Electronic Journal of Probability

Electron. J. Probab. **20** (2015), no. 112, 1–30.
ISSN: 1083-6489 DOI: 10.1214/EJP.v20-4190

# Local central limit theorem for diffusions in a degenerate and unbounded random medium

Alberto Chiarini* Jean-Dominique Deuschel*

**Abstract**

We study a symmetric diffusion $X$ on $\mathbb{R}^d$ in divergence form in a stationary and ergodic environment, with measurable unbounded and degenerate coefficients. We prove a quenched local central limit theorem for $X$, under some moment conditions on the environment; the key tool is a local parabolic Harnack inequality obtained with Moser iteration technique.

**Keywords:** local central limit theorem; Harnack inequality; Moser iteration; diffusions in random environment.
**AMS MSC 2010:** 31B05; 60K37.
Submitted to EJP on March 17, 2015, final version accepted on October 23, 2015.
Supersedes arXiv:1501.03476.

## 1 Description of the main result

We model the stationary and ergodic random environment by a probability space $(\Omega, \mathcal{G}, \mu)$, on which we define a measure-preserving group of transformations $\tau_x : \Omega \to \Omega$, $x \in \mathbb{R}^d$. One can think about $\tau_x\omega$ as a translation of the environment $\omega \in \Omega$ in direction $x \in \mathbb{R}^d$. The function $(x, \omega) \to \tau_x\omega$ is assumed to be $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}$-measurable, being $\mathcal{B}(\mathbb{R}^d)$ the usual Borel $\sigma$-algebra on $\mathbb{R}^d$, and ergodic, namely if $\tau_x A = A$ for all $x \in \mathbb{R}^d$, then $\mu(A) \in \{0, 1\}$. Given the random environment $(\Omega, \mathcal{G}, \mu, \{\tau_x\}_{x\in\mathbb{R}^d})$ we can construct a stationary and ergodic random field simply by taking a random variable $f : \Omega \to \mathbb{R}$ and defining $f^\omega(x) := f(\tau_x\omega)$, $x \in \mathbb{R}^d$.

We are given a $\mathcal{G}$-measurable function $a : \Omega \to \mathbb{R}^{d\times d}$ with values in the set of symmetric matrices such that

$(a.1)$ there exist $\mathcal{G}$-measurable non-negative functions $\lambda, \Lambda : \Omega \to \mathbb{R}$ such that for $\mu$-almost all $\omega \in \Omega$ and all $\xi \in \mathbb{R}^d$

$$\lambda(\omega)|\xi|^2 \le \langle a(\omega)\xi, \xi\rangle \le \Lambda(\omega)|\xi|^2,$$

$(a.2)$ there exist $p, q \in [1, \infty]$ satisfying $1/p + 1/q < 2/d$ such that

$$\mathbb{E}_\mu[\Lambda^p] < \infty, \quad \mathbb{E}_\mu[\lambda^{-q}] < \infty.$$

*Institut für Mathematik, Technische Universität Berlin, Straße des 17. Juni 136, 10623 Berlin, Germany.
E-mail: chiarini,deuschel@math.tu-berlin.de

Our diffusion process is formally associated with the following generator in divergence form

$$L^\omega u(x) = \frac{1}{\Lambda^\omega(x)}\nabla\cdot(a^\omega(x)\nabla u(x)). \tag{1.1}$$

Since $a^\omega(x)$ is modeling a random field, it is not natural to assume its differentiability in $x\in\mathbb{R}^d$. Therefore, the operator defined in (1.1) does not make sense and the standard techniques from the stochastic differential equations theory or Itô calculus are not helpful either in the construction of the diffusion process or in performing the relevant computations.

We will exploit Dirichlet forms theory to construct the diffusion process formally associated with (1.1). Instead of the operator $L^\omega$ we consider the bilinear form obtained by $L^\omega$ formally integrating by parts,

$$\mathcal{E}^\omega(u,v) = \sum_{i,j}\int_{\mathbb{R}^d} a^\omega_{ij}(x)\partial_i u(x)\partial_j u(x)dx \tag{1.2}$$

for a proper class of functions $u,v\in\mathcal{F}^{\Lambda,\omega}\subset L^2(\mathbb{R}^d,\Lambda^\omega dx)$, more precisely $\mathcal{F}^{\Lambda,\omega}$ is the completion of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d,\Lambda^\omega dx)$ with respect to $\mathcal{E}^\omega+(\cdot,\cdot)_\Lambda$. It is a classical result of Fukushima [18, Theorem 7.2.2] that it is possible to associate a diffusion process $(X^\omega,\mathbb{P}^\omega_x)$ to (1.2) as soon as $(\lambda^\omega)^{-1}$ and $\Lambda^\omega$ are locally integrable. As a drawback, the process cannot in general start from every $x\in\mathbb{R}^d$ but only from almost all, and the set of exceptional points may depend on the realization of the environment.

In [9] we show that under $(a.1)$, $(a.2)$ and if $\lambda^\omega(\cdot)^{-1},\Lambda^\omega(\cdot)\in L^\infty_{loc}(\mathbb{R}^d)$ for $\mu$-almost all $\omega\in\Omega$, then a quenched invariance principle holds for $X^\omega$, namely the scaled process $X^{\epsilon,\omega}_\cdot := \epsilon X^\omega_{\cdot/\epsilon^2}$ converges in distribution under $\mathbb{P}^\omega_0$ to a Brownian motion with a non-trivial deterministic covariance structure as $\epsilon\to 0$. In that work local boundness was assumed in order to get some regularity for the density of the process $X^\omega$ and avoid technicalities due to exceptional sets arising from Dirichlet forms theory.

In this paper we show that if a quenched invariance principle holds, then under $(a.1)$ and $(a.2)$, the density of $X^{\epsilon,\omega}$ converges uniformly on compacts to the gaussian density. Hence, to state the theorem we need the following assumption.

$(a.3)$ Assume that there exists a positive definite symmetric $d$-dimensional matrix $\Sigma$ such that for $\mu$-almost all $\omega\in\Omega$ we have that for almost all $o\in\mathbb{R}^d$, all balls $B\subset\mathbb{R}^d$ and all compact intervals $I\subset(0,\infty)$

$$\lim_{\epsilon\to 0}\mathbb{P}^\omega_o(\epsilon X^\omega_{t/\epsilon^2}\in B) = \frac{1}{\sqrt{(2\pi t)^d\det\Sigma}}\int_B \exp\Big(-\frac{x\cdot\Sigma^{-1}x}{2t}\Big)\,dx$$

uniformly in $t\in I$.

**Remark 1.1.** If $\lambda^\omega(\cdot)^{-1},\Lambda^\omega(\cdot)\in L^\infty_{loc}(\mathbb{R}^d)$ for $\mu$-almost all $\omega\in\Omega$, then assumption $(a.3)$ is satisfied for all $o\in\mathbb{R}^d$, $\mu$-almost surely as a consequence of [9, Theorem 1.1].

Set

$$k^\Sigma_t(x) := \frac{1}{\sqrt{(2\pi t)^d\det\Sigma}}\exp\Big(-\frac{x\cdot\Sigma^{-1}x}{2t}\Big).$$

**Theorem 1.2.** *Let $d\geq 2$. Assume $(a.1)$, $(a.2)$ and $(a.3)$. Let $p^\omega_t(\cdot,\cdot)$ be the density with respect to $\Lambda^\omega(x)dx$ of the semigroup $P^\omega_t$ associated to $(\mathcal{E}^\omega,\mathcal{F}^{\Lambda,\omega})$ on $L^2(\mathbb{R}^d,\Lambda^\omega dx)$. Let $r>0$ and $I\subset(0,\infty)$ be compact. Then, for $\mu$-almost all $\omega\in\Omega$ we have that for almost all $o\in\mathbb{R}^d$*

$$\lim_{\epsilon\to 0}\sup_{|x-o|\leq r}\sup_{t\in I}\Big|\epsilon^{-d}p^\omega_{t/\epsilon^2}(o,x/\epsilon) - \mathbb{E}_\mu[\Lambda]^{-1}k^\Sigma_t(x)\Big| = 0. \tag{1.3}$$

*If we further assume that $\lambda^\omega(\cdot)^{-1},\Lambda^\omega(\cdot)\in L^\infty_{loc}(\mathbb{R}^d)$ for $\mu$-almost all $\omega\in\Omega$, then (1.3) is satisfied for all $o\in\mathbb{R}^d$.*

**The method.** The proof of Theorem 1.2 relies on a priori bounds for solutions to the “formal” parabolic equation

$$\partial_t u(t,x) - \frac{1}{\Lambda^\omega(x)} \nabla \cdot (a^\omega(x) \nabla u(t,x)) = 0, \qquad t \in (0,\infty),\ x \in \mathbb{R}^d. \tag{1.4}$$

It is well known that when $x \mapsto a^\omega(x)$ and $x \mapsto \Lambda^\omega(x)$ are bounded and bounded away from zero, uniformly in $\omega \in \Omega$, then a parabolic Harnack's inequality holds for solutions to (1.4), this is a celebrated result due to Moser [24]. He showed that there is a positive constant $C_{PH}$, which depends only on the uniform bounds on $a$ and $\Lambda$, such that for any positive weak solution of (1.4) in $(t, t+r^2) \times B(x,r)$ we have

$$\sup_{(s,z)\in Q_-} u(s,z) \le C_{PH} \inf_{(s,z)\in Q_+} u(s,z),$$

where $Q_- = (t + 1/4r^2, t + 1/2r^2) \times B(x, r/2)$ and $Q_+ = (t + 3/4r^2, t + r^2) \times B(x, r/2)$. The parabolic Harnack inequality plays a prominent role in the theory of partial differential equations, in particular to prove Hölder continuity for solutions to parabolic equations, as it was observed by Nash [26] and De Giorgi [12], or to prove Gaussian bounds for the fundamental solution $p_t^\omega(x,y)$ of (1.4) as done by Aronson [4]. It is remarkable that such results do not depend either on the regularity of $a$ or of $\Lambda$.

In this paper we exploit the robustness of Moser's method to derive a parabolic Harnack inequality also in the case of degenerate and possibly unbounded coefficients. Many authors successfully applied this technique to obtain a priori bounds. In the field of diffusion in random environments we mention [15, 16], for discrete space models for which we refer to [2, 3].

Moser's method is based on two steps. One wants first to get a Sobolev inequality to control some $L^\rho$-norm in terms of the Dirichlet form and then control the Dirichlet form of any caloric function by a lower moment. In the uniform elliptic case this is rather standard and it is possible to control the $L^{2d/(d-2)}$-norm by the $L^2$-norm. In our case the coefficients are neither bounded from above nor from below and we need to work with a weighted Sobolev inequality, which was already established in [9] by means of Hölder's inequality. Doing so we are able to control, locally on balls, the $L^\rho$-norm by means of the $L^{2p/(p-1)}$-norm, with $\rho = 2qd/(q(d-2)+d)$. In order to start the iteration we need $\rho > 2p/(p-1)$ which is equivalent to $1/p + 1/q < 2/d$. As a result we are able to bound the $L^\infty$-norm of a caloric function $u$ by its $L^\alpha$-norm for some finite $\alpha > 0$, on a slightly larger ball. Since the same holds for $u^{-1}$, what is left to do is to link the $L^\alpha$-norm of $u$ and the $L^\alpha$-norm of $u^{-1}$. In the uniformly elliptic case this is achieved by means of the exponential integrability of BMO functions, hence with John-Nirenberg inequality. In the present work we exploit an abstract lemma due to Bombieri and Giusti [8] (see Lemma C.1 below) for which application, in addition to the maximum inequality for $u$, we will need to establish weighted Poincaré inequalities.

The integrability assumption with exponents satisfying $1/p+1/q < 2/d$ firstly appeared in [14] to extend De Giorgi and Nash's results to degenerate elliptic equations, however the authors focus on weights belonging to the Muckenhaupt's class. A similar condition was also recently exploited in [29] to obtain a Gaussian upper bound for solutions to degenerate parabolic equations, the same type of conclusion holds also in the discrete setting, for which we refer to [1].

Following the proof of Moser, with some extra care due to the different exponents, we get a local parabolic Harnack inequality for solutions to (1.4) in our setting. In the uniformly elliptic and bounded case the constant in front of the Harnack inequality depends only on the uniform bounds on $a$ and $\Lambda$. Under our assumptions we cannot expect that to be true for general weights, and the constant will possibly depend on the center and the radius of the ball. In particular, we don't have any control of the constant for small balls, so that a genuine Hölder's continuity result like the one of Nash is not given. Luckily, in the diffusive limit, the ergodic theorem helps to control constants and to prove Theorem 1.2.

**Remark 1.3.** Let us consider $\theta : \Omega \to (0, +\infty)$ and $\theta^\omega(x) := \theta(\tau_x\omega)$ such that $\theta^\omega$ and $1/\theta^\omega$ are in $L^1_{loc}(\mathbb{R}^d)$ almost surely. One can then consider the Dirichlet form $(\mathcal{E}^\omega, \mathcal{F}^{\theta,\omega})$ on $L^2(\mathbb{R}^d, \theta^\omega dx)$ where $\mathcal{E}^\omega$ is given by (1.2) and $\mathcal{F}^{\theta,\omega}$ is the closure of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, \theta^\omega dx)$ with respect to $\mathcal{E}^\omega + (\cdot,\cdot)_\theta$. This corresponds to the formal generator

$$L^\omega u(x) = \frac{1}{\theta^\omega(x)} \nabla \cdot (a^\omega(x) \nabla u(x)). \tag{1.5}$$

One can show along the same lines of the proof for $\theta = \Lambda$ that if

$$\mathbb{E}_\mu[\theta^r] < \infty,\ \mathbb{E}_\mu[\lambda^{-q}] < \infty,\ \mathbb{E}_\mu[\Lambda^p \theta^{1-p}] < \infty,$$

where $p, q, r \in (1, \infty]$ are such that

$$\frac{1}{q} + \frac{1}{r} + \frac{r-1}{r} \cdot \frac{1}{p} < \frac{2}{d},$$

then the parabolic Harnack inequality still works, in particular a quenched local central limit theorem can still be derived in this situation for the density with respect to $\theta^\omega(x)dx$ of the semigroup $P_t^{\theta,\omega}$ associated to $(\mathcal{E}^\omega, \mathcal{F}^{\theta,\omega})$ on $L^2(\mathbb{R}^d, \theta^\omega dx)$ (see Appendix A for more details).

Observe that in the case $\theta = \Lambda$ we recover the familiar condition $1/r + 1/q < 2/d$ and for $\theta \equiv 1$, since $r = \infty$, the condition reads again $1/p + 1/q < 2/d$. This is an improvement of the condition previously given for the variable speed conductance model in [2], the idea for the proof was given to us by Martin Slowik.

**Remark 1.4.** One particular example which arises from the general form (1.5) is the one with $\theta \equiv e^{V^\omega(x)}$ and $a^\omega(x) = e^{V^\omega(x)} I_d$, being $I_d$ the $d$-dimensional identity matrix. The generator corresponding to this choice reads

$$L^\omega u(x) = e^{-V^\omega(x)} \nabla \cdot (e^{V^\omega(x)} \nabla u(x)).$$

In [5], the authors proved an “individual” invariance principle in the case that $V$ is a deterministic periodic function and $e^V, e^{-V}$ are integrable on the $d$-dimensional torus. This corresponds to $(a.2)$ with $p = q = 1$, the problem of showing invariance principle for general random environment under this condition remains open.

**Remark 1.5.** The condition $1/p + 1/q < 2/d$ is morally optimal to state Theorem 1.2. Indeed in the discrete space setting it was shown that if $1/p + 1/q > 2/d$, then there is an ergodic environment for which the quenched local central limit theorem does not hold [2, see Theorem 5.4]. One could possibly construct a counterexample also in the continuous by exploiting the same ideas given in [2].

A summary of the paper is the following.

In Section 2 we present a deterministic model obtained by looking at a fixed realization of the environment. We derive Sobolev, Poincaré and Nash inequalities for such a model.

In Section 3 we prove a priori estimates, on-diagonal bounds and Hölder continuity type estimates for caloric functions. The main result of this section is a local parabolic Harnack inequality.

In Section 4 we prove a local Central Limit Theorem for the deterministic model which we apply to derive Theorem 1.2.

## 2 Deterministic model and local inequalities

Since we want to prove a quenched result we will develop a collection of inequalities for a deterministic model, we will be fixing a particular environment $\omega \in \Omega$. With a slight abuse of notation we will note with $a(x)$, $\lambda(x)$ and $\Lambda(x)$ the deterministic versions of $a(\tau_x\omega)$, $\lambda(\tau_x\omega)$ and $\Lambda(\tau_x\omega)$.

We are given a symmetric matrix $a : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ such that

$(b.1)$ there exist $\lambda, \Lambda : \mathbb{R}^d \to \mathbb{R}$ non-negative functions such that for almost all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$

$$\lambda(x)|\xi|^2 \le \langle a(x)\xi, \xi\rangle \le \Lambda(x)|\xi|^2,$$

$(b.2)$ there exist $p, q \in [1, \infty]$ satisfying $1/p + 1/q < 2/d$ such that

$$\limsup_{r\to\infty} \frac{1}{|B(0,r)|} \int_{B(0,r)} \Lambda^p + \lambda^{-q}\, dx < \infty.$$

Assumption $(b.2)$ plays the role of ergodicity in the random environment model.

