Multi-level pinning problems for random walks and self-avoiding lattice paths[*]

Pietro Caputo[†]   Fabio Martinelli[†]   Fabio Lucio Toninelli[‡]

Abstract

We consider a generalization of the classical pinning problem for integer-valued random walks conditioned to stay non-negative. More specifically, we take pinning potentials of the form $\sum_{j \geqslant 0} \varepsilon_j N_j$, where N_j is the number of visits to the state j and $\{\varepsilon_j\}$ is a non-negative sequence. Partly motivated by similar problems for low-temperature contour models in statistical physics, we aim at finding a sharp characterization of the threshold of the wetting transition, especially in the regime where the variance σ^2 of the single step of the random walk is small. Our main result says that, for natural choices of the pinning sequence $\{\varepsilon_j\}$, localization (respectively delocalization) occurs if $\sigma^{-2} \sum_{j \geqslant 0} (j+1)\varepsilon_j \geqslant \delta^{-1}$ (respectively $\leqslant \delta$), for some universal $\delta < 1$. Our finding is reminiscent of the classical Bargmann-Jost-Pais criteria for the absence of bound states for the radial Schrödinger equation. The core of the proof is a recursive argument to bound the free energy of the model. Our approach is rather robust, which allows us to obtain similar results in the case where the random walk trajectory is replaced by a self-avoiding path γ in $\mathbb{Z}^2$ with weight $\exp(-\beta|\gamma|)$, $|\gamma|$ being the length of the path and $\beta > 0$ a large enough parameter. This generalization is directly relevant for applications to the above mentioned contour models.

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1 Introduction and motivations

Consider a one-dimensional integer-valued symmetric random walk starting at zero, conditioned to stay non-negative. If the walk has a reward $\varepsilon > 0$ for each return to zero, it is a classical fact that there exists a critical value ε_c such that for $\varepsilon > \varepsilon_c$ the random walk has a positive density of returns to the origin while for $\varepsilon < \varepsilon_c$ entropic repulsion prevails and the density of returns is zero; see e.g. [12] and references therein. This is often called a wetting transition [11]. The critical parameter ε_c depends crucially on the single-step variance σ^2; in simple examples such as the symmetric walk with increments in $\{-1, 0, +1\}$ one finds that ε_c scales linearly in σ^2 as $\sigma^2 \to 0$ [15].

Multi-level pinning problems

In this work we consider a natural generalization where the pinning at the origin is replaced by a long range pinning potential $\boldsymbol{\varepsilon} = \{\varepsilon_j\}_{j \geqslant 0}$, where $\varepsilon_j \geqslant 0$ is the reward for a visit to the state $j \geqslant 0$. To be specific, for a trajectory γ of length L, define

$$\Phi(\gamma) = \sum_{j=0}^{\infty} \varepsilon_j N_j(\gamma), \tag{1.1}$$

where $N_j(\gamma)$ is the number of visits to state j. Define also the free energy

$$f(\boldsymbol{\varepsilon}) = \lim_{L \to \infty} \frac{1}{L} \log \mathbb{E}_{0,L}^+ (e^{\Phi}),$$

where $\mathbb{E}_{0,L}^+(\cdot)$ stands for the expectation w.r.t. to the path measure conditioned to $\gamma \geqslant 0$ and $\gamma_0 = \gamma_L = 0$. The existence of the limit follows by sub-additivity. With this notation the localized (resp. delocalized) phase is characterized by $f(\boldsymbol{\varepsilon}) > 0$ (resp. $f(\boldsymbol{\varepsilon}) = 0$). Under mild assumptions on the random walk kernel, we prove a result of the following flavour: given a pinning potential $\boldsymbol{\varepsilon}$ and $\lambda > 0$, the wetting transition associated to the potential $\lambda \boldsymbol{\varepsilon}$ occurs at a critical value $\lambda_c = \lambda_c(\boldsymbol{\varepsilon})$ such that $\lambda_c \, \rho \in (a, b)$, where

$$\rho := \frac{1}{\sigma^2} \sum_{j=0}^{\infty} (j+1)\varepsilon_j$$

and $a, b > 0$ are universal constants. In view of the definition of ρ it is natural to restrict to pinning potentials satisfying

$$\sum_j j\varepsilon_j < +\infty, \tag{1.2}$$

a condition that we will assume henceforth. As it will be shown later on, under this assumption the localization/delocalization defined in terms of the free energy corresponds to a natural pathwise localization/delocalization.

As far as we know, this is the first analysis of the wetting transition for a multi-level pinning problem of the general form (1.1). We refer to [5, 17] for previous studies of certain specific models of random walks with pinning on several layers. It is interesting to note the analogy between our condition for delocalization and the classical Bargmann-Jost-Pais [16, 1, 18] criteria for the absence of bound states for the radial Schroedinger equation; cf. Remark 2.4 below for a discussion of this point.

The most challenging part of the proof is to show delocalization for ρ small. That requires establishing an upper bound on the partition function

$$Z_{0,L}^{\Phi,+} := \sum_{\gamma \geqslant 0} w(\gamma)e^{\Phi(\gamma)},$$

where the sum runs over non-negative trajectories returning to the origin at time L and $w(\gamma)$ is the probability of γ. Using the strategy outlined below we prove that

$$Z_{0,L}^{\Phi,+} \leqslant C \, Z_{0,L}, \tag{1.3}$$

where C is a universal constant and $Z_{0,L} = \sum_{\gamma} w(\gamma)$, the sum being over all trajectories returning to the origin at time L. Clearly,

$$\mathbb{E}_{0,L}^+ (e^{\Phi}) = \frac{Z_{0,L}^{\Phi,+}}{Z_{0,L}} \, \mathbb{P}_{0,L}(\gamma \geqslant 0)^{-1}$$

where $\mathbb{P}_{0,L}(\gamma \geqslant 0)$ is the probability that a path returning to the origin after L steps remains non-negative. Well known bounds show that $\mathbb{P}_{0,L}(\gamma \geqslant 0)^{-1} = O(L)$, so that the estimate (1.3) establishes delocalization.

To prove (1.3) we argue as follows. The first step is to decouple the problem into a collection of independent pinning problems, one for each height level $j = 0, 1, \ldots$. More precisely, let $\rho_j := (\rho\sigma^2)^{-1}(j+1)\varepsilon_j$ so that $\sum_{j \geqslant 0} \rho_j = 1$. Then, Jensen's inequality implies that

$$Z_{0,L}^{\Phi,+} = \sum_{\gamma \geqslant 0} w(\gamma) e^{\sum_{j=0}^{\infty} \varepsilon_j N_j(\gamma)} \leqslant \sum_{j=0}^{\infty} \rho_j Z_{0,L}^{+,\kappa_j} \,,$$

where $Z_{0,L}^{+,\kappa_j} = \sum_{\gamma \geqslant 0} w(\gamma) e^{\rho\sigma^2 N_j(\gamma)/(j+1)}$ is the partition function of a random walk returning to the origin after L steps, pinned at height j with pinning strength $\kappa_j = \rho\sigma^2/(j+1)$.

The next step is to show that, if the parameter ρ is small enough, then *uniformly* in the law of the random walk and in the height j, one has $Z_{0,L}^{+,\kappa_j} \leqslant C Z_{0,L}$. Using $\sum_j \rho_j = 1$, this bound implies (1.3).

The main idea of the proof goes as follows. With a natural inductive argument we show that, for all L larger than a critical "diffusive" scale $L_c(j) \sim (j+1)^2/\sigma^2$, one has $Z_{0,L}^{+,\kappa_j} \leqslant C Z_{0,L}$, provided that the same holds for $L = L_c(j)$. The base case of the induction is solved by a fine analysis based on careful local limit theorem estimates. It is only at this stage, that is when $L = L_c(j)$, that we need to take the parameter ρ small enough.

The method outlined above is rather accurate in finding the threshold for the wetting transition. In this respect, we remark that a direct "energy vs. entropy" argument such as the one used in [2] would fail to capture the right dependence in the parameter σ^2 for instance. Moreover, our method is robust enough to admit an extension to the setting of self-avoiding paths, as we discuss below.

One of the motivations for this work stems from the mathematical analysis of contour models arising in low-temperature two-dimensional spin systems and related interface models. In this context the random walk is replaced by a self-avoiding and weakly self-interacting random lattice path with an effective diffusion constant $\sigma^2 \sim e^{-\beta}$, where β is the inverse temperature. Here the analog of the pinning strength ε_j above typically decays like $e^{-\alpha\beta(j+1)}$ for $j \to \infty$ with $\alpha > 1$. Whether such long range potential is able to localize the contour is a key question in the analysis of large deviations problems such as e.g. the Wulff construction for the 2D Ising model [8] and for the $(2+1)$-dimensional Solid-on-Solid model [3]. We refer the interested reader to [14] for more details.

Our approach can be applied in principle to this setting. In Section 4 below we work out the details of this extension in the simplified case where the self-avoiding path has no additional self-interaction. The general case of contour models has been recently solved in [14], with a stronger bound on the partition function in the delocalized phase and with a very different approach. Let us however remark that [14] does not cover the random walk setting of the present work, and that for contour models it provides bounds on the location of the localization/delocalization threshold that are considerably less sharp than the ones given by Theorems 3 and 4 below. The main idea of [14] consists in constructing, out of the self-interacting contour path, an effective random walk together with a renewal structure and then prove delocalization for the latter.

2 Models and results

2.1 Random walks and pinning

We consider a class of symmetric and irreducible random walk kernels on $\mathbb{Z}$ with variance σ^2. Since we are interested in the regime of small σ^2, we will make the assumption $\sigma^2 \in (0, \frac{1}{2}]$.

Definition 2.1. *For a fixed constant $c_0 \in (0,1]$ and $\sigma^2 \in (0, \frac{1}{2}]$, we call $\mathcal{P}(c_0, \sigma^2)$ or simply $\mathcal{P}(\sigma^2)$ the set of all symmetric probabilities $p : \mathbb{Z} \mapsto [0,1]$ with variance σ^2 such that*

$$p(1) \geqslant \tfrac{1}{2} c_0 \sigma^2 \quad and \quad \sum_{k \in \mathbb{Z}} |k|^3 p(k) \leqslant c_0^{-1} \sigma^2.$$

In particular, any $p \in \mathcal{P}(\sigma^2)$ satisfies $p(0) \geqslant 1 - \sigma^2 \geqslant \frac{1}{2}$, and the associated random walk is irreducible. Below we shall restrict ourselves to random walk kernels in the class $\mathcal{P}(\sigma^2)$. While we do not believe this to be the largest possible class for our results to hold, the above assumptions turn out to be very convenient from the technical point of view. At the same time, they include a wide range of interesting models. Two key examples to keep in mind are:

1) the symmetric nearest neighbor walk with $p(\pm 1) = \frac{\sigma^2}{2}$, $p(0) = 1 - \sigma^2$ and $p(k) = 0$ otherwise, referred to as the *binomial walk*, and

2) the geometric walk with $p(k) = \frac{1}{Z_\beta} e^{-\beta |k|}$, $k \in \mathbb{Z}$, where $\beta \in (0, \infty)$ is the unique positive solution of

$$\sigma^2 = \frac{2e^{-\beta}}{(1 - e^{-\beta})^2},$$

and $Z_\beta = (1 + e^{-\beta})(1 - e^{-\beta})^{-1}$. We refer to this as the *SOS walk* at inverse temperature β, from its relation with the so-called Solid-On-Solid model.

We call $\mathbb{P}_i$ the law on trajectories of the random walk starting at $i \in \mathbb{Z}$. Let

$$\Omega_{0,L} = \{\gamma = (\gamma_0, \ldots, \gamma_L) : \gamma_i \in \mathbb{Z}, \gamma_0 = \gamma_L = 0\},$$

denote the set of trajectories which start and end at zero, and define the partition function

$$Z_{0,L} = \sum_{\gamma \in \Omega_{0,L}} w(\gamma), \qquad w(\gamma) = \prod_{i=1}^{L} p(\gamma_i - \gamma_{i-1}). \tag{2.1}$$

We write $\mathbb{P}_{0,L}$ for the law of the walk conditioned to $\Omega_{0,L}$, that is for any $\gamma \in \Omega_{0,L}$:

$$\mathbb{P}_{0,L}(\gamma) = \frac{\mathbb{P}_0(\gamma)}{\mathbb{P}_0(\Omega_{0,L})} = \frac{w(\gamma)}{Z_{0,L}}.$$

For a fixed integer $j \geqslant 0$, consider the paths $\Omega_{0,L}^{+,j}$ that stay above height $-j$:

$$\Omega_{0,L}^{+,j} = \{\gamma \in \Omega_{0,L} : \gamma_i \geqslant -j \ \forall i\}, \qquad Z_{0,L}^{+,j} = \sum_{\gamma \in \Omega_{0,L}} w(\gamma) \mathbf{1}(\gamma \in \Omega_{0,L}^{+,j}). \tag{2.2}$$

The number of contacts with level zero is given by

$$N(\gamma) = \sum_{i=1}^{L-1} \mathbf{1}(\gamma_i = 0). \tag{2.3}$$

For any $\varepsilon > 0$ we consider the probability measures

$$\mathbb{P}_{0,L}^{+,j}(\gamma) \propto w(\gamma) \mathbf{1}(\gamma \in \Omega_{0,L}^{+,j}), \qquad \mathbb{P}_{0,L}^{\varepsilon,+,j}(\gamma) \propto e^{\varepsilon N(\gamma)} w(\gamma) \mathbf{1}(\gamma \in \Omega_{0,L}^{+,j}),$$

and the corresponding expectations $\mathbb{E}_{0,L}^{+,j}$, $\mathbb{E}_{0,L}^{\varepsilon,+,j}$.

Theorem 1. There exist constants $a \geqslant b > 0$ and $c > 0$, such that the following holds for any integer $j \geqslant 0$, any $\sigma^2 \in (0, \frac{1}{2}]$ and any random walk $p \in \mathcal{P}(\sigma^2)$:

i) If $\varepsilon \geqslant a\sigma^2/(j+1)$, then for all L large enough

$$\mathbb{E}_{0,L}^{+,j}\left[e^{\varepsilon N}\right] \geqslant \exp\left(c\,\sigma^2 L\,(j+1)^{-2}\right). \tag{2.4}$$

ii) If $\varepsilon \leqslant b\sigma^2/(j+1)$, then:

$$\mathbb{E}_{0,L}^{+,j}\left[e^{\varepsilon N}\right] \leqslant 2\frac{Z_{0,L}}{Z_{0,L}^{+,j}}. \tag{2.5}$$

Remark 2.2. It is well known that the ratio $Z_{0,L}/Z_{0,L}^{+,j}$ appearing in (2.5) is $O(L)$ as $L \to \infty$; see (3.14) below. In particular, the bounds in Theorem 1 imply that a wetting transition occurs at a critical value ε_c that satisfies $b \leqslant \varepsilon_c\sigma^{-2}(j+1) \leqslant a$, with constants a, b that are independent of σ^2 and j and independent of $p \in \mathcal{P}(\sigma^2)$. This extends well known results in the case $j = 0$; see e.g. [15, 12, 7, 4].

