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THE LAWS OF CHUNG AND HIRSCH FOR CAUCHY'S PRINCIPAL VALUES RELATED TO BROWNIAN LOCAL TIMES

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Abstract Two Chung-type and Hirsch-type laws are established to describe the liminf asymptotic behaviours of the Cauchy's principal values related to Brownian local times. These results are generalized to a class of Brownian additive functionals.

Keywords Principal values, Brownian additive functional, liminf asymptotic behaviours.

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1 Introduction

Let $(B(t), t \geq 0)$ be a one-dimensional Brownian motion starting from 0, and denote $(L_t^x, t \geq 0, x \in \mathbb{R})$ a continuous version of its local times such that $x \to L_t^x$ is Hölder continuous of order β for every $\beta < 1/2$ uniformly in t on each compact interval, and for every bounded Borel function $f : \mathbb{R} \to \mathbb{R}$,

$$\int_0^t f(B(s))ds = \int_{\mathbb{R}} f(x) L_t^x \, dx, \qquad t \geq 0.$$

For every $-\infty < \alpha < 3/2$, we define an additive functional $X_\alpha(t) \equiv$ p.v. $\int_0^t ds/\widetilde{B(s)^\alpha}$ as Cauchy's principal values related to (L_t^x) ($\widetilde{x}^\alpha \stackrel{\text{def}}{=} |x|^\alpha \, \text{sgn}(x)$ for $x \in \mathbb{R}$) by:

$$X_\alpha(t) \stackrel{\text{def}}{=} \lim_{\epsilon \to 0} \int_0^t \frac{ds}{\widetilde{B(s)^\alpha}} \mathbb{1}_{(|B(s)| \geq \epsilon)} = \int_0^\infty \frac{dx}{x^\alpha}(L_t^x - L_t^{-x}), \qquad t \geq 0, \tag{1.1}$$

(in the case $\alpha < 1$, the integral $\int_0^t ds/\widetilde{B(s)^\alpha}$ is absolutely convergent). The principal values have been a subject of many recent works. For motivations and studies on X_α and related topics, see e.g. Biane and Yor [4], Fitzsimmons and Getoor [14,15] and Bertoin [1,3], Csáki et al. [7,8,10], Yamada [34] and Yor [36] together with their references. We only mention the facts that the process X_α admits a (Hölder) continuous version, and inherits from the Brownian motion the self-similarity of order $(1 - \alpha/2)$.

Here, we shall study some path properties of X_α. Recall from [18] ($\alpha = 1$) and from Csáki et al. [8] that for all $-\infty < \alpha < 3/2$, we have

$$\limsup_{t \to \infty} \frac{X_\alpha(t)}{t^{1-\frac{\alpha}{2}} \, (\log \log t)^{|\frac{\alpha}{2}|}} = C_1(\alpha), \qquad \text{a.s.,} \tag{1.2}$$

where $C_1(\alpha)$ is some positive constant. In fact, Csáki et al. [8] stated (1.2) for all $\alpha \in (0, 3/2)$; in the case $\alpha = 1$ (cf. [18]), the exact value of the constant $C_1(\alpha)$ has been explicitly known as $\sqrt{8}$. The remaining case $\alpha \leq 0$ of (1.2) is a straightforward consequence of Strassen's laws of iterated logarithm (cf. Strassen [33]). For further and deep studies related to the limsup properties of X_α, see Csáki et al. [8] and [7] (modulus of continuity and increments in the case $\alpha = 1$) .

In constrast with the deep understanding of the limsup asymptotic behaviors of X_α, relatively little is known about its liminf properties, to our best knowledge. This paper aims at giving two Chung-type and Hirsch-type laws for X_α. The first one reads as follows:

Theorem 1.1 *There exists some constant $0 < C_2(\alpha) < \infty$ such that for $-\infty < \alpha \leq 1$,*

$$\liminf_{t \to \infty} \Big(\frac{\log \log t}{t}\Big)^{1-\alpha/2} \sup_{0 \leq s \leq t} |X_\alpha(s)| = C_2(\alpha), \qquad \text{a.s.} \tag{1.3}$$

When $1 < \alpha < 3/2$, we have

$$\liminf_{t \to \infty} \frac{\Big(\log \log t\Big)^{\alpha/2}}{t^{1-\alpha/2}} \sup_{0 \leq s \leq t} |X_\alpha(s)| = C_2(\alpha), \qquad \text{a.s.} \tag{1.4}$$

The exact value of $C_2(\alpha)$ remains unknown. When $\alpha < 1$, Theorem 1.1 yields some interesting examples which *a priori* do not involve principal values:

Example Put $\alpha = -1$ in (1.3), we recover the following Chung's law due to Khoshnevisan and Shi [22] for the **integrated Brownian motion**:

$$\liminf_{t\to\infty} \Big(\frac{\log\log t}{t}\Big)^{3/2} \sup_{0\le s\le t} \Big| \int_0^s B(u)\,du \Big| = C_2(-1), \qquad \text{a.s.} \tag{1.5}$$

More generally, let $\theta \geq 0$, we obtain:

$$\liminf_{t\to\infty} \Big(\frac{\log\log t}{t}\Big)^{1+\theta/2} \sup_{0\le s\le t} \Big| \int_0^s |B(u)|^\theta \operatorname{sgn}(B(u))\,du \Big| = C_2(-\theta), \qquad \text{a.s.} \tag{1.6}$$

Whereas Donsker and Varadhan's LIL ([12, (4.5)]) yields that

$$\liminf_{t\to\infty} \frac{\Big(\log\log t\Big)^{\theta/2}}{t^{1+\theta/2}} \int_0^t |B(u)|^\theta\,du = \tilde{C}_2(-\theta), \qquad \text{a.s.} \tag{1.7}$$

for some constant $\tilde{C}_2(-\theta) > 0$.

It is also of interest to describe how small $\sup_{0\le s\le t} X_\alpha(s)$ can be:

Theorem 1.2 *Fix* $-\infty < \alpha < 3/2$. *Let* $f(t) > 0$ *be a nondecreasing function, we have*

$$\mathbb{P}\Big(\sup_{0\le s\le t} X_\alpha(s) < \frac{t^{1-\alpha/2}}{f(t)};\ i.o.\ \Big) = \begin{cases} 0 \\ 1 \end{cases} \iff \int^\infty \frac{dt}{t(f(t))^{1/(2(2-\alpha))}} \begin{cases} = \infty \\ < \infty \end{cases}, \tag{1.8}$$

where here and in the sequel, i.o. means "infinitely often" as the relevant index goes to infinity.

Remark In view of the invariance principle for additive functionals established by Csáki et al. [8, Theorem 1.3], the above two LILs also hold for the principal values related to the simple symmetric random walk on $\mathbb{Z}$.