We are interested in finding a priori estimates for solutions to the formal parabolic equation

$$\partial_t u(t,x) - \frac{1}{\Lambda(x)} \nabla \cdot (a(x)\nabla u(t,x)) = 0, \tag{2.1}$$

for $t \in (0,\infty)$ and $x \in \mathbb{R}^d$.

Clearly, in the way it is stated (2.1) is not well defined because we only assume $a(\cdot)$ to be measurable. In order to make sense of (2.1) we shall exploit the Dirichlet form framework, see [18] for an exhaustive treatment of the subject.

### 2.1 Caloric functions

For this section we follow closely [6]. Let $\theta : \mathbb{R}^d \to \mathbb{R}$ be a non-negative function such that $\theta^{-1}, \theta$ are locally integrable on $\mathbb{R}^d$. Consider the symmetric form $\mathcal{E}$ on $L^2(\mathbb{R}^d, \theta dx)$ with domain $C_0^\infty(\mathbb{R}^d)$ defined by

$$\mathcal{E}(u,v) := \sum_{i,j} \int_{\mathbb{R}^d} a_{ij}(x)\partial_i u(x)\partial_j v(x)\, dx. \tag{2.2}$$

Then, $(\mathcal{E}, C_0^\infty(\mathbb{R}^d))$ is *closable* in $L^2(\mathbb{R}^d, \theta dx)$ thanks to [27, see Chapter II, example 3b], since $\lambda^{-1}, \Lambda \in L^1_{loc}(\mathbb{R}^d)$ by $(b.2)$. We shall denote by $(\mathcal{E}, \mathcal{F}^\theta)$ its closure; it is clear that $\mathcal{F}^\theta$ is the completion of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d, \theta dx)$ with respect to $\mathcal{E}_1 := \mathcal{E} + (\cdot,\cdot)_\theta$. Observe that $(\mathcal{E}, \mathcal{F}^\theta)$ is a strongly local regular Dirichlet form, having $C_0^\infty(\mathbb{R}^d)$ as a core. In the case that $\theta \equiv 1$ we will simply write $\mathcal{F}$. Given an open subset $G$ of $\mathbb{R}^d$ we will denote by $\mathcal{F}^\theta_G$ the completion of $C_0^\infty(G)$ in $L^2(G, \theta dx)$ with respect to $\mathcal{E}_1$.

**Definition 2.1** (Caloric functions)**.** *Let $I \subset \mathbb{R}$ and $G \subset \mathbb{R}^d$ be an open set. We say that a function $u : I \to \mathcal{F}^\theta$ is a subcaloric (supercaloric) function in $I \times G$ if $t \mapsto (u(t,\cdot),\phi)_\theta$ is differentiable in $t \in I$ for any $\phi \in L^2(G, \theta dx)$ and*

$$\frac{d}{dt}(u,\phi)_\theta + \mathcal{E}(u,\phi) \le 0, \quad (\ge) \tag{2.3}$$

*for all non-negative $\phi \in \mathcal{F}^\theta_G$. We say that a function $u : I \to \mathcal{F}^\theta$ is a caloric function in $I \times G$ if it is both sub- and supercaloric.*

It is clear from the definition that if a function is subcaloric on $I \times G$, then it is subcaloric on $I' \times G'$ whenever $I' \subset I$ and $G' \subset G$.

Moreover, observe that if $P_t^G$ is the semigroup associated to $(\mathcal{E}, \mathcal{F}^\theta)$ on $L^2(G, \theta dx)$ and $f \in L^2(G, \theta dx)$, for a given open set $G \subset \mathbb{R}^d$, then the function $u(t, \cdot) = P_t^G f(\cdot)$ is caloric on $(0, \infty) \times G$. To complete the picture we state the following maximum principle which appeared in [20]. For a real number $x$ denote by $x_+ = x \vee 0$.

**Lemma 2.2.** *Fix $T \in (0, \infty]$, a set $G \subset \mathbb{R}^d$ and let $u : (0, T) \to \mathcal{F}_G^\theta$ be a subcaloric function in $(0, T) \times G$ which satisfies the boundary condition $u_+(t, \cdot) \in \mathcal{F}_G^\theta$, $\forall t \in (0, T)$ and $u_+(t, \cdot) \to 0$ in $L^2(G, \theta dx)$ as $t \to 0$. Then $u \le 0$ on $(0, T) \times G$.*

As a corollary of this lemma we have a super mean value inequality for subcaloric functions.

**Corollary 2.3.** *Fix $T \in (0, \infty]$, an open set $G \subset \mathbb{R}^d$ and $f \in L^2(G, \theta dx)$ non-negative. Let $u : (0, T) \to \mathcal{F}_G^\theta$ be a non-negative subcaloric function on $(0, T) \times G$ such that $u(t, \cdot) \to f$ in $L^2(G, \theta dx)$ as $t \to 0$. Then for any $t \in (0, T)$*

$$u(t, \cdot) \ge P_t^G f, \text{ in } G.$$

*In particular for $0 < s < t < T$*

$$u(t, \cdot) \ge P_{t-s}^G u(s, \cdot), \text{ in } G.$$

## 2.2 Sobolev inequalities

In this section we will state local inequalities on the flat space $L^2(\mathbb{R}^d, dx)$ and on the weighted space $L^2(\mathbb{R}^d, \Lambda dx)$. We are interested in Sobolev, Poincaré and Nash type inequalities. The first and the second provide an effective tool for deriving local bounds of solutions to elliptic and parabolic degenerate partial differential equations, while the latter will be used to prove the existence of a kernel for the semigroup $P_t$ associated to $(\mathcal{E}, \mathcal{F}^\Lambda)$ on $L^2(R^d, \Lambda dx)$.

We shall see that the constants appearing in the inequalities are strongly dependent on averages of $\lambda$ and $\Lambda$ and in particular on the ball on which we focus our analysis.

It will be clear in Proposition 2.4 below that the case $d = 2$ and $q = \infty$ is special since the classical Sobolev inequality cannot be applied. Nevertheless it is always possible to find $q' < \infty$ such that $\lambda$ and $\Lambda$ satisfy $(b.1)$, $(b.2)$ with $q'$ replacing $q$. Therefore, for the rest of this article the case $d = 2$ and $q = \infty$ will be excluded.

**Notation.** Let $B \subset \mathbb{R}^d$ be a bounded set. For a function $u : B \to \mathbb{R}$, $s \ge 1$ and a weight $\theta : B \to \mathbb{R}$ we note

$$\|u\|_{s,\theta} := \left( \int_{\mathbb{R}^d} |u(x)|^s \theta(x) dx \right)^{\frac{1}{s}}, \quad \|u\|_{s,B} := \left( \frac{1}{|B|} \int_B |u(x)|^s \, dx \right)^{\frac{1}{s}}.$$

and

$$\|u\|_{s,B,\theta} := \left( \frac{1}{|B|} \int_B |u(x)|^s \, \theta(x) dx \right)^{\frac{1}{s}}.$$

In the sequel we shall use the symbol $\lesssim$ to say that the inequality $\le$ holds up to a multiplicative constant depending only on $p$, $q$ and the dimension $d \ge 2$.

In the next proposition it is enough to assume $\Lambda \in L^1_{loc}(\mathbb{R}^d)$ and $\lambda^{-1} \in L^q_{loc}(\mathbb{R}^d)$.

**Proposition 2.4** (Local Sobolev inequality)**.** *Fix a ball $B \subset \mathbb{R}^d$. Then for all $u \in \mathcal{F}_B$*

$$\|u\|_{\rho,B}^2 \lesssim C_S^B |B|^{\frac{2}{d}} \, \frac{\mathcal{E}(u,u)}{|B|}, \tag{2.4}$$

*where $C_S^B := \|\lambda^{-1}\|_{q,B}$, and*

$$\rho(q, d) := \frac{2qd}{q(d-2) + d}, \tag{2.5}$$

*is the Sobolev conjugate of* $2q/(q+1)$.

*Proof.* We start proving (2.4) for $u \in C_0^\infty(B)$. Since $\rho$ as defined in (2.5) is the Sobolev conjugate of $2q/(q+1)$, by the classical Sobolev's inequality [28, Theorem 1.5.2]

$$\|u\|_\rho \lesssim \|\nabla u\|_{2q/(q+1)},$$

where it is clear that we are integrating over $B$. By Hölder's inequality and $(b.1)$ we can estimate the right hand side as follows

$$\|\nabla u\|_{2q/(q+1)}^2 = \Big( \int_B |\nabla u|^{\frac{2q}{q+1}} \lambda^{\frac{q}{q+1}} \lambda^{-\frac{q}{q+1}} \, dx \Big)^{\frac{q+1}{q}} \leq \|1_B \lambda^{-1}\|_q \, \mathcal{E}(u,u),$$

which leads to (2.4) for $u \in C_0^\infty(B)$, after averaging over the ball $B$. By approximation, the inequality is easily extended to $u \in \mathcal{F}_B$. □

**Proposition 2.5** (Local weighted Sobolev inequality)**.** *Fix a ball* $B \subset \mathbb{R}^d$. *Then for all* $u \in \mathcal{F}_B^\Lambda$

$$\|u\|_{\rho/p^*,B,\Lambda}^2 \lesssim C_S^{B,\Lambda} |B|^{\frac{2}{d}} \, \frac{\mathcal{E}(u,u)}{|B|}, \tag{2.6}$$

*being* $C_S^{B,\Lambda} := \|\lambda^{-1}\|_{q,B} \|\Lambda\|_{p,B}^{2p^*/\rho}$ *and* $p^* = p/(p-1)$.

*Proof.* The proof readily follows from Hölder's inequality

$$\|u\|_{\rho/p^*,B,\Lambda}^2 \leq \|u\|_{\rho,B}^2 \|\Lambda\|_{p,B}^{2p^*/\rho}$$

and the previous proposition. □

**Remark 2.6.** From these two Sobolev's inequalities it follows that the domains $\mathcal{F}_B$ and $\mathcal{F}_B^\Lambda$ coincide for all balls $B \subset \mathbb{R}^d$. Indeed, from (2.4) and (2.6), since $\rho, \rho/p^* > 2$, $(\mathcal{F}_B, \mathcal{E})$ and $(\mathcal{F}_B^\Lambda, \mathcal{E})$ are two Hilbert spaces; therefore $\mathcal{F}_B, \mathcal{F}_B^\Lambda$ coincide with their extended Dirichlet space, which by [17, page 324], is the same, hence $\mathcal{F}_B = \mathcal{F}_B^\Lambda$.

**Inequalities with cutoffs.** Since assumptions $(b.1)$ and $(b.2)$ only assure local integrability of $\lambda^{-1}$ and $\Lambda$, we will need to work with functions that are locally in $\mathcal{F}$ or $\mathcal{F}^\Lambda$ and with cutoff functions. We say that $u \in \mathcal{F}_{loc}^\theta$, if for all balls $B \subset \mathbb{R}^d$ there exists $u_B \in \mathcal{F}^\theta$ such that $u \equiv u_B$ almost surely on $B$. It clearly follows from the previous remark that $\mathcal{F}_{loc}^\Lambda = \mathcal{F}_{loc}$ whenever the condition $(b.2)$ is satisfied.

Let $B \subset \mathbb{R}^d$ be a ball, a cutoff on $B$ is a function $\eta \in C_0^\infty(B)$, such that $0 \leq \eta \leq 1$. For $u, v \in \mathcal{F}_{loc}$ we define the bilinear form

$$\mathcal{E}_\eta(u,v) = \sum_{i,j} \int_{\mathbb{R}^d} a_{ij}(x) \partial_i u(x) \partial_j v(x) \eta^2(x) \, dx. \tag{2.7}$$

**Proposition 2.7** (Local Sobolev inequality with cutoff)**.** *Fix a ball* $B \subset \mathbb{R}^d$ *and a cutoff function* $\eta \in C_0^\infty(B)$ *as above. Then for all* $u \in \mathcal{F}_{loc}^\Lambda$

$$\|\eta u\|_{\rho,B}^2 \lesssim C_S^B |B|^{\frac{2}{d}} \Big[ \frac{\mathcal{E}_\eta(u,u)}{|B|} + \|\nabla \eta\|_\infty^2 \|u\|_{2,B,\Lambda}^2 \Big] \tag{2.8}$$

*and*

$$\|\eta u\|_{\rho/p^*,B,\Lambda}^2 \lesssim C_S^{B,\Lambda} |B|^{\frac{2}{d}} \Big[ \frac{\mathcal{E}_\eta(u,u)}{|B|} + \|\nabla \eta\|_\infty^2 \|u\|_{2,B,\Lambda}^2 \Big]. \tag{2.9}$$

*Proof.* We prove only (2.8), (2.9) being analogous. Take $u \in \mathcal{F}_{loc}$, by Lemma B.1 in the appendix, $\eta u \in \mathcal{F}_B$, therefore we can apply (2.4) and get

$$\|\eta u\|_{\rho,B}^2 \lesssim C_S^B |B|^{\frac{2-d}{d}} \mathcal{E}(\eta u, \eta u).$$

To get (2.8) we compute $\nabla(\eta u) = u\nabla\eta + \eta\nabla u$ and we easily estimate

$$\begin{aligned}\mathcal{E}(\eta u, \eta u) &= \int_{\mathbb{R}^d} \langle a\nabla(\eta u), \nabla(\eta u)\rangle dx \\ &\leq 2\int_{\mathbb{R}^d} \langle a\nabla u, \nabla u\rangle \eta^2 dx + 2\int_{\mathbb{R}^d} \langle a\nabla \eta, \nabla\eta\rangle |u|^2 dx \\ &\leq 2\mathcal{E}_\eta(u,u) + 2\|\nabla\eta\|_\infty^2 \|1_B u\|_{2,\Lambda}^2.\end{aligned}$$

Concatenating the two inequalities and averaging over $B$ we get the result. □

### 2.3 Nash inequalities

Local Nash inequalities follow as an easy corollary of the Sobolev inequalities (2.4) and (2.6).

**Proposition 2.8** (Nash inequality)**.** *Let $B \subset \mathbb{R}^d$ be a ball. Then for all $u \in \mathcal{F}_B$ we have*

$$\|u\|_{2,B}^{2+\frac{2}{\mu}} \lesssim C_S^B |B|^{\frac{2-d}{d}} \mathcal{E}(u,u) \|u\|_{1,B}^{\frac{2}{\mu}}, \tag{2.10}$$

*where $\mu := (\frac{2}{d} - \frac{1}{q})^{-1} > 0$, and*

$$\|u\|_{2,\Lambda,B}^{2+\frac{2}{\gamma}} \lesssim C_S^{B,\Lambda} |B|^{\frac{2-d}{d}} \mathcal{E}(u,u) \|u\|_{1,\Lambda,B}^{\frac{2}{\gamma}}, \tag{2.11}$$

*where $\gamma := \frac{p-1}{p}\,(\frac{2}{d} - \frac{1}{p} - \frac{1}{q})^{-1} > 0$.*

*Proof.* We prove only (2.10) being the other completely analogous. By Hölder's inequality

$$\|u\|_{2,B} \leq \|u\|_{\rho,B}^\theta \|u\|_{1,B}^{1-\theta}$$

with $\theta \in (0,1)$ and

$$\frac{1}{2} = (1-\theta) + \frac{\theta}{\rho}.$$

Now it suffices to solve for $\theta$, use (2.4) to estimate $\|u\|_{\rho,B}$ and the result is obtained. □

Note that the condition $1/p + 1/q < 2/d$ is important to have and $\gamma$ positive while to ensure $\mu > 0$ it is enough $q > d/2$. Moreover, $\gamma \geq d/2$, with the equality holding if $p = q = \infty$. It is well known [11, Section 2.4], [28, Theorem 4.1.1] that Nash inequality (2.11) for the Dirichlet form $(\mathcal{E}, \mathcal{F}_B^\Lambda)$ implies the ultracontractivity of the semigroup $P_t^B$ associated to $\mathcal{E}$ on $L^2(B, \Lambda dx)$, in particular there exists a density $p_t^B(x,y)$ with respect to $\Lambda(x)dx$ which satisfies

$$\sup_{x,y\in B} p_t^B(x,y) \lesssim t^{-\gamma}\left[C_S^{B,\Lambda}|B|^{\frac{2}{d}-\frac{1}{\gamma}}\right]^\gamma,$$

where it is once more worthy to notice that $2/d - 1/\gamma \geq 0$, with the equality holding for the non-degenerate situation.

Furthermore, we have just seen that the semigroup $P_t$ associated to $\mathcal{E}$ on $L^2(\mathbb{R}^d, \Lambda dx)$ is locally ultracontractive, being $P_t^B$ ultracontractive for all balls $B \subset \mathbb{R}^d$. It follows by Theorem 2.12 of [21] that $P_t$ admits a symmetric transition kernel $p_t(x,y)$ on $(0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$ with respect to $\Lambda(x)dx$.

There is an increasing interest in deriving Nash inequalities for degenerate operators, mostly because they allow to obtain bounds on the transition probability densities. We mention the recent work [25] where the authors are able to state anchored versions of the Nash inequality, which they use to control the $L^2$-norm of a function by Dirichlet forms that are not uniformly elliptic.

