcal{T}}$ (thin line) and the process $\underline{K}$ (dashed line). The crosses and thick lines represent visits to vertices of the spine.

Note that our "artifact" was designed to give an exact equality in Lemma 8. Removing the artifact to make processes càdlàg can only change the processes involved by ± 1, which will not affect our subsequent asymptotic arguments.

Figure 4 shows the height process of the tree of Figure 3, with U such that the spine is the same. We also draw the process $\underline{K}$. As noted before, the unmarked walk associated with the image of the last tree by the Joyal map depends only on the spine, and so this walk is that of Figure 2. The next figure depicts the process $\mathbf{J}^U(K)$.

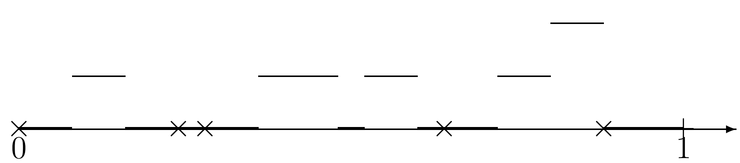

Fig. 5: The process $\mathbf{J}^U(K)$ (compare with Fig. 2).

3.4 J transforms B^{exc} to $B^{|\text{br}|}$

Lemma 9 *Let B^{exc} be standard Brownian excursion, and let U be uniform independent on $[0,1]$. Then $\mathbf{J}^U(B^{\text{exc}})$ is distributed as $B^{|\text{br}|}$, reflecting Brownian bridge on $[0,1]$.*

Proof. By [19], the reflecting Brownian bridge is obtained from the family of its excursions by concatenating them in exchangeable random order. Precisely, let $(\varepsilon_1, \varepsilon_2, \ldots)$ be the excursions of $B^{|\text{br}|}$ away from 0, ranked by decreasing order of their durations $\ell_1 \geq \ell_2 \geq \ldots > 0$, and let $\preceq$ be a random order ($\prec$ is then the associated strict order) on $\mathbb{N}$ independent of the excursions, such that for every k, each one of the $k!$ possible strict orderings on the set $[k]$ are equally likely. Then the process

$$X_s = \varepsilon_i \left(s - \sum_{j \prec i} \ell_j \right) \quad \text{for } \sum_{j \prec i} \ell_j \leq s \leq \sum_{j \preceq i} \ell_j$$

has the same law as $B^{|\text{br}|}$.

By [7, Theorem 3.2], the excursions away from 0 of $\mathbf{J}^U(B^{\text{exc}})$ are those of a reflecting Brownian bridge. It thus remains to show that the different ordering of the excursions used to define the process $\mathbf{J}^U(B^{\text{exc}})$ is an independent exchangeable order. Now, by a conditioned form of Bismut's decomposition (see e.g. Biane [8]), conditionally on U and B^{exc}_U, the paths $(B^{\text{exc}}_{U-s}, 0 \leq s \leq U)$ and $(B^{\text{exc}}_{s+U}, 0 \leq s \leq 1-U)$ are independent Brownian paths starting at B^{exc}_U, conditioned to first hit 0 at time U and $1-U$ respectively, and killed at these times. Still conditionally on (U, B^{exc}_U), consider the excursions $(\varepsilon^1_1, \varepsilon^1_2, \ldots)$ of $(B^{\text{exc}}_s, 0 \leq s \leq U)$ above its future infimum process, ordered in decreasing lifetimes order, and their respective heights $(h^1_1, h^1_2, \ldots)$. Let also $(\varepsilon^2_1, \varepsilon^2_2, \ldots)$ be the excursions of $(B^{\text{exc}}_{s+U}, 0 \leq s \leq 1-U)$ above its infimum process, also ordered in decreasing lifetimes order, and denote their respective heights by $(h^2_1, h^2_2, \ldots)$. Then we have from [19, Proposition 6.2] that $(h^1_1/B^{\text{exc}}_U, h^1_2/B^{\text{exc}}_U, \ldots)$ and $(h^2_1/B^{\text{exc}}_U, h^2_2/B^{\text{exc}}_U, \ldots)$ are independent conditionally on B^{exc}_U, and are two sequences of i.i.d. uniform$[0,1]$ r.v.'s. Hence, the concatenation of these two sequences is again a sequence of independent uniform$[0,1]$ r.v.'s. So this holds also unconditionally on (U, B^{exc}_U). Now, by definition, the order of the excursions of $\mathbf{J}^U(B^{\text{exc}})$ is that induced by this concatenated family, meaning that the excursion ε^i_k appears before excursion $\varepsilon^j_{k'}$ if and only if $h^i_k < h^j_{k'}$ for $k, k' \geq 1$, $i, j \in \{1,2\}$. The excursions are thus in exchangeable random order, and the claim follows. $\square$