Remark 2.3. In Proposition 3.5 below we show that the upper bound (2.5) can be upgraded to the following bound independent of L, but with possibly non-optimal dependence on j, σ^2:

$$\mathbb{E}_{0,L}^{+,j}\left[e^{\varepsilon N}\right] \leqslant K, \tag{2.6}$$

for some constant $K = K(j,p)$, $p \in \mathcal{P}(\sigma^2)$, whenever $\varepsilon \leqslant b\sigma^2/(j+1)$.

Next, we consider a more general interaction with the wall. For any $\gamma \in \Omega_{0,L}^{+,0}$, define the potential

$$\Phi(\gamma) = \sum_{j=0}^{\infty}\varepsilon_j N_j(\gamma), \qquad N_j(\gamma) = \sum_{i=1}^{L-1}\mathbf{1}(\gamma_i = j),$$

where $\{\varepsilon_j\}$ is a given nonnegative sequence satisfying (1.2), and let

$$Z_{0,L}^{\Phi,+,0} = \sum_{\gamma \in \Omega_{0,L}^{+,0}} w(\gamma)e^{\Phi(\gamma)}. \tag{2.7}$$

Theorem 2. There exist absolute constant $a, b, c > 0$ such that, for any $\sigma^2 \in (0,\frac{1}{2}]$ and any random walk $p \in \mathcal{P}(\sigma^2)$, the following holds:

i) For any integer $d \geqslant 0$ such that

$$\frac{1}{d+1}\sum_{j=0}^{d/2}(j+1)^2\varepsilon_j \geqslant a\sigma^2, \tag{2.8}$$

we have

$$\mathbb{E}_{0,L}^{+,0}\left[e^{\Phi(\gamma)}\right] \geqslant \exp\left(c\,\sigma^2 L(d+1)^{-2}\right), \tag{2.9}$$

for all L large enough.

ii) If the sequence $\{\varepsilon_j\}$ satisfies

$$\sum_{j=0}^{\infty}(j+1)\varepsilon_j \leqslant b\sigma^2, \tag{2.10}$$

then

$$Z_{0,L}^{\Phi,+,0} \leqslant 4\,Z_{0,L}. \tag{2.11}$$

Notice that since $j \leqslant d$ in the summation (2.8), the condition for localization is slightly stronger than the bound $\sum_{j=0}^{\infty}(j+1)\varepsilon_j \geqslant a\sigma^2$ that would be sharp in view point (ii). However, in many natural cases of interest, condition (2.8) is actually rather sharp; see Corollary 1 below.

Remark 2.4. Our criterion (2.10) for the absence of a localized phase bears some similarity with the one derived in [16] by Jost and Pais (cf. also the more general Bargmann's bounds in [1, 18]) to exclude bound states for the Schrödinger equation

$$-\Delta\psi + V(r)\psi = E\psi$$

in an attractive central potential $V \leqslant 0$ in $\mathbb{R}^3$ or, after moving to radial coordinates, for the Sturm-Liouville problem on the half-line

$$-\frac{d^2}{dx^2}f(x) + V(x)f(x) = \lambda f(x), \quad f(0) = 0.$$

In [16] it was proved in fact that if $\int_0^\infty dr\, r|V(r)| < 1$ then there are no bound states. The connection between the pinning problem and the bound state problem goes as follows. Let $\gamma(\cdot)$ be the random walk on $\mathbb{Z}$ with law $p \in \mathcal{P}(\sigma^2)$ and let $\mathbb{E}_0(\cdot)$ denote the average over the trajectories of $\gamma(\cdot)$ starting at the origin. If τ denotes the hitting time of the half-line $(-\infty, -1]$ then we can write

$$\frac{Z_{0,L}^{\Phi,+,0}}{Z_{0,L}^+} = \mathbb{E}_0\left(e^{\sum_{s=0}^{L} F(\gamma(s))} \mid \gamma(L) = 0;\, \tau > L\right),$$

where $F(x) = \sum_{j \geqslant 0} \varepsilon_j \mathbf{1}(x = j)$. If we pretend that the random walk behaves like a Brownian motion with the correct diffusion constant and we replace F with $V(x) := -F(\lfloor x \rfloor)$, then, using the Feynman-Kac formula, we get that the r.h.s. above has the form

$$e^{-L\mathcal{H}}(0,0)/p(0,0;L), \tag{2.12}$$

where $\mathcal{H} = -\sigma^2\frac{d^2}{dx^2} + V(x)$ acts on $\mathbb{L}^2((-1,+\infty))$ with Dirichlet boundary conditions and $p(x,y;L)$ is the transition probability density for the Brownian motion killed at -1. If the equation $\mathcal{H}f = \lambda f$ has a solution bounded in $\mathbb{L}^2$ (of course with a negative eigenvalue λ) then the above ratio should diverge exponentially fast in L. The Jost-Pais criterium says that this cannot be the case if $\int_0^{+\infty} dr\, r|V(r)| < \sigma^2$, which is indeed analogous to our condition $\sum_{j \geqslant 0} \varepsilon_j(j+1) < b\sigma^2$. Notice that the absence of bound states does not guarantee that the ratio (2.12) stays bounded in L. In this sense our result is stronger.

The above theorems allow us to identify rather precisely the critical point of the wetting transition for the generalized pinning problem described by (2.7) for natural sequences ε_j. For the sake of definiteness we mention only two types of sequences below, the power law and the exponential law. It is immediate to deduce the following corollary.

Corollary 1. Let the sequence ε_j^0 be either the power law $\varepsilon_j^0 = (j+1)^{-2-\delta}$ or the exponential law $\varepsilon_j^0 = e^{-\delta j}$, for some $\delta > 0$. Then there exist constants $a > b > 0$ and $c > 0$ such that the following holds for any $\sigma^2 \in (0, \frac{1}{2}]$ and any $p \in \mathcal{P}(\sigma^2)$:

i) If the pinning sequence is $\varepsilon_j := a\,\sigma^2\varepsilon_j^0$, then the walk is localized: for L large enough,

$$\mathbb{E}_{0,L}^{+,0}\left[e^{\Phi(\gamma)}\right] \geqslant \exp\left(c\,\sigma^2 L\right);$$

ii) If the pinning sequence is $\varepsilon_j := b\,\sigma^2\varepsilon_j^0$, then the walk is delocalized:

$$Z_{0,L}^{\Phi,+,0} \leqslant 4\,Z_{0,L}.$$

We conclude this part by discussing pathwise localization/delocalization. Assume first that $f(\varepsilon) > 0$. Let j_0 be such that $\sum_{j \geq j_0} (j+1)\varepsilon_j \leqslant b\sigma^2$, where b is the universal constant appearing in Theorem 2, and let $\varepsilon^* = \max_j \varepsilon_j$. Then, for any $\alpha > 0$,

$$\frac{\mathbb{E}^+_{0,L}\left(e^{\Phi(\gamma)}\mathbb{1}(\sum_{j<j_0} N_j \leqslant \alpha L)\right)}{\mathbb{E}^+_{0,L}\left(e^{\Phi(\gamma)}\right)} \leqslant e^{\varepsilon^* \alpha L} \frac{\mathbb{E}^+_{0,L}\left(e^{\hat{\Phi}(\gamma)}\right)}{\mathbb{E}^+_{0,L}\left(e^{\Phi(\gamma)}\right)},$$

where $\hat{\Phi}(\gamma) = \sum_{j \geq j_o} \varepsilon_j N_j(\gamma)$. Using Theorem 2 (ii) the numerator in the r.h.s. above does not grow exponentially fast in L while, by the assumption $f(\varepsilon) > 0$, the denominator does. Thus, for $\alpha > 0$ small enough, the left hand side is exponentially small *i.e.* a pathwise localization. Conversely, assume that $f((1+\delta)\varepsilon) = 0$ for $\delta > 0$ small enough. For any event $A \subset \Omega_{0,L}$ we have

$$\frac{\mathbb{E}^+_{0,L}\left(e^{\Phi(\gamma)}\mathbb{1}(A)\right)}{\mathbb{E}^+_{0,L}\left(e^{\Phi(\gamma)}\right)} \leqslant \left[\frac{\mathbb{E}^+_{0,L}\left(e^{(1+\delta)\Phi(\gamma)}\right)}{\mathbb{E}^+_{0,L}\left(e^{\Phi(\gamma)}\right)}\right]^{1/(1+\delta)} \left[\frac{\mathbb{P}^+_{0,L}(A)}{\mathbb{E}^+_{0,L}\left(e^{\Phi(\gamma)}\right)}\right]^{\delta/(1+\delta)}$$

$$\leqslant \mathbb{E}^+_{0,L}\left(e^{(1+\delta)\Phi(\gamma)}\right)^{1/(1+\delta)} \mathbb{P}^+_{0,L}(A)^{\delta/(1+\delta)}.$$

By assumption, for any $\delta > 0$ small enough, the first term in the r.h.s. does not grow exponentially fast in L (actually in many cases it will only grow polynomially fast). Thus, if A is any event with an exponentially small (in L) probability w.r.t. the unpinned walk, the same will hold for the pinned case. This is e.g. the case when A is the event that the number of contacts at a given level j is proportional to L.

2.2 Self-avoiding lattice paths interacting with a wall

We turn to the description of the lattice path model. We first define the class of lattice paths to be considered.

Definition 2.5. *We call $V := (\mathbb{Z} + \frac{1}{2}) \times \mathbb{Z}$ the vertex set of our lattice paths. An edge is an unordered pair of points $e = \{x, y\}$, $x, y \in V$, with euclidean distance $d(x, y) = 1$. An edge can be horizontal if $x = y \pm e_1$ or vertical if $x = y \pm e_2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. A self-avoiding lattice path (for short a path in the sequel) joining $x \in V$ and $y \in V$ is a sequence $f_0, \ldots, f_n$ of edges such that:*

1. *for every $i = 0, \ldots, n-1$, f_i and f_{i+1} have one common vertex $z_i \in V$;*

2. *$f_0 = \{z_0, z_1\}$, $f_n = \{z_n, z_{n+1}\}$ with $z_0 = x$ and $z_{n+1} = y$;*

3. *all vertices z_i, $i = 0, \ldots, n+1$, are distinct.*

We denote the length of a lattice path γ, that is the number of edges in γ, by $|\gamma|$. Given $x, y \in V$, $x \neq y$, we call $\Omega(x, y)$ the set of all paths joining x and y.

Notice that for $x = (x_1, x_2)$ and $y = (y_1, y_2)$, all $\gamma \in \Omega(x, y)$ have at least $|y_1 - x_1|$ horizontal edges. Paths that have exactly $|y_1 - x_1|$ horizontal edges are in one to one correspondence with random walk trajectories from $x - \frac{1}{2}e_1$ to $y + \frac{1}{2}e_1$; see Figure 1.

Next, we define the ensemble of lattice paths. Given $x, y \in V$, $x \neq y$, we write

$$\mathcal{Z}(x, y) = \sum_{\gamma \in \Omega(x, y)} e^{-\beta|\gamma|},$$

where $\beta > 0$ is the inverse temperature parameter. With slight abuse of notation, when $x = (\frac{1}{2}, 0)$ and $y = (L - \frac{1}{2}, 0)$, we write $\Omega_{0,L}$ instead of $\Omega(x, y)$ and $\mathcal{Z}_{0,L}$ instead of $\mathcal{Z}(x, y)$.
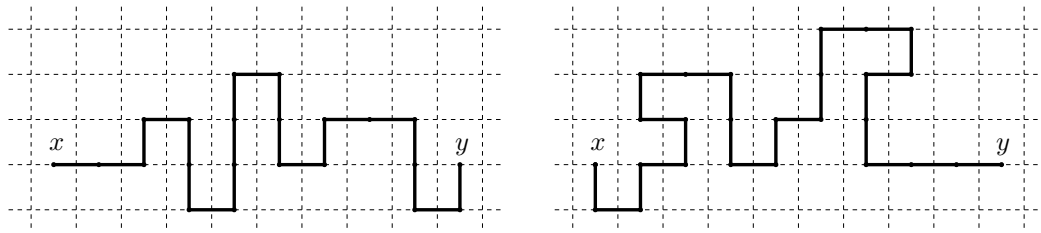
Figure 1: Two examples of lattice paths joining the vertices $x = (\frac{1}{2}, 0)$ and $y = (10 - \frac{1}{2}, 0)$. In the background, the integer grid $\mathbb{Z}^2$.

Observe that if we restrict to the set $\hat{\Omega}_{0,L}$ of paths with minimal number of horizontal edges then we obtain the following relation with the partition function $Z_{0,L}$ of the SOS walk with parameter β:

$$\hat{\mathcal{Z}}_{0,L} := \sum_{\gamma \in \hat{\Omega}_{0,L}} e^{-\beta|\gamma|} = e^{-\beta(L-1)} (Z_\beta)^L Z_{0,L}, \tag{2.13}$$

where $Z_\beta = (1 + e^{-\beta})(1 - e^{-\beta})^{-1}$.

For any $\gamma \in \Omega_{0,L}$, define

$$\Phi(\gamma) = \sum_{j=0}^{\infty} \varepsilon_j N_j(\gamma), \qquad N_j(\gamma) = \sum_{i=1}^{L-2} \mathbf{1}((i + \tfrac{1}{2}, j) \in \gamma), \tag{2.14}$$

where $\{\varepsilon_j\}$ is a nonnegative sequence. Define also

$$\mathcal{Z}_{0,L}^{\Phi,+,0} = \sum_{\gamma \in \Omega_{0,L}^{+,0}} e^{\Phi(\gamma) - \beta|\gamma|},$$

where $\Omega_{0,L}^{+,0}$ denotes the set of paths $\gamma \in \Omega_{0,L}$ with $\gamma \geqslant 0$. With slight abuse of notation we write again $\mathbb{P}_{0,L}$ and $\mathbb{E}_{0,L}$ (resp. $\mathbb{P}_{0,L}^{+,0}$ and $\mathbb{E}_{0,L}^{+,0}$) for the probability and expectation over paths $\gamma \in \Omega_{0,L}$ (resp. $\gamma \in \Omega_{0,L}^{+,0}$) with weight $e^{-\beta|\gamma|}$.