### 2.4 Poincaré inequalities

Let $B \subset \mathbb{R}^d$ be a ball. Given a weight $\theta : B \to [0,\infty]$, we denote by

$$(u)_B^\theta := \int_B u\,\theta dx \Big/ \int_B \theta dx,$$

if $\theta \equiv 1$ we simply write $(u)_B$ instead of $(u)_B^\theta$. Moreover, for $u \in \mathcal{F}_{loc}$ we denote

$$\mathcal{E}_B(u,u) := \int_B a\nabla u \cdot \nabla u\, dx.$$

**Proposition 2.9** (Poincaré inequalities). *Let $B \subset \mathbb{R}^d$ be a ball. If $u \in \mathcal{F}_{loc}$, then*

$$\|u - (u)_B\|_{2,B}^2 \lesssim C_P^B |B|^{\frac{2-d}{d}} \mathcal{E}_B(u,u), \tag{2.12}$$

*being $C_P^B := \|\lambda^{-1}\|_{d/2,B}$, and*

$$\|u - (u)_B^\Lambda\|_{2,B,\Lambda}^2 \lesssim C_P^{B,\Lambda} |B|^{\frac{2-d}{d}} \mathcal{E}_B(u,u), \tag{2.13}$$

*being $C_P^{B,\Lambda} := \|\Lambda\|_{\bar{p},B} \|\lambda^{-1}\|_{\bar{q},B}$ with $\bar{p},\bar{q} \in [1,\infty]$ such that $1/\bar{p} + 1/\bar{q} = 2/d$.*

*Proof.* For (2.12) use Hölder's inequality for the standard Sobolev inequality [28, Theorem 1.5.2]. We now prove (2.13) for $u \in C^\infty(B)$, the final result can be obtained by approximation. As first remark, notice that

$$\begin{aligned}\|u - (u)_B^\Lambda\|_{2,B,\Lambda}^2 &= \inf_{a\in\mathbb{R}} \|u - a\|_{2,B,\Lambda}^2 \\ &\le \|\Lambda\|_{\bar{p},B} \inf_{a\in\mathbb{R}} \|u-a\|_{2\bar{p}^*,B}^2 \le \|\Lambda\|_{\bar{p},B} \|u - (u)_B\|_{2\bar{p}^*,B}^2.\end{aligned}$$

We have by Theorem 1.5.2 in [28].

$$\|u - (u)_B\|_{2\bar{p}^*,B}^2 \lesssim |B|^{\frac{2}{d}} \|\nabla u\|_{\beta,B}^2 \le \|\lambda^{-1}\|_{\bar{q},B} |B|^{\frac{2-d}{d}} \mathcal{E}_B(u,u).$$

where $\beta$ is such that $2\bar{p}^* d/(d+2\bar{p}^*) = \beta = 2\bar{q}/(\bar{q}+1)$, which is true whenever $1/\bar{p}+1/\bar{q} = 2/d$. Concatenating the two inequalities leads to the result. □

In order to get mean value inequalities for the logarithm of caloric functions and, given that, the parabolic Harnack inequality, we will need a Poincaré inequality with a radial cutoff. The cutoff function $\eta : \mathbb{R}^d \to [0,\infty)$ supported in a ball $B = B(x_0,r)$, is a radial function, $\eta(x) := \Phi(|x-x_0|/r)$ where $\Phi$ is some non-increasing, non-negative càdlàg function non identically zero on $(r/2, r]$.

**Proposition 2.10** (Poincaré inequalities with radial cutoff). *Let $B \subset \mathbb{R}^d$ be a ball of radius $r > 0$ and center $x_0$ and let $\eta$ be a radial cutoff as above. If $u \in \mathcal{F}_{loc}$, then*

$$\|u - (u)_B^{\eta^2}\|_{2,B,\eta^2} \lesssim M^B C_P^B |B|^{\frac{2-d}{d}} \mathcal{E}_\eta(u,u) \tag{2.14}$$

*where $M^B = \Phi(0)/\Phi(1/2)$, and*

$$\|u - (u)_B^{\Lambda\eta^2}\|_{2,B,\Lambda\eta^2} \lesssim M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2-d}{d}} \mathcal{E}_\eta(u,u), \tag{2.15}$$

*where $M^{B,\Lambda} := M^B \|\Lambda\|_{1,B}/\|\Lambda\|_{1,B/2}$.*

*Proof.* We give the proof only for (2.15), since (2.14) follows by a similar argument. We apply Theorem 1 in [13]. Accordingly we define a functional $F(u,s) : L^2(\mathbb{R}^d, \Lambda dx) \times (r/2, r] \to [0,\infty]$ by

$$F(u,s) \equiv C_P^{B_s,\Lambda} |B_s|^{\frac{2}{d}} \int_{B_s} a\nabla u \cdot \nabla u dx.$$

for $u \in \mathcal{F}^\Lambda$, and $F(u,s) = \infty$ otherwise, where $B_s$ is the ball of center $x_0$ and radius $s \in (r/2, r]$.

Such functional satisfies $F(u+a,s) = F(u,s)$ for all $a \in \mathbb{R}$ and $u \in L^2(\mathbb{R}^d, \Lambda dx)$, moreover

$$\|u - (u)_{B_s}\|^2_{2,B_s,\Lambda} \lesssim |B_s|^{-1} F(u,s)$$

for every $s \in (r/2, r]$ and $u \in \mathcal{F}^\Lambda$ by the Poincaré inequality (2.13). It follows from [13, Theorem 1] that there exists $M > 0$, explicitly given by $(\|\Lambda\|_{1,B}\Phi(0))/(\|\Lambda\|_{1,B/2}\Phi(1/2))$, such that for all $u \in \mathcal{F}^\Lambda$

$$\begin{aligned}
\|u - (u)_B^{\Lambda\eta^2}\|_{2,B,\Lambda\eta^2} &\lesssim M|B|^{-1} \int_{r/2}^r F(u,s)\nu(ds) \\
&\lesssim MC_P^{B,\Lambda}|B|^{\frac{2-d}{d}} \int_{r/2}^r \int_B a\nabla u \cdot \nabla u 1_{B_s} dx\, \gamma(ds) \\
&= MC_P^{B,\Lambda}|B|^{\frac{2-d}{d}} \int_B \eta^2 a \nabla u \cdot \nabla u dx.
\end{aligned}$$

Here $\gamma(ds)$ is a non-zero positive $\sigma$-finite Borel measure on $(r/2, r]$ such that

$$\eta^2(x) = \int_{r/2}^r 1_{B_s}(x)\, \nu(ds)$$

as in [13]. Of course such an inequality is local and we can extend it for $u \in \mathcal{F}_{loc}$. □

### 2.5 Remark on the constants

In the previous sections we introduced several constants. We recall them here:

- $C_S^{B,\Lambda} := \|\lambda^{-1}\|_{q,B}\|\Lambda\|_{p,B}^{2p^*/\rho}$,
- $C_P^{B,\Lambda} := \|\lambda^{-1}\|_{\bar{q},B}\|\Lambda\|_{\bar{p},B}$ where $\bar{p} \le p$ and $\bar{q} \le q$ are such that $1/\bar{p} + 1/\bar{q} = 2/d$,
- $M^{B,\Lambda} := \|\Lambda\|_{1,B}/\|\Lambda\|_{1,B/2}$.

The following lemmas show that it is possible to bound the aforementioned constants for very large balls $B$.

**Lemma 2.11.** *Under $(b.2)$, for all $x \in \mathbb{R}^d$*

$$\limsup_{r\to\infty} \frac{1}{|B(x,r)|} \int_{B(x,r)} \Lambda^p + \lambda^{-q}\, dz < \infty,$$

*and the limit does not depend on $x$.*

*Proof.* Observe that for $r > |x|$, $B(0, r-|x|) \subset B(x,r) \subset B(0, r+|x|)$, therefore the following inequality holds

$$\begin{aligned}
&\left(\frac{r-|x|}{r}\right)^d \frac{1}{|B(0,r-|x|)|} \int_{B(0,r-|x|)} \Lambda^p + \lambda^{-q}\, dz \\
&\qquad \le \frac{1}{|B(x,r)|} \int_{B(x,r)} \Lambda^p + \lambda^{-q}\, dz \\
&\qquad\qquad \le \left(\frac{r+|x|}{r}\right)^d \frac{1}{|B(0,r+|x|)|} \int_{B(0,r+|x|)} \Lambda^p + \lambda^{-q}\, dz,
\end{aligned}$$

and taking the limit $r \to +\infty$ both sides gives the result. □

**Lemma 2.12.** *Assume* $(b.2)$*. Then, there exist finite positive constants* $C_S^{*,\Lambda}$*,* $C_P^{*,\Lambda}$ *and* $M^{*,\Lambda}$*, independent of* $x \in \mathbb{R}^d$ *such that*

$$\limsup_{r\to\infty} C_S^{B(x,r),\Lambda} = C_S^{*,\Lambda}, \quad \limsup_{r\to\infty} C_P^{B(x,r),\Lambda} = C_P^{*,\Lambda},$$

$$\sup_{x\in\mathbb{R}^d} \limsup_{r\to\infty} M^{B(x,r),\Lambda} = M^{*,\Lambda}.$$

*In particular, for all* $\delta > 0$ *and* $x \in \mathbb{R}^d$ *there exists* $s(x,\delta) \geq 1$ *such that for all* $r > s(x,\delta)$

$$C_S^{B(x,r),\Lambda} < C_S^{*,\Lambda}(1+\delta), \quad C_P^{B(x,r),\Lambda} < C_P^{*,\Lambda}(1+\delta), \quad M^{B(x,r),\Lambda} < M^{*,\Lambda}(1+\delta).$$

*Proof.* The existence of a finite limit for $C_S^{B(x,r),\Lambda}$ and $C_P^{B(x,r),\Lambda}$ as $r \to \infty$ is a direct consequence of $(b.2)$. In the case of $M^{B(x,r),\Lambda}$ one must be slightly more careful since $\|\Lambda\|_{1,B/2}$ appears in the denominator. It suffices to observe that

$$\limsup_{r\to\infty} M^{B(x,r),\Lambda} \leq \limsup_{r\to\infty} \frac{\|\Lambda\|_{1,B(x,r)}}{\|\Lambda\|_{1,B(x,r/2)}} \leq \limsup_{r\to\infty} \|\Lambda\|_{1,B(x,r)} \|\Lambda^{-1}\|_{1,B(x,r/2)} < \infty.$$

The independence of the limits from $x \in \mathbb{R}^d$ can be obtained as in Lemma 2.11. The second statement is an immediate consequence of the first part. □

# 3 Estimates for caloric functions

## 3.1 Mean value inequalities for subcaloric functions

To avoid the same type of technical problems which we faced in [9, Section 2.3], we shall assume that our positive subcaloric functions $u$ are locally bounded. It turns out that any positive subcaloric function is locally bounded; this can be proved repeating the argument below with some additional technicalities similar to what we did in the proof of [9, Proposition 2.4].

**Proposition 3.1.** *Consider* $I = (t_1, t_2) \subset \mathbb{R}$ *and a ball* $B \subset \mathbb{R}^d$*. Let* $u$ *be a locally bounded positive subcaloric function in* $Q = I \times B$*. Take cutoffs* $\eta \in C_0^\infty(B)$*,* $0 \leq \eta \leq 1$ *and* $\zeta \in C^\infty(\mathbb{R})$*,* $\zeta \equiv 0$ *on* $(-\infty, t_1]$*,* $0 \leq \zeta \leq 1$*. Set* $\nu = 2 - 2p^*/\rho$*. Then for all* $\alpha \geq 1$

$$\|\zeta\eta^2 u^{2\alpha}\|_{\nu,I\times B,\Lambda}^{\nu} \lesssim C_S^{B,\Lambda} \frac{|B|^{\frac{2}{d}}}{|I|^{1-\nu}} \Big[\alpha(\|\zeta'\|_\infty + \|\nabla\eta\|_\infty^2)\Big]^{\nu} \|u^{2\alpha}\|_{1,I\times B,\Lambda}^{\nu}. \tag{3.1}$$

*Proof.* Since $u_t > 0$ is locally bounded, the power function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = |x|^{2\alpha}$ with $\alpha \geq 1$ satisfies the assumptions of Lemma B.3. Thus, for $\eta \in C_0^\infty(B)$ as above we have

$$\frac{d}{dt}(u_t^{2\alpha}, \eta^2)_\Lambda + 2\alpha\,\mathcal{E}(u_t, u_t^{2\alpha-1}\eta^2) \leq 0, \quad t \in I. \tag{3.2}$$

We can estimate

$$\begin{aligned}\mathcal{E}(u_t, u_t^{2\alpha-1}\eta^2) &= 2\int_{\mathbb{R}^d} \eta u_t^{2\alpha-1}\langle a\nabla u_t, \nabla\eta\rangle\, dx + (2\alpha-1)\int_{\mathbb{R}^d} \eta^2 u_t^{2\alpha-2}\langle a\nabla u_t, \nabla u_t\rangle\, dx \\ &\geq \frac{2\alpha-1}{\alpha^2}\,\mathcal{E}_\eta(u_t^\alpha, u_t^\alpha) - \frac{2\|\nabla\eta\|_\infty}{\alpha}\,\mathcal{E}_\eta(u_t^\alpha, u_t^\alpha)^{1/2}\|1_B u_t^{2\alpha}\|_{1,\Lambda}^{1/2},\end{aligned}$$

by Young's inequality $2ab \leq (\epsilon a^2 + b^2/\epsilon)$ with $a = \mathcal{E}_\eta(u_t^\alpha, u_t^\alpha)^{1/2}$, $b = \|\nabla\eta\|_\infty \|1_B u_t^{2\alpha}\|_{1,\Lambda}^{1/2}$ and $\epsilon = 1/2\alpha$, we get, using that $\alpha \geq 1$,

$$\mathcal{E}(u_t, u_t^{2\alpha-1}\eta^2) \geq (1/2\alpha)\mathcal{E}_\eta(u_t^\alpha, u_t^\alpha) - 2\|\nabla\eta\|_\infty^2 \|1_B u_t^{2\alpha}\|_{1,\Lambda}.$$

Going back to (3.2) we deduce

$$\frac{d}{dt}\|(u_t^\alpha\eta)^2\|_{1,\Lambda}+\mathcal{E}_\eta(u_t^\alpha,u_t^\alpha)\le 4\alpha\|\nabla\eta\|_\infty^2\|1_B u_t^{2\alpha}\|_{1,\Lambda}.$$

We now take a smooth cutoff in time $\zeta:\mathbb{R}\to[0,1]$, $\zeta\equiv 0$ on $(-\infty,t_1]$, where $I=(t_1,t_2)$. We multiply the inequality above by $\zeta$ and integrate in time. This yields

$$\zeta(t)\|(u_t^\alpha\eta)^2\|_{1,\Lambda}+\int_{t_1}^t\zeta(s)\mathcal{E}_\eta(u_s^\alpha,u_s^\alpha)\,ds\le 4\alpha\Big[\|\zeta'\|_\infty+\|\nabla\eta\|_\infty^2\Big]\int_{t_1}^t\|1_B u_s^{2\alpha}\|_{1,\Lambda}\,ds,$$

after averaging in space and taking the supremum for $t\in I$, we obtain

$$\sup_{t\in I}\zeta(t)\|(\eta u_t^\alpha)^2\|_{1,B,\Lambda}+\int_I\zeta(s)\frac{\mathcal{E}_\eta(u_s^\alpha,u_s^\alpha)}{|B|}\,ds\lesssim\alpha\Big[\|\zeta'\|_\infty+\|\nabla\eta\|_\infty^2\Big]\int_I\|u_s^{2\alpha}\|_{1,B,\Lambda}\,ds. \quad (3.3)$$

The idea is to use (3.3) together with (2.9) to get (3.1). Observe that $\nu=2-2p^*/\rho$ is greater than one, since $\rho>2p^*$ by $1/p+1/q<2/d$. Using Hölder's inequality and some easy manipulation

$$\|(\eta u_s^\alpha)^2\|_{\nu,B,\Lambda}^\nu\le\|\eta u_s^\alpha\|_{\rho/p^*,B,\Lambda}^2\|(\eta u_s^\alpha)^2\|_{1,B,\Lambda}^{\nu-1}. \quad (3.4)$$

We can now integrate this inequality against $\zeta(s)^\nu$ over $I$ and obtain

$$\frac{1}{|I|}\int_I\zeta(s)^\nu\|\eta^2u_s^{2\alpha}\|_{\nu,B,\Lambda}^\nu\,ds\le\Big(\sup_{s\in I}\zeta(s)\|(\eta u_s^\alpha)^2\|_{1,B,\Lambda}\Big)^{\nu-1}\frac{1}{|I|}\int_I\zeta(s)\|\eta u_s^\alpha\|_{\rho/p^*,B,\Lambda}^2\,ds.$$

In view of the Sobolev inequality (2.9) we deduce that

$$\|\eta u_s^\alpha\|_{\rho/p^*,B,\Lambda}^2\lesssim C_S^{B,\Lambda}|B|^{\frac{2}{d}}\Big[\frac{\mathcal{E}_\eta(u_s^\alpha,u_s^\alpha)}{|B|}+\|\nabla\eta\|_\infty^2\|u_s^{2\alpha}\|_{1,B,\Lambda}\Big].$$

By (3.3) we can bound each of the two factors. We end up with the following iterative step

$$\|\zeta\eta^2u^{2\alpha}\|_{\nu,I\times B,\Lambda}^\nu\lesssim C_S^{B,\Lambda}\frac{|B|^{\frac{2}{d}}}{|I|^{1-\nu}}\Big[\alpha(\|\zeta'\|_\infty+\|\nabla\eta\|_\infty^2)\Big]^\nu\|u^{2\alpha}\|_{1,I\times B,\Lambda}^\nu,$$

which is what we wanted to prove. □

The main idea is to use Moser's iteration technique on a sequence of parabolic balls; Proposition 3.1 with a suitable choice of the cutoffs $\eta$, $\zeta$ and of the parameter $\alpha$ is the iteration step. Fix a parameter $\tau>0$, let $x\in\mathbb{R}^d$, $r>0$ and $\delta\in(0,1)$. Then, we define the parabolic balls

$$\begin{aligned}Q(\tau,x,s,r)=Q&=(s-\tau r^2,s)\times B(x,r),\\ Q_\delta&=(s-\delta\tau r^2,s)\times B(x,\delta r).\end{aligned}$$

Clearly $Q_\delta\subset Q$ for all $\delta\in(0,1)$.