It is worth noticing that the above results hold for a more general class of additive functionals, see Section 5. The proofs of Theorems 1.1 and 1.2 rely on the estimates of $\mathbb{P}(\sup_{0\le s\le 1} |X_\alpha(s)| < \epsilon)$ and $\mathbb{P}(\sup_{0\le s\le 1} X_\alpha(s) < \epsilon)$, which we shall give in Section 3 with aid of Brownian excursions and Biane and Yor [4]'s representation of stable processes. In Section 4, we prove Theorems 1.1 and 1.2.

Throughout the whole paper, we adopt the notation that $f(x) \sim g(x)$ (resp: $f(x) \asymp g(x)$) as $x \to x_0 \in [0, \infty]$ when $\lim_{x\to x_0} f(x)/g(x) = 1$ (resp: $0 < \liminf_{x\to x_0} f(x)/g(x) \leq \limsup_{x\to x_0} f(x)/g(x) < \infty$).

For notational convenience, we write in the rest of this paper $\nu \stackrel{\text{def}}{=} 1/(2-\alpha) \in (0, 2)$.

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2 Preliminaries

Define

$$\tau_r \stackrel{\text{def}}{=} \inf\{t > 0 : L_t^0 > r\}, \qquad r \geq 0. \tag{2.1}$$

The following important fact is due to Biane and Yor [4], see also Fitzsimmons and Getoor [14] and Bertoin [1,3] for generalizations to Lévy processes.

Fact 2.1 For each $\alpha \in (-\infty, 3/2)$, the process $r \to C_3(\alpha)\, X_\alpha(\tau_r)$ is a standard symmetric stable process of index $\nu = 1/(2-\alpha)$. The positive constant $C_3(\alpha)$ is explicit.

Using the well-known estimates for a stable process (see Getoor [16] and Bertoin [2, Chap.VIII]), we obtain:

Fact 2.2 There exists two positive constants $C_4(\alpha)$ and $C_5(\alpha)$ such that $(\nu \equiv 1/(2-\alpha))$

$$\mathbb{P}\Big(\sup_{0\leq s\leq 1} X_\alpha(\tau_s) < \epsilon \Big) \;\sim\; C_4(\alpha)\, \epsilon^{\nu/2}, \qquad \epsilon \to 0, \tag{2.2}$$

$$\log \mathbb{P}\Big(\sup_{0\leq s\leq 1} |X_\alpha(\tau_s)| < \epsilon \Big) \;\sim\; -C_5(\alpha)\, \epsilon^{-\nu}, \qquad \epsilon \to 0. \tag{2.3}$$

Define a Brownian meander $(m(u), 0 \leq u \leq 1)$ and a Brownian normalized excursion $(e(u), 0 \leq u \leq 1)$ by

$$m(u) \;\stackrel{\text{def}}{=}\; \frac{1}{\sqrt{1-g_1}}\, |B(g_1 + u(1-g_1))|, \qquad 0 \leq u \leq 1, \tag{2.4}$$

$$e(u) \;\stackrel{\text{def}}{=}\; \frac{1}{\sqrt{d_1-g_1}}\, |B(g_1 + u(d_1-g_1))|, \qquad 0 \leq u \leq 1, \tag{2.5}$$

where here and in the sequel, $g_t \stackrel{\text{def}}{=} \sup\{s \leq t : B(s) = 0\}$ and $d_t \stackrel{\text{def}}{=} \inf\{s > t : B(s) = 0\}$ for $t > 0$. The densities of $\int_0^1 \frac{du}{m(u)}$ and $\int_0^1 \frac{du}{e(u)}$ are evaluated by Biane and Yor [4, pp.68], who obtained the following identities in law (cf. [4, pp. 69, formulas (5c) and (5d)]):

$$\int_0^1 \frac{du}{m(u)} \stackrel{\text{law}}{=} 2 \sup_{0\leq s\leq 1} |p(s)| \stackrel{\text{law}}{=} \sup_{0\leq s\leq 1} m(s), \qquad \int_0^1 \frac{du}{e(u)} \stackrel{\text{law}}{=} 2 \sup_{0\leq s\leq 1} e(s), \tag{2.6}$$

where $(p(s), 0 \leq s \leq 1)$ denotes a standard Brownian bridge from 0 to 0. The distribution functions of $\sup_{0\leq s\leq 1} |p(s)|$ and $\sup_{0\leq s\leq 1} e(s)$ have been evaluated respectively by Kolmogorov [24] and Smirnov [32] and by Chung [6, (4.5), (4.9)–(4.10) and Theorem 7]. Using these two distribution functions together with (2.6), we obtain (see also Shorack and Wellner [31, pp. 34–39]):

Fact 2.3 As $\epsilon \to 0$, we have

$$\mathbb{P}\Big(\int_0^1 \frac{du}{m(u)} < \epsilon \Big) \;=\; \mathbb{P}\Big(\sup_{0\leq u\leq 1} m(u) < \epsilon \Big) \;\sim\; \frac{\sqrt{8\pi}}{\epsilon} \exp\Big(-\frac{\pi^2}{2\epsilon^2} \Big), \tag{2.7}$$

$$\mathbb{P}\Big(\int_0^1 \frac{du}{e(u)} < \epsilon \Big) \;=\; \mathbb{P}\Big(\sup_{0\leq u\leq 1} e(u) < \frac{\epsilon}{2} \Big) \;\sim\; \frac{\sqrt{2^7\pi^5}}{\epsilon^3} \exp\Big(-\frac{2\pi^2}{\epsilon^2} \Big). \tag{2.8}$$

As $\lambda \to \infty$, we have

$$\mathbb{P}\Big(\sup_{0\le s\le 1} m(s) > \lambda\Big) = \mathbb{P}\Big(\sup_{0\le s\le 1} |p(s)| > \frac{\lambda}{2}\Big) \quad \sim \quad 2\,\exp\Big(-\frac{\lambda^2}{2}\Big), \tag{2.9}$$

$$\mathbb{P}\Big(\sup_{0\le s\le 1} e(s) > \lambda\Big) \quad \sim \quad 8\,\lambda^2\,\exp\Big(-2\,\lambda^2\Big). \tag{2.10}$$

We end this section by presenting some estimates of the small deviations probabilities for $\int_0^1 du/m^{\alpha}(u)$ and $\int_0^1 du/e^{\alpha}(u)$, whose Laplace transforms are associated with the solutions of a class of Sturm-Liouville equations (see Jeanblanc et al. [21]):

Lemma 2.1 *For all $-\infty < \alpha < 3/2$ and $\alpha \neq 0$, there exists some constant $C_6(\alpha) > 0$ such that for all $x > 0$, we have*

$$\mathbb{P}\Big(\int_0^1 \frac{du}{m^{\alpha}(u)} < x\Big) \quad \le \quad \exp\Big(C_6 - \frac{1}{2}\,x^{-2/|\alpha|}\Big), \tag{2.11}$$

$$\mathbb{P}\Big(\int_0^1 \frac{du}{e^{\alpha}(u)} < x\Big) \quad \le \quad \exp\Big(C_6 - \frac{1}{3}\,x^{-2/|\alpha|}\Big). \tag{2.12}$$