Remark 2.6 (Alternative definitions of the number of contacts). In analogy with the random walk case one could have considered the number $\hat{N}_j(\gamma)$ of *horizontal* contacts of γ with the j-th level rather than the number $N_j(\gamma)$ defined in (2.14). By horizontal contact with the j-th level here we mean an horizontal edge $f \in \gamma$ at height j. For instance the path in Figure 1 (left) has $N_0(\gamma) = 7$ and $\hat{N}_0(\gamma) = 3$. It is easy to check that all our results apply with no modifications to the potential $\hat{\Phi}$ obtained as in (2.14) with N_j replaced by $\hat{N}_j$. As we will see in the proof of Theorem 3, our results actually extend to a model where one takes into account also possible contacts of γ with the j-th level occurring outside of the horizontal interval $[0, L]$.

Theorem 3 (Delocalized phase). There exist universal constants $b > 0$, $\beta_0 > 0$ such that the following holds for any $\beta \geqslant \beta_0$. If $\sum_{j=0}^{\infty}(j + 1)\varepsilon_j \leqslant b e^{-\beta}$, then for all $L \in \mathbb{N}$

$$\mathcal{Z}_{0,L}^{\Phi,+,0} \leqslant 4 \mathcal{Z}_{0,L}.$$

Theorem 4 (Localized phase). There exist universal constants $a, c, \beta_0 > 0$ such that the following holds: For any sequence $\{\varepsilon_j\}$, any integer $d \geqslant 0$ and any $\beta \geqslant \beta_0$ such that

$$\frac{1}{d+1} \sum_{j=0}^{d} (j + 1)^2 \varepsilon_j \geqslant a e^{-\beta}, \qquad e^{-\beta} \leqslant c(d + 1)^{-2},$$

one has

$$\mathbb{E}_{0,L}^{+,0}\big[e^{\Phi(\gamma)}\big] \geqslant \exp\big(c\,e^{-\beta}L(d+1)^{-2}\big),$$

for all L large enough.

As in Corollary 1 one can immediately infer from Theorem 3 and Theorem 4 the following facts.

Corollary 2. Let the sequence ε_j^0 be either the power law $\varepsilon_j^0 = (j+1)^{-2-\delta}$ or the exponential law $\varepsilon_j^0 = e^{-\delta j}$, for some $\delta > 0$. Then there exist constants $a > b > 0, c > 0$ and $\beta_0 > 0$ such that the following holds for any $\beta \geqslant \beta_0$:

i) If the pinning sequence is $\varepsilon_j := a\,e^{-\beta}\varepsilon_j^0$, then the path is localized with

$$\mathbb{E}_{0,L}^{+,0}\big[e^{\Phi(\gamma)}\big] \geqslant \exp\big(c\,e^{-\beta}L\big).$$

ii) If the pinning sequence is $\varepsilon_j := b\,e^{-\beta}\varepsilon_j^0$, then the path is delocalized with

$$\mathcal{Z}_{0,L}^{\Phi,+,0} \leqslant 4\mathcal{Z}_{0,L}.$$

3 Random walks

Here we prove the main results for the random walk model.

3.1 Proof of the lower bounds

The high level idea of the proof is to restrict the path to a strip $[0,d] \times [0,L]$, where d is a free parameter, and to impose that the density of contacts N_j/L at level $j \in [0,d]$ is given by a probability vector $\{\nu_j\}_{j=0}^d$. Thus

$$\mathbb{E}_{0,L}^+\left(e^{\Phi(\gamma)}\right) \geq e^{L\sum_{j=0}^d \varepsilon_j \nu_j}\,\mathbb{P}_{0,L}^+(N_j/L \sim \nu_j \ \forall j = 0,\dots,d).$$

Large deviations considerations suggest that

$$\mathbb{P}_{0,L}^+(N_j/L \sim \nu_j \ \forall j = 0,\dots,d) \sim e^{-\frac{L}{2}\sum_{i,j} P_{i,j}(\sqrt{\nu_j} - \sqrt{\nu_i})^2},$$

with $P_{ij} = p(i-j)1_{\{i,j \geq 0\}}$. One is then left with the problem of maximizing over ν the expression

$$\sum_j \varepsilon_j \nu_j - \frac{1}{2}\sum_{i,j} P_{i,j}(\sqrt{\nu_j} - \sqrt{\nu_i})^2.$$

A natural variational choice, inspired by the quasi-stationary distribution of the binomial walk on the interval $[0,d]$, is $\nu_j \propto \sin^2(\pi(j+1)/(d+2))$ and this leads to the desired conclusion.

We now turn to the actual proof. We start with some considerations that apply to both the lower bound (2.4) in Theorem 1 and to (2.9) in Theorem 2. First of all, we can assume that $\varepsilon_j \leqslant \log 2$ for each j. Otherwise if $\varepsilon_{j^*} > \log 2$, one gets immediately exponential growth of the partition function using irreducibility of the walk and the fact that $p(0) \geqslant 1/2$ (just consider the trajectory that reaches height j^* and sticks there: its weight grows like $p(0)^L e^{\varepsilon_{j^*} L}$).

Letting

$$W_L(i,j) = \sum_{\gamma \geq 0, \gamma_0 = i, \gamma_L = j} w(\gamma)e^{\Phi(\gamma)}$$

one checks immediately that

$$W_{L+1}(i,j) = \sum_{z \geq 0} W_L(i,z) e^{\varepsilon z} p(z - j).$$

Thus, viewing W_L as a matrix and observing that $W_1 = P$, with P the symmetric matrix $P_{ij} = p(i - j) 1_{i,j \geq 0}$, we have

$$W_L = e^{-V}(e^V P)^L,$$

with V the diagonal operator $V(i,j) = 1_{i=j} \varepsilon_j$. Let $\varphi = (\varphi_i)_{i \geq 0}$ be a vector with $\varphi_i \geq 0$, set $\psi = e^{-V/2} \varphi$ and assume that $(\psi, \psi) = \sum_{i \geq 0} \varphi_i^2 e^{-\varepsilon_i} = 1$. We first claim

$$(\varphi, W_L \varphi) \geqslant (\varphi, P \varphi)^L, \tag{3.1}$$

where $(\cdot, \cdot)$ is the scalar product in $\ell^2(\mathbb{Z}_+)$. To see this observe that, if $\tilde{P}$ is the self-adjoint operator $\tilde{P} = e^{V/2} P e^{V/2}$, then

$$(\varphi, e^{-V}(e^V P)^L \varphi) = (\psi, (\tilde{P})^L \psi).$$

Since $p(0) \geqslant 1/2$, the matrix P is non-negative definite, and so is $\tilde{P}$. Letting $\mu_\psi(dE)$ denote the spectral measure of $\tilde{P}$ associated with ψ (the total mass is 1 since ψ has unit $\ell^2(\mathbb{Z}_+)$-norm), we have

$$(\psi, (\tilde{P})^L \psi) = \int_0^\infty E^L \mu_\psi(dE) \geqslant \left(\int_0^\infty E \mu_\psi(dE) \right)^L = (\psi, \tilde{P} \psi)^L = (\varphi, P \varphi)^L$$

where we used convexity of x^L on $[0, \infty)$, for $L \geqslant 1$, and Jensen's inequality.

Given an integer $d \geqslant 0$, let $s(i) = \sin(\pi \frac{i+1}{d+2})$ and

$$\varphi_i = \frac{s(i)}{\sqrt{K}} = \frac{s(i)}{\sqrt{\sum_0^d s(i)^2 \exp(-\varepsilon_i)}} \quad 0 \leqslant i \leqslant d,$$

and $\varphi_i = 0$ if $i < 0$ or $i > d$. We have then

$$(\varphi, P \varphi) = (\varphi, \varphi) - (\varphi, (1 - P)\varphi) = \frac{1}{K} \left[\sum_0^d s(i)^2 - \frac{1}{2} \sum_{i,j \,\leqslant\, d} P_{ij}(s(i) - s(j))^2 \right]$$

where we used the symmetry of P. Since

$$|s(i) - s(j)| \leqslant \pi \frac{|i - j|}{d + 1}$$

we have

$$\frac{1}{2} \sum_{i,j \,\leqslant\, d} P_{ij}(s(i) - s(j))^2 \leqslant \frac{\pi^2}{2} \frac{\sigma^2}{(d + 1)}.$$

On the other hand, recalling that $0 \leqslant \varepsilon_j \leqslant \log 2$ and using $\exp(-x) \leqslant 1 - x/4$ for $0 \leqslant x \leqslant \log 2$,

$$K \leqslant \sum_0^{d/2} s(i)^2 (1 - \varepsilon_i/4) + \sum_{d/2+1}^d s(i)^2.$$

Finally, since $s(i) \geqslant c(i + 1)/(d + 1)$ for $0 \leqslant i \leqslant d/2$ for some positive constant c, we conclude that

$$(\varphi, P \varphi) \geq \frac{\sum_0^d s(i)^2 - \frac{\pi^2}{2} \frac{\sigma^2}{(d+1)}}{\sum_0^d s(i)^2 - \frac{c}{4(d+1)^2} \sum_0^{d/2} (i+1)^2 \varepsilon_i}. \tag{3.2}$$

Proof of (2.9). Assume by monotonicity that we have equality in (2.8). Then we go back to (3.2), we observe that $\sum_0^d s(i)^2$ grows linearly in d and that we can assume that d is much larger than σ^2/d. Then we obtain, for some universal positive constant C,

$$(\varphi, P\varphi) \geqslant 1 - C \frac{\sigma^2}{(d+1)^2} + \frac{a}{C} \frac{\sigma^2}{(d+1)^2} \geqslant 1 + \frac{a}{2C} \frac{\sigma^2}{(d+1)^2}$$

if the value of a in Theorem 2 is chosen large enough. Recalling (3.1) we see that there exist $i, j \leqslant d$ and positive constants $c(d), a'$ such that

$$W_L(i,j) \geqslant c(d) \exp\left[La' \frac{\sigma^2}{(d+1)^2}\right].$$

In turn, using the assumption $p(1) > 0$, we obtain

$$W_L(0,0) \geqslant p(1)^{i+j} c'(d) \exp\left[La' \frac{\sigma^2}{(d+1)^2}\right],$$

for some new constant $c'(d) > 0$. This implies the desired lower bound (2.9). $\qquad\square$

Proof of (2.4). By vertical translation invariance, we can assume that the walk is conditioned to stay non-negative (instead of $\gamma \geqslant -j$) and that the pinning is at height j, i.e. $\varepsilon_i = \varepsilon 1_{i=j}$. The proof of (2.4) is then essentially identical to the proof of (2.9), once the choice $d = 2j$ is made in the definition of $s(\cdot)$ above. $\qquad\square$

3.2 Proof of the upper bound in Theorem 1

Here we prove (2.5). Define

$$Z_{0,L}^{\varepsilon,+,j} = \sum_{\gamma \in \Omega_{0,L}} e^{\varepsilon N(\gamma)} w(\gamma) \mathbf{1}(\gamma \in \Omega_{0,L}^{+,j}).$$

Thus, (2.5) can be restated as follows

Proposition 3.1. *There exists a universal constant $b > 0$ such that if $\varepsilon \leqslant b\sigma^2/(j+1)$, then uniformly in L, σ^2, j one has*

$$Z_{0,L}^{\varepsilon,+,j} \leqslant 2 Z_{0,L}.$$

The proof of Proposition 3.1 is divided in two steps. We start with a general lemma for the walk with no wall constraint. This will allow us to cover the region $L \leqslant C(j+1)^2/\sigma^2$ for any constant $C > 0$.

Lemma 3.2. *There exists $c > 0$ such that for any L and $\sigma^2 \in (0, \frac{1}{2}]$, any $p \in \mathcal{P}(\sigma^2)$, for any $b < 1/c$ one has*

$$\mathbb{E}_{0,L}\left[e^{b\sigma N/\sqrt{L}}\right] \leqslant 1 + c\,b. \tag{3.4}$$

Proof. Writing N as in (2.3), for $\gamma \in \Omega_{0,L}$ one has the expansion

$$e^{b\sigma N(\gamma)/\sqrt{L}} = \prod_{i=1}^{L-1} e^{b\sigma \mathbf{1}(\gamma_i=0)/\sqrt{L}}$$

$$\leqslant \prod_{i=1}^{L-1} (1 + 2b\sigma \mathbf{1}(\gamma_i=0)/\sqrt{L})$$

$$= 1 + \sum_{n=1}^{L-1} (2b\sigma L^{-1/2})^n \sum_{x_1 < \cdots < x_n} \mathbf{1}(\gamma_{x_1} = \ldots \gamma_{x_n} = 0),$$

where we use $e^a \leqslant 1 + 2a$ for $a \in (0, 1)$, and the sum ranges over all possible positions of the internal zeros. We also set $x_0 = 0, x_{n+1} = L$. We can assume that $\sigma^2 L$ is large, since otherwise the statement becomes obvious by estimating $N \leqslant L$. Observe that

$$\mathbb{P}_{0,L}(\gamma_{x_1} = \ldots \gamma_{x_n} = 0) = \frac{Z_{0,x_1} \cdots Z_{x_n,L}}{Z_{0,L}},$$

where we write $Z_{u,v} = Z_{0,v-u}$. From Proposition B.1 in the appendix one has the local CLT estimates

$$Z_{x_i,x_{i+1}} \leqslant C \, \sigma^{-1}(x_{i+1} - x_i)^{-1/2}, \tag{3.5}$$

and $Z_{0,L} \geqslant C^{-1}\sigma^{-1}L^{-1/2}$, for some absolute constant $C > 0$. It follows that

$$\mathbb{P}_{0,L}(\gamma_{x_1} = \ldots \gamma_{x_n} = 0) \leqslant C^n \, \sigma\sqrt{L} \prod_{i=1}^{n+1} \sigma^{-1}(x_i - x_{i-1})^{-1/2}$$

$$= C^n \sigma^{-n}\sqrt{L} \prod_{i=1}^{n+1} (x_i - x_{i-1})^{-1/2},$$

for some absolute constant $C > 0$. Therefore,

$$\mathbb{E}_{0,L}\big[e^{b\sigma N/\sqrt{L}}\big] \leqslant 1 + \sum_{n=1}^{L-1} (2bC)^n L^{-\frac{n-1}{2}} \sum_{x_1 < \cdots < x_n} \prod_{i=1}^{n+1} (x_i - x_{i-1})^{-1/2}.$$

It remains to check that for any $n \geqslant 1$, for some new absolute constant $C > 0$ one has:

$$\sum_{x_1 < \cdots < x_n} \prod_{i=1}^{n+1} (x_i - x_{i-1})^{-1/2} \leqslant C^n L^{\frac{n-1}{2}}. \tag{3.6}$$

Once (3.6) is available, it is immediate to conclude that (3.4) holds if b is small.