**Theorem 3.2.** *Fix $\tau>0$ and let $1/2\le\sigma'<\sigma\le 1$. Assume that $1/p+1/q<2/d$ and let $u_t$ be a positive subcaloric function on $Q=Q(\tau,x,s,r)$. Then there exists a positive constant $C_1:=C_1(d,p,q)$ such that*

$$\sup_{Q_{\sigma'}}u\le C_1(C_S^{B,\Lambda})^{\frac{1}{2\nu-2}}\tau^{\frac{1}{2}}\left[\frac{1+\tau^{-1}}{(\sigma-\sigma')^2}\right]^{\frac{\nu}{2\nu-2}}\|u\|_{2,Q_\sigma,\Lambda}, \quad (3.5)$$

*where $\nu=2-2p^*/\rho$.*

*Proof.* We want to apply (3.1) with a suitable sequence of cutoffs $\eta_k$ and $\zeta_k$. Set

$$\sigma_k = \sigma' + 2^{-k}(\sigma - \sigma'), \quad \delta_k = 2^{-k-1}(\sigma - \sigma')$$

then $\sigma_k - \sigma_{k+1} = \delta_k$. Consider a cutoff in space $\eta_k : \mathbb{R}^d \to [0,1]$, such that $\operatorname{supp} \eta_k \subset B(\sigma_k r)$ and $\eta_k \equiv 1$ on $B(\sigma_{k+1} r)$, moreover assume that $\|\nabla \eta_k\|_\infty \le 2/(r\delta_k)$. Take also a cutoff in time $\zeta_k : \mathbb{R} \to [0,1]$, $\zeta_k \equiv 1$ on $I_{\sigma_{k+1}} = (s - \sigma_{k+1}\tau r^2, s)$, $\zeta_k \equiv 0$ on $(-\infty, s - \sigma_k \tau r^2)$ and $\|\zeta_k'\|_\infty \le 2/(r^2\tau\delta_k)$. Let $\alpha_k = \nu^k$ with $\nu = 2 - 2p^*/\rho$ as above. Then, an application of (3.1) and using the fact that $\alpha_{k+1} = \nu\alpha_k$ yields

$$\|u\|_{2\alpha_{k+1}, Q_{\sigma_{k+1}}, \Lambda} \le \left\{ c(d) C_S^{B,\Lambda} \tau^{\nu-1} \Big[ \frac{\alpha_k (1+\tau^{-1}) 2^{2k}}{(\sigma - \sigma')^2} \Big]^\nu \right\}^{\frac{1}{2\alpha_{k+1}}} \|u\|_{2\alpha_k, Q_{\sigma_k}, \Lambda}.$$

where we used the fact that $\sigma_k/\sigma_{k+1} < 2$, and that $\sigma_k \in [1/2, 1]$. This is the starting point for Moser's iteration. Iterating the inequality from $k = 0$ up to $i$ we derive at the price of a constant $C_1 > 0$ which depends on $p, q$ and $d$

$$\|u\|_{2\alpha_i, Q_{\sigma_i}, \Lambda} \le C_1 (C_S^{B,\Lambda})^{\frac{1}{2\nu-2}} \tau^{\frac{1}{2}} \left[ \frac{1+\tau^{-1}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{2\nu-2}} \|u\|_{2, Q_\sigma, \Lambda}. \tag{3.6}$$

where we exploited the fact that $\sum_{k=0}^\infty 1/\alpha_k = \nu/(\nu-1)$ and that $\sum_{k=0}^\infty k/\alpha_k < \infty$. Increasing $C_1$ if needed, from (3.6) we easily derive

$$\|u\|_{2\alpha_i, Q_{\sigma'}, \Lambda} \le C_1 (C_S^{B,\Lambda})^{\frac{1}{2\nu-2}} \tau^{\frac{1}{2}} \left[ \frac{1+\tau^{-1}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{2\nu-2}} \|u\|_{2, Q_\sigma, \Lambda}.$$

Taking the limit as $i \to \infty$ gives the result

$$\sup_{Q_{\sigma'}} u \le C_1 (C_S^{B,\Lambda})^{\frac{1}{2\nu-2}} \tau^{\frac{1}{2}} \left[ \frac{1+\tau^{-1}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{2\nu-2}} \|u\|_{2, Q_\sigma, \Lambda}.$$

□

**Corollary 3.3.** *Fix $\tau > 0$ and let $1/2 \le \sigma' < \sigma \le 1$. Assume that $1/p + 1/q < 2/d$ and let $u$ be a subcaloric function in $Q = Q(\tau, x, s, r)$. Then, there exists a positive constant $C_2 := C_2(q, p, d)$, which depends only on $p, q$ and $d$, such that for all $\alpha > 0$*

$$\sup_{Q_{\sigma'}} u \lesssim C_2 2^{\frac{2}{\alpha^2}\frac{\nu}{\nu-1}} (C_S^{B,\Lambda})^{\frac{1}{\alpha\nu-\alpha}} \tau^{\frac{1}{\alpha}} \left[ \frac{1+\tau^{-1}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{\alpha\nu-\alpha}} \|u\|_{\alpha, Q_\sigma, \Lambda}. \tag{3.7}$$

*Proof.* To prove (3.7) one can follow the same approach in [28, Theorem 2.2.3] with the only difference that we will consider parabolic balls $Q_\sigma$ instead of balls. Observe that for $\alpha > 2$ this is just an application of Jensen's inequality. □

We remark that (3.7) is not good for an application of Bombieri-Giusti's lemma (see Lemma C.1 below) since $2^{\frac{2}{\alpha}\frac{\nu}{\nu-1}}$ is exploding as $\alpha$ approaches zero. To get rid of this problem, in the next section we derive bounds for supercaloric functions.

Theorem 3.2 can be applied to obtain a global on-diagonal heat kernel upper bound, as it is done in the next proposition following in spirit the proof of Zhikov in [29].

**Proposition 3.4.** *Let $f \in L^2(\mathbb{R}^d, \Lambda dx)$, and assume that $(b.1)$ and $(b.2)$ are satisfied, then there exists a constant $C_3 = C_3(q, p, d, C_S^{*,\Lambda}) > 0$ such that for all $x \in \mathbb{R}^d$ and $t > 0$ the following inequality holds*

$$P_t f(x) \le C_3 t^{-\gamma} (s(0,1) + |x| + \sqrt{t})^{\gamma - d/2} \int_{\mathbb{R}^d} (s(0,1) + |y| + \sqrt{t})^{\gamma - d/2} |f(y)| \Lambda(y) dy.$$

*where $\gamma$ was defined in (2.11) and $s(x,\delta)$ was defined in Lemma 2.12.*

*Proof.* We want to apply Theorem 3.2. Fix $\tau \in (0,2]$, $x = 0$, $r > 0$, $s = \tau r^2$, $\sigma = 1$ and $\sigma' = 1/2$. It follows that

$$Q_1 = (0, \tau r^2) \times B(0,r), \quad Q_{1/2} = \tau r^2(1/2,1) \times B(0,r/2).$$

We choose $r := s(0,1) + 2|z| + \sqrt{t}$ where $s(0,1)$ was defined in Lemma 2.12. In this way $C_S^{B(0,r),\Lambda} \le 2C_S^{*,\Lambda}$ and we can read inequality (3.5) for $u(s,z) := P_s^\Lambda f(z)$ as follows

$$\sup_{Q_{1/2}} P_s f(z) \le c(C_S^{*,\Lambda})^{\gamma/2} \frac{\tau^{-\gamma/2}}{r^{d/2}} \|f\|_{2,\Lambda},$$

with $c = c(p,q,d)$ changing throughout the proof. By definition of $r$ we can find $\tau \in (0,2]$ such that $3/4\tau r^2 = t$ and in particular such that $(t,z) \in Q_{1/2}$. This gives

$$P_t f(z) \le c t^{-\gamma/2}(s(0,1) + |z| + \sqrt{t})^{\gamma - d/2} \|f\|_{2,\Lambda},$$

for all $z \in \mathbb{R}^d$ and $t > 0$, where now $c = c(p,q,d,C_S^{*,\Lambda})$ depends on $C_S^{*,\Lambda}$ as well. Set $b_t(z) = (s(0,1) + |z| + \sqrt{t})^{\gamma - d/2}$. It follows that

$$\|b_t^{-1} P_t f\|_\infty \le c t^{-\gamma/2} \|f\|_{2,\Lambda},$$

from which $\|b_t^{-1} P_t\|_{2\to\infty} \le c t^{-\gamma/2}$. By duality we get $\|P_t b_t^{-1}\|_{1\to 2} \le c t^{-\gamma/2}$. Hence

$$\|P_t f\|_{2,\Lambda} \le c t^{-\gamma/2} \|b_t f\|_{1,\Lambda}.$$

Now it is left to use the semigroup property and classical techniques [11, Chapter 2] to finally get the bound. ☐

It is now standard to get global on-diagonal estimates for the kernel $p_t(x,y)$ of the semigroup $P_t$ associated to $(\mathcal{E}, \mathcal{F}^\Lambda)$ on $L^2(\mathbb{R}^d, \Lambda dx)$. Namely we obtain that for almost all $x, y \in \mathbb{R}^d$ and for all $t > 0$

$$p_t(x,y) \le C_3 t^{-\gamma}(s(0,1) + |x| + \sqrt{t})^{\gamma - d/2}(s(0,1) + |y| + \sqrt{t})^{\gamma - d/2}. \tag{3.8}$$

### 3.2 Mean value inequalities for supercaloric functions

**Theorem 3.5.** *Fix $\tau > 0$ and let $1/2 \le \sigma' < \sigma \le 1$. Assume that $1/p + 1/q < 2/d$ and let $u_t$ be a positive supercaloric function of on $Q = Q(\tau, x, s, r)$. Then there exists a positive constant $C_4 := C_4(p,q,d)$ which depends only on the dimension and on $p,q$ such that for all $\alpha \in (0,\infty)$*

$$\sup_{Q_{\sigma'}} u^{-\alpha} \le C_4 (C_S^{B,\Lambda})^{\frac{1}{\nu-1}} \tau \left[\frac{1+\tau^{-1}}{(\sigma-\sigma')^2}\right]^{\frac{\nu}{\nu-1}} \|u^{-1}\|^\alpha_{\alpha, Q_\sigma, \Lambda}. \tag{3.9}$$

*where $\nu = 2 - 2p^*/\rho$.*

*Proof.* We can always assume that $u > \epsilon$ by considering the supercaloric function $u + \epsilon$ and then sending $\epsilon$ to zero at the end of the argument. Applying Lemma B.3 with the function $F(x) := -|x|^{-\beta}$ and $\beta > 0$ we get

$$-\frac{d}{dt}\|\eta^2 u_t^{-\beta}\|_{1,\Lambda} + \beta\, \mathcal{E}(u_t^{-\beta-1}\eta^2, u_t) \ge 0$$

which after some manipulation gives

$$-\frac{d}{dt}\|\eta^2 u_t^{-\beta}\|_{1,\Lambda} - 4\frac{\beta+1}{\beta}\, \mathcal{E}_{\eta^2}(u_t^{-\beta/2}, u_t^{-\beta/2}) - 4\int_{\mathbb{R}^d} a\nabla\eta \cdot \nabla(u_t^{-\beta/2})\eta u_t^{-\beta/2} dx \ge 0$$

by means of Young's inequality $4ab \leq 3a^2+2b^2/3$ and using the simple fact that $(\beta+1)/\beta > 1$, we get, after averaging

$$\frac{d}{dt}\|\eta^2 u_t^{-\beta}\|_{1,B,\Lambda} + \frac{\mathcal{E}_{\eta^2}(u_t^{-\beta/2}, u_t^{-\beta/2})}{|B|} \lesssim \|\nabla\eta\|_\infty^2 \|u_t^{-\beta}\|_{1,B,\Lambda}$$

We now integrate against a time cutoff $\zeta : \mathbb{R} \to [0,1]$ to obtain something similar to (3.3). The same approach as in Proposition 3.1 applies and we get

$$\|\zeta\eta^2 u^{-\beta}\|_{\nu,I\times B,\Lambda}^\nu \lesssim C_S^{B,\Lambda} \frac{|B|^{\frac{2}{d}}}{|I|^{1-\nu}} \Big[\|\zeta'\|_\infty + \|\nabla\eta\|_\infty^2\Big]^\nu \|u^{-\beta}\|_{1,I\times B,\Lambda}^\nu.$$

Moser's iteration technique with $\beta_k = \nu^k\alpha$ and $\alpha > 0$ and the same argument of Theorem 3.2 will finally give

$$\sup_{Q_{\sigma'}} u^{-\alpha} \leq C_4 (C_S^{B,\Lambda})^{\frac{1}{\nu-1}} \tau \left[\frac{1+\tau^{-1}}{(\sigma-\sigma')^2}\right]^{\frac{\nu}{\nu-1}} \|u^{-1}\|_{\alpha,Q_\sigma,\Lambda}^\alpha.$$

□

We introduce the following parabolic ball. Given $x \in \mathbb{R}^d$, $r, \tau > 0$ and $s \in \mathbb{R}$, $\delta \in (0,1)$, we note

$$Q'_\delta = Q'_\delta(\tau, x, s, r) = (s - \tau r^2, s - (1-\delta)\tau r^2) \times B(x, \delta r).$$

**Theorem 3.6.** *Fix $\tau > 0$ and let $1/2 \leq \sigma' < \sigma \leq 1$. Assume that $1/p + 1/q < 2/d$ and let $u$ be a positive supercaloric function on $Q = Q(\tau, x, s, r)$. Fix $0 < \alpha_0 < \nu$. Then there exists a positive constant $C_5 := C_5(q, p, d, \alpha_0)$ which depends only on the dimension, on $p, q$ and on $\alpha_0$ such that for all $0 < \alpha < \alpha_0\nu^{-1}$ we have*

$$\|u\|_{\alpha_0,Q'_{\sigma'},\Lambda} \leq \left\{ C_5 \tau (1+\tau^{-1})^{\frac{\nu}{\nu-1}} \left[\frac{1 \vee C_S^{B,\Lambda}}{(\sigma-\sigma')^2}\right]^{\frac{\nu}{\nu-1}} \right\}^{(1+\nu)(1/\alpha - 1/\alpha_0)} \|u\|_{\alpha,Q'_\sigma,\Lambda} \tag{3.10}$$

*where $\nu = 2 - 2p^*/\rho$.*

*Proof.* Assume $u$ is supercaloric on $Q = I \times B$. Applying Lemma B.3 with the function $F(x) := |x|^\beta$ with $\beta \in (0,1)$ we get

$$\frac{d}{dt}\|\eta^2 u_t^\beta\|_{1,\Lambda} + \beta\,\mathcal{E}(u_t^{\beta-1}\eta^2, u_t) \geq 0$$

which after some manipulation gives

$$\frac{d}{dt}\|\eta^2 u_t^\beta\|_{1,\Lambda} + 4\frac{\beta-1}{\beta}\,\mathcal{E}_\eta(u_t^{\beta/2}, u_t^{\beta/2}) + 4\int_{\mathbb{R}^d} a\nabla\eta\cdot\nabla(u_t^{\beta/2})\eta u_t^{\beta/2} dx \geq 0$$

Note that $(\beta - 1)$ is negative. If we take $0 < \beta < \alpha_0\nu^{-1}$ then we have

$$\frac{1-\beta}{\beta} > 1 - \beta > 1 - \alpha_0/\nu =: \epsilon,$$

this yields after Young's inequality

$$-\frac{d}{dt}\|\eta^2 u_t^\beta\|_{1,\Lambda} + \epsilon\,\mathcal{E}_\eta(u_t^{\beta/2}, u_t^{\beta/2}) \leq A\|\nabla\eta\|_\infty^2 \|1_B u_t^\beta\|_{1,\Lambda},$$

where $A$ is a constant possibly depending on $q, p, \alpha_0$ and $d$ which will be changing throughout the proof. Here we introduce a difference, the time cutoff $\zeta : \mathbb{R} \to [0,1]$, $\zeta \equiv 0$

on $(t_2, \infty]$, where $I = (t_1, t_2)$, is zero at the top of the time interval and not at the bottom. This gives after integrating,

$$\zeta(t)\|\eta^2 u_t^\beta\|_{1,\Lambda} + \int_t^{t_2} \zeta(s)\mathcal{E}_\eta(u_s^{\beta/2}, u_s^{\beta/2})\, ds \le A\left[\|\zeta'\|_\infty + \|\nabla\eta\|_\infty^2\right]\int_t^{t_2} \|1_B u_s^\beta\|_{1,\Lambda}\, ds$$

which has the same flavor as (3.3). Starting from this inequality, and repeating the argument we used for subcaloric functions, we end up with

$$\|\zeta\eta^2 u^\beta\|_{\nu,I\times B,\Lambda}^\nu \le A\, C_S^{B,\Lambda}\frac{|B|^{\frac{2}{d}}}{|I|^{1-\nu}}\left[\|\zeta'\|_\infty + \|\nabla\eta\|_\infty^2\right]^\nu \|u^\beta\|_{1,I\times B,\Lambda}^\nu. \tag{3.11}$$

The idea is now to iterate inequality (3.11) with an appropriate choice of exponents parabolic balls and cutoffs. We follow closely the iteration argument in Theorem 2.2.5 of [28].