Notice also from [19] that for the reflected Brownian bridge $B^{|\text{br}|} = \mathbf{J}^U(B^{\text{exc}})$, one can extend the fact that $L_1 = 2B^{\text{exc}}_U$ to

$$L_s = 2h_s \tag{12}$$

if s is not a zero of $B^{|\mathrm{br}|}$, where h_s is the height of the starting point of the excursion of B^{exc} that is matched to the excursion of $\mathbf{J}^U(B^{\mathrm{exc}})$ straddling s, and L is then defined on all $[0,1]$ by continuity.

3.5 Completing the proof of Theorem 1

As in Proposition 3, we may take the $\mathbf{p}^{(n)}$-mapping M_n in its representation $M_n = J(\mathcal{T}_n, X_{1,n})$, where $\mathcal{T}_n$ is a $\mathbf{p}^{(n)}$-tree and $X_{1,n}$ is a $\mathbf{p}^{(n)}$ sample from $\mathcal{T}_n$. By Theorem 4(b) and the Skorokhod representation Theorem, we may suppose that we have a.s. convergence of $c(\mathbf{p}^{(n)})H^{\mathcal{T}_n}$ to $2B^{\mathrm{exc}}$. (Here and below, convergence is $*$-convergence in general, and uniform convergence in the special case of uniform $\mathbf{p}^{(n)}$). For each n we may use the same U to define $X_{1,n}$. From the definition (10) of K^n we also have a.s. convergence of $c(\mathbf{p}^{(n)})K^n$ to $2B^{\mathrm{exc}}$. Then by Lemmas 7 and 9, the process $c(\mathbf{p}^{(n)})\mathbf{J}^U(K^n)$ converges to $2B^{|\mathrm{br}|}$. Hence, so does $c(\mathbf{p}^{(n)})H^{M_n}$ according to Lemma 8. This is assertion (i) of the Theorem.

For (ii), the assertion about the marks $(Z_1^n, Z_2^n, \ldots)$ follows easily by incorporating the representation of section 3.1 into the argument above (the only possible trouble is when a U_i falls on a zero of H^n, but this happens with probability going to 0).

To obtain (iii) we observe that the number of cyclic points visited in depth-first order before the vertex coded by $s \in [0,1]$ is equal (except for an unimportant possible error of 1) to the starting height of some excursion of K^n above $\underline{K}^n$. Now suppose that s is not a zero of $B^{|\mathrm{br}|}$, so that it is strictly included in the excursion interval of, say the k-th longest-lifetime excursion of $2B^{|\mathrm{br}|}$ away from 0. Then for n sufficiently big, s also belongs to the excursion interval of the k-th longest-lifetime excursion of H^{M_n} away from 0, which corresponds to the k-th longest-lifetime excursion of K^n above $\underline{K}^n$. But this excursion's starting height, once multiplied by $c(\mathbf{p}^{(n)})$, converges to the starting height of the k-th longest-lifetime excursion of $2B^{\mathrm{exc}}$ above $2\underline{B}^{\mathrm{exc}}$. It now follows from classical considerations (see e.g. [19]) that this last height is equal to L_s (note this is consistent with (12) wherein $2h_s$ is the height for $2B^{\mathrm{exc}}$). We can now conclude, since the limiting process L is continuous and increasing on $[0,1]$, and since the lengths of excursions of $2B^{\mathrm{exc}}$ above $2\underline{B}^{\mathrm{exc}}$ sum to 1, that the convergence of $c(\mathbf{p}^{(n)})\ell^{M_n}$ to L holds uniformly and not only pointwise. $\square$