To prove (3.6) we change variables to $\xi_i = x_i - x_{i-1} \in \mathbb{N}$, $i = 1, \ldots, n+1$. Thus

$$\sum_{x_1 < \cdots < x_n} \prod_{i=1}^{n+1} (x_i - x_{i-1})^{-1/2} = \sum_{\xi_1, \ldots, \xi_{n+1} : \sum_{j=1}^{n+1} \xi_j = L} (\xi_1 \cdots \xi_{n+1})^{-1/2}. \tag{3.7}$$

One easily checks that there is $C > 0$ independent of n, L such that for any fixed values of $\xi_1, \ldots, \xi_{n-1}$ one has

$$\sum_{\xi_n, \xi_{n+1} : \xi_n + \xi_{n+1} = L - \sum_{i=1}^{n-1} \xi_i} (\xi_n \xi_{n+1})^{-1/2} \leqslant C.$$

Therefore the sum in (3.7) can be bounded by

$$C \prod_{j=1}^{n-1} \sum_{1 \leqslant \xi_j \leqslant L} \xi_j^{-1/2} \leqslant (C')^n L^{\frac{n-1}{2}},$$

for some new constant $C' > 0$. $\qquad\square$

Lemma 3.2 implies that for any small constant b, taking $L \leqslant (j+1)^2/\sigma^2$, and $\varepsilon \leqslant b\sigma^2/(j+1)$, then

$$Z_{0,L}^{\varepsilon,+,j} \leqslant Z_{0,L}\mathbb{E}_{0,L}\big[e^{\varepsilon N}\big] \leqslant (1 + c\,b)Z_{0,L}, \tag{3.8}$$

where the first inequality follows by dropping the wall constraint $\gamma \geqslant -j$, while the second one is implied by (3.4).

Definition 3.3. *Fix $L_0 = \frac{C}{2}(j+1)^2/\sigma^2$ (with C the same constant as in Proposition B.1) and set $L_n := 2^n L_0$. For $n \in \mathbb{N}$, $\delta > 0$, and $b > 0$, let $\mathcal{H}_n(\delta, b)$ denote the following statement:*

$$Z_{0,L}^{\varepsilon,+,j} \leqslant (1+\delta)Z_{0,L} \quad \forall L \in [1, L_n],$$

where $\varepsilon = b\sigma^2/(j+1)$.

From (3.8) we know that for any $\delta > 0$, there is some $b_0(\delta) > 0$ such that $\mathcal{H}_1(\delta, b)$ holds for all $b \leqslant b_0(\delta)$. The proof of Proposition 3.1 is then completed by the following induction.

Lemma 3.4. *There exist $\delta \in (0,1)$, and $b \in (0, b_0(\delta))$ such that, for all $n \geqslant 1$, $\mathcal{H}_n(\delta, b)$ implies $\mathcal{H}_{n+1}(\delta, b)$.*

Proof. Fix $\delta > 0$ and $L \in (L_n, L_{n+1}]$, and assume the validity of $\mathcal{H}_n(\delta, b)$. Define ξ as the last zero of the walk up to $L/2$, and η as the first zero of the walk beyond $L/2$:

$$\xi(\gamma) = \max\{i \leqslant L/2 : \gamma_i = 0\}, \quad \eta(\gamma) = \min\{i > L/2 : \gamma_i = 0\}.$$

Then

$$
\begin{aligned}
Z_{0,L}^{\varepsilon,+,j} &= \sum_{x \leqslant L/2} \sum_{y > L/2} \sum_{\gamma \in \Omega_{0,L}} e^{\varepsilon N(\gamma)} w(\gamma) \mathbf{1}(\gamma \in \Omega_{0,L}^{+,j}) \mathbf{1}(\xi(\gamma) = x, \eta(\gamma) = y) \\
&\leqslant \sum_{x \leqslant L/2} \sum_{y > L/2} e^{2\varepsilon} Z_{0,x}^{\varepsilon,+,j} \Xi_{x,y} Z_{y,L}^{\varepsilon,+,j},
\end{aligned}
\tag{3.9}
$$

where we use the notation $Z_{y,L}^{\varepsilon,+,j} = Z_{0,L-y}^{\varepsilon,+,j}$ with the convention that $Z_{0,0}^{\varepsilon,+,j} = Z_{L,L}^{\varepsilon,+,j} = 1$ and we define

$$\Xi_{x,y} := \sum_{\gamma : x \to y} \mathbf{1}(N(\gamma) = 0, \gamma \geqslant -j) \prod_{i=x+1}^{y} p(\gamma_i - \gamma_{i-1}). \tag{3.10}$$

Here $\gamma : x \to y$ means that $\gamma = \{\gamma_i, \ i = x, \dots, y\}$ is a path such that $\gamma_x = \gamma_y = 0$. Since $x \in [0, L/2]$, $L - y \in [0, L/2]$, using $\mathcal{H}_n$ it follows that $Z_{0,x}^{\varepsilon,+,j} \leqslant (1+\delta)Z_{0,x}$ and $Z_{y,L}^{\varepsilon,+,j} \leqslant (1+\delta)Z_{y,L}$. Therefore (3.9) implies

$$Z_{0,L}^{\varepsilon,+,j} \leqslant (1+\delta)^2 e^{2\varepsilon} \sum_{x \leqslant L/2} \sum_{y > L/2} Z_{0,x} \Xi_{x,y} Z_{y,L}. \tag{3.11}$$

Let E denote the event that $\gamma_{\lfloor L/2 \rfloor} \geqslant -j$ and $\gamma_{\lfloor L/2 \rfloor + 1} \geqslant -j$. Clearly,

$$\frac{1}{Z_{0,L}} \sum_{x \leqslant L/2} \sum_{y > L/2} Z_{0,x} \Xi_{x,y} Z_{y,L} \leqslant \mathbb{P}_{0,L}(E). \tag{3.12}$$

From the estimate (b) in Proposition B.1 in the appendix, $\mathbb{P}_{0,L}(E) \leqslant 3/4$, uniformly in $L \geqslant L_1 = C(j+1)^2/\sigma^2$, $j \geqslant 0$ and $\sigma^2 > 0$. Thus, (3.11) and (3.12) imply

$$Z_{0,L}^{\varepsilon,+,j} \leqslant \tfrac{3}{4}(1+\delta)^2 e^{2\varepsilon} Z_{0,L}.$$

Since $\varepsilon = b\sigma^2/(j+1) \leqslant b$, if δ and b are small enough one has $\frac{3}{4}(1+\delta)^2 e^{2\varepsilon} \leqslant 1+\delta$. This implies $\mathcal{H}_{n+1}(\delta, b)$. $\qquad\square$

Next, we turn to the proof of the upper bound (2.6) announced in Remark 2.3. It can be restated as follows.

Proposition 3.5. *Taking $\varepsilon \leqslant b\sigma^2/(j+1)$ as in Proposition 3.1 one has, for all L, j, for all random walk kernel $p \in \mathcal{P}(\sigma^2)$:*

$$Z_{0,L}^{\varepsilon,+,j} \leqslant K\, Z_{0,L}^{+,j}, \tag{3.13}$$

where $K = K(j, p)$ is independent of L.

Proof. We first recall the well known bounds (see e.g. [9]):

$$c_1 \frac{1}{L} \leqslant \frac{Z_{0,L}^{+,j}}{Z_{0,L}} = \mathbb{P}_{0,L}(\gamma \geqslant -j) \leqslant c_2 \frac{1}{L}, \tag{3.14}$$

for some constants $c_i = c_i(j, p) > 0$ independent of L.

From Proposition 3.1 one has $Z_{0,L}^{\varepsilon,+,j} \leqslant 2Z_{0,L}$ for all L. By Hölder's inequality, for $k \in \mathbb{N}$ and $\varepsilon' = \varepsilon/k$,

$$\frac{Z_{0,L}^{\varepsilon',+,j}}{Z_{0,L}} = \mathbb{E}_{0,L}\left[\mathbf{1}(\gamma \geqslant -j)e^{\varepsilon' N(\gamma)}\right] \leqslant 2^{\frac{1}{k}}\,\mathbb{P}_{0,L}(\gamma \geqslant -j)^{\frac{k-1}{k}}.$$

If $k = 3$, using (3.14) to estimate $\mathbb{P}_{0,L}(\gamma \geqslant -j)$ from above, we obtain

$$\frac{Z_{0,L}^{\varepsilon/3,+,j}}{Z_{0,L}} \leqslant \kappa L^{-2/3}, \tag{3.15}$$

with $\kappa = \kappa(j, p) > 0$. We claim that we can bootstrap the bound (3.15) to

$$\frac{Z_{0,L}^{\varepsilon/3,+,j}}{Z_{0,L}} \leqslant \kappa' L^{-1}, \tag{3.16}$$

for some new constant $\kappa' = \kappa'(j, p)$. Once this is achieved, the proposition follows by using the left side in (3.14).

To prove (3.16), we replace $\varepsilon/3$ by ε for ease of notation. Consider the decomposition (3.9). The bound (3.15) implies

$$Z_{0,x}^{\varepsilon,+,j} Z_{y,L}^{\varepsilon,+,j} \leqslant \kappa^2 (x+1)^{-2/3}(L-y+1)^{-2/3} Z_{0,x} Z_{y,L}.$$

Moreover, by neglecting the constraint $\gamma \geqslant -j$ in (3.10), $\Xi_{x,y} \leqslant 2Z_{x,y}^{+,0}$. Therefore by (3.14), now with $j = 0$, one has

$$\Xi_{x,y} \leqslant \kappa_0 (y-x)^{-1} Z_{x,y},$$

for some $\kappa_0 = \kappa_0(p) > 0$. Summarizing, we obtain

$$Z_{0,L}^{\varepsilon,+,j} \leqslant \kappa^2 \kappa_0 \sum_{x \leqslant L/2} \sum_{y > L/2} Z_{0,x} Z_{x,y} Z_{y,L}\, (x+1)^{-2/3}(L-y+1)^{-2/3}(y-x)^{-1}.$$

From the local CLT estimates in (3.5) one has

$$\frac{Z_{0,x} Z_{x,y} Z_{y,L}}{Z_{0,L}} = \mathbb{P}_{0,L}(\gamma_x = \gamma_y = 0) \leqslant \kappa_1 \frac{1}{\sqrt{x}} \max\left(\frac{1}{\sqrt{y-x}}, \frac{1}{\sqrt{L-y}}\right),$$

for some $\kappa_1 = \kappa_1(p) > 0$. In conclusion, for $\kappa_2 := \kappa^2 \kappa_0 \kappa_1$,

$$\frac{Z_{0,L}^{\varepsilon,+,j}}{Z_{0,L}} \leqslant \kappa_2 \sum_{x \leqslant L/2} \sum_{y > L/2} \frac{1}{x^{2/3}(L-y)^{2/3}(y-x)} \frac{1}{\sqrt{x}} \max\left(\frac{1}{\sqrt{y-x}}, \frac{1}{\sqrt{L-y}}\right).$$

We need to show that the r.h.s. above is $O(1/L)$. To this end, it suffices to consider the two sums

$$A := \sum_{x \leqslant L/2} \sum_{y > L/2} \frac{1}{x^{2/3}(L-y)^{2/3}(y-x)\sqrt{x}\sqrt{y-x}},$$

$$B := \sum_{x \leqslant L/2} \sum_{y > L/2} \frac{1}{x^{2/3}(L-y)^{2/3}(y-x)\sqrt{x}\sqrt{L-y}}.$$

It is not hard to see that the sum in A is always $o(L^{-1})$. On the other hand, the sum in B behaves as $1/L$ as one easily sees by restricting to either $x \in [1, L/4]$ or $y \in [3L/4, L]$. $\quad\square$

3.3 Proof of Theorem 2

Let $\rho_j := (b\sigma^2)^{-1}(j+1)\varepsilon_j$. By monotonicity we may assume that $\sum_{j=0}^{\infty} \rho_j = 1$. Then, using Jensen's inequality

$$Z_{0,L}^{\Phi,+,0} \leqslant \sum_{\gamma \in \Omega_{0,L}^{+,0}} w(\gamma) \sum_{j=0}^{\infty} \rho_j e^{\kappa_j N_j(\gamma)}$$

$$= \sum_{j=0}^{\infty} \rho_j Z_{0,L}^{+,0} \, \mathbb{E}_{0,L}^{+,0} \left[e^{\kappa_j N_j} \right], \tag{3.17}$$

where $\kappa_j := \varepsilon_j / \rho_j = b\sigma^2/(j+1)$.

Next, we show that the upper bound in Proposition 3.1 implies

$$\mathbb{E}_{0,L}^{+,0} \left[e^{\kappa_j N_j} \right] \leqslant 4 \frac{Z_{0,L}}{Z_{0,L}^{+,0}}, \tag{3.18}$$

if b is small enough, uniformly in j, σ^2. Notice that this and (3.17) imply the desired estimate (2.11).