*Exponents:* we define the exponents $\alpha_i := \alpha_0\nu^{-i}$ and $\beta_j = \alpha_i\nu^{j-1}$ for $j = 1, \dots, i$. Observe that $0 < \beta_j < \alpha_0\nu^{-1}$ and thus we are in a setting where (3.11) is applicable.

*Parabolic balls:* we define the parabolic balls. We fix $\sigma_0 = \sigma$, $\sigma_j - \sigma_{j+1} = 2^{-j-1}(\sigma - \sigma')$, and set for $j = 1, \dots, i$

$$I_{\sigma_j} = (s - \tau r^2, s - (1-\sigma_j)\tau r^2), \quad Q'_{\sigma_j} = I_{\sigma_j} \times B(\sigma_j r).$$

*Cutoffs:* for $j = 1, \dots, i$ we define the cutoffs $\eta_j : \mathbb{R}^d \to [0,1]$, such that $\operatorname{supp}\eta_j \subset B(\sigma_j r)$, $\eta_j \equiv 1$ on $B(\sigma_{j+1} r)$ and $\|\nabla\eta_j\|_\infty \le 2/(r\delta_j)$, and the cutoffs $\zeta_j : \mathbb{R} \to [0,1]$, $\zeta_j \equiv 1$ on $I_{\sigma_{j+1}}$, $\zeta_j \equiv 0$ on $(s - (1-\sigma_j)\tau r^2, \infty)$ and $\|\zeta_j'\|_\infty \le 2/(r^2\tau\delta_j)$.

We are ready to apply (3.11) for $j = 1, \dots, i$ and the choices above.

$$\|u^{\alpha_i\nu^j}\|_{1,Q'_{\sigma_j},\Lambda} \le A\, C_S^{B,\Lambda}\tau^{\nu-1}\left[\frac{(1+\tau^{-1})2^{2j}}{(\sigma-\sigma')^2}\right]^\nu \|u^{\alpha_i\nu^{j-1}}\|_{1,Q'_{\sigma_{j-1}},\Lambda}^\nu,$$

which after an iteration from $j = 1$ to $j = i$ gives

$$\|u\|_{\alpha_0,Q'_{\sigma_i},\Lambda}^{\alpha_0} \le 2^{2\sum_{j=0}^{i-1}(i-k)\nu^j}\left\{A\, C_S^{B,\Lambda}\tau^{\nu-1}\left[\frac{1+\tau^{-1}}{(\sigma-\sigma')^2}\right]^\nu\right\}^{\sum_{j=0}^{i-1}\nu^j} \|u^{\alpha_i}\|_{1,Q'_\sigma,\Lambda}^{\nu^i}.$$

Now observe that

$$\sum_{j=0}^{i-1}(i-j)\nu^j \le C(\nu)(\alpha_0/\alpha_i - 1), \quad \sum_{j=0}^{i-1}\nu^j = \frac{\nu^i - 1}{\nu - 1} = \frac{\alpha_0/\alpha_i - 1}{\nu - 1}$$

where $C(\nu) > 0$ is a constant which depends on $\nu$ but not on $i$. This yields the following inequality

$$\|u\|_{\alpha_0,Q'_{\sigma'},\Lambda} \le \left\{A\,\tau(1+\tau^{-1})^{\frac{\nu}{\nu-1}}\left[\frac{1 \vee C_S^{B,\Lambda}}{(\sigma-\sigma')^2}\right]^{\frac{\nu}{\nu-1}}\right\}^{1/\alpha_i - 1/\alpha_0} \|u\|_{\alpha_i,Q'_\sigma,\Lambda}$$

where the constant $A$ depends only on $\alpha_0, q, p$ and the dimension $d \ge 2$. Replacing $A$ by $A \vee 1$, we can assume it greater than one. Finally we extend the inequality for $\alpha \in (0, \alpha_0\nu^{-1})$. Let $i \ge 2$ be an integer such that $\alpha_i \le \alpha < \alpha_{i-1}$, then $1/\alpha_i - 1/\alpha_0 \le (1+\nu)(1/\alpha - 1/\alpha_0)$ and by means of Jensen's inequality we obtain

$$\|u\|_{\alpha_0,Q'_{\sigma'},\Lambda} \le \left\{A\,\tau(1+\tau^{-1})^{\frac{\nu}{\nu-1}}\left[\frac{1 \vee C_S^{B,\Lambda}}{(\sigma-\sigma')^2}\right]^{\frac{\nu}{\nu-1}}\right\}^{(1+\nu)(1/\alpha - 1/\alpha_0)} \|u\|_{\alpha,Q'_\sigma,\Lambda},$$

which is what we wanted to prove. □

### 3.3 Mean value inequalities for the logarithm

In this section we get mean value inequalities for $\log u_t$ where $u_t$ is a positive supercaloric function on $Q = (s-\tau r^2, s)\times B(x,r)$, with $\tau > 0$ fixed. We denote by $m^\Lambda := \Lambda dx$ and by $\gamma^\Lambda := dt \otimes m^\Lambda$.

**Theorem 3.7.** *Fix $\tau > 0$ and $\kappa \in (0,1)$, $\delta \in [1/2, 1)$. For any $s \in \mathbb{R}$ and $r > 0$ and any positive supercaloric function $u$ on $Q = (s-\tau r^2, s)\times B(x,r)$, there exist a positive constant $C_6 := C_6(q,p,d,\delta)$ and a constant $k := k(u,\kappa) > 0$ such that*

$$\gamma^\Lambda\{(t,z) \in K^+ \mid \log u_t < -\ell - k\} \le C_6\, m^\Lambda(B)\Big[M^{B,\Lambda}|B|^{\frac{2}{d}}(C_P^{B,\Lambda} \vee \tau^2)\Big]\ell^{-1}, \tag{3.12}$$

*and*

$$\gamma^\Lambda\{(t,z) \in K^- \mid \log u_t > \ell - k\} \le C_6\, m^\Lambda(B)\Big[M^{B,\Lambda}|B|^{\frac{2}{d}}(C_P^{B,\Lambda} \vee \tau^2)\Big]\ell^{-1}, \tag{3.13}$$

*where $K^+ = (s-\kappa\tau r^2, s)\times B(x,\delta r)$ and $K^- = (s-\tau r^2, s-\kappa\tau r^2)\times B(x,\delta r)$.*

*Proof.* We follow closely the strategy adopted in [28, Theorem 5.4.1]. We can always assume $u_t \ge \epsilon$ and then send $\epsilon$ to zero in our estimates, since $u_t+\epsilon$ is still a supercaloric function. We denote as usual $B := B(x,r)$. By Lemma B.3,

$$\begin{aligned}\frac{d}{dt}(\eta^2, -\log u_t)_\Lambda &\le \mathcal{E}(u_t^{-1}\eta^2, u_t) = -\mathcal{E}_\eta(\log u_t, \log u_t) + 2\int \langle a\nabla\eta, \nabla u_t\rangle \eta u_t^{-1} dx \\ &\le -\mathcal{E}_\eta(\log u_t,\log u_t) + 2\mathcal{E}_\eta(\log u_t,\log u_t)^{1/2}\|\nabla\eta\|_\infty \|1_B\|_{1,\Lambda}^{1/2} \\ &\le -\frac{1}{2}\mathcal{E}_\eta(\log u_t,\log u_t) + 2m^\Lambda(B)\|\nabla\eta\|_\infty^2,\end{aligned} \tag{3.14}$$

where in the last inequality we exploited Young's inequality $2ab \le (1/2a^2 + 2b^2)$. The cutoff function $\eta$ must be in the form introduced in (2.15). Namely, we take

$$\eta(z) := (1 - |x-z|/r)_+$$

where $x, r$ are the center and the radius of the ball $B$. We note

$$w_t(z) := -\log u_t(z), \quad W_t := (w_t)_B^{\Lambda\eta^2},$$

then (2.15) reads

$$\frac{|B|}{\|\eta^2\Lambda\|_1}\|w_t - W_t\|_{2,B,\Lambda\eta^2}^2 \lesssim M^{B,\Lambda}C_P^{B,\Lambda}|B|^{\frac{2}{d}}\frac{\mathcal{E}(w_t,w_t)}{2\|\eta^2\Lambda\|_1},$$

rewriting (3.14) we get

$$\partial_t W_t + \frac{|B|}{\|\eta^2\Lambda\|_1}\Big(M^{B,\Lambda}C_P^{B,\Lambda}|B|^{\frac{2}{d}}\Big)^{-1}\|w_t - W_t\|_{2,B,\Lambda\eta^2}^2 \lesssim \|\nabla\eta\|_\infty^2 \frac{m^\Lambda(B)}{\|\eta^2\Lambda\|_1}.$$

By the fact that $(1-\delta)^2 m^\Lambda(B(x,\delta r)) \le \|\eta^2\Lambda\|_1 \le m^\Lambda(B)$ and $\|\nabla\eta\|_\infty^2 \lesssim |B|^{-\frac{2}{d}}$, it follows

$$\partial_t W_t + \Big(m^\Lambda(B)M^{B,\Lambda}C_P^{B,\Lambda}|B|^{\frac{2}{d}}\Big)^{-1}\int_{\delta B}|w_t - W_t|^2\,\Lambda dx \le c\,M^{B,\Lambda}|B|^{-\frac{2}{d}} \tag{3.15}$$

for some constant $c > 0$ depending only on the dimension and $\delta$. Observe that we fixed $\delta \in [1/2, 1)$ to stay away from the boundary. What we have above resembles closely what is given in [28, Theorem 5.4.1], except for the dependence of the constant on $B$. Let us introduce the following auxiliary functions

$$\bar{w}_t := w_t - c\,M^{B,\Lambda}|B|^{-\frac{2}{d}}(t-s'), \quad \bar{W}_t := W_t - c\,M^{B,\Lambda}|B|^{-\frac{2}{d}}(t-s'),$$

where $s' = s - \kappa\tau r^2$. We can now rewrite (3.15) as

$$\partial_t \bar{W}_t + \Big(m^\Lambda(B) M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2}{d}}\Big)^{-1} \int_{\delta B} |\bar{w}_t - \bar{W}_t|^2 \, \Lambda dx \le 0. \tag{3.16}$$

Now set $k(u,\kappa) := \bar{W}_{s'}$ and define the two sets

$$D_t^+(\ell) := \{z \in B(x,\delta r) \,|\, \bar{w}(t,z) > k + \ell\},$$

$$D_t^-(\ell) := \{z \in B(x,\delta r) \,|\, \bar{w}(t,z) < k - \ell\}.$$

Since $\partial_t \bar{W}_t \le 0$ we have that, for $t > s'$, $\bar{w}_t - \bar{W}_t > \ell + k(u) - \bar{W}_t \ge \ell$ on $D_t^+(\ell)$. Using this in (3.16) we obtain

$$\partial_t \bar{W}_t + \Big(m^\Lambda(B) M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2}{d}}\Big)^{-1} |\ell + k - \bar{W}_t|^2 \, m^\Lambda(D_t^+(\ell)) \le 0. \tag{3.17}$$

or equivalently

$$-\Big(m^\Lambda(B) M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2}{d}}\Big) \partial_t |\ell + k - \bar{W}_t|^{-1} \ge m^\Lambda(D_t^+(\ell)). \tag{3.18}$$

Integrating from $s'$ to $s$ yields, for $\gamma^\Lambda = dt \otimes m^\Lambda$,

$$\gamma^\Lambda\{(t,z) \in K^+ \,|\, \bar{w}(t,z) > k + \ell\} \le m^\Lambda(B)\Big(M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2}{d}}\Big)\ell^{-1}.$$

Recall that $-\log u_t = \bar{w}_t + c\, M^{B,\Lambda} |B|^{-\frac{2}{d}}(t - s')$, therefore

$$\gamma^\Lambda\{(t,z) \in K^+ \mid \log u_t + c\, M^{B,\Lambda}|B|^{-\frac{2}{d}}(t-s') < -k - \ell\} \le m^\Lambda(B)\Big(M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2}{d}}\Big)\ell^{-1}.$$

Finally,

$$\begin{aligned}
\gamma^\Lambda\{(t,z) \in K^+ \mid \log u_t < -k(u) - \ell\} \\
&\le \gamma^\Lambda\{(t,z) \in K^+ \mid \log u_t + c\, M^{B,\Lambda}|B|^{-\frac{2}{d}}(t-s') < -k - \ell/2\} \\
&\quad + \gamma^\Lambda\{(t,z) \in K^+ \,|\, c\, M^{B,\Lambda}|B|^{-\frac{2}{d}}(t-s') > \ell/2\} \\
&\lesssim m^\Lambda(B)\Big(M^{B,\Lambda} C_P^{B,\Lambda} |B|^{\frac{2}{d}}\Big)\ell^{-1} + m^\Lambda(B)\Big(\tau^2 M^{B,\Lambda} |B|^{\frac{2}{d}}\Big)\ell^{-1} \\
&\lesssim m^\Lambda(B)\Big[M^{B,\Lambda}|B|^{\frac{2}{d}}(C_P^{B,\Lambda} \vee \tau^2)\Big]\ell^{-1}.
\end{aligned}$$

where in the second but last step we used Markov's inequality and the fact that $\kappa < 1$. Working with $D_t^-(\ell)$ and $K^-$ and using similar arguments proves the second inequality. □

### 3.4 Parabolic Harnack's inequality

We have all the tools to apply Lemma C.1 effectively to a positive function $u$ which is caloric in the parabolic ball $Q(\tau, s, x, r) = (s - \tau r^2, s) \times B(x, r)$. This will finally give us the parabolic Harnack inequality. Fix $\delta \in (0,1)$ and $\tau > 0$. For $x \in \mathbb{R}^d$, $s \in \mathbb{R}$ and $r > 0$ denote

$$\begin{aligned}
Q_- &= (s - (3+\delta)\tau r^2/4, s - (3-\delta)\tau r^2/4) \times \delta B, \qquad (3.19)\\
Q'_- &= (s - \tau r^2, s - (3-\delta)\tau r^2/4) \times \delta B, \\
Q_+ &= (s - (1+\delta)\tau r^2/4, s) \times \delta B.
\end{aligned}$$

**Theorem 3.8.** *Fix $\tau > 0$, $\delta \in [1/2, 1)$ and $\alpha_0 \in (0, \nu)$. Let $u$ be any positive caloric function on $Q = (s - \tau r^2, s) \times B(x, r)$. Then*

$$\|u\|_{\alpha_0, Q'_-, \Lambda} \le C_7 \inf_{Q_+} u \tag{3.20}$$

*where the constant $C_7$ depends increasingly on $C_S^{B,\Lambda}, C_P^{B,\Lambda}, M^{B,\Lambda}$, and on $\tau, p, q, \alpha_0, d, \delta$.*

*Proof.* For the proof we follow [28, Theorem 5.4.2]. Take $k := k(u, \kappa)$ corresponding to $\kappa = 1/2$ in Theorem 3.7. Set $v = e^k u$ and

$$U = (s - \tau r^2, s - 1/2\tau r^2) \times B(x, r), \quad U_\sigma = (s - \tau r^2, s - (3 - \sigma)\tau r^2/4) \times B(x, \sigma r).$$

By Theorem 3.6 it follows that

$$\|v\|_{\alpha_0, U_{\sigma'}, \Lambda} \le \left\{ C_5 \tau (1 + \tau^{-1})^{\frac{\nu}{\nu - 1}} \left[ \frac{1 \vee C_S^{B,\Lambda}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{\nu - 1}} \right\}^{(1+\nu)(1/\alpha - 1/\alpha_0)} \|v\|_{\alpha, U_\sigma, \Lambda}$$

for all $1/2 \le \sigma' < \sigma \le 1$ and all $\alpha \in (0, \alpha_0 \nu^{-1})$, in particular notice that $\alpha_0 \nu^{-1} > \alpha_0/2$ and that $\alpha_0/2 < \nu/2 < 1$ since $\nu \in (1, 2)$. By Theorem 3.7 we have that

$$\gamma^\Lambda\{(t, z) \in U \mid \log v > \ell\} \le C_6\, \gamma^\Lambda(U) \tau^{-1} \Big[ M^{B,\Lambda} (C_P^{B,\Lambda} \vee \tau^2) \Big] \ell^{-1}.$$

Bombieri-Giusti's Lemma C.1 is applicable and we obtain

$$\|e^\kappa u\|_{\alpha_0, Q'_-, \Lambda} \lesssim C_{BG}^B \tag{3.21}$$

where $C_{BG}^B$ depends increasingly on $C_S^{B,\Lambda}, C_P^{B,\Lambda}, M^{B,\Lambda}$, and on $\tau, p, q, \alpha_0, d$.