Proof of Lemma 2.1. We shall only show (2.11), the proof of (2.12) is similar. We can replace 1/3 in (2.12) by any constant smaller than 1/2. Let us begin with the case $0 < \alpha < 3/2$. Observe the simple inequality:

$$\int_0^1 \frac{ds}{m^{\alpha}(s)} \ge \inf_{0\le s\le 1}\Big(m^{-\alpha}(s)\Big) = \Big(\sup_{0\le s\le 1} m(s)\Big)^{-\alpha},$$

which in view of (2.9) yields (2.11). Now, suppose $\alpha < 0$. Applying Hölder's inequality with $p \stackrel{\text{def}}{=} 1 + |\alpha|$ and $q \stackrel{\text{def}}{=} 1 + 1/|\alpha|$ gives

$$1 = \int_0^1 ds\, m^{1/q}(s)\, m^{-1/q}(s) \le \Big(\int_0^1 ds\, m^{p/q}(s)\Big)^{1/p}\Big(\int_0^1 \frac{ds}{m(s)}\Big)^{-1/q}.$$

This implies that

$$\int_0^1 \frac{ds}{m^{\alpha}(s)} = \int_0^1 ds m^{|\alpha|}(s) \ge \Big(\int_0^1 \frac{ds}{m(s)}\Big)^{-|\alpha|} \stackrel{\text{law}}{=} \Big(\sup_{0\le s\le 1} m(s)\Big)^{-|\alpha|},$$

yielding (2.11) by means of (2.9). □

We prove a counterpart of (2.11):

Lemma 2.2 *Fix $1 \le \alpha < 3/2$. There exists some constant $C_7(\alpha) > 0$ such that for all $0 < \epsilon < 1$,*

$$\mathbb{P}\Big(\int_0^1 \frac{ds}{m^{\alpha}(s)} < \epsilon\Big) \ge C_7\,\exp\Big(-\frac{8}{(2-\alpha)^2}\,\epsilon^{-2/\alpha}\Big). \tag{2.13}$$

Proof of Lemma 2.2. The case $\alpha = 1$ follows from (2.7). Let us consider $1 < \alpha < 3/2$ and small $\epsilon > 0$. Using Imhof [20]'s absolute continuity between the law of Brownian meander m and that of Bessel process R of dimension 3 starting from 0:

$$\mathbb{P}\Big(\int_0^1 \frac{ds}{m^\alpha(s)} < \epsilon \Big) = \sqrt{\frac{\pi}{2}}\, \mathbb{E}\Big(\frac{1}{R(1)}\, 1\!\!1_{(\int_0^1 R^{-\alpha}(s)\, ds < \epsilon)} \Big). \tag{2.14}$$

Using the time-change formula between two Bessel processes of different dimensions (cf. [4, Lemma 3.1]), we obtain a Bessel process $\widehat{R}$ of index $\nu \equiv 1/(2-\alpha) > 0$, starting from 0 such that

$$\widehat{R}\Big(\int_0^t R^{-\alpha}(s)\, ds \Big) = 2\nu\, R^{1/(2\nu)}(t), \qquad t \ge 0, \tag{2.15}$$

and by time-change

$$\inf\{ s > 0 : \int_0^s R^{-\alpha}(u) du > t \} = (2\nu)^{-2\alpha\nu} \int_0^t (\widehat{R}(u))^{2\alpha\nu}\, du \stackrel{\text{def}}{=} (2\nu)^{-2\alpha\nu}\, \widehat{A}(t), \quad t \ge 0. \tag{2.16}$$

Using (2.15), on $\{ \int_0^1 R^{-\alpha}(s)\, ds < \epsilon \}$, we have $R(1) \le (2\nu)^{-2\nu} \sup_{0 \le s \le \epsilon} \widehat{R}^{2\nu}(s)$. This fact together with (2.16) imply that the RHS of (2.14) is larger than

$$\begin{aligned} & C_8 \;\; \mathbb{E}\Big(\frac{1}{\sup_{0 \le s \le \epsilon} \widehat{R}^{2\nu}(s)}\, 1\!\!1_{(\widehat{A}(\epsilon) > (2\nu)^{2\alpha\nu})} \Big) \\ & \ge C_8\, \mathbb{P}\Big(\widehat{A}(\epsilon) > (2\nu)^{2\alpha\nu} \Big) - C_8\, \mathbb{P}\Big(\sup_{0 \le s \le \epsilon} \widehat{R}^{2\nu}(s) \ge 1 \Big) \\ & \stackrel{\text{def}}{=} C_8\, I_1 - C_8\, I_2, \end{aligned} \tag{2.17}$$

with $C_8 \stackrel{\text{def}}{=} \sqrt{\frac{\pi}{2}}\, (2\nu)^{2\nu}$. For $1 < \alpha < 3/2$, we have $\alpha\nu = \alpha/(2-\alpha) > 1$, and it follows from scaling property for $\widehat{R}$ and from Hölder inequality that $((1+\alpha\nu)/(\alpha\nu) = 2/\alpha))$

$$I_1 = \mathbb{P}\Big(\int_0^1 \widehat{R}^{2\alpha\nu}(u) du > (2\nu)^{2\alpha\nu}\, \epsilon^{-(1+\alpha\nu)} \Big) \ge \mathbb{P}\Big(\int_0^1 \widehat{R}^2(u) du > (2\nu)^2\, \epsilon^{-2/\alpha} \Big). \tag{2.18}$$

Since the dimension of the Bessel process $\widehat{R}$ is $2 + 2\nu > 2$, the integral $\int_0^1 \widehat{R}^2(u) du$ is stochastically larger than $\int_0^1 (B^2(u) + \widehat{B}^2(u)) du$, where $\widehat{B}$ is an independent Brownian motion. Notice that $\int_0^1 (B^2(u) + \widehat{B}^2(u)) du \stackrel{\text{law}}{=} \Theta_1 \stackrel{\text{def}}{=} \inf\{ t > 0 : |B(t)| > 1 \}$ (by e.g. comparing their Laplace transforms). It follows that there exists some constant $C_9 > 0$ such that

$$I_1 \ge \mathbb{P}\Big(\Theta_1 > (2\nu)^2\, \epsilon^{-2/\alpha} \Big) \ge C_9 \exp\Big(-2\, (2\nu)^2\, \epsilon^{-2/\alpha} \Big), \qquad 0 < \epsilon < 1. \tag{2.19}$$

by using the well-known fact that $\mathbb{P}(\Theta_1 > x) \asymp \exp(-\pi^2\, x/8)$ as $x \to \infty$. To bound I_2 above, we have

$$I_2 = \mathbb{P}\Big(\sup_{0 \le s \le 1} \widehat{R}(s) \ge \epsilon^{-\nu} \Big) \le C_{10} \exp\Big(-\frac{1}{3}\, \epsilon^{-2\nu} \Big),$$

see e.g. Gruet and Shi [17] for the tail of $\sup_{0 \le s \le 1} \widehat{R}(s)$. The above estimate together with (2.14), (2.17) and (2.19) imply (2.13) for all sufficiently small $\epsilon > 0$, since $2\nu > 2/\alpha$. □