To prove (3.18), observe that by a vertical translation, Proposition 3.1 refers to the case where the walk starts and ends at level j, with a wall at zero. Thus, writing $Z_{u,v}^{\kappa_j,+,j}$ for the partition function of the walk with pinning strength κ_j at level j, that starts at j at time u and ends at j at time v, with wall at zero, Proposition 3.1 yields the bound

$$Z_{u,v}^{\kappa_j,+,j} \leqslant 2 Z_{u,v}, \tag{3.19}$$

for any $0 \leqslant u \leqslant v \leqslant L$. Therefore (3.18) follows by writing

$$Z_{0,L}^{+,0} \mathbb{E}_{0,L}^{+,0} \left[e^{\kappa_j N_j} \right] = \sum_{\gamma_1} w(\gamma_1) + \sum_{\gamma_2} w(\gamma_2) e^{\kappa_j N_j(\gamma_2)}, \tag{3.20}$$

where the first sum is over all paths $\gamma_1 \geqslant 0$ joining the vertices $(0,0)$ and $(L,0)$ that never reach level j, while the second sum is over all paths $\gamma_2 \geqslant 0$ joining the same vertices and that touch level j at least once. The first sum is trivially bounded by $Z_{0,L}$. The second sum is bounded by summing over the first and last contact with the level j, so that

$$\sum_{\gamma_2} w(\gamma_2) e^{\kappa_j N_j(\gamma_2)} \leqslant \sum_{u \leqslant v} e^{2\kappa_j} \hat{Z}_{0,u}^{+,0} Z_{u,v}^{\kappa_j,+,j} \hat{Z}_{v,L}^{+,0}, \tag{3.21}$$

where $\hat{Z}_{0,u}^{+,0}$ stands for the sum of $w(\gamma)$ over all $\gamma \geqslant 0$ joining $(0,0)$ and (u,j) that never touch height j before the ending point. Similarly, $\hat{Z}_{v,L}^{+,0}$ stands for the sum of $w(\gamma)$ over all $\gamma \geqslant 0$ joining (v,j) and $(L,0)$ that never touch height j after the starting point. Using (3.19) one obtains immediately that (3.21) is bounded by $2e^{2\kappa_j} Z_{0,L}$. Since $\kappa_j \leqslant b$ it follows that (3.18) holds as soon as $1 + 2e^{2b} \leqslant 4$.

4 Self-avoiding paths

In this section we prove Theorem 3 and Theorem 4. The strategy follows closely the corresponding arguments in the random walk case.

4.1 Lower bound

Proof of Theorem 4. Recall that if we restrict to the set $\hat{\Omega}_{0,L}$ of paths with minimal number of horizontal edges then we obtain the SOS walk with parameter β; see (2.13). We write $Z_{0,L}, Z_{0,L}^{+,0}$ for the usual partition functions of the SOS walk with weight $w(\gamma)$; see (2.1) and (2.2). By restricting to $\hat{\Omega}_{0,L}$ we may write

$$\mathbb{E}_{0,L}^{+,0}\big[e^{\Phi}\big] = \frac{\mathcal{Z}_{0,L}^{\Phi,+,0}}{\mathcal{Z}_{0,L}^{+,0}} \geqslant \frac{e^{-\beta(L-1)}(Z_\beta)^L}{\mathcal{Z}_{0,L}^{+,0}} \sum_{\gamma \in \hat{\Omega}_{0,L}} w(\gamma)e^{\Phi(\gamma)}$$

$$\geqslant \frac{Z_{0,L}^{+,0}}{Z_{0,L}} \frac{Z_{0,L}}{\mathcal{Z}_{0,L}} e^{-\beta(L-1)}(Z_\beta)^L \, \mathbb{E}_{0,L}^{+,0,\mathrm{SOS}}\big[e^{\Phi}\big],$$

where the second line follows from the obvious bound $\mathcal{Z}_{0,L}^{+,0} \leqslant \mathcal{Z}_{0,L}$ and the expectation $\mathbb{E}_{0,L}^{+,0,\mathrm{SOS}}$ refers now to the SOS walk. From [8, Eq. (4.8.4)],

$$\hat{\mathcal{Z}}_{0,L} \geqslant \mathcal{Z}_{0,L} \exp\left(-ce^{-2\beta}L\right),$$

for some absolute constant $c > 0$ and L large enough, where $\hat{\mathcal{Z}}_{0,L}$ is defined in (2.13). Therefore,

$$\frac{Z_{0,L}}{\mathcal{Z}_{0,L}} e^{-\beta(L-1)}(Z_\beta)^L = \frac{\hat{\mathcal{Z}}_{0,L}}{\mathcal{Z}_{0,L}} \geqslant \exp\left(-ce^{-2\beta}L\right),$$

for some absolute constant $c > 0$ and L large enough. Moreover, by (3.13) one has $Z_{0,L}^{+,0} \geqslant c(\beta)L^{-1}Z_{0,L}$ for some constant $c(\beta) > 0$. Finally, the conclusion follows from estimating from below $\mathbb{E}_{0,L}^{+,0,\mathrm{SOS}}\big[e^{\Phi}\big]$ as in (2.9). $\square$

4.2 Upper bound

We consider the probability measure $\mathbb{P}_{0,L}$ on $\Omega_{0,L}$ defined by the weight $e^{-\beta|\gamma|}$, and write $\mathbb{E}_{0,L}$ for the expectation w.r.t. $\mathbb{P}_{0,L}$. Call $N(\gamma) = N_0(\gamma)$ the number of contacts with the zero line, as in the definifion (2.14). For lightness of notation, below we set

$$\sigma^2 := e^{-\beta}.$$

We have the following version of Lemma 3.2.

Lemma 4.1. *There exists $c > 0, \beta_0 > 0$ such that for any L and $\beta \geqslant \beta_0$, for any $0 < b < 1/c$ one has*

$$\mathbb{E}_{0,L}\big[e^{b\,\sigma N(\gamma)/\sqrt{L}}\big] \leqslant 1 + c\,b.$$

Proof. We proceed as in Lemma 3.2. Define $\chi_i(\gamma) = \mathbf{1}((i + \frac{1}{2}, 0) \in \gamma)$. We have

$$e^{b\sigma N(\gamma)/\sqrt{L}} = \prod_{i=1}^{L-1} e^{b\sigma\chi_i(\gamma)/\sqrt{L}}$$

$$\leqslant 1 + \sum_{n=1}^{L-1} (2b\sigma L^{-1/2})^n \sum_{x_1 < \cdots < x_n} \chi_{x_1} \cdots \chi_{x_n},$$

where the sum ranges over all values of the integers $1 \leqslant x_1 < \cdots < x_n \leqslant L - 2$. Let us prove that

$$\mathbb{E}_{0,L}(\chi_{x_1} \cdots \chi_{x_n}) \leqslant C^n \, \sigma^{-n} \sqrt{L} \prod_{i=1}^{n+1} (x_i - x_{i-1})^{-1/2}, \tag{4.1}$$

where C is a universal constant. Once (4.1) is available, the rest of the proof is exactly as in Lemma 3.2.

Given a permutation $\pi = (\pi(1), \ldots, \pi(n))$, call E_π the event that in going from $(\frac{1}{2}, 0)$ to $(L - \frac{1}{2}, 0)$, the path γ visits the zeros at $\{x_1 + \frac{1}{2}, \ldots, x_n + \frac{1}{2}\}$ in the order $(x_{\pi(1)} + \frac{1}{2}, \ldots, x_{\pi(n)} + \frac{1}{2})$. One has

$$\mathbb{E}_{0,L}(\chi_{x_1} \cdots \chi_{x_n}; E_\pi) \leqslant \frac{\mathcal{Z}_{0,x_{\pi(1)}+1} \mathcal{Z}_{x_{\pi(1)},x_{\pi(2)}+1} \cdots \mathcal{Z}_{x_{\pi(n)},L}}{\mathcal{Z}_{0,L}}.$$

Using Proposition B.2 in the appendix it follows that

$$\frac{\mathcal{Z}_{0,x_{\pi(1)}+1} \mathcal{Z}_{x_{\pi(1)},x_{\pi(2)}+1} \cdots \mathcal{Z}_{x_{\pi(n)},L}}{\mathcal{Z}_{0,L}} \leqslant \frac{C^n \sigma^{-n} \sqrt{L}}{\Xi_L} \prod_{i=1}^{n+1} \frac{\Xi_{|x_{\pi(i)} - x_{\pi(i-1)}|}}{|x_{\pi(i)} - x_{\pi(i-1)}|^{1/2}},$$

where Ξ_k denotes the grand-canonical partition function (the one where the horizontal coordinate of the endpoint of the path is L while the vertical coordinate is free), and we set $x_{\pi(0)} = 0$ and $x_{\pi(n+1)} = L$. Since for any permutation π one has

$$\prod_{i=1}^{n+1} (x_{\pi(i)} - x_{\pi(i-1)})^{-1/2} \leqslant \prod_{i=1}^{n+1} (x_i - x_{i-1})^{-1/2},$$

to establish (4.1) it is sufficient to prove that for some constant $C > 0$:

$$\sum_\pi \frac{1}{\Xi_L} \prod_{i=1}^{n+1} \Xi_{x_{\pi(i)} - x_{\pi(i-1)}} \leqslant C^n$$

As in [8, Eq. (4.8.6)],

$$\Xi_L \leqslant C_1 e^{-c(\beta)L}, \quad \Xi_L \geqslant C_2 e^{-c(\beta)L},$$

for some absolute constants $C_1, C_2 > 0$, where $c(\beta) > 0$ is a constant such that $c(\beta) \to \infty$ as $\beta \to \infty$. It follows that for any π:

$$\frac{1}{\Xi_L} \prod_{i=1}^{n+1} \Xi_{x_{\pi(i)} - x_{\pi(i-1)}} \leqslant C^n e^{-c(\beta)\ell(\pi)},$$

where we define the excess length associated to a permutation π by

$$\ell(\pi) := -L + \sum_{i=0}^{n} |x_{\pi(i+1)} - x_{\pi(i)}|.$$

Notice that $\ell(\pi) \geqslant 0$ and $\ell(\pi) = 0$ iff π is the identity $\pi(i) \equiv i$. To conclude, we show that

$$\sum_\pi e^{-c(\beta)\ell(\pi)} \leqslant (1 + \delta(\beta))^n, \tag{4.2}$$

where $\delta(\beta) \to 0$ when $\beta \to \infty$. The statement (4.2) can be obtained by induction over n, as follows. Fix $x_0 = 0$ and $x_{n+1} = L$ and let $x_1, \ldots, x_n$ be arbitrary integers satisfying

$x_i \neq 0$ and $x_1 < \cdots < x_n < L$. Here we do not assume that $x_1 > 0$. Let $\phi(n)$ be defined as the sum in (4.2) for this choice of points $\{x_i\}$. We claim that

$$\phi(n) \leqslant (1 + \delta(\beta))^n e^{-c(\beta)x_1^-}, \tag{4.3}$$

where $x_1^- := \max(-x_1, 0)$ is the negative part of x_1. Clearly, (4.3) is sufficient to prove (4.2) which corresponds to the case $x_1 > 0$. To prove (4.3) notice that for $n = 1$ one has $\pi(1) = 1$ and $\ell(\pi) = 2x_1^-$ so that (4.3) is satisfied in this case. For $k \in \mathbb{N}$ and $\delta > 0$, assume $\phi(k) \leqslant (1 + \delta)^k e^{-c(\beta)x_1^-}$ and consider the case $n = k + 1$. Decomposing along the value of $\pi(1)$ one has

$$\phi(k + 1) = \sum_{j=1}^{n} \sum_{\pi:\pi(1)=j} e^{-c(\beta)\ell(\pi)}.$$

By the inductive assumption one has: if j is such that $x_j > 0$, then

$$\sum_{\pi:\pi(1)=j} e^{-c(\beta)\ell(\pi)} \leqslant e^{-c(\beta)(x_j - x_1)}(1 + \delta)^k,$$

while if j is such that $x_j < 0$ (and thus $x_1 < 0$), then

$$\sum_{\pi:\pi(1)=j} e^{-c(\beta)\ell(\pi)} \leqslant e^{-c(\beta)(x_1^- + |x_j|)}(1 + \delta)^k.$$

To conclude observe that if $x_1 > 0$ (and thus $x_j > 0$ for all $j \geqslant 1$), then one has

$$\phi(k + 1) \leqslant \sum_{j=1}^{n} e^{-c(\beta)(x_j - x_1)}(1 + \delta)^k \leqslant (1 + \delta)^{k+1}$$

if $\delta \geqslant \sum_{h=1}^{\infty} e^{-c(\beta)h}$. If instead $x_1 < 0$ then

$$\phi(k + 1) \leqslant (1 + \delta)^k \sum_{j=1}^{n} [e^{-c(\beta)(x_j - x_1)}\mathbf{1}(x_j > 0) + e^{-c(\beta)(x_1^- + |x_j|)}\mathbf{1}(x_j < 0)],$$

which is bounded by $2\delta(1 + \delta)^k e^{-c(\beta)x_1^-}$. This implies the claim (4.3). $\qquad\square$

We turn to the proof of Theorem 3. We need a version of Proposition 3.1 for lattice paths. For integers $j \geqslant 0$, define

$$\Omega_{0,L}^{+,j} = \{\gamma \in \Omega_{0,L} : \gamma \geqslant -j\}, \qquad \mathcal{Z}_{0,L}^{+,j} = \sum_{\gamma \in \Omega_{0,L}} e^{-\beta|\gamma|}\mathbf{1}(\gamma \in \Omega_{0,L}^{+,j}),$$

where the condition $\gamma \geqslant -j$ means that all vertices of γ have vertical coordinate at least $-j$. For the purpose of the recursive argument we have to consider also the number of external contacts of the path with the zero line. Namely, for any $\gamma \in \Omega_{0,L}$, define

$$N_{\text{ext}}(\gamma) = \sum_{i \in \mathbb{Z}} \mathbf{1}((i + \tfrac{1}{2}) \in \gamma)\mathbf{1}(i \notin [0, L - 1]). \tag{4.4}$$

If $N(\gamma)$ denotes the number of internal contacts with level zero as in (2.14), for $\varepsilon > 0$ we write

$$\mathcal{Z}_{0,L}^{\varepsilon,+,j} = \sum_{\gamma \in \Omega_{0,L}} e^{-\beta|\gamma| + \varepsilon N(\gamma) + \varepsilon N_{\text{ext}}(\gamma)}\mathbf{1}(\gamma \in \Omega_{0,L}^{+,j}). \tag{4.5}$$

Definition 4.2. *Fix the constant $C > 0$ as in Proposition B.2, an integer $j \geqslant 0$ and $\beta > 0$. Let $\sigma^2 = e^{-\beta}$, and define $L_0 = \frac{1}{2}C(j+1)^2/\sigma^2$, $L_n := 2^n L_0$. For $n \in \mathbb{N}$, $\delta > 0$, and $b > 0$, let $\mathcal{H}_n(\delta, b)$ denote the following statement:*

$$\mathcal{Z}_{0,L}^{\varepsilon,+,j} \leqslant (1+\delta)\mathcal{Z}_{0,L} \quad \forall L \in [1, L_n],$$

where $\varepsilon = b\sigma^2/(j+1)$.