On the other hand, we can now fix

$$V = (s - 1/2\tau r^2, s) \times B(x, r), \quad V_\sigma = (s - (1 + \sigma)\tau r^2/4, s) \times B(x, \sigma r)$$

and apply Theorem 3.5 to $v = e^{-k} u^{-1}$ where $k$ is the same constant as above, this produces

$$\sup_{V_{\sigma'}} v \le \left\{ C_4 (C_S^{B,\Lambda})^{\frac{1}{\nu - 1}} \tau \left[ \frac{1 + \tau^{-1}}{(\sigma - \sigma')^2} \right]^{\frac{\nu}{\nu - 1}} \right\}^{1/\alpha} \|v\|_{\alpha, V_\sigma, \Lambda},$$

for all $\alpha > 0$ and $1/2 \le \sigma' < \sigma \le 1$. Since by Theorem 3.7 we have

$$\gamma^\Lambda\{(t, z) \in V \mid \log v > \ell\} \le C_6\, \gamma^\Lambda(V) \tau^{-1} \Big[ M^{B,\Lambda} (C_P^{B,\Lambda} \vee \tau^2) \Big] \ell^{-1},$$

then Bombieri-Giusti's lemma is applicable and yields

$$\sup_{Q_+} e^{-\kappa} u^{-1} \lesssim C_{BG}^B \tag{3.22}$$

for some $C_{BG}^B$ which we can assume to be the same as before taking the maximum of the two. Putting (3.21) and (3.22) together gives the result. □

**Theorem 3.9** (Parabolic Harnack inequality)**.** *Fix $\tau > 0$ and $\delta \in [1/2, 1)$. Let $u$ be any positive caloric function in $Q = (s - \tau r^2, s) \times B(x, r)$. Then we have*

$$\sup_{Q_-} u \le C_H^{B,\Lambda} \inf_{Q_+} u, \tag{3.23}$$

*where the constant $C_H^{B,\Lambda}$ depends increasingly on $C_S^{B,\Lambda}, C_P^{B,\Lambda}, M^{B,\Lambda}$, and on $\tau, p, q, d, \delta$.*

*Proof.* It follows from the previous theorem for positive supercaloric functions and Corollary 3.3. □

We have to remark that the constant appearing in (3.23) is strongly dependent on the ball $B$ we are considering, in particular depends on its center and its radius. In the next proposition we use assumption $(b.2)$ to get rid of this dependence for balls which are large enough as it was discussed in Section 2.5.

**Remark 3.10.** By Lemma 2.12, there exist constants $C_H^{*,\Lambda} < \infty$ and $r_H(x) \in [1, \infty)$ such that for all $r \geq r_H(x)$ we have $C_H^{B(x,r),\Lambda} \leq C_H^{*,\Lambda}$.

**Theorem 3.11** (Hölder continuity)**.** *Let $x \in \mathbb{R}^d$, and $r_H(x) \geq 1$ as above. Let $r > r_H(x)$ and $\sqrt{t} \geq r$. Define $t_0 := t + 1$ and $r_0 := \sqrt{t_0}$. If $u$ is a positive caloric function on $(0, t_0) \times B(x, r_0)$, then for all $z, y \in B(x, r)$ we have*

$$u(t,z) - u(t,y) \leq c\left(\frac{r}{\sqrt{t}}\right)^{\theta} \sup_{[3t_0/4, t_0] \times B(x, \sqrt{t_0}/2)} u, \tag{3.24}$$

*where $\theta, c$ are constants which depend only on $C_H^{*,\Lambda}$.*

*Proof.* Set $r_k := 2^{-k} r_0$ and let

$$Q_k := (t_0 - r_k^2, t_0) \times B(x, r_k),$$

$Q_k^-$ and $Q_k^+$ be accordingly defined as in (3.19) with $\delta = 1/2$ and $\tau = 1$,

$$Q_k^- := (t_0 - 7/8 r_k^2, t_0 - 5/8 r_k^2) \times B(x, 1/2 r_k), \quad Q_k^+ := (t_0 - 1/4 r_k^2, t_0) \times B(x, 1/2 r_k).$$

Notice that $Q_{k+1} \subset Q_k$ and actually $Q_{k+1} = Q_k^+$. We set

$$v_k = \frac{u - \inf_{Q_k} u}{\sup_{Q_k} u - \inf_{Q_k} u}.$$

Clearly $v_k$ is a caloric on $Q_k$, in particular $0 \leq v_k \leq 1$ and

$$\operatorname{osc}(v_k, Q_k) := \sup_{Q_k} v_k - \inf_{Q_k} v_k = 1.$$

This implies that, replacing $v_k$ by $1 - v_k$ if necessary, $\sup_{Q_k^-} v_k \geq 1/2$. Now, for all $k$ such that $r_k \geq r_H(x)$ we can apply the parabolic Harnack inequality with common constant $C_H^{*,\Lambda}$ and get

$$\frac{1}{2} \leq \sup_{Q_k^-} v_k \leq C_H^{*,\Lambda} \inf_{Q_k^+} v_k.$$

Since by construction $Q_k^+ = Q_{k+1}$, we deduce that

$$\begin{aligned} \operatorname{osc}(u, Q_{k+1}) &= \frac{\sup_{Q_{k+1}} u - \inf_{Q_{k+1}} u}{osc(u, Q_k)} \operatorname{osc}(u, Q_k) \\ &= \left(\frac{\sup_{Q_{k+1}} u - \inf_{Q_k} u}{\operatorname{osc}(u, Q_k)} - \inf_{Q_{k+1}} v_k\right) \operatorname{osc}(u, Q_k), \end{aligned}$$

which yields $\operatorname{osc}(u, Q_{k+1}) \leq (1 - \delta) \operatorname{osc}(u, Q_k)$ with $\delta^{-1} = 2C_H^{*,\Lambda}$. We can now iterate the inequality up to $k_0$ such that $r_{k_0} \geq r > r_{k_0+1}$ and obtain

$$\operatorname{osc}(u, Q_{k_0}) \leq (1 - \delta)^{k_0 - 1} \operatorname{osc}(u, Q_0^+).$$

Finally since $B(x, r) \subset B(x, r_{k_0})$, $t \in (t_0 - r_{k_0}^2, t_0)$ and $-k_0 \leq \log_2(r/\sqrt{t})$ the claim is proved. □

Starting from (3.24) and knowing that $p_t(z,\cdot)$ is caloric on the whole $\mathbb{R}^d$ for almost all $z \in \mathbb{R}^d$ we get the following corollary.

**Corollary 3.12.** *Let $x \in \mathbb{R}^d$ and $r_H(x) \geq 1$ as above. Let $r > r_H(x)$ and $\sqrt{t} \geq r$. Then we have that for almost all $o \in \mathbb{R}^d$*

$$\sup_{z,y\in B(x,r)} |p_t(o,z) - p_t(o,y)| \leq c\left(\frac{r}{\sqrt{t}}\right)^{\theta} t^{-d/2}, \tag{3.25}$$

*where $\theta, c$ are positive constants which depends only on $C_H^{*,\Lambda}$.*

*Proof.* We have just to bound the right hand side of (3.24). Define $t_0 = t+1$ as in the previous theorem. By the Harnack's inequality applied to the caloric function $p_t(o,\cdot)$ we have

$$\begin{aligned}\sup_{[3t_0/4,t_0]\times B(x,\sqrt{t_0}/2)} p_s(o,u) &\leq C_H^{*,\Lambda} \inf_{[3/2t_0,7/4t_0]\times B(x,\sqrt{t_0}/2)} p_s(o,u)\\ &\leq C_H^{*,\Lambda}\Big[|B(x,\sqrt{t_0}/2)|\|\Lambda\|_{1,B(x,\sqrt{t_0}/2)}\Big]^{-1}\int_{B(x,\sqrt{t_0}/2)} p_{\bar{t}}(o,u)\Lambda(u)\,du\end{aligned}$$

where $\bar{t} \in [3/2t_0, 7/4t_0]$.

Clearly $\int_{B(x,\sqrt{t_0}/2)} p_{\bar{t}}(o,u)\Lambda(u)\,du \leq 1$. Therefore, for $\sqrt{t_0} > r > r_H(x)$, we can bound $\|\Lambda\|_{1,B(x,\sqrt{t_0}/2)}$ by a constant which does not depend on $x$ or $t_0$, hence we finally get the desired estimate. □

We want to stress that Corollary (3.12) is not a true Hölder's continuity result, since we cannot bound the oscillations for arbitrarily small balls, and indeed it is not even possible to prove continuity of the density with this technique.

We are interested in finding Hölder's continuity bounds for $p_{t/\epsilon^2}(o,\cdot/\epsilon)$ for almost all $x \in \mathbb{R}^d$, for small $\epsilon$ . In order to do that we need the following assumption, which accounts for a control of moving averages.

$(b.3)$ there exist $p, q \in [1,\infty]$ satisfying $1/p + 1/q < 2/d$ such that

$$\sup_{x\in\mathbb{R}^d} \limsup_{\epsilon\to 0} \frac{1}{|B(x/\epsilon,1/\epsilon)|}\int_{B(x/\epsilon,1/\epsilon)} \Lambda^p + \lambda^{-q}\,dx < \infty.$$

It is clear that assumption $(b.3)$ implies $((b.2))$; Indeed the latter can be obtained by the former choosing $x = 0$.

**Lemma 3.13.** *Let $F : \mathbb{R}^d \to [0,+\infty)$ and let $\delta, r_0 > 0$. Assume that*

$$\sup_{x\in\mathbb{R}^d} \limsup_{\epsilon\to 0} \frac{1}{|B(x/\epsilon,1/\epsilon)|}\int_{B(x/\epsilon,1/\epsilon)} F\,dz =: K < \infty. \tag{3.26}$$

*Then, there exists a constant $\epsilon_1(x,R_0,\delta) > 0$ such that for all $x \in \mathbb{R}^d$ and all $\epsilon < \epsilon_1(x,r_0,\delta)$*

$$\sup_{r\geq r_0} \frac{1}{|B(x/\epsilon,r/\epsilon)|}\int_{B(x/\epsilon,r/\epsilon)} F\,dz < K(1+\delta).$$

*Proof.* Fix $x \in \mathbb{R}^d$. First we observe that it is enough to prove the statement for $r_0 = 1$ and $x \neq 0$, being the case $r_0 \neq 1$ completely analogous and the case $x = 0$ immediate. Let $0 < \delta_0 < |x|$. We split the supremum into two parts

$$\begin{aligned}&\sup_{r\geq 1}\frac{1}{|B(x/\epsilon,r/\epsilon)|}\int_{B(x/\epsilon,r/\epsilon)} F\,dz\\ &\quad= \sup_{1\leq r\leq |x|/\delta_0} \frac{1}{|B(x/\epsilon,r/\epsilon)|}\int_{B(x/\epsilon,r/\epsilon)} F\,dz \vee \sup_{r\geq |x|/\delta_0} \frac{1}{|B(x/\epsilon,/\epsilon)|}\int_{B(x/\epsilon,r/\epsilon)} F\,dz.\end{aligned}$$

First we deal with the second part.

$$\sup_{r\geq|x|/\delta_0}\frac{1}{|B(x/\epsilon,r/\epsilon)|}\int_{B(x/\epsilon,R/\epsilon)}F\,dz$$
$$\leq\sup_{r\geq|x|/\delta_0}\left(1+\frac{|x|}{r}\right)^d\frac{1}{|B(0,(r+|x|)/\epsilon)|}\int_{B(0,(r+|x|)/\epsilon)}F\,dz.$$

Recall that by (3.26), for all $r>0$ there exists $\epsilon_2(x,r,\delta_0)>0$ such that for all $\epsilon<\epsilon_2(x,r,\delta_0)$

$$\frac{1}{|B(x/\epsilon,r/\epsilon)|}\int_{B(x/\epsilon,r/\epsilon)}F\,dz<K(1+\delta_0). \tag{3.27}$$

In particular, recalling that $r\geq|x|/\delta_0>1$, for all $\epsilon<\epsilon_2(0,1,\delta_0)$

$$\sup_{r\geq|x|/\delta_0}\frac{1}{|B(x/\epsilon,r/\epsilon)|}\int_{B(x/\epsilon,r/\epsilon)}F\,dz\leq K(1+\delta_0)^{d+1}.$$

The first part is a bit more delicate. For all $\epsilon>0$ define $\Psi_\epsilon:[1,\infty)\to[0,\infty)$ by

$$\Psi_\epsilon(r):=\frac{1}{|B(x/\epsilon,r/\epsilon)|}\int_{B(x/\epsilon,r/\epsilon)}F\,dz.$$

Let $1\leq r_-<r_+\leq\bar r:=|x|/\delta_0$ and $r\in[r_-,r_+]$, then we have

$$\Psi_\epsilon(r)-\Psi_\epsilon(r_+)\leq d\bar r^{2d-1}\cdot\frac{1}{|B(x/\epsilon,\bar r/\epsilon)|}\int_{B(x/\epsilon,\bar r/\epsilon)}Fdz\cdot(r_+-r_-)$$

and for all $x\in\mathbb{R}^d$ we can find $\epsilon_2(x,\bar r,\delta_0)>0$ such that for all $\epsilon<\epsilon_2(x,\bar r,\delta_0)$

$$\Psi_\epsilon(r)-\Psi_\epsilon(r_+)\leq dK(1+\delta_0)\bar r^{2d-1}(r_+-r_-).$$

Now, take a partition $1=r_0,...,r_m=:\bar r$ of $[1,\bar r]$ in such a way that

$$|r_i-r_{i-1}|\leq\delta_0/(d\bar r^{2d-1}(1+\delta_0))$$

for all $i=1,...,m$. Define $\epsilon_3(x,\delta_0):=\epsilon_2(x,\bar r,\delta_0)\wedge\min_{i=1,\dots,m}\epsilon_2(x,r_i,\delta_0)>0$. Then for all $x\in\mathbb{R}^d$ and all $\epsilon\leq\epsilon_3(x,\delta_0)$, we have that for all $r\in[1,\bar r]$

$$\frac{1}{|B(x/\epsilon,r/\epsilon)|}\int_{B(x/\epsilon,r/\epsilon)}F\,dz=\Psi_\epsilon(r)$$
$$=\Psi_\epsilon(r_{i(r)})+(\Psi_\epsilon(r)-\Psi_\epsilon(r_{i(r)}))\leq K(1+\delta_0)+K\delta_0=K(1+2\delta_0), \tag{3.28}$$

where $i(r)$ is such that $0\leq r_{i(r)}-r\leq\delta_0/(d\bar r^{2d-1}(1+\delta_0))$. Putting together (3.27) and (3.28), and defining $\epsilon_1(x,\delta_0):=\epsilon_2(0,1,\delta_0)\wedge\epsilon_3(x,\delta_0)$, we can deduce that for all $x\in\mathbb{R}^d$ and all $\epsilon<\epsilon_1(x,\delta_0)$

$$\sup_{r\geq1}\frac{1}{|B(x/\epsilon,r/\epsilon)|}\int_{B(x/\epsilon,r/\epsilon)}F\,dz\leq K(1+2\delta_0)\wedge K(1+\delta_0)^{d+1}.$$

Finally, the statement follows by an appropriate choice of $\delta_0$ small enough and taking into account the dependence on $r_0>0$ for the case $r_0\neq1$. □

**Remark 3.14.** Assumption $(b.3)$ and Lemma 3.13 allow to control $C_S^{B,\Lambda}$, $C_P^{B,\Lambda}$, $M^{B,\Lambda}$ uniformly on balls $B=B(x/\epsilon,r/\epsilon)$ when $\epsilon$ is small enough. In particular, for all $x\in\mathbb{R}^d$ and $r_0>0$ we can find $\epsilon_H(x,r_0)>0$ and a finite constant $C_H^{*,\Lambda}$ independent of $x$, $r_0$ such that for all $\epsilon<\epsilon_H(x,r_0)$ and $r\geq r_0$

$$C_H^{B(x/\epsilon,r/\epsilon),\Lambda}\leq C_H^{*,\Lambda}.$$

**Lemma 3.15.** *Fix $r > 0$, $\sqrt{t} \geq r$ and $x \in \mathbb{R}^d$. Then for all $\epsilon < \epsilon_H(x,r)$ we have that for almost all $o \in \mathbb{R}^d$*

$$\sup_{z,y\in B(x,r)} \epsilon^{-d}|p_{t/\epsilon^2}(o,z/\epsilon) - p_{t/\epsilon^2}(o,y/\epsilon)| \leq c\Big(\frac{r}{\sqrt{t}}\Big)^{\theta} t^{-d/2} \tag{3.29}$$

*where $\theta, c$ are positive constants which depend only on $C_H^{*,\Lambda}$.*

*Proof.* The proof is the same as in Theorem 3.11 since given a caloric function $u(t,x)$, $u(t/\epsilon^2, x/\epsilon)$ is also caloric with respect to the Dirichlet form with coefficients given by $a(x/\epsilon)$. Assumption $(b.3)$ is used to have uniform constants for moving averages. □

From the estimate (3.29) we can prove the following key lemma.