3 Small Deviations

The main results of this section are the following Propositions 3.1 and 3.2:

Proposition 3.1 *As* $\epsilon \to 0$*, we have*

$$\log \mathbb{P}\Big(\sup_{0\le s\le 1} |X_\alpha(s)| < \epsilon \Big) \asymp \begin{cases} -\epsilon^{-2/(2-\alpha)}, & \text{if } -\infty < \alpha \le 1, \\ -\epsilon^{-2/\alpha}, & \text{if } 1 < \alpha < 3/2. \end{cases} \tag{3.1}$$

Proposition 3.2 *For all* $-\infty < \alpha < 3/2$*, we have*

$$\mathbb{P}\Big(\sup_{0\le s\le 1} X_\alpha(s) < \epsilon \Big) \asymp \epsilon^{1/(2(2-\alpha))}, \qquad \epsilon \to 0. \tag{3.2}$$

Recall $\nu \equiv 1/(2-\alpha) \in (0,2)$. Throughout the proofs, we shall constantly use the simple observation that X_α is monotone on each Brownian excursion interval (τ_{s-}, τ_s). We begin with the proof of Proposition 3.1:

Proof of Proposition 3.1 (lower bound). Only small ϵ needs to be considered. Pick up a large $r > 0$ whose value will be determined later. It follows from (2.3) and the self-similarity that

$$\begin{aligned} \mathbb{P}\Big(\sup_{0\le s\le 1} |X_\alpha(s)| < \epsilon \Big) &\ge \mathbb{P}\Big(\sup_{0\le s\le \tau_r} |X_\alpha(s)| < \epsilon\, ;\, \tau_r \ge 1 \Big) \\ &\ge \mathbb{P}\Big(\sup_{0\le s\le \tau_r} |X_\alpha(s)| < \epsilon \Big) - \mathbb{P}\Big(\tau_r < 1 \Big) \\ &= \mathbb{P}\Big(\sup_{0\le t\le 1} |X_\alpha(\tau_t)| < r^{-1/\nu}\epsilon \Big) - \mathbb{P}\Big(|\mathcal{N}| > r \Big) \\ &\ge \exp\Big(-\frac{C_5}{2}\, r\, \epsilon^{-\nu} \Big) - 2\exp\Big(-r^2/2 \Big) \\ &\ge \frac{1}{2}\exp\Big(-C_5^2\, \epsilon^{-2\nu} \Big), \end{aligned} \tag{3.3}$$

by choosing $r = 2C_5\epsilon^{-\nu}$ and using the fact that $\tau_1 \overset{\text{law}}{=} 1/\mathcal{N}^2$ for a standard Gaussian law $\mathcal{N}$. This implies the desired lower bound in the case $\alpha \le 1$.

In the case $1 < \alpha < 3/2$, we prove the lower bound by showing:

$$\liminf_{\epsilon\to 0} \epsilon^{2/\alpha}\, \mathbb{P}\Big(\sup_{0\le s\le 1} |X_\alpha(s)| < \epsilon \Big) \ge \frac{2^{3+2/\alpha}}{(2-\alpha)^2}. \tag{3.4}$$

To this end, recall the definition (2.4) of m. Let p be the Brownian bridge:

$$p(u) \overset{\text{def}}{=} \frac{1}{\sqrt{g_1}} B(ug_1), \qquad 0 \le u \le 1.$$

The Brownian path–decomposition (cf. [29, Exercise (XII.3.8)]) shows that the quadruplet $(p, m, g_1, \text{ sgn } (B(1)))$ are independent. Since the Brownian motion B does not change the sign over $(g_1, 1)$, it follows that

$$\begin{aligned}\sup_{0\le s\le 1}|X_\alpha(s)| &= \max\Big(g_1^{1-\frac{\alpha}{2}}\sup_{0\le u\le 1}|Y(u)|, \\ &\quad |g_1^{1-\frac{\alpha}{2}}Y(1) + \text{ sgn }(B(1))\,(1-g_1)^{1-\frac{\alpha}{2}}\textstyle\int_0^1\frac{du}{m^\alpha(u)}|\Big),\end{aligned} \tag{3.5}$$

where Y denotes the principal value related to the local times of the Brownian bridge p, which can be defined in the same way of (1.1):

$$Y(u) \stackrel{\text{def}}{=} \text{p.v.} \int_0^u \frac{ds}{\widetilde{p}^\alpha(s)} \stackrel{\text{def}}{=} \lim_{\epsilon\to 0}\int_0^u \frac{ds}{|p(s)|^\alpha}\,\text{sgn}(p(s))\,\mathbb{1}_{(|p(s)|\ge\epsilon)}, \qquad 0\le u\le 1.$$

It follows from (3.5) and from the independence between Y, m, g_1 and $\text{sgn}(B(1))$ that

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\le s\le 1}|X_\alpha(s)|<\epsilon\Big) &\ge \mathbb{P}\Big(g_1\le \epsilon^{2/(2-\alpha)};\ \sup_{0\le u\le 1}|Y(u)|<1/2;\ \int_0^1\frac{du}{m^\alpha(u)}<\epsilon/2\Big) \\ &= \mathbb{P}\Big(\sup_{0\le u\le 1}|Y(u)|<1/2\Big)\,\mathbb{P}\Big(g_1\le\epsilon^{2/(2-\alpha)}\Big)\,\mathbb{P}\Big(\int_0^1\frac{du}{m^\alpha(u)}<\epsilon/2\Big).\end{aligned} \tag{3.6}$$

Applying Lemma 2.2 and using Lévy's first arcsine law: $\mathbb{P}(g_1\in dx) = \frac{dx}{\pi\sqrt{x(1-x)}}$ for $0<x<1$, we obtain (3.4), as desired. □

Proof of Proposition 3.1 (upper bound). Let us introduce the ranked Brownian excursion lengths: For $t>0$, denote by

$$V_1(t)\ge V_2(t)\ge ...\ge V_n(t)\ge ... \tag{3.7}$$

the ordered lengths of the countable excursion intervals of B over $[0,t]$ (including the incomplete meander length $t-g_t$). Therefore, $\sum_{n\ge 1}V_n(t)=t$. For detailed studies on excursion lengths, see Pitman and Yor [27, 28] together with their references. Denote by $\{(a_i,b_i), i\ge 1\}$ the corresponding excursions intervals of B over $[0,g_1]$ such that $V_i(g_1)=b_i-a_i, i\ge 1$. Define

$$e_i(u)\stackrel{\text{def}}{=}\frac{1}{\sqrt{V_i(g_1)}}\,|B(a_i+uV_i(g_1))|, \qquad 0\le u\le 1. \tag{3.8}$$