Lemma 4.3. *For any $\delta > 0$, there exist constants $b_0 > 0, \beta_0 > 0$ such that $\mathcal{H}_1(b, \delta)$ holds for all $b \leqslant b_0$ and $\beta \geqslant \beta_0$.*

Proof. Dropping the wall constraint, we have

$$\mathcal{Z}_{0,L}^{\varepsilon,+,j} \leqslant \sum_{\gamma \in \Omega_{0,L}} e^{-\beta|\gamma|+\varepsilon N(\gamma)+\varepsilon N_{\text{ext}}(\gamma)}$$

$$= \mathcal{Z}_{0,L}\, \mathbb{E}_{0,L}\left[e^{\varepsilon N(\gamma)+\varepsilon N_{\text{ext}}(\gamma)}\right].$$

From Corollary A.5 we know that $\mathbb{E}_{0,L}\left[e^{2\varepsilon N_{\text{ext}}(\gamma)}\right] \leqslant 1 + u_\beta$ for some constant $u_\beta \to 0$ as $\beta \to \infty$. Thus, using Schwarz' inequality we have

$$\mathcal{Z}_{0,L}^{\varepsilon,+,j} \leqslant (1+u_\beta)^{1/2}\mathcal{Z}_{0,L}\left(\mathbb{E}_{0,L}\left[e^{2\varepsilon N(\gamma)}\right]\right)^{1/2}.$$

Now, if $L \leqslant L_1$ then $2\varepsilon \leqslant 2b\sqrt{C}\sigma/\sqrt{L}$. Therefore, Lemma 4.1 implies

$$\mathbb{E}_{0,L}\left[e^{2\varepsilon N(\gamma)}\right] \leqslant 1 + 2\sqrt{C}cb.$$

It remains to take b, β such that $(1+u_\beta)^{1/2}(1+2c\sqrt{C}b)^{1/2} \leqslant 1 + \delta$. $\qquad \square$

The next proposition establishes that $\mathcal{H}_n(\delta, b)$ holds in fact for all $n \in \mathbb{N}$ if we take e.g. $\delta = 1$, b small enough, and β large enough.

Proposition 4.4. *There exist constants $b > 0, \beta_0 > 0$ such that if $\varepsilon \leqslant b\, e^{-\beta}/(j+1)$, then uniformly in L, j and $\beta \geqslant \beta_0$ one has*

$$\mathcal{Z}_{0,L}^{\varepsilon,+,j} \leqslant 2\mathcal{Z}_{0,L}.$$

Proof. We repeat the very same argument of Lemma 3.4. The variable $\xi(\gamma)$ is now defined as the horizontal coordinate of the last contact of γ with the zero line in the interval $[0, L/2]$, while $\eta(\gamma)$ denotes the horizontal coordinate of the first contact with the zero line in the interval $(L/2, L]$. Equation (3.11) now becomes: for any $L \in (L_n, L_{n+1}]$,

$$\mathcal{Z}_{0,L}^{\varepsilon,+,j} \leqslant (1+\delta)^2 e^{2\varepsilon} \sum_{x \leqslant L/2} \sum_{y > L/2} \mathcal{Z}_{0,x}\Xi_{x,y}^\varepsilon \mathcal{Z}_{y,L}, \qquad (4.6)$$

where

$$\Xi_{x,y}^\varepsilon := \sum_{\gamma \in \Omega((x,0),(y,0))} e^{-\beta|\gamma|+\varepsilon N_{\text{ext}}(\gamma)}\mathbf{1}(N(\gamma) = 0, \gamma \geqslant -j). \qquad (4.7)$$

Note that the above sum includes the contribution of possible external zeros. This comes from the event that the path γ intersects the zero line outside of the interval $[x, y]$. Moreover, in (4.6) we have used the inductive assumption $\mathcal{H}_n(\delta, b)$. Notice that due to the possible presence of overhangs in the path γ it is crucial to take into account the external contacts N_{ext} in order to exploit the induction.

From Corollary A.5 in the appendix we know that $\Xi_{x,y}^\varepsilon \leqslant (1+u_\beta)\Xi_{x,y}^0$ for some constant $u_\beta \to 0$ as $\beta \to \infty$. Moreover, Proposition A.2 in the appendix shows that

$\Xi^0_{x,y} \leqslant (1 + u_\beta)\widetilde{\Xi}^0_{x,y}$, where $\widetilde{\Xi}^0_{x,y}$ denotes the sum $\Xi^0_{x,y}$ in (4.7) restricted to paths that are regular at the endpoints, see Definition A.1. From Corollary A.7 one has $\mathscr{Z}_{0,x} \leqslant (1 + u_\beta)\widehat{\mathscr{Z}}_{0,x}$ and $\mathscr{Z}_{y,L} \leqslant (1 + u_\beta)\widehat{\mathscr{Z}}_{y,L}$ where $\widehat{\mathscr{Z}}_{u,v}$ denotes the partition function $\mathscr{Z}_{u,v}$ restricted to paths that start and end with horizontal edges and that are regular at the points $x = u + 1$ and $x = v - 1$. Notice that if γ_1 is a path appearing in $\widehat{\mathscr{Z}}_{0,x}$, γ_2 is a path from $\widetilde{\Xi}^0_{x,y}$, and γ_3 is a path from $\widehat{\mathscr{Z}}_{y,L}$, then the composition $\gamma_1 \circ \gamma_2 \circ \gamma_3$ defines a valid path from 0 to L. That is, the above restrictions allow us to avoid complications due to the self-avoiding constraint when we reconstruct the global partition function $\mathscr{Z}_{0,L}$.

From (4.6) we obtain, for some constant $v_\beta \to 0$ as $\beta \to \infty$,

$$\mathscr{Z}^{\varepsilon,+,j}_{0,L} \leqslant (1 + \delta)^2 (1 + v_\beta)e^{2\varepsilon} \sum_{x \leqslant L/2} \sum_{y > L/2} \widehat{\mathscr{Z}}_{0,x} \widetilde{\Xi}^0_{x,y} \widehat{\mathscr{Z}}_{y,L}. \tag{4.8}$$

Define $\gamma_{\max}(\ell)$ as the maximal vertical height of γ at points with horizontal coordinate $\lfloor \ell \rfloor$. Then it is not hard to see that (4.8) implies

$$\frac{\mathscr{Z}^{\varepsilon,+,j}_{0,L}}{\mathscr{Z}_{0,L}} \leqslant (1 + \delta)^2 (1 + v_\beta)e^{2\varepsilon}\, \mathbb{P}_{0,L}(\gamma_{\max}(L/2) \geqslant -j).$$

As in the proof of Proposition 3.1, it remains to show that $\mathbb{P}_{0,L}(\gamma_{\max}(L/2) \geqslant -j) \leqslant 1 - \delta_0$ for some $\delta_0 > 0$, uniformly in $L \geqslant L_1 = C(j+1)^2 e^\beta$, $j \geqslant 0$ and $\beta > \beta_0$. This follows from Proposition B.2 (b) in the appendix. □

Proof of Theorem 3. We proceed exactly as in (3.17). Thus,

$$\mathscr{Z}^{\Phi,+,0}_{0,L} \leqslant \sum_{j=0}^\infty \rho_j \sum_{\gamma \in \Omega^{+,0}_{0,L}} e^{-\beta|\gamma|} e^{\kappa_j N_j(\gamma)},$$

where $\kappa_j := \varepsilon_j / \rho_j = b\sigma^2/(j+1)$, and $\sum_{j=0}^\infty \rho_j = 1$. The estimate (3.18) now takes the form

$$\sum_{\gamma \in \Omega^{+,0}_{0,L}} e^{-\beta|\gamma|} e^{\kappa_j N_j(\gamma)} \leqslant 4\mathscr{Z}_{0,L}. \tag{4.9}$$

To prove (4.9), thanks to Proposition 4.4 the same reasoning as in (3.21) can be applied with minor modifications. The only difference is that to conclude one needs to handle the partition functions in (3.21) with some care in order to restore the final partition function $\mathscr{Z}_{0,L}$. More precisely, fix $u, v \in \mathbb{Z}$, with $0 \leqslant u \leqslant v \leqslant L$; let $\hat{\mathscr{Z}}^{+,0}_{0,u}$ denote the partition function corresponding to paths $j \geqslant \gamma \geqslant 0$ from $(\frac{1}{2}, 0)$ to $(u + \frac{1}{2}, j)$ that never touch level j before the endpoint; similarly, let $\hat{\mathscr{Z}}^{+,0}_{v,L}$ denote the partition function corresponding to paths $j \geqslant \gamma \geqslant 0$ from $(v - \frac{1}{2}, j)$ to $(L - \frac{1}{2}, 0)$ that never touch level j after the staring point. Then, reasoning as in (3.20) and (3.21), the sum in (4.9) is bounded above by

$$\mathscr{Z}_{0,L} + e^{2\kappa_j} \sum_{u \leqslant v} \hat{\mathscr{Z}}^{+,0}_{0,u} \mathscr{Z}^{\varepsilon,+,j}_{u,v} \hat{\mathscr{Z}}^{+,0}_{v,L},$$

where $\mathscr{Z}^{\varepsilon,+,j}_{u,v} = \mathscr{Z}^{\varepsilon,+,j}_{0,v-u}$ is defined as in (4.5) above. Note that for this estimate to hold it is crucial that $\mathscr{Z}^{\varepsilon,+,j}_{u,v}$ takes into account the interaction with external zeros. By Proposition 4.4 one has $\mathscr{Z}^{\varepsilon,+,j}_{u,v} \leqslant 2\mathscr{Z}_{u,v}$. Moreover, let $\widetilde{\mathscr{Z}}^{+,0}_{0,u}, \widetilde{\mathscr{Z}}^{+,0}_{v,L}$ denote the partition functions $\hat{\mathscr{Z}}^{+,0}_{0,u}$ and $\hat{\mathscr{Z}}^{+,0}_{v,L}$ respectively defined with the restriction that the path is regular at the endpoints, see Definition A.1. Then, from Proposition A.2 in the appendix, $\hat{\mathscr{Z}}^{+,0}_{0,u} \hat{\mathscr{Z}}^{+,0}_{v,L} \leqslant (1 + u_\beta)\widetilde{\mathscr{Z}}^{+,0}_{0,u} \widetilde{\mathscr{Z}}^{+,0}_{v,L}$, for some $u_\beta \to 0$ as $\beta \to \infty$. Furthermore, from Corollary

A.7 one has $\mathcal{Z}_{u,v} \leqslant (1 + u_\beta)\widehat{\mathcal{Z}}_{u,v}$, where $\widehat{\mathcal{Z}}_{u,v}$ denotes the partition function $\mathcal{Z}_{u,v}$ restricted to paths that start and end with horizontal edges and that are regular at the points $x = u + 1$ and $x = v - 1$. It follows that

$$\hat{\mathcal{Z}}_{0,u}^{+,0}\, \mathcal{Z}_{u,v}\, \hat{\mathcal{Z}}_{v,L}^{+,0} \leqslant (1 + v_\beta)\widetilde{\mathcal{Z}}_{0,u}^{+,0}\, \widehat{\mathcal{Z}}_{u,v}\, \widetilde{\mathcal{Z}}_{v,L}^{+,0},$$

for some $v_\beta \to 0$ as $\beta \to \infty$ As discussed before Eq. (4.8), the regularity constraint at the endpoints together with the restriction to horizontal edges allows us to reconstruct the partition function $\mathcal{Z}_{0,L}$ as an upper bound so that, taking β large enough, one can conclude in the same way as in the proof of (3.18). □

A

In this first appendix we prove few technical results which, roughly speaking, say that, even in the presence of a wall, the lattice path ensemble for β large is likely to intersect a given vertical line only once, not to make excursions to the left of its starting point (or to the right of the final point) and to start and to end with a horizontal bond[1]. These very intuitive results are useful when trying to concatenate together different pieces of the path in our recursive method.

Consider the lattice path model defined in Section 2.2. We introduce a bit more notation. Given $H \in \mathbb{Z}$ we will denote by $\Omega_{0,L}^H$ the space of self-avoiding lattice paths γ connecting $(1/2, 0)$ to $(L - 1/2, H)$. The corresponding ensemble will be denoted by $\mathbb{P}_{0,L}^H$. If $H = 0$ we will simply write $(\Omega_{0,L}, \mathbb{P}_{0,L})$ as usual. The vertical line through the point $(u, 0)$, $u \in \mathbb{Z}$, will be denoted by $\mathcal{L}_u$ and the cardinality of a finite set S will be denoted by $|S|$. We write $\gamma \geqslant -j$ if the height of $\gamma \in \Omega_{0,L}^H$ is everywhere at least $-j$. Below, j is always a non-negative integer.

Definition A.1 (Local regularity). *Given $u \in [1, L - 1] \cap \mathbb{Z}$ we say that $\gamma \in \Omega_{0,L}^H$ is regular at u if γ intersects* only once *the line $\mathcal{L}_u$, i.e. if $|\gamma \cap \mathcal{L}_u| = 1$. For $u \in \{0, L\}$, we say that $\gamma \in \Omega_{0,L}^H$ is regular at u if γ does not* intersect *$\mathcal{L}_u$, i.e. if $|\gamma \cap \mathcal{L}_u| = 0$. If $\gamma \in \Omega_{0,L}^H$ is regular at both 0 and L we say that γ is regular at the endpoints.*

As an example, both paths in Figure 1 are regular at the endpoints, the left path is regular at $u = 2$, while the right path is not.

Proposition A.2. *We have*

$$\lim_{\beta \to +\infty} \sup_{j,L} \max_{\substack{H \geqslant -j \\ |H| \leqslant L}} \max_{u \in [0,L]} \mathbb{P}_{0,L}^H\left(\gamma \text{ is not regular at } u \mid \gamma \geqslant -j\right) = 0. \tag{A.1}$$

For simplicity we only treat the case $u = 0$ but the same strategy with minor modifications works for other values of u. Let $\hat{\mathbb{P}}_{0,L}^H(\cdot) = \mathbb{P}_{0,L}^H(\cdot \mid \gamma \cap \mathcal{L}_L = \emptyset)$ and let $\mathcal{A}$ denote the event that $\gamma \in \Omega_{0,L}^H$ is not regular at 0. We will first show that

$$\lim_{\beta \to +\infty} \sup_{j,L} \max_{\substack{H \geqslant -j \\ |H| \leqslant L}} \hat{\mathbb{P}}_{0,L}^H\left(\mathcal{A} \mid \gamma \geqslant -j\right) = 0. \tag{A.2}$$

Later on we will show that (A.2) implies the same bound for $\mathbb{P}_{0,L}^H$.