**Lemma 3.16.** *Let $I \subset (0,\infty)$ be a compact interval. Then, for almost all $o \in \mathbb{R}^d$ and all $r > 0$*

$$\lim_{r_0\to 0} \limsup_{\epsilon\to 0} \sup_{\substack{x,y\in B(o,r)\\|x-y|<r_0}} \sup_{t\in I} \epsilon^{-d}|p_{t/\epsilon^2}(o,x/\epsilon) - p_{t/\epsilon^2}(o,y/\epsilon)| = 0$$

*Proof.* Let us denote by $t_1 := \inf I$. Fix $\delta > 0$ and set

$$r_0 := \frac{\sqrt{t_1}}{2} \wedge \Big(\frac{t_1^{d/2}\delta}{2c}\Big)^{1/\theta}\sqrt{t_1}.$$

Since $B(o,r)$ is compact we can cover it by a finite set of balls $\{B(x,r_0/2)\}_{x\in\mathcal{X}}$ of radius $r_0/2$ and centers $x \in \mathcal{X} \subset B(o,r)$. Set $\bar\epsilon := \min_{x\in\mathcal{X}} \epsilon_H(x,r_0)$, then an application of (3.29) gives for all $\epsilon < \bar\epsilon$

$$\sup_{x\in\mathcal{X}} \sup_{|x-y|<r_0} \sup_{t\in I} \epsilon^{-d}|p_{t/\epsilon^2}(o,x/\epsilon) - p_{t/\epsilon^2}(o,y/\epsilon)| \leq c\Big(\frac{r_0}{\sqrt{t_1}}\Big)^{\theta} t_1^{-d/2} \leq \frac{\delta}{2}.$$

Next we can use this bound to conclude, namely take $z \in B(o,r)$, and $x \in \mathcal{X}$ such that $|z-x| < r_0/2$, then

$$\begin{aligned}\sup_{|z-y|\leq r_0/2} \sup_{t\in I} &\epsilon^{-d}|p_{t/\epsilon^2}(o,z/\epsilon) - p_{t/\epsilon^2}(o,y/\epsilon)| \\ &\leq \sup_{t\in I} \epsilon^{-d}|p_{t/\epsilon^2}(o,x/\epsilon) - p_{t/\epsilon^2}(o,z/\epsilon)| \\ &+ \sup_{|y-x|\leq r_0} \sup_{t\in I} \epsilon^{-d}|p_{t/\epsilon^2}(o,x/\epsilon) - p_{t/\epsilon^2}(o,y/\epsilon)| \leq \delta\end{aligned}$$

and this ends the proof since we show that the bound is uniform in $z \in B(o,r)$. □

## 4 Local Central Limit Theorem

We finally give the main application of the computations we have developed in the preceding sections. The approach we exploit is the one in [7, 10], in particular [10, Assumption (4)] must be compared with our inequality (3.24).

We denote by $k_t^{\Sigma}(x)$, $x \in \mathbb{R}^d$ the gaussian kernel with covariance matrix $\Sigma$, namely

$$k_t^{\Sigma}(x) := \frac{1}{\sqrt{(2\pi t)^d \det\Sigma}} \exp\Big(-\frac{x\cdot\Sigma^{-1}x}{2t}\Big).$$

We need here two further assumptions

$(b.4)$ for almost all $x \in \mathbb{R}^d$ and all $r > 0$

$$\lim_{\epsilon \to 0} \frac{1}{|B(x/\epsilon, r/\epsilon)|} \int_{B(x/\epsilon, r/\epsilon)} \Lambda \, dx =: a_\Lambda < \infty.$$

$(b.5)$ there exists a positive define symmetric matrix $\Sigma$ such that for almost all $o \in \mathbb{R}^d$, for any compact interval $I \subset (0, \infty)$, almost all $x \in \mathbb{R}^d$ and $r > 0$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^d} \int_{B(x,r)} p_{t/\epsilon^2}(o, y/\epsilon) \, \Lambda(y/\epsilon) dy \to \int_{B(x,r)} k_t^\Sigma(y) \, dy,$$

uniformly in $t \in I$.

**Theorem 4.1.** *Fix a compact interval $I \subset (0, \infty)$ and $r > 0$. Assume $(b.1)$–$(b.5)$, then for almost all $o \in \mathbb{R}^d$ and for all $r > 0$*

$$\lim_{\epsilon \to 0} \sup_{x \in B(o,r)} \sup_{t \in I} |\epsilon^{-d} p_{t/\epsilon^2}(o, x/\epsilon) - a_\Lambda^{-1} k_t^\Sigma(x)| = 0.$$

*Proof.* The proof presented here is a slight variation of the one in [7, 10], the only difference is that we work on $\mathbb{R}^d$ rather than on graphs. For $x \in B(o, r)$ and $r_0 > 0$ we denote

$$J(t, \epsilon) := \frac{1}{\epsilon^d} \int_{B(x,r_0)} p_{t/\epsilon^2}(o, y/\epsilon) \, \Lambda(y/\epsilon) dy - \int_{B(x,r_0)} k_t^\Sigma(y) \, dy,$$

where $k_t^\Sigma$ is the gaussian kernel with covariance matrix $\Sigma$. Now, we can split $J(t, \epsilon) = J_1(t, \epsilon) + J_2(t, \epsilon) + J_3(t, \epsilon) + J_4(t, \epsilon)$ where

$$\begin{aligned}
J_1(t, \epsilon) &:= \int_{\frac{1}{\epsilon} B(x,r_0)} \Big( p_{t/\epsilon^2}(o, y) - p_{t/\epsilon^2}(o, x/\epsilon) \Big) \Lambda(y) dy, \\
J_2(t, \epsilon) &:= \int_{\frac{1}{\epsilon} B(x,r_0)} \Lambda(y) dy \, \Big( p_{t/\epsilon^2}(o, x/\epsilon) - \epsilon^d a_\Lambda^{-1} k_t^\Sigma(x) \Big), \\
J_3(t, \epsilon) &:= k_t^\Sigma(x) \Big( \epsilon^d a_\Lambda^{-1} \int_{\frac{1}{\epsilon} B(x,r_0)} \Lambda(y) dy - |B(x, r_0)| \Big), \\
J_4(t, \epsilon) &:= \int_{B(x,r_0)} (k_t^\Sigma(x) - k_t^\Sigma(y)) dy.
\end{aligned}$$

Fix $\delta > 0$. By the continuity of $k_t^\Sigma$ we can choose $r_0 \in (0, 1)$ small enough such that

$$\sup_{\substack{x,y \in B(0,r+1) \\ |x-y| \le r_0}} \sup_{t \in I} |k_t^\Sigma(y) - k_t^\Sigma(x)| \le \delta, \tag{4.1}$$

from which we can easily obtain the bound $\sup_{t \in I} |J_4(t, \epsilon)| \le \delta |B(x, r_0)|$. Taking $r_0$ smaller if needed, thanks to Lemma 3.16, we can find $\bar{\epsilon} > 0$ such that for all $\epsilon < \bar{\epsilon}$

$$\sup_{\substack{x,y \in B(o,r+1) \\ |x-y| \le r_0}} \sup_{t \in I} \epsilon^{-d} |p_{t/\epsilon^2}(o, y/\epsilon) - p_{t/\epsilon^2}(o, x/\epsilon)| \le \delta, \tag{4.2}$$

which immediately implies that $\sup_{t \in I} |J_1(t, \epsilon)| \le \delta |B(x, r_0)|$. Furthermore, by assumption $(b.4)$ taking $\bar{\epsilon}$ smaller if needed we get $\sup_{t \in I} |J_3(t, \epsilon)| \le \delta |B(x, r_0)|$ for all $\epsilon \le \bar{\epsilon}$. Finally, assumption $(b.5)$ readily gives $\sup_{t \in I} |J(t, \epsilon)| \le \delta |B(x, r_0)|$ for $\epsilon$ small enough.

These estimates can then be used to control $|J_2(t, \epsilon)|$ for $\epsilon \le \bar{\epsilon}$ uniformly in $t \in I$. Namely, one gets

$$\sup_{t \in I} |\epsilon^{-d} p_{t/\epsilon^2}(o, x/\epsilon) - a_\Lambda^{-1} k_t^\Sigma(x)| \le 4\delta \bigg( \frac{\epsilon^d}{|B(x, r_0)|} \int_{\frac{1}{\epsilon} B(x,r_0)} \Lambda(y) dy \bigg)^{-1}$$

and we can take $\bar{\epsilon}$ even smaller to obtain, using assumption $(b.4)$,

$$\left(\frac{\epsilon^d}{|B(x,r_0)|}\int_{\frac{1}{\epsilon}B(x,r_0)}\Lambda(y)dy\right)^{-1}\le(\delta+a_\Lambda).$$

This implies that for almost all $x\in R^d$

$$\lim_{\epsilon\to 0}\sup_{t\in I}|\epsilon^{-d}p_{t/\epsilon^2}(o,x/\epsilon)-a_\Lambda^{-1}k_t^\Sigma(x)|=0.$$

Consider now $r>0$, $\delta>0$ and let $r_0\in(0,1)$ be chosen as before. Since $B(o,r)$ is compact there exists a finite covering $\{B(z,r_0)\}_{z\in\mathcal{X}}$ of $B(o,r)$ with $\mathcal{X}\subset B(o,r)$. Since $\mathcal{X}$ is finite, there exists $\bar{\epsilon}>0$ such that for all $\epsilon\le\bar{\epsilon}$

$$\sup_{z\in\mathcal{X}}\sup_{t\in I}|\epsilon^{-d}p_{t/\epsilon^2}(o,z/\epsilon)-a_\Lambda^{-1}k_t^\Sigma(z)|\le\delta.$$

Next, for a general $x\in B(o,r)$, there exists some $z\in\mathcal{X}$ such that $x\in B(z,r_0)$. Thus, we can write

$$\begin{aligned}\sup_{t\in I}|\epsilon^{-d}p_{t/\epsilon^2}(o,x/\epsilon)-a_\Lambda^{-1}k_t^\Sigma(x)|\le&\sup_{t\in I}\epsilon^{-d}|p_{t/\epsilon^2}(o,x/\epsilon)-p_{t/\epsilon^2}(o,z/\epsilon)|\\&+\sup_{t\in I}|\epsilon^{-d}p_{t/\epsilon^2}(o,z/\epsilon)-a_\Lambda^{-1}k_t^\Sigma(z)|\\&+a_\Lambda^{-1}\sup_{t\in I}|k_t^\Sigma(x)-k_t^\Sigma(z)|.\end{aligned}$$

Since $x,z\in B(o,r+1)$ and $|x-z|\le r_0$, inequality (4.1) implies that the last addendum is bounded by $\delta$, the second addendum is also bounded uniformly by $\delta$ since $z\in\mathcal{X}$. We can bound the first term uniformly by $\delta$ by means of (4.2). This ends the proof. □

### 4.1 Application to Diffusions in Random Environment

In this section we finally apply Theorem 4.1 to obtain Theorem 1.2.

*Proof of Theorem 1.2.* It is enough to show that assumptions $(b.1)$-$(b.5)$ are satisfied for $\mu$-almost all realizations of the environment, then Theorem 4.1 gives the result.

By construction $(a.1)$ implies $(b.1)$ for $\mu$-almost all $\omega\in\Omega$. Assumption $(a.2)$ together with the ergodic theorem [22, Theorem 11.18] gives easily $(b.2)$-$(b.4)$ $\mu$-almost surely, in particular the constant $a_\Lambda$ equals $\mathbb{E}_\mu[\Lambda]$. Finally $(b.5)$ for $\mu$-almost all $\omega\in\Omega$ can be deduced directly from $(a.3)$.

The second part of the statement follows readily since, if we assume $\lambda^\omega(\cdot)^{-1},\Lambda^\omega(\cdot)\in L^\infty_{loc}(\mathbb{R}^d)$ for $\mu$-almost all $\omega\in\Omega$, then the density $p_t^\omega(x,y)$ is a continuous function of $x$ and $y$ by classical results in PDE theory [19]. Thus, Theorem 4.1 holds for all $o\in\mathbb{R}^d$, $\mu$-almost surely. □

## A On the moment condition for the time-changed model

Let us take $\theta:\Omega\to(0,+\infty)$ and $\theta^\omega(x):=\theta(\tau_x\omega)$ such that $\theta^\omega$ and $1/\theta^\omega$ are in $L^1_{loc}(\mathbb{R}^d)$ almost surely. One can then consider the Dirichlet form $(\mathcal{E}^\omega,\mathcal{F}^{\theta,\omega})$ on $L^2(\mathbb{R}^d,\theta^\omega dx)$ where $\mathcal{E}^\omega$ is given by (1.2) and $\mathcal{F}^{\theta,\omega}$ is the closure of $C_0^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d,\theta^\omega dx)$ with respect to $\mathcal{E}^\omega+(\cdot,\cdot)_\theta$. This corresponds to the formal generator

$$L^\omega u(x)=\frac{1}{\theta^\omega(x)}\nabla\cdot(a^\omega(x)\nabla u(x)).$$

What are the conditions on $\theta$, $\lambda$ and $\Lambda$ in order to obtain a quenched local central limit theorem for the diffusion associated to $(\mathcal{E}^\omega,\mathcal{F}^{\theta,\omega})$? It turns out that if

$$\mathbb{E}_\mu[\theta^r]<\infty,\ \mathbb{E}_\mu[\lambda^{-q}]<\infty,\ \mathbb{E}_\mu[\Lambda^p\theta^{1-p}]<\infty,$$

where $p, q, r \in (1, \infty]$ are such that

$$\frac{1}{q} + \frac{1}{r} + \frac{r-1}{r} \cdot \frac{1}{p} < \frac{2}{d}, \tag{A.1}$$

then we can still derive a parabolic Harnack inequality of the type (3.23) and thus obtain a quenched local central limit theorem also in this situation. Let us motivate this claim.

In the same spirit of sections 2 and 3 let us fix the environment $\omega \in \Omega$ and, abusing notation, consider $\theta(x)$, $\lambda(x)$, $\Lambda(x)$ and $a(x)$ in place of $\theta^\omega(x)$, $\lambda^\omega(x)$, $\Lambda^\omega(x)$ and $a^\omega(x)$. We look for a priori estimates for solutions to the formal parabolic equation

$$\partial_t u(t,x) - \frac{1}{\theta(x)} \nabla \cdot (a(x) \nabla u(t,x)) = 0.$$

Following the strategy of Section 3 it is clear that the iterative step (3.1) in Proposition 3.1 is central to obtain the parabolic Harnack inequality. One could ask whether the argument given in such a proposition still works for the case $\theta = \Lambda$. By redoing the very same computations of Proposition 3.1 with the same notation, we get in place of (3.3)

$$\begin{aligned} \sup_{t \in I} \zeta(t) \|(\eta u_t^\alpha)^2\|_{1,B,\theta} + \int_I \zeta(s) \frac{\mathcal{E}_\eta(u_s^\alpha, u_s^\alpha)}{|B|} \, ds \\ \lesssim \alpha \Big[ \|\zeta'\|_\infty + \|\nabla \eta\|_\infty^2 \Big] \int_I \|u_s^{2\alpha}\|_{1,B,\theta} \vee \|u_s^{2\alpha}\|_{1,B,\Lambda} \, ds. \end{aligned} \tag{A.2}$$

On the right hand side we will need the weight $\theta$, this is easily achieved with Hölder inequality

$$\|u_s^{2\alpha}\|_{1,B,\Lambda} \le \|u_s^{2\alpha}\|_{p^*,B,\theta} \|\Lambda \theta^{1/(p-1)}\|_{p,B}, \qquad \|u_s^{2\alpha}\|_{1,B,\theta} \le \|u_s^{2\alpha}\|_{p^*,B,\theta},$$

provided that $\Lambda \theta^{1/(p-1)} \in L^p_{loc}(\mathbb{R}^d)$, which explains the condition $\mathbb{E}_\mu[\Lambda^p \theta^{1-p}] < \infty$. Notice that whenever $\theta \ge \Lambda$ the condition $\mathbb{E}_\mu[\Lambda^p \theta^{1-p}] < \infty$ is satisfied for all $p$, so that in (A.1) it is possible to take $p = \infty$.

At this point of the argument, we realized that (3.4) does not give an optimal moment condition, and one should better play with space-time norms. With this in mind let us introduce some notation following [23].

For any non-empty, compact interval $I \subset \mathbb{R}$ and any $B \subset \mathbb{R}^d$ finite, we introduce for arbitrary $\alpha, \beta \in [1, \infty)$ the Banach space, $L^{\alpha,\beta}(I \times B)$, of measurable functions $u : I \times B \to \mathbb{R}$ with norm

$$\|u\|_{\alpha,\beta,I\times B,\theta} := \left( \frac{1}{|I|} \int_I \left( \frac{1}{|B|} \int_B |u(t,x)|^\alpha \, \theta(x) dx \right)^{\beta/\alpha} dt \right)^{1/\beta}$$

and

$$\|u\|_{\alpha,\infty,I\times B,\theta} := \max_{t \in I} \left( \frac{1}{|B|} \int_B |u(t,x)|^\alpha \, \theta(x) dx \right)^{1/\alpha}.$$

The lemma which follows takes the role of (3.4) in Proposition 3.1.