Therefore the processes $\{e_i(\cdot), i\ge 1\}$ are i.i.d., with the common law of a Brownian normalized excursion e defined by (2.5). Furthermore, these normalized excursions $\{e_i, i\ge 1\}$, the meander m defined by (2.4) and the excursion lengths $\{V_i(1), i\ge 1\}$ are mutually independent. Recall $\nu\equiv 1/(2-\alpha)$. By the change of variable, we have that for $i\ge 1$,

$$|X_\alpha(b_i)-X_\alpha(a_i)| = (V_i(g_1))^{1/(2\nu)}\int_0^1\frac{du}{e_i^\alpha(u)}, \tag{3.9}$$

$$|X_\alpha(1)-X_\alpha(g_1)| = (1-g_1)^{1/(2\nu)}\int_0^1\frac{du}{m^\alpha(u)}. \tag{3.10}$$

Now, we distinguish three different cases:

Case $\alpha = 0$: It follows from (3.9) and (3.10) that ($\nu = 1/2$)

$$\mathbb{P}\Big(\sup_{0\le s\le 1} |X_0(s)| \le \epsilon \Big) \le \mathbb{P}\Big(V_1(1) < 2\epsilon \Big). \tag{3.11}$$

Using the Laplace transform of $1/V_1(1)$ (cf. Pitman and Yor [28, Proposition 7 and Corollary 12]), we obtain by applying the analytical continuation that $\mathbb{E}\exp\Big(1/(2V_1(1))\Big) < \infty$ (in fact, $1/V_1(1)$ has a tail of exponential decay, a fact already known for the simple random walk (cf. Csáki et al. [9]). It follows that

$$\mathbb{P}\Big(\sup_{0\le s\le 1} |X_0(s)| \le \epsilon \Big) \le \mathbb{E}\Big(e^{1/(2V_1(1))} \Big)\, e^{-1/\epsilon},$$

yielding the desired upper bound in the case $\alpha = 0$.

Case $\alpha \neq 0$ and $\alpha \le 1$: Pick up a constant $\lambda > 0$ and a large integer $n = n(\lambda, \epsilon)$, whose values will be determined ultimately. Let $E_\lambda \stackrel{\text{def}}{=} \{\sum_{i=1}^n V_i(1) \ge \frac{1}{1+\lambda}\}$, we have

$$\begin{aligned} \mathbb{P}\Big(\sup_{0\le s\le 1} |X_\alpha(s)| \le \epsilon \Big) &\le \mathbb{P}\Big(E_\lambda \cap \{ \sup_{0\le s\le 1} |X_\alpha(s)| \le \epsilon \} \Big) + \mathbb{P}\Big(E_\lambda^c \Big) \\ &\stackrel{\text{def}}{=} I_3 + I_4. \end{aligned} \tag{3.12}$$

Let us bound at first I_3. Using Lemma 2.1 and (3.9)-(3.10) and conditioning on $\{V_i(g_1), i \ge 1\}$, we obtain:

$$\begin{aligned} I_3 &\le \mathbb{P}\Big(E_\lambda \cap \bigcap_{i=1}^n \{ (V_i(g_1))^{\frac{1}{2\nu}} \int_0^1 \frac{du}{e_i^\alpha(u)} < 2\epsilon \} \bigcap \{ (1-g_1)^{\frac{1}{2\nu}} \int_0^1 \frac{du}{m^\alpha(u)} < 2\epsilon \} \Big) \\ &= \mathbb{E}\Big[\prod_{i=1}^n \mathbb{P}\Big(\int_0^1 \frac{du}{e^\alpha(u)} < x_i \Big)\Big|_{x_i = 2\epsilon\, V_i^{-\frac{1}{2\nu}}(g_1)} \mathbb{P}\Big(\int_0^1 \frac{du}{m^\alpha(u)} < y \Big)\Big|_{y = 2\epsilon\, (1-g_1)^{-\frac{1}{2\nu}}} \mathbb{1}_{E_\lambda} \Big] \\ &\le \mathbb{E}\Big[\exp\Big(C_6\,(n+1) - \frac{1}{3} 2^{-2/|\alpha|}\, \epsilon^{-2/|\alpha|}\, \big(\sum_{i=1}^n V_i^{\frac{1}{\nu|\alpha|}}(g_1) + (1-g_1)^{\frac{1}{\nu|\alpha|}} \big) \Big) \mathbb{1}_{E_\lambda} \Big]. \end{aligned} \tag{3.13}$$

Since $\frac{1}{\nu|\alpha|} = (2-\alpha)/|\alpha| \ge 1$, we have from the convexity of the function $x^{\frac{1}{\nu|\alpha|}}$ that

$$\sum_{i=1}^n V_i^{\frac{1}{\nu|\alpha|}}(g_1) + (1-g_1)^{\frac{1}{\nu|\alpha|}} \ge \sum_{i=1}^n V_i^{\frac{1}{\nu|\alpha|}}(1) \ge n^{1-\frac{1}{\nu|\alpha|}} \Big(\sum_{i=1}^n V_i(1) \Big)^{\frac{1}{\nu|\alpha|}}. \tag{3.14}$$

In view of (3.13) and the definition of E_λ, we obtain:

$$I_3 \le \exp\Big(C_6\,(n+1) - \frac{1}{3} 2^{-2/|\alpha|}\, (1+\lambda)^{\frac{-1}{\nu|\alpha|}}\, \epsilon^{-2/|\alpha|}\, n^{1-\frac{1}{\nu|\alpha|}} \Big). \tag{3.15}$$

The probability term I_4 has been estimated by [19, Theorem 4.4]: For fixed $\lambda > 0$, we have

$$\log I_4 = \log \mathbb{P}\Big(\sum_{i=n+1}^{\infty} V_i(1) > \frac{\lambda}{1+\lambda}\Big) \sim -(\log \Xi(\lambda))\, n, \qquad n \to \infty, \tag{3.16}$$

where

$$\Xi(\lambda) \overset{\text{def}}{=} \sup_{a \ge 0} \Big[\Big(1 - \sum_{j=1}^{\infty} \frac{a^j}{j!\,(2j-1)}\Big) \Big/ \Big(\int_1^{\infty} \frac{e^{-\lambda\, a\, x} dx}{2x^{3/2}}\Big)\Big].$$

It is elementary to prove that

$$\log \Xi(\lambda) \ge \lambda/2, \qquad \lambda \ge \lambda_0,$$

for some constant $\lambda_0 > 0$. Therefore, we obtain that for all large $n \ge n_0$,

$$I_4 \le \exp\Big(-\frac{1}{4}\lambda\, n\Big), \qquad \lambda \ge \lambda_0. \tag{3.17}$$

Under the two estimates (3.15) and (3.17), by choosing $\lambda = \max(12C_6, \lambda_0, 1)$ and $n = 2^{-2\nu}\, \lambda^{-1-\nu|\alpha|}\, \epsilon^{-2/(2-\alpha)}$, we establish that

$$I_3 + I_4 \le 2\exp\Big(-\frac{1}{8}\, 2^{-2\nu}\, \lambda^{-\nu|\alpha|}\, \epsilon^{-2/(2-\alpha)}\Big),$$

which in view of (3.12) implies the desired upper bound in the case that $\alpha \neq 0$ and $\alpha \le 1$.