To prove (A.2) fix $\delta \leqslant 1/10$ (independent of β) and let $\mathcal{G}_L$ be the event that there exists $y \in [L/2 - \delta^2 L^{1-\delta}, L/2 + \delta^2 L^{1-\delta}] \cap \mathbb{Z}$ such that $\gamma \cap \mathcal{L}_y = (y, H_y)$ and H_y is within $\delta^2 L^{1-\delta}$ from the average height $\bar{H}_y := Hy/L$.

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[1]It might be possible that these regularity results could be derived using the Ornstein-Zernike techniques (see e.g. [6, 13]). However the presence of the wall poses quite annoying technical problems which we decided to overcome following a different route.

Lemma A.3. *For all β large enough and uniformly in $j \geqslant 0$, $|H| \leqslant 3L/2$ and L,*

$$\hat{\mathbb{P}}_{0,L}^{H}\left(\mathcal{G}_L \mid \gamma \geqslant -j\right) \geqslant 1 - \exp(-cL^{1-2\delta}),$$

for some constant $c = c(\delta, \beta)$ with $\lim_{\beta \to \infty} c(\delta, \beta) = +\infty$. The same bound applies to
$\mathbb{P}_{0,L}^{H}\left(\mathcal{G}_L \mid \gamma \geqslant -j\right)$.

Proof of the Lemma. We prove the lemma for $\hat{\mathbb{P}}_{0,L}^{H}$ but the same arguments apply to
$\mathbb{P}_{0,L}^{H}$. Clearly,

$$\hat{\mathbb{P}}_{0,L}^{H}\left(\mathcal{G}_L^c \mid \gamma \geqslant -j\right) \leqslant \hat{\mathbb{P}}_{0,L}^{H}\left(\mathcal{G}_L^c\right) / \hat{\mathbb{P}}_{0,L}^{H}\left(\gamma \geqslant -j\right).$$

Using [8]*Section 4.14 the probability that there exists $y \in [L/2 - \delta^2 L^{1-\delta}, L/2 + \delta^2 L^{1-\delta}] \cap \mathbb{Z}$
such that $\gamma \cap \mathcal{L}_y$ contains a point whose height differs from $\bar{H}_y$ by more than $\delta^2 L^{1-\delta}$
is smaller than $e^{-cL^{1-2\delta}}$ for some constant $c = c(\delta)$. If for all $y \in [L/2 - \delta^2 L^{1-\delta}, L/2 +$
$\delta^2 L^{1-\delta}] \cap \mathbb{Z}$ the set $\gamma \cap \mathcal{L}_y$ is *not* a singleton and it is contained in the interval $[\bar{H}_y \pm \delta^2 L^{1-\delta}]$,
then in the interval $[L/2 - \delta^2 L^{1-\delta}, L/2 + \delta^2 L^{1-\delta}]$ the path γ has length at least $10\delta^2 L^{1-\delta}$
i.e. an excess length (w.r.t. to its minimal length) of at least $(6 - 2H/L)\delta^2 L^{1-\delta} \geqslant 3\delta^2 L^{1-\delta}$.
Therefore a Peierls argument shows that the above event has probability not larger
than $e^{-c'\beta L^{1-\delta}}$ for some constant $c' = c'(\delta)$. In conclusion, by renaming the constants if
necessary, $\hat{\mathbb{P}}_{0,L}^{H}\left(\mathcal{G}_L^c\right) \leqslant e^{-cL^{1-2\delta}}$ for some constant $c = c(\delta, \beta)$ diverging as $\beta \to +\infty$.

We conclude with a rough lower bound on $\hat{\mathbb{P}}_{0,L}^{H}\left(\gamma \geqslant -j\right)$ of the form

$$\hat{\mathbb{P}}_{0,L}^{H}\left(\gamma \geqslant -j\right) = \sum_{\substack{\gamma \geqslant -j \\ \gamma \cap \mathcal{L}_L = \emptyset}} e^{-\beta|\gamma|} / \sum_{\gamma \cap \mathcal{L}_L = \emptyset} e^{-\beta|\gamma|} \geq e^{-c''\beta L^{1-3\delta}} \tag{A.3}$$

for some constant $c'' = c''(\delta)$. To prove this, let $\ell = \lfloor L^{1-3\delta} \rfloor$ with $\delta < 1/6$. We can restrict
the sum in the numerator above to paths which, while staying above $-j$ and never
intersecting $\mathcal{L}_L$, first go straight to the point $(0, \ell)$, then reach the point $(L, H + \ell)$ and
finally go straight to the point (L, H). Since $\ell \gg L^{1/2}$, [8] implies that the extra constraint
of staying above level $-j$ is irrelevant for this restricted sum which is therefore greater
than e.g. $e^{-3\beta\ell} \sum_{\gamma \cap \mathcal{L}_L = \emptyset} e^{-\beta|\gamma|}$. Hence the claimed bound. $\qquad\square$

We return to the proof of (A.2) let

$$p_L := \sup_{\substack{L' \\ |L - L'| \leqslant L^{1-\delta}}} \sup_j \max_{\substack{H \geqslant -j \\ |H| \leqslant L + L^{1-\delta}}} \hat{\mathbb{P}}_{0,L'}^{H}\left(\mathcal{A} \mid \gamma \geqslant -j\right).$$

We claim that

$$p_L \leqslant p_{L/2} + \exp(-cL^{1-2\delta}) \tag{A.4}$$

with $c = c(\delta)$. Fix $L' \in [L - L^{1-\delta}, L + L^{1-\delta}]$. On the event $\mathcal{G}_{L'}$ we can condition on the
rightmost point $\xi \in [L'/2 - \delta^2 (L')^{1-\delta}, L'/2 + \delta^2 (L')^{1-\delta}] \cap \mathbb{Z}$ satisfying the requirements
of $\mathcal{G}_{L'}$ and on the height H_ξ of the path there. By construction, for δ small enough and L
large enough independent of β,

$$\xi \in [L/2 - (L/2)^{1-\delta}, L/2 + (L/2)^{1-\delta}] \quad \text{and} \quad |H_\xi| \leqslant (1 + (L/2)^{-\delta})L/2. \tag{A.5}$$

Notice that the law of the part of the path γ joining the origin to $(\xi - 1/2, H_\xi)$ is exactly
$\hat{\mathbb{P}}_{0,\xi}^{H}$. Hence, using Lemma A.3 and (A.5), we get

$$\hat{\mathbb{P}}_{0,L'}^{H}\left(\mathcal{A} \mid \gamma \geqslant -j\right) \leqslant \max_{\xi, H_\xi \sim \mathcal{G}_{L'}} \hat{\mathbb{P}}_{0,L'}^{H}\left(\mathcal{A} \mid \xi, H_\xi, \gamma \geqslant -j\right) + \hat{\mathbb{P}}_{0,L'}^{H}\left(\mathcal{G}_{L'}^c \mid \gamma \geqslant -j\right)$$

$$\leqslant p_{L/2} + e^{-cL^{1-2\delta}},$$

i.e. (A.4). To finish the proof of (A.2) it is sufficient to observe that, for any fixed L_0,

$$\lim_{\beta \to +\infty} \sup_j \max_{\substack{H \geqslant -j \\ |H| \leqslant L_0}} \hat{\mathbb{P}}_{0,L_0}^H \left(\mathcal{A} \mid \gamma \geqslant -j \right) = 0,$$

because the event $\mathcal{A}$ forces the path to have an excess length (w.r.t. to the minimal lenght) of at least 2. Hence (A.4) implies that, for any L_0,

$$\limsup_{\beta \to +\infty} \sup_{j,L} \max_{\substack{H \geqslant -j \\ |H| \leqslant 2L}} \hat{\mathbb{P}}_{0,L}^H \left(\mathcal{A} \mid \gamma \geqslant -j \right) \leqslant \sum_{n \geqslant 0} e^{-c(L_0 \, 2^n)^{1-2\delta}}$$

and (A.2) follows.

We finally observe that (A.1) follows at once from (A.2). Fix some large L_0 independent of β. For $L \leqslant L_0$ a simple Peierls argument shows that

$$\lim_{\beta \to +\infty} \sup_j \max_{\substack{H \geqslant -j \\ |H| \leqslant 2L}} \mathbb{P}_{0,L}^H \left(\mathcal{A} \mid \gamma \geqslant -j \right) = 0.$$

For $L > L_0$ write as before

$$\mathbb{P}_{0,L}^H \left(\mathcal{A} \mid \gamma \geqslant -j \right) \leqslant \max_{\xi, H_\xi \sim \mathcal{G}_L} \mathbb{P}_{0,L}^H \left(\mathcal{A} \mid \xi, H_\xi, \gamma \geqslant -j \right) + \mathbb{P}_{0,L}^H \left(\mathcal{G}_L^c \mid \gamma \geqslant -j \right)$$

In the first term in the r.h.s. the path to the left of $\mathcal{L}_\xi$ has exactly the distribution $\hat{\mathbb{P}}_{0,\xi}^H$. Hence its $\beta \to \infty$ limit is zero by (A.2). The second term in the r.h.s. is smaller than $e^{-cL^{1-2\delta}} \leqslant e^{-cL_0^{1-2\delta}}$ by the previous lemma. Since L_0 was arbitrary the result follows. This ends the proof of Proposition A.2.

Proposition A.4. *Given $\gamma \in \Omega_{0,L}^H$ let $X(\gamma)$ be the position of the leftmost zero of γ. There exist C and β_0 such that, for all $\beta \geqslant \beta_0$ and all $\ell > 0$,*

$$\sup_{j,L} \max_{\substack{H \geqslant -j \\ |H| \leqslant L}} \mathbb{P}_{0,L}^H \left(X(\gamma) = -\ell \mid \gamma \geqslant -j \right) \leqslant C e^{-m_\beta \ell},$$

with $\lim_{\beta \to \infty} m_\beta = +\infty$.

Corollary A.5. *Let $N_{\text{ext}}(\gamma)$ denote the external zeros as defined in (4.4). Then, for any fixed constant $a > 0$:*

$$\lim_{\beta \to +\infty} \sup_{j,L} \max_{\substack{H \geqslant -j \\ |H| \leqslant L}} \mathbb{E}_{0,L}^H \left(e^{a N_{\text{ext}}(\gamma)} \mid \gamma \geqslant -j \right) = 1 \tag{A.6}$$

Proof of the Corollary. We may write $N_{\text{ext}} = N_-(\gamma) + N_+(\gamma)$, where $N_+(\gamma)$ (resp. $N_-(\gamma)$) denotes the number of zeros to the right (resp. left) of the interval $[0, L]$. By Schwarz' inequality and using symmetry it suffices to prove the statement (A.6) with N_{ext} replaced by $N_-(\gamma)$. Clearly $N_-(\gamma) \leqslant |X(\gamma)|$. The statement then follows at once from Proposition A.4. $\qquad\square$

Proof of Proposition A.4. Fix $\delta \leqslant 1/10$ and assume first that $\ell \geqslant L^{1-\delta}$. In this case, using (A.3), it suffices to prove the required estimate for the unconditioned probability. If by traveling along the path the last intersection with the column $\mathcal{L}_1$ is at height Y with $|Y| \leqslant \ell$ then the part of the path between the origin and the vertex $(1, Y)$ has an excess length of at least ℓ. A standard Peierls argument shows that the unconditioned probability of such an event is bounded from above by $e^{-m_\beta \ell}$ with $\lim_{\beta \to \infty} m_\beta = +\infty$. If instead $|Y| \geqslant \ell$ we can appeal to the large deviation bounds in [8]*Section 4 to get the same result.

Suppose now that $\ell \leqslant L^{1-\delta}$. The argument leading to (A.4), with $L_0 = \ell^{1/(1-2\delta)}$, implies the estimate

$$\sup_{j,L} \max_{\substack{H \geqslant -j \\ |H| \leqslant L}} \mathbb{P}_{0,\ell}^H \left(X(\gamma) = -\ell \mid \gamma \geqslant -j \right)$$

$$\leqslant \sup_j \max_{\substack{H \geqslant -j \\ |H| \leqslant 2L_0}} \mathbb{P}_{0,L_0}^H \left(X(\gamma) = -\ell \mid \gamma \geqslant -j \right) + \sum_{n \geqslant 0} e^{-c_\beta (L_0 \, 2^n)^{1-2\delta}}.$$

Using the first part of the proof we get that both terms in the r.h.s. above can be bounded from above by $e^{-m_\beta \ell}$ with $\lim_{\beta \to \infty} m_\beta = +\infty$. $\qquad \square$

The last result says that, under $\mathbb{P}_{0,L}$, the path γ is likely to start with a horizontal bond.

Lemma A.6. *Let f_0 be the first edge of the path $\gamma \in \Omega_{0,L}$. Then*

$$\lim_{\beta \to \infty} \sup_L \mathbb{P}_{0,L} \left(f_0 \text{ is vertical} \right) = 0$$

Proof. We will first prove the result in the "grand canonical" ensemble $\{\mathcal{O}_L, \mathcal{P}_L\}$, where $\mathcal{O}_L$ is the set of self-avoiding paths starting at $(1/2, 0)$ and ending at $(L - 1/2, H)$ for *some* $H \in \mathbb{Z}$ and $\mathcal{P}_L(\gamma) = e^{-\beta|\gamma|}/\Xi_L$ with $\Xi_L = \sum_{\gamma \in \mathcal{O}_L} e^{-\beta|\gamma|}$. Indeed, if we decompose over the number of the first consecutive vertical edges, we immediately get that $\mathcal{P}_L(f_0 \text{ is vertical}) \leqslant 2\sum_{n \geqslant 1} e^{-\beta n}$. The result for the canonical ensemble $\mathbb{P}_{0,L}(\cdot) = \mathcal{P}_L(\cdot \mid \gamma \text{ ends at zero height})$ follows from Lemma B.4 in Appendix B. $\qquad \square$

From Lemma A.6 and Proposition A.2 one has the following

Corollary A.7. *Let $\widehat{\mathcal{Z}}_{0,L}$ denote the partition function obtained by restricting $\mathcal{Z}_{0,L}$ to paths $\gamma \in \Omega_{0,L}$ that are regular at both $x = 1$ and $x = L - 1$ and such that the first and last edge of γ is horizontal. Then there exists $u_\beta > 0$ with $u_\beta \to 0$ as $\beta \to \infty$ such that $\mathcal{Z}_{0,L} \leqslant (1 + u_\beta)\widehat{\mathcal{Z}}_{0,L}$ for any $L \geqslant 1$.*

B

Here, we prove two estimates on moderate deviations for random walks and self-avoiding paths. In the random walk setting the arguments that we use are rather standard but we decided to detail them, on one hand in order to pave the way for the self-avoiding paths setting, and on the other hand to get estimates that hold uniformly on all scales and for all random walks in our class.