**Lemma A.1.** *Let $v : I \times B \to \mathbb{R}$. Then, for any $\gamma_1 \ge \varrho \ge 1$ and $\gamma_2 \ge 1$ such that $\varrho/\gamma_1 + (\varrho - 1)/\gamma_2 = \varrho$ it holds that*

$$\|v\|_{\gamma_1,\gamma_2,I\times B,\theta} \le \|v\|_{1,\infty,I\times B,\theta} + \|v\|_{\varrho,1,I\times B,\theta}.$$

*Proof.* This is [23, Lemma 1]. □

We apply Lemma A.1 to $v = \zeta\eta^2 u^{2\alpha}$ with $\gamma_1 = \kappa p^*$, $\gamma_2 = \kappa$ and $\varrho = \rho/2r^*$. This yields

$$\|\zeta\eta^2 u^{2\alpha}\|_{\kappa p^*,\kappa,I\times B,\theta} \le \|\zeta\eta^2 u^{2\alpha}\|_{1,\infty,I\times B,\theta} + \|\zeta\eta^2 u^{2\alpha}\|_{\rho/2r^*,1,I\times B,\theta}. \tag{A.3}$$

for $\kappa = 2 - 1/p - 2r^*/\rho$. The choice of $\varrho$ is done to apply Sobolev inequality (2.8) and obtain

$$\begin{aligned}\|\zeta\eta^2 u^{2\alpha}\|_{\rho/2r^*,1,I\times B,\theta} =& \frac{1}{|I|}\int_I \|\zeta\eta^2 u_s^{2\alpha}\|_{\rho/2r^*,B,\theta} ds\\ &\lesssim \|\theta\|_{r,B}^{2r^*/\rho} C_S^{B,\Lambda} \frac{|B|^{\frac{2}{d}}}{|I|}\int_I \frac{\mathcal{E}_\eta(u_s^\alpha, u_s^\alpha)}{|B|} + \|\nabla\eta\|_\infty^2 \|u_s^{2\alpha}\|_{1,B,\Lambda}\, ds.\end{aligned}$$

Notice that $\mathbb{E}_\mu[\theta^r] < \infty$ is needed to control $\|\theta\|_{r,B}$ for large balls.

Using (A.2) and (A.3) it is now possible to deduce the following iterative step

$$\begin{aligned}&\|\zeta\eta^2 u^{2\alpha}\|_{\kappa p^*,\kappa,I\times B,\theta}\\ &\lesssim \alpha \|1+\Lambda\theta^{1/(p-1)}\|_{p,B}\Big(|I| + \|\theta\|_{r,B}^{2r^*/\rho} C_S^{B,\Lambda}|B|^{\frac{2}{d}}\Big)\Big[\|\zeta'\|_\infty + \|\nabla\eta\|_\infty^2\Big]\|u^{2\alpha}\|_{p^*,1,I\times B,\theta}.\end{aligned}$$

To be able to perform Moser iteration as in Theorem 3.2, we will need $\kappa = 2-1/p-2r^*/\rho > 1$, which is equivalent to

$$\frac{1}{r} + \frac{1}{q} + \frac{r-1}{r}\cdot\frac{1}{p} < \frac{2}{d}.$$

## B Dirichlet Forms

Let $X$ be a locally compact metric separable space, and $m$ a positive Radon measure on $X$ such that $\mathrm{supp}[X] = m$. Consider the Hilbert space $L^2(X, m)$ with scalar product $\langle\cdot,\cdot\rangle$. We call a *symmetric* form, a non-negative definite bilinear form $\mathcal{E}$ defined on a dense subset $\mathcal{D}(\mathcal{E}) \subset L^2(X, m)$. Given a symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, m)$, the form $\mathcal{E}_\beta := \mathcal{E} + \beta\langle\cdot,\cdot\rangle$ defines a new symmetric form on $L^2(X, m)$ for each $\beta > 0$. Note that $\mathcal{D}(\mathcal{E})$ is a pre-Hilbert space with inner product $\mathcal{E}_\beta$. If $\mathcal{D}(\mathcal{E})$ is complete with respect to $\mathcal{E}_\beta$, then $\mathcal{E}$ is said to be *closed*.

A closed symmetric form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, m)$ is called a *Dirichlet form* if it is Markovian, namely if for any given $u \in \mathcal{D}(\mathcal{E})$, then $v = (0 \vee u) \wedge 1$ belongs to $\mathcal{D}(\mathcal{E})$ and $\mathcal{E}(v, v) \le \mathcal{E}(u, u)$.

We say that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(X, m)$ is *regular* if there is a subset $\mathcal{H}$ of $\mathcal{D}(\mathcal{E}) \cap C_0(X)$ dense in $\mathcal{D}(\mathcal{E})$ with respect to $\mathcal{E}_1$ and dense in $C_0(X)$ with respect to the uniform norm. $\mathcal{H}$ is called a *core* for $\mathcal{D}(\mathcal{E})$.

We say that the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is *local* if for all $u, v \in \mathcal{D}(\mathcal{E})$ with disjoint compact support $\mathcal{E}(u, v) = 0$. $\mathcal{E}$ is said *strongly local* if $u, v \in \mathcal{D}(\mathcal{E})$ with compact support and $v$ constant on a neighborhood of $\mathrm{supp}\, u$ implies $\mathcal{E}(u, v) = 0$.

**Lemma B.1.** *Let $B \subset \mathbb{R}^d$ and consider a cutoff $\eta \in C_0^\infty(B)$. Then, $u \in \mathcal{F}_{loc} \cup \mathcal{F}_{loc}^\Lambda$ implies $\eta u \in \mathcal{F}_B$.*

*Proof.* Take $u \in \mathcal{F}_{loc}^\Lambda$, then there exists $\bar{u} \in \mathcal{F}^\Lambda$ such that $u = \bar{u}$ on $2B$. Let $\{f_n\}_{\mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)$ be such that $f_n \to \bar{u}$ with respect to $\mathcal{E} + \langle\cdot,\cdot\rangle_\Lambda$. Clearly $\eta f_n \in \mathcal{F}_B^\Lambda$ and $\eta f_n \to \eta\bar{u} = \eta u$ in $L^2(B, \Lambda dx)$. Moreover

$$\mathcal{E}(\eta f_n - \eta f_m) \le 2\mathcal{E}(f_n - f_m) + \|\nabla\eta\|_\infty^2 \int_B |f_n - f_m|^2 \Lambda dx.$$

Hence $\eta f_n$ is Cauchy in $L^2(B, \Lambda dx)$ with respect to $\mathcal{E} + \langle\cdot,\cdot\rangle_\Lambda$, which implies that $\eta u \in \mathcal{F}_B^\Lambda = \mathcal{F}_B$. If $u \in \mathcal{F}_{loc}$ the proof is similar and one has only to observe that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $W^{2q/(q+1)}(B)$, which by Sobolev's embedding theorem implies that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $L^2(B, \Lambda dx)$. ☐

**Lemma B.2.** *Let $\psi : \mathbb{R} \to \mathbb{R}$ be a globally Lipschitz function with constant $L$ and with $\psi(0) = 0$. If $u \in \mathcal{F}$ ($\mathcal{F}_{loc}$), then $\psi(u) \in \mathcal{F}$ ($\mathcal{F}_{loc}$).*

*Proof.* Let $B \subset \mathbb{R}^d$ be a ball, and $\tilde{u} \in \mathcal{F}$ such that $u = \tilde{u}$ on $B$. Then, it is easy to verify that the function $\psi(\tilde{u})/L$ is a normal contraction of $\tilde{u}$, since

$$|\psi(\tilde{u}(x)) - \psi(\tilde{u}(y))| \le L|\tilde{u}(x) - \tilde{u}(y)|, \quad |\psi(\tilde{u}(x))| \le L|\tilde{u}(x)|.$$

Hence $\psi(\tilde{u}) \in \mathcal{F}$ as can be seen in [18, Chapter 1]. In particular $\psi(\tilde{u}) = \psi(u)$ on $B$, which ends the proof. □

**Lemma B.3.** *Let $F : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with bounded second derivative and positive first derivative. Assume that $F'(0) = 0$. Then for any caloric (subcaloric, supercaloric) function $u$ we have*

$$\frac{d}{dt}(F(u_t), \phi)_\Lambda + \mathcal{E}(u_t, F'(u_t)\phi) = 0, \quad (\le, \ge)$$

*for all $\phi \in C_0^\infty(\mathbb{R}^d)$, $\phi > 0$ and $t > 0$.*

*Proof.* Observe that $F''$ bounded and $F'(0) = 0$ implies that $F'(u_t) \in \mathcal{F}^\Lambda$ by Lemma B.2.

$$\begin{aligned}\frac{d}{dt}(F(u_t), \phi)_\Lambda &= \lim_{h\downarrow 0} \frac{1}{h}(F(u_{t+h}) - F(u_t), \phi)_\Lambda \\ &= \lim_{h\downarrow 0} \frac{1}{h}(F'(u_t)(u_{t+h} - u_t), \phi)_\Lambda + \frac{1}{h}(R(u_{t+h} - u_t), \phi)_\Lambda,\end{aligned}$$

where $|R(x)| \le \|F''\|_\infty |x|^2$. The first summand converges to $\mathcal{E}(u_t, F'(u_t)\phi)$ since $u_t$ solves (2.3). It remains to show that the second summand goes to zero. It is enough to see that

$$\frac{1}{h}|(R(u_{t+h} - u_t), \phi)_\Lambda| \le h\|\phi\|_\infty \|F''\|_\infty \|(u_{t+h} - u_t)h^{-1}\|_{2,\Lambda}^2 \to 0$$

as $h \to 0$. For subcaloric and supercaloric functions the proof follows the same lines. □

## C Bombieri-Giusti's Lemma

In order to obtain an Harnack inequality for positive weak solutions to an elliptic or parabolic equation we will make use of the following lemma due to Bombieri and Giusti, whose proof can be found in [28] or in the original paper [8].

Consider a collection of measurable subsets $U_\sigma$, $0 < \sigma \le 1$, of a fixed measure space $(\mathcal{X}, \mathcal{M})$ endowed with a measure $\gamma$, such that $U_{\sigma'} \subset U_\sigma$ whenever $\sigma' < \sigma$. In our application, $U_\sigma$ will be $B(x, \sigma r)$ for some fixed ball $B(x, r) \subset \mathbb{R}^d$.

**Lemma C.1** (Bombieri-Giusti [8])**.** *Fix $\delta \in (0, 1)$. Let $\kappa$ and $K_1, K_2$ be positive constants and $0 < \alpha_0 \le \infty$. Let $u$ be a positive measurable function on $U := U_1$ which satisfies*

$$\left(\int_{U_{\sigma'}} |u|^{\alpha_0}\, d\gamma\right)^{\frac{1}{\alpha_0}} \le \Big(K_1(\sigma - \sigma')^{-\kappa}\gamma(U)^{-1}\Big)^{\frac{1}{\alpha} - \frac{1}{\alpha_0}} \left(\int_{U_\sigma} |u|^{\alpha}\, d\gamma\right)^{\frac{1}{\alpha}} \tag{C.1}$$

*for all $\sigma, \sigma'$ and $\alpha$ such that $0 < \delta \le \sigma' < \sigma \le 1$ and $0 < \alpha \le \min\{1, \alpha_0/2\}$. Assume further that $u$ satisfies*

$$\gamma(\log u > \ell) \le K_2\, \gamma(U)\ell^{-1} \tag{C.2}$$

*for all $\ell > 0$. Then*

$$\left(\int_{U_\delta} |u|^{\alpha_0}\, d\gamma\right)^{\frac{1}{\alpha_0}} \le C_{BG}\, \gamma(U)^{\frac{1}{\alpha_0}},$$

*where $C_{BG}$ depends only on $K_1, K_2, \delta, \kappa$ and a lower bound on $\alpha_0$.*

## References

[1] Andres, S., Deuschel, J.-D., and Slowik, M. (2015). *Heat kernel estimates for random walks with degenerate weights*, arXiv:1412.4338.

[2] Andres, S., Deuschel, J.-D., and Slowik, M. (2015). Harnack inequalities on weighted graphs and some applications to the random conductance model, Probability Theory and Related Fields, 1–47.

[3] Andres, S., Deuschel, J.-D., and Slowik, M. (2015). Invariance principle for the random conductance model in a degenerate ergodic environment. *Ann. Probab.* **43**, 4, 1866–1891. MR-3353817

[4] Aronson, D. G. (1967). Bounds for the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.* **73**, 890–896. MR-0217444

[5] Ba, M. and Mathieu, P. (2015). A Sobolev inequality and the individual invariance principle for diffusions in a periodic potential. *SIAM J. Math. Anal.* **47**, 3, 2022–2043. MR-3348123

[6] Barlow, M. T., Grigor'yan, A., and Kumagai, T. (2012). On the equivalence of parabolic Harnack inequalities and heat kernel estimates. *J. Math. Soc. Japan* **64**, 4, 1091–1146. MR-2998918

[7] Barlow, M. T. and Hambly, B. M. (2009). Parabolic Harnack inequality and local limit theorem for percolation clusters. *Electron. J. Probab. 14*, no. 1, 1–27. MR-2471657

[8] Bombieri, E. and Giusti, E. (1972). Harnack's inequality for elliptic differential equations on minimal surfaces. *Invent. Math. 15*, 24–46. MR-0308945

[9] Chiarini, A. and Deuschel, J.-D. (2015) Invariance Principle for symmetric diffusions in a degenerate and unbounded stationary and ergodic Random Medium. To appear in Annales de l'Institut Henri Poincaré (B).

[10] Croydon, D. A. and Hambly, B. M. (2008). Local limit theorems for sequences of simple random walks on graphs. *Potential Anal.* **29**, 4, 351–389. MR-2453564

[11] Davies, E. B. (1990). *Heat kernels and spectral theory*. Cambridge Tracts in Mathematics, Vol. **92**. Cambridge University Press, Cambridge. MR-1103113

[12] De Giorgi, E. (1957). Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari.. Mem. Accad. Sci. Torino, P. I., III. Ser. 3, pp. 23–43.

[13] Dyda, B. and Kassmann, M. (2013). On weighted Poincaré inequalities. *Ann. Acad. Sci. Fenn. Math.* **38**, 2, 721–726. MR-3113104

[14] Edmunds, D. E. and Peletier, L. A. (1972). A Harnack inequality for weak solutions of degenerate quasilinear elliptic equations. *J. London Math. Soc. (2) 5*, 21–31. MR-0298217

[15] Fannjiang, A. and Komorowski, T. (1997). A martingale approach to homogenization of unbounded random flows. *Ann. Probab.* **25**, 4, 1872–1894. MR-1487440

[16] Fannjiang, A. and Komorowski, T. (1999). An invariance principle for diffusion in turbulence. *Ann. Probab.* **27**, 2, 751–781. MR-1698963

[17] Fukushima, M., Nakao, S., and Takeda, M. (1987). On Dirichlet forms with random data—recurrence and homogenization. In *Stochastic processes—mathematics and physics, II (Bielefeld, 1985)*. Lecture Notes in Math., Vol. **1250**. Springer, Berlin, 87–97. MR-897799

[18] Fukushima, M., Ōshima, Y., and Takeda, M. (1994). *Dirichlet forms and symmetric Markov processes*. de Gruyter Studies in Mathematics, Vol. **19**. Walter de Gruyter & Co., Berlin. MR-1303354

[19] Gilbarg, D. and Trudinger, N. S. (2001). *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin. Reprint of the 1998 edition. MR-1814364

[20] Grigor'yan, A., Hu, J. and Lau, K. (2009). Heat Kernels on Metric Spaces with Doubling measure, Fractal Geometry and Stochastics IV, Progress in Probability, vol. 61, Birkhäuser Basel, pp. 3–44.

[21] Grigor'yan, A. and Telcs, A. (2012). Two-sided estimates of heat kernels on metric measure spaces. *Ann. Probab.* **40**, 3, 1212–1284. MR-2962091

[22] Komorowski, T., Landim, C., and Olla, S. (2012). *Fluctuations in Markov processes*. Vol. **345**. Springer, Heidelberg. Time symmetry and martingale approximation. MR-2952852

[23] Kružkov, S. N. and Kolodiĭ, Ī. M. (1977). A priori estimates and Harnack's inequality for generalized solutions of degenerate quasilinear parabolic equations. *Sibirsk. Mat. Ž.* **18**, 3, 608–628, 718. MR-0470492

[24] Moser, J. (1964). A Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math. 17*, 101–134. MR-0159139

[25] Mourrat, J.-C. and Otto, F. (2015). Anchored Nash inequalities and heat kernel bounds for static and dynamic degenerate environments. arXiv:1503.08280.

[26] Nash, J. (1958). Continuity of solutions of parabolic and elliptic equations. *Amer. J. Math. 80*, 931–954. MR-0100158

[27] Röckner, M. (1993). General theory of Dirichlet forms and applications. In *Dirichlet forms (Varenna, 1992)*. Lecture Notes in Math., Vol. **1563**. Springer, Berlin, 129–193. MR-1292279

[28] Saloff-Coste, L. (2002). *Aspects of Sobolev-type inequalities*. London Mathematical Society Lecture Note Series, Vol. **289**. Cambridge University Press, Cambridge. MR-1872526

[29] Zhikov, V. V. (2013). Estimates of the Nash – Aronson type for degenerating parabolic equations. Journal of Mathematical Sciences **190**. *1, 66–79. MR-2830677*

**Acknowledgments.** We kindly thank Martin Slowik for fruitful discussions and in particular for suggesting the interpolation argument in Appendix A. The first author thanks the Research Training Group 1845 for financing his research.