4 Final comments

1. At the start of the proof of Lemma 9 we used the result from [7] that the excursions of $\mathbf{J}^U(B^{\mathrm{exc}})$ away from 0 are those of a reflecting Brownian bridge. Here is a way to rederive that result. First, by the well-known formula for the entrance law of the Brownian excursion, one easily gets that the law of (U, B_U^{exc}) has the same law as $(T_{R/2}, R/2)$ given $T_R = 1$, where T is the first-passage subordinator associated with Brownian motion and R is an independent r.v. with Rayleigh distribution. By Bismut's decomposition, one deduces that the process Y defined by $Y_t = B_{U-t}^{\mathrm{exc}}$ for $0 \leq t \leq U$ and $Y_t = B_t^{\mathrm{exc}} - 2B_U^{\mathrm{exc}}$ for $U \leq t \leq 1$ is, conditionally on B_U^{exc} but unconditionally on U, a first-passage bridge of the Brownian motion, i.e. a Brownian motion conditioned to first hit $-2B_U^{\mathrm{exc}}$ at time 1. By [19], its associated reflected process above its infimum is a reflecting Brownian bridge conditioned to have local time $2B_U^{\mathrm{exc}}$ at level 0, and we can uncondition on B_U^{exc}, since $2B_U^{\mathrm{exc}}$ has the Rayleigh law, which is that of the local time at 0 of $B^{|\mathrm{br}|}$.

2. Our work implicitly answers a question of Pitman [18]. Let

$$C_n^k = |\{i \in [n] : M_n^{k-1}(i) \notin \mathcal{C}(M_n), M_n^k(i) \in \mathcal{C}(M_n)\}|$$

be the number of vertices at distance k of the set of cyclic points of the uniform random mapping M_n (for the distance induced by the digraph of M_n). In particular, $C_n^0 = \ell_1^{M_n}$ with our previous notations. Drmota and Gittenberger [11] show that the process $(n^{-1/2} C_n^{[2s\sqrt{n}]}, s \geq 0)$ converges in law to the process $(L_1^s(B^{|\mathrm{br}|}), s \geq 0)$ of local times of $B^{|\mathrm{br}|}$ (with our choice of normalization as half the occupation density). One of the question raised in Pitman [18] is whether this convergence holds jointly with the convergences of our main theorem. To show this is true, first note that from the tightness of each individual component, we get that the pair $(n^{-1/2} H^{M_n}, n^{-1/2} C_n^{[2\sqrt{n}\cdot]})$ is tight. Call $(2B^{|\mathrm{br}|}, L')$ its weak limit through some subsequence, and suppose that the convergence in law holds a.s. by Skorokhod's representation theorem. If we prove that $L'(s) = L_1^s(B^{|\mathrm{br}|})$ for every s, we will have shown that $(2B^{|\mathrm{br}|}, L_1^{\cdot}(B^{|\mathrm{br}|}))$ is the only possible limit, hence that $(n^{-1/2} H^{M_n}, n^{-1/2} C_n^{[2\sqrt{n}\cdot]})$ jointly converges to this limit. Now for every $s \geq 0$ one has that

$$\int_0^1 dt \mathbb{1}\{H_t^{M_n} \leq [2\sqrt{n}s]\} = n^{-1} \sum_{k=0}^{[2\sqrt{n}s]} C_n^k \to 2 \int_0^s du L'(u),$$

whereas the left-hand term converges to $\int_0^1 dt \mathbb{1}\{B_t^{|\mathrm{br}|} \leq s\}$, which equals $2 \int_0^s du L_1^u(B^{|\mathrm{br}|})$. Hence the result by identification.

Acknowledgements We thank Jennie Hansen and an anonymous referee for careful reading.

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