Case $1 < \alpha < 3/2$: We use the same arguments above. The only difference comes from (3.14) and (3.15). In fact, since $\frac{1}{\nu\alpha} = (2-\alpha)/\alpha < 1$, we have

$$\sum_{i=1}^{n} V_i^{\frac{1}{\nu|\alpha|}}(g_1) + (1-g_1)^{\frac{1}{\nu|\alpha|}} \ge \sum_{i=1}^{n} V_i(g_1) + (1-g_1) \ge \sum_{i=1}^{n} V_i(1). \tag{3.18}$$

In view of (3.13), we obtain:

$$I_3 \le \exp\Big(C_6(n+1) - \frac{1}{3(1+\lambda)} 2^{-2/\alpha}\epsilon^{-2/\alpha}\Big). \tag{3.19}$$

Under (3.19) and (3.17), we choose $\lambda = \max(12C_6, \lambda_0, 1)$ and $n = 2^{-2/\alpha}\lambda^{-2}\epsilon^{-2/\alpha}$ and arrive to

$$I_3 + I_4 \le 2\exp\Big(-\frac{1}{8\lambda} 2^{-2/\alpha}\epsilon^{-2/\alpha}\Big).$$

The proof of the upper bound of Proposition 3.1 is complete. □

Proof of Proposition 3.2. Recall $\nu \equiv 1/(2-\alpha) \in (0,2)$. We prove the lower bound by showing the following stronger statement which we shall make use of in the proof of Theorem 1.2:

$$\mathbb{P}\Big(\sup_{0 \le s \le \tau_1} X_\alpha(s) < \epsilon\,; X_\alpha(\tau_1) > -2\,; \tau_1 > 1\Big) \ge C_{11}\epsilon^{\nu/2}, \qquad 0 < \epsilon < 1. \tag{3.20}$$

Let $\hat{B}(t) \overset{\text{def}}{=} B(t+\tau_{1/2})$ for $t \geq 0$ be a Brownian motion independent of $\mathcal{F}_{\tau_{1/2}}$, where $(\mathcal{F}_t, t \geq 0)$ denotes the natural filtration generated by B. Define $\hat{\tau}_r$ and $\hat{X}_\alpha$ relate to $\hat{B}$ the same way τ_r and X_α do to B. Therefore, $X_\alpha(t+\tau_{1/2}) = X_\alpha(\tau_{1/2}) + \hat{X}_\alpha(t)$ for $t > 0$ and $\tau_1 = \tau_{1/2} + \hat{\tau}_{1/2}$. Consider the events

$$\begin{aligned} E_1 &\overset{\text{def}}{=} \Big\{ \sup_{0 \leq s \leq \tau_{1/2}} X_\alpha(s) < \epsilon;\, -2 < X_\alpha(\tau_{1/2}) < -1 \Big\}, \\ E_2 &\overset{\text{def}}{=} \Big\{ \sup_{0 \leq s \leq \hat{\tau}_{1/2}} \hat{X}_\alpha(s) < 1;\, \hat{X}_\alpha(\hat{\tau}_{1/2}) > 0;\, \hat{\tau}_{1/2} > 1 \Big\}. \end{aligned}$$

Observe that $E_1 \cap E_2 \subset \{\sup_{0 \leq s \leq \tau_1} X_\alpha(s) < \epsilon \,; X_\alpha(\tau_1) > -2 \,; \tau_1 > 1\}$, it follows from the independence between E_1 and E_2 that the probability term of (3.20) is bigger than

$$\begin{aligned} \mathbb{P}(E_1 \cap E_2) &= \mathbb{P}(E_1)\mathbb{P}(E_2) \\ &= C_{12} \mathbb{P}\Big(-2 < X_\alpha(\tau_{1/2}) < -1 \mid \sup_{0 \leq u \leq 1/2} X_\alpha(\tau_u) < \epsilon \Big) \mathbb{P}\Big(\sup_{0 \leq u \leq 1/2} X_\alpha(\tau_u) < \epsilon \Big), \end{aligned} \tag{3.21}$$

where $C_{12} = \mathbb{P}(E_2) > 0$, and we have used the fact that $\sup_{0 \leq s \leq \tau_\ell} X(s) = \sup_{0 \leq r \leq \ell} X(\tau_r)$ for $\ell > 0$. Using Fact 2.1 and applying Chaumont's result [5] (see also Bertoin [2, Theorem VIII.18]) to the stable process $r \to -X_\alpha(\tau_r)$ give:

$$\lim_{\epsilon \to 0} \mathbb{P}\Big(-2 < X_\alpha(\tau_{1/2}) < -1 \mid \sup_{0 \leq u \leq 1/2} X_\alpha(\tau_u) < \epsilon \Big) = \mathbb{P}\Big(1 < \mathcal{M}_\nu(1) < 2\Big) > 0, \tag{3.22}$$

where $\mathcal{M}_\nu(\cdot)$ denotes a stable meander of index ν. Assembling (3.22), (2.2) and (3.21), we obtain (3.20) and prove the lower bound part of Proposition 2.2.

To obtain the upper bound, we consider an independent exponentially distributed variable $\mathbf{e}$, with parameter $1/2$. According to the result of Brownian path–decomposition at $g_{\mathbf{e}}$ (see e.g. Yor [35, Proposition 3.2]), we obtain:

$$\begin{aligned} \mathbb{P}\Big(\sup_{0 \leq s \leq g_{\mathbf{e}}} X(s) < \epsilon \Big) &= \int_0^\infty d\ell \; \mathbb{E}\Big(e^{-\frac{1}{2}\tau_\ell} \, \mathbb{1}_{(\sup_{0 \leq s \leq \tau_\ell} X_\alpha(s) < \epsilon)} \Big) \\ &\leq \int_0^\infty d\ell \; \mathbb{E}\Big(e^{-\frac{1}{2}(\tau_\ell - \tau_{\ell/2})} \, \mathbb{1}_{(\sup_{0 \leq s \leq \ell/2} X_\alpha(\tau_s) < \epsilon)} \Big) \\ &= \int_0^\infty d\ell \; \mathbb{E}\Big(e^{-\frac{1}{2}(\tau_\ell - \tau_{\ell/2})} \Big) \mathbb{P}\Big(\sup_{0 \leq s \leq \ell/2} X_\alpha(\tau_s) < \epsilon \Big), \end{aligned} \tag{3.23}$$

where we have used the Markov property at $\tau_{\ell/2}$ to obtain (3.23). It follows from (2.2) and scaling that the above integral of (3.23) is less than

$$C_{13} \int_0^\infty d\ell e^{-\ell/2} \min\left(\epsilon^{\nu/2} \, \ell^{-1/2}, 1\right) \leq C_{14} \, \epsilon^{\nu/2}, \tag{3.24}$$

with some constants $C_{13}, C_{14} > 0$. It follows that

$$\mathbb{P}\Big(\sup_{0 \leq s \leq g_1} X_\alpha(s) < \epsilon \Big) \leq \frac{1}{\mathbb{P}(\mathbf{e} < 1)} \mathbb{P}\Big(\sup_{0 \leq s \leq g_{\mathbf{e}}} X_\alpha(s) < \epsilon \Big) \leq \frac{C_{14}}{1 - e^{-1/2}} \, \epsilon^{\nu/2},$$

yielding the desired upper bound part of Proposition 2.2. □

4 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Let us only prove (1.3), the rest can be shown in the same way. Using the upper bound of Proposition 3.1, it is routine (see e.g. Erdős [13] and Csörgő and Révész [11]) to prove that

$$\liminf_{t\to\infty}\Big(\frac{\log\log t}{t}\Big)^{1-\alpha/2}\sup_{0\le s\le t}|X_\alpha(s)|\;\ge\;C_{15},\qquad\text{a.s.,}\tag{4.1}$$

for some constant $C_{15}=C_{15}(\alpha)>0$. We omit here the details. We shall make use of (3.3) to prove that

$$\liminf_{t\to\infty}\Big(\frac{\log\log t}{t}\Big)^{1-\alpha/2}\sup_{0\le s\le t}|X_\alpha(s)|\;\le\;(C_5)^{2-\alpha},\qquad\text{a.s..}\tag{4.2}$$

To this end, fix $c>(C_5)^{2-\alpha}$ and choose a small δ such that $0<\delta<(c^{2/(2-\alpha)}C_5^{-2}-1)/2$. Define

$$\begin{aligned}
t_n &\stackrel{\text{def}}{=} \exp\Big(n^{1+\delta}\Big),\qquad n\ge 2,\\
d_t &\stackrel{\text{def}}{=} \inf\{s>t: B(s)=0\},\qquad t>0,\\
A_n &\stackrel{\text{def}}{=} \Big\{\sup_{0\le u\le (t_n-d_{t_{n-1}})^+}|X_\alpha(u+d_{t_{n-1}})-X_\alpha(d_{t_{n-1}})|<c\Big(\frac{t_n}{\log\log t_n}\Big)^{1-\alpha/2}\Big\},
\end{aligned}$$

with $y^+\stackrel{\text{def}}{=}\max(y,0)$ for $y\in\mathbb{R}$. Let $(\mathcal{F}_t,t\ge 0)$ be the natural filtration generated by the Brownian motion B. Remark that A_n is $\mathcal{F}_{t_n}$-measurable. Let us prove that

$$\sum_n{}^{\prime}\,\mathbb{P}(A_n\,|\,\mathcal{F}_{t_{n-1}})=\infty,\qquad\text{a.s.,}\tag{4.3}$$

which, according to Lévy's version of Borel-Cantelli's lemma (cf. [30]), yields that

$$\mathbb{P}\Big(A_n\ \text{ is realized infinitely often as } n\to\infty\Big)=1.\tag{4.4}$$

To arrive to (4.3), let $\widehat{B}(u)\stackrel{\text{def}}{=}B(u+d_{t_{n-1}})$ for $u\ge 0$. Then $\widehat{B}$ is a standard Brownian motion, independent of $\mathcal{F}_{d_{t_{n-1}}}$. Define $\widehat{X}_\alpha(\cdot)$ from $\widehat{B}$ the same way $X_\alpha(\cdot)$ does from B. Therefore, $X_\alpha(u+d_{t_{n-1}})-X_\alpha(d_{t_{n-1}})=\widehat{X}_\alpha(u)$ for $u\ge 0$. It follows from (3.3) that

$$\begin{aligned}
\mathbb{P}(A_n\,|\,\mathcal{F}_{t_{n-1}}) &\ge \mathbb{P}\Big(\sup_{0\le u\le t_n}|\widehat{X}_\alpha(u)|<c\Big(\frac{t_n}{\log\log t_n}\Big)^{1-\alpha/2}\,\Big|\,\mathcal{F}_{t_{n-1}}\Big)\\
&= \mathbb{P}\Big(\sup_{0\le u\le t_n}|\widehat{X}_\alpha(u)|<c\Big(\frac{t_n}{\log\log t_n}\Big)^{1-\alpha/2}\Big)\\
&\ge n^{-\frac{1}{(1+\delta/2)}},\qquad\text{for all large } n,
\end{aligned}$$

implying (4.3); hence (4.4) is proven. Now, observe that $d_{t_{n-1}}\stackrel{\text{law}}{=}t_{n-1}(1+\mathcal{C}^2)$, with $\mathcal{C}$ is a symmetric Cauchy variable: $\mathbb{P}(\mathcal{C}\in dx)=\frac{dx}{\pi(1+x^2)}$. It follows that

$$\mathbb{P}\Big(d_{t_{n-1}}>\frac{t_n}{n}\Big)=\mathbb{P}\Big(1+\mathcal{C}^2>\frac{t_n}{nt_{n-1}}\Big)\le e^{-\frac13 n^\delta},\qquad n\ge n_0,$$

for some large n_0. The convergence part of Borel-Cantelli's lemma yields that almost surely for all large n, we have $d_{t_{n-1}} \le \frac{1}{n} t_n$. Applying (1.2) to X and to $-X$, we establish that for all large n,

$$\sup_{0\le s\le d_{t_{n-1}}} |X_\alpha(s)| \le \sup_{0\le s\le \frac{t_n}{n}} |X_\alpha(s)| \le 2C_2(\frac{t_n}{n})^{1-\frac{\alpha}{2}}(\log\log(\frac{t_n}{n}))^{\frac{|\alpha|}{2}} \le \delta\Big(\frac{t_n}{\log\log t_n}\Big)^{1-\frac{\alpha}{2}}.$$

This fact together with (4.4) show that almost surely, there are infinite n such that the event A_n realizes; and on A_n,

$$\begin{aligned}\sup_{0\le s\le t_n} |X_\alpha(s)| &\le \sup_{0\le s\le d_{t_{n-1}}} |X_\alpha(s)| + \sup_{0\le u\le (t_n-d_{t_{n-1}})^+} |X_\alpha(u+d_{t_{n-1}}) - X_\alpha(d_{t_{n-1}})| \\ &\le \le (\delta + c)\Big(\frac{t_n}{\log\log t_n}\Big)^{1-\alpha/2}.\end{aligned}$$

Hence (4.2) is proven by letting $\delta \to 0$ and $c \to (C_5)^{2-\alpha}$. In view of (4.1) and (4.2), (1.3) follows from the usual Kolmogorov's 0-1 law for Brownian motion, with $C_{15} \le C_2 \le (C_5)^{2-\alpha}$. □

Proof of Theorem 1.2. By using scaling and the upper bound of Proposition 3.2, the easy part of Borel-Cantelli's lemma immediately implies the convergence part of Theorem 1.2. We omit the details.

To show the divergence part of Theorem 1.2, we can suppose without any loss of generality that for some large t_0,

$$(\log t)^{1/\nu} \le f(t) \le (\log t)^{4/\nu}, \qquad t \ge t_0, \tag{4.5}$$

see e.g. Erdős [13] for a rigorous justification. Recall $\nu \equiv 1/(2-\alpha)$. Consider large n. Define

$$\begin{aligned} t_n &\overset{\text{def}}{=} 2^n, \\ \hat f(t) &\overset{\text{def}}{=} f(t^2), \\ F_n &\overset{\text{def}}{=} \Big\{ \sup_{0\le u\le \tau_{t_n}} X_\alpha(u) < \frac{t_n^{1/\nu}}{\hat f(t_n)}\,;\, \tau_{t_n} > t_n^2\,;\, X_\alpha(\tau_{t_n}) > -2\,t_n^{1/\nu}\Big\}. \end{aligned}$$

Let us show that

$$\mathbb{P}\Big(F_n \text{ is realized infinitely often as } n \to \infty\Big) > 0. \tag{4.6}$$

Assuming that we have proven (4.6), we deduce from our definition of F_n, t_n and $\hat f(t)$ that $\mathbb{P}\Big(\sup_{0\le s\le t} X_\alpha(s) < t^{1/(2\nu)}/f(t)\,,\ \text{i. o.}\Big) > 0$. Kolmogorov's 0-1 law implies that this probability equals in fact 1 and proves the divergence part of Theorem 1.2.

It remains to show (4.6). to this end, using scaling and Proposition 3.2 that implies:

$$\mathbb{P}(F_n) \asymp \hat f^{-\nu/2}(t_n), \tag{4.7}$$

whose sum on n diverges thanks to the divergence of the integral of (1.7). We shall estimate the second moment $\mathbb{P}(F_i \cap F_j)$ for large $j > i \ge i_0$. Define $\hat B(t) \overset{\text{def}}{=} B(t+\tau_{t_i})$ for $t \ge 0$ be a Brownian motion independent of F_i. Define $\hat X_\alpha$ and $\hat\tau$ from $\hat B$ the same way X_α and τ do from B. Observe that on $F_i \cap F_j$, we have $\sup_{0\le s\le t_j-t_i} \hat{X}_\alpha(\hat{\tau}_s) = \sup_{\tau_{t_i}\le u\le \tau_{t_j}} X_\alpha(u) - X_\alpha(\tau_{t_i}) < \frac{t_j^{1/\nu}}{\hat{f}(t_j)} + 2t_i^{1/\nu}$. This remark together with (2.2) yield:

$$
\begin{aligned}
\mathbb{P}(F_i \cap F_j) &\le \mathbb{P}(F_i)\,\mathbb{P}\Big(\sup_{0\le s\le t_j-t_i} \hat{X}_\alpha(\hat{\tau}_s) < \frac{t_j^{1/\nu}}{\hat{f}(t_j)} + 2t_i^{1/\nu}\Big) \\
&\le C_{16}\,\mathbb{P}(F_i)\,\Big((\frac{t_j}{t_j-t_i})^{1/\nu} \hat{f}^{-1}(t_j) + 2(\frac{t_i}{t_j-t_i})^{1/\nu} \Big)^{\nu/2}, \qquad (4.8)
\end{aligned}
$$

for some constant $C_{16} = C_{16}(\nu) > 0$. From (4.7) and (4.8), it is elementary to show there exists some constant $C_{17} > 0$ such that

$$
\liminf_{n\to\infty} \frac{\sum_{1\le i,j\le n} \mathbb{P}(F_i \cap F_j)}{\Big(\sum_{1\le i\le n} \mathbb{P}(F_i)\Big)^2} \le C_{17} < \infty,
$$

which according to Kochen and Stone's version of Borel-Cantelli's lemma (see [23]) yields (4.6) and completes the whole proof. $\square$

5 Some Generalizations

Notice that the main ingredients to obtain Theorems 1.1 and 1.2 are the self-similarity and the fact that the process X_α is monotone over each excursion interval (τ_{r-}, τ_r), it turns out that the same method can be applied to the processes of the following type:

$$
Z(t) \stackrel{\text{def}}{=} \int_0^t \frac{ds}{|B(s)|^\alpha} \,(a\mathbb{1}_{(B(s)>0)} - b\mathbb{1}_{(B(s)<0)}), \qquad t>0, \qquad (5.1)
$$

where $a, b > 0$ are two constants and we restrict our intention to the case $-\infty < \alpha < 1$. Then, the process $Z(\tau_\cdot)$ is again a stable process of index $\nu = 1/(2-\alpha)$. Put

$$
\rho \stackrel{\text{def}}{=} \mathbb{P}\Big(Z(\tau_1) \ge 0\Big) = \mathbb{P}\Big(\Lambda_\nu / \hat{\Lambda}_\nu < a/b\Big), \qquad (5.2)
$$

where $\Lambda_\nu > 0$ denotes a stable variable of index ν, $\hat{\Lambda}_\nu$ is an independent copy of Λ_ν, and we have used the Ray-Knight theorem to obtain (5.2) (for details, see Pitman and Yor [26, pp.435]). The density function of the ratio $\Lambda_\nu/\hat{\Lambda}_\nu$ is explicitly known (I learn it from M. Yor, see Lamperti [25, (3.17)] for an equivalent statement; see also Barlow, Pitman and Yor: "Une extension multidimensionnelle de la loi de l'arc sinus" (1989) *Séminaire de Probabilités XXIII*):

$$
\rho = \mathbb{P}\Big((\Lambda_\nu/\hat{\Lambda}_\nu)^\nu < (a/b)^\nu\Big) = \frac{\sin(\pi\nu)}{\pi\nu} \int_0^{(a/b)^\nu} \frac{dx}{1 + 2x\cos(\pi\nu) + x^2}. \qquad (5.3)
$$

From these, we can make use of the same method together the known estimates for stable processes (see Bertoin [2, Chap. VIII]) to obtain the following results. The proofs are omitted.

Theorem 5.1 *Fix $-\infty < \alpha < 1$. There exists some constant $0 < C_{18}(\alpha, a, b) < \infty$ such that*

$$\liminf_{t\to\infty} \Big(\frac{\log\log t}{t}\Big)^{1-\alpha/2} \sup_{0\le s\le t} |Z(s)| = C_{18}, \qquad \text{a.s.} \tag{5.4}$$

Theorem 5.2 *Fix $-\infty < \alpha < 1$. Let $f(t) > 0$ be a nondecreasing function, we have*

$$\mathbb{P}\Big(\sup_{0\le s\le t} Z(s) < \frac{t^{1-\alpha/2}}{f(t)};\ i.o.\ \Big) = \begin{cases} 0 \\ 1 \end{cases} \iff \int^{\infty} \frac{dt}{t(f(t))^{\rho/(2-\alpha)}} \begin{cases} = \infty \\ < \infty \end{cases}, \tag{5.5}$$

where $0 < \rho < 1$ is given by (5.3).

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