Proposition B.1. *There exists $C \geqslant 1$ such that, for any $\sigma^2 \in (0, 1/2]$ and for any random walk kernel $p \in \mathcal{P}(\sigma^2)$, the following holds.*
(a) For any L,

$$\frac{C^{-1}}{\max\left(1, \sqrt{\sigma^2 L}\right)} \leqslant Z_{0,L} \leqslant \frac{C}{\sqrt{\sigma^2 L}}. \tag{B.1}$$

(b) For all $j \geqslant 0$ and for all $L \geqslant C(j+1)^2/\sigma^2$,

$$\mathbb{P}_{0,L}(\gamma(L/2) \geqslant -j) \leqslant 3/4, \tag{B.2}$$

where $\gamma(L/2) = \gamma_{\lfloor L/2 \rfloor}$.

Proof.
(a) Let $S_n = X_1 + \cdots + X_n$, $S_0 = 0$, where $\{X_i\}_{i=1}^\infty$ are i.i.d. random variables with law $p \in \mathcal{P}(\sigma^2)$. In the sequel we will write $f_n(t)$ for the characteristic function of S_n/σ_n

where $\sigma_n = \sqrt{n}\sigma$. Clearly $f_n(t) = \varphi(t/\sigma_n)^n$ where $\varphi(\cdot)$ is the characteristic function of the variable X_1. Let (S_n, S_n') be two independent copies of the same random variable. Using the identity

$$\mathbb{1}_{\{S_n = S_n'\}} = \frac{1}{2\pi\sigma_n} \int_{-\pi\sigma_n}^{\pi\sigma_n} e^{i\frac{t}{\sigma_n}(S_n - S_n')} dt$$

we can write

$$\mathbb{P}(S_n = S_n') = \frac{1}{2\pi\sigma_n} \int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |f_n(t)|^2 \tag{B.3}$$

Notice that $|\varphi(t/\sigma_n)| \leqslant e^{\frac{1}{2}(|\varphi(t/\sigma_n)|^2 - 1)}$ and that, using the assumption $p \in \mathcal{P}(\sigma^2)$,

$$|\varphi(t/\sigma_n)|^2 - 1 = -\operatorname{Var}\left(\cos(tX_1/\sigma_n)\right) - \operatorname{Var}\left(\sin(tX_1/\sigma_n)\right)$$
$$\leqslant -2p(0)p(1)(1 - \cos(t/\sigma_n))$$
$$\leqslant -c_0(1 - \sigma^2)\frac{t^2}{2n} \leqslant -\frac{c_0}{4n}t^2$$

for all $|t| \leqslant \pi\sigma_n$. Above we used the formula $\operatorname{Var}(Z) = \frac{1}{2}\mathbb{E}((Z - Z')^2)$ where Z, Z' are two independent copies of the same random variable to get

$$\operatorname{Var}\left(\cos(tX_1/\sigma_n)\right) + \operatorname{Var}\left(\sin(tX_1/\sigma_n)\right) \geqslant p(0)p(1)\left[(1 - \cos(t/\sigma_n))^2 + \sin(t/\sigma_n)^2\right].$$

In conclusion $|f_n(t)| \leqslant e^{-c_0 t^2/8}$ and

$$\mathbb{P}(S_n = S_n') \leqslant \frac{C}{\sigma_n}$$

for some constant C depending on c_0. To prove a lower bound we use

$$|e^{it} - 1 - it + \frac{t^2}{2}| \leqslant \min\left(|t|^2, |t|^3/6)\right],$$

to write

$$|\varphi(t/\sigma_n) - 1 + \frac{t^2}{2n}| \leqslant \frac{t^2}{\sigma_n^2}\mathbb{E}\left(X_1^2 \wedge \frac{|t|}{6\sigma_n}|X_1|^3\right) \leqslant c\frac{|t|^3}{6n\sigma_n},$$

where we used the assumptions $\mathbb{E}(X_1) = 0$ and $\mathbb{E}(|X_1|^3) \leqslant c\sigma^2$. Thus, if $\sigma_n \geqslant 1$ and by choosing δ small enough independent of σ^2, we obtain

$$\mathbb{P}(S_n = S_n') = \frac{1}{2\pi\sigma_n} \int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |f_n(t)|^2$$
$$\geqslant \frac{1}{2\pi\sigma_n} \int_{-\delta}^{\delta} dt \, |f_n(t)|^2 \geqslant \frac{1}{2\pi\sigma_n} \int_{-\delta}^{\delta} dt \, e^{-t^2/4} \geqslant \frac{C_\delta}{\sigma_n}.$$

If instead $\sigma_n \leqslant 1$ we simply write

$$\mathbb{P}(S_n = S_n') \geqslant \mathbb{P}(S_n = 0)^2 \geqslant p(0)^{2n} \geqslant (1 - \sigma^2)^{2n} \geq e^{-4}.$$

Equation (B.1) follows if we observe that (without loss of generality we assume L even)

$$Z_{0,L} = \mathbb{P}(S_{L/2} = S_{L/2}').$$

We turn to the proof of part (b). With the previous notation we can write ($n = L/2$)

$$\mathbb{P}(\gamma(n) \geqslant -j \,|\, \gamma(2n) = 0) = \mathbb{P}(S_n' \geqslant -j \,|\, S_n = S_n')$$
$$= \frac{\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, f_n(t)\mathbb{E}\left(\mathbb{1}_{S_n' \geqslant -j} e^{-i\frac{t}{\sigma_n}S_n'}\right)}{\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |f_n(t)|^2}$$
$$\leqslant \frac{\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |f_n(t)|}{\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |f_n(t)|^2} \, \mathbb{P}(S_n' \geqslant -j). \tag{B.4}$$

We now claim that, given $0 < \delta \leqslant 0.02$, we can choose $C = C(\delta)$ so large that, for $n \geqslant C/\sigma^2$,

$$\frac{\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |f_n(t)|}{\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |f_n(t)|^2} \leqslant \sqrt{2} + \delta, \tag{B.5}$$

and for $n \geqslant C(j+1)^2/\sigma^2$

$$\mathbb{P}(S_n' \geqslant -j) \leqslant \frac{1}{2} + \delta. \tag{B.6}$$

These bounds imply (B.2).

We begin by discussing (B.5). Consider first $\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |f_n(t)|$. For any $A > 0$ we write

$$\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |f_n(t)| \leqslant \int_{-\infty}^{\infty} dt \, e^{-t^2/2} + J_1 + J_2 + J_3 = \sqrt{2\pi} + J_1 + J_2 + J_3,$$

where

$$J_1 = \int_{-A}^{A} dt \, |f_n(t) - e^{-t^2/2}|, \quad J_2 = \int_{|t| \geqslant A} dt \, e^{-t^2/2}, \quad J_3 = \int_{A \leqslant |t| \leqslant \pi\sigma_n} dt \, |f_n(t)|.$$

Using the bound $|f_n(t)| \leqslant e^{-c_0 t^2/4}$ we can always choose A in such a way that $J_2, J_3 \leqslant \delta/6$. Given A, we can use as before the second order Taylor expansion and choose C so large (independent of σ^2) that $J_1 \leqslant \delta/6$. In conclusion

$$\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |f_n(t)| \leqslant \sqrt{2\pi} + \delta/2.$$

To lower bound $\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |f_n(t)|^2$ we may simply restrict the integral to $|t| \leqslant A$ and get

$$\int_{-\pi\sigma}^{\pi\sigma} dt \, |f(t)|^2 \geqslant \int_{-A}^{A} dt \, e^{-t^2} - \int_{-A}^{A} dt \, |\, |f(t)|^2 - e^{-t^2} |$$

$$= \sqrt{\pi} - \int_{|t| \geqslant A} dt \, e^{-t^2} - \int_{-A}^{A} dt \, |\, |f(t)|^2 - e^{-t^2} |$$

and choose again A, C large enough to make the two error terms smaller than $\delta/2$. Thus the l.h.s. of (B.5) is smaller than $\frac{\sqrt{2\pi} + \delta/2}{\sqrt{\pi} - \delta/2} \leqslant \sqrt{2} + \delta$ for $\delta \leqslant 1$.

Next we prove (B.6). The Berry-Essen theorem (see e.g. [10]) gives

$$\sup_x |\mathbb{P}(S_n/\sigma_n \leqslant x) - \Phi(x)| \leqslant 3\frac{\mathbb{E}(|X_1|^3)}{\sigma^3 \sqrt{n}},$$

where $\Phi(\cdot)$ is the distribution function of the standard normal. Thus, using $\mathbb{E}(|X_1|^3) \leqslant c\sigma^2$,

$$\mathbb{P}(S_n \geqslant -j) \leqslant \Phi(j/\sigma_n) + \frac{c}{2\sigma_n} \leqslant 1/2 + \delta$$

for all $n \geqslant C(j+1)^2/\sigma^2$ with C large enough depending on δ. $\qquad\square$

Proposition B.1 holds also for the ensemble of self-avoiding paths introduced in Section 2.2. We need few additional notation. We will denote by $\gamma_{\max}(L/2)$ the highest intersection of $\gamma \in \Omega_{0,L}$ with the vertical line through the point $(\lfloor L/2 \rfloor, 0)$. Recall that $\mathcal{Z}_{0,L} = \sum_{\gamma \in \Omega_{0,L}} e^{-\beta|\gamma|}$ and let Ξ_L be the partition function of the grand canonical ensemble $\{\mathcal{O}_L, \mathcal{P}_L\}$ defined in the proof of Lemma A.6. Finally, denote by $\{\hat{\mathcal{O}}_L, \hat{\mathcal{P}}_L\}$ the restricted grand canonical ensemble in which the intersection of the path $\gamma \in \mathcal{O}_L$ with the vertical line through the point $(L, 0)$ is empty.

Proposition B.2. *There exists C and β_0 such that, for all $\beta \geqslant \beta_0$, the following holds.*
(a) There exists $\hat{\sigma}_\beta$ with $|\hat{\sigma}_\beta^2 - \frac{1}{\cosh(\beta)-1}| \leqslant e^{-2(\beta-\beta_0)}$ such that, for all $L \geqslant 1$,

$$\frac{C^{-1}}{\max\left(1, \sqrt{\hat{\sigma}_\beta^2 L}\right)} \leqslant \frac{\mathcal{Z}_{0,L}}{\Xi_L} \leqslant \frac{C}{\sqrt{\hat{\sigma}_\beta^2 L}}. \tag{B.7}$$

(b) For all $j \geqslant 0$ and all $L \geqslant C(j+1)^2 e^\beta$,

$$\mathbb{P}_{0,L}(\gamma_{\max}(L/2) \geqslant -j) \leqslant 3/4.$$

Before proving the lemma we recall some key results from [8]. For any $\gamma \in \mathcal{O}_L$ let $h_L(\gamma)$ be the height of its final point. Let also $\sigma_L^2, f_L(t)$ be the variance of $h_L(\gamma)$ and the characteristic function of $h_L(\gamma)/\sigma_L$ respectively in the ensemble $\mathcal{P}_L$.

Lemma B.3 ([8]*Sections 4.9,4.10,4.10.20,4.10.29)**.** *There exists $\beta_0 > 0$ such that, for all $\beta \geqslant \beta_0$, the following holds.*

(a) There exists $\hat{\sigma}(\beta)$ with $|\hat{\sigma}^2(\beta) - \frac{1}{\cosh(\beta)-1}| \leqslant e^{-2(\beta-\beta_0)}$ and such that

$$|\sigma_L^2 - \hat{\sigma}(\beta)^2 L| \leqslant e^{-2(\beta-\beta_0)} \quad \forall L.$$

(b) There exist two constants λ and $\alpha \leqslant \pi$ independent of β, L such that

    (i) $\log(f_L(t)) = -\frac{t^2}{2} + \frac{t^3}{6}\frac{R_L(t)}{\sigma_L^3}$ for all $t \in \mathbb{R}$ with $\sup_t |R_L(t)| \leqslant \lambda \sigma_L^2$.

    (ii) $|f_L(t)| \leqslant e^{-t^2/4}$ for all $|t| \leqslant \alpha \sigma_L$.

    (iii) $|f_L(t)| \leqslant (1 - \frac{\alpha^2}{4e^\beta})^L \leqslant e^{-c(\alpha)\sigma_L^2}$ for all $|t| \in (\alpha\sigma_L, \pi\sigma_L]$.

(c) $1/2 \leqslant \Xi_L/(e^{-\beta}\Xi_{L/2}^2) \leqslant 1$ for L even and $1/2 \leqslant \Xi_L/\Xi_{\lceil L/2 \rceil}^2 \leqslant 1$ for L odd.

Similar bounds hold for the quantities $\hat{\sigma}_L^2, \hat{f}_L(t)$ computed in the restricted ensemble $\hat{\mathcal{P}}_L$.

Proof of Proposition B.2. Without loss of generality we assume that L is odd and we let $n = \lceil L/2 \rceil$.

Using (c) of Lemma B.3 it is enough to prove (B.7) for the ratio $\mathcal{Z}_{0,L}/\Xi_n^2$. Using Proposition A.2, for any β large enough uniformly in L, $\mathcal{Z}_{0,L} \leqslant 2\mathcal{Z}_{0,L}^{\mathrm{reg}}$ where $\mathcal{Z}_{0,L}^{\mathrm{reg}} := \sum_{\gamma \in \Omega_{0,L}^{\mathrm{reg}}} e^{-\beta|\gamma|}$ and $\Omega_{0,L}^{\mathrm{reg}} \subset \Omega_{0,L}$ consists of those paths that are regular at n. In turn the ratio $\mathcal{Z}_{0,L}^{\mathrm{reg}}/\Xi_n^2$ is at most the probability (in the $\hat{\mathcal{P}}_n$ ensemble) that two independent copies of $\gamma \in \hat{\mathcal{O}}_n$ have the same final height, a quantity which can be written (cf. (B.3))

$$\hat{\mathcal{P}}_n \otimes \hat{\mathcal{P}}_n(h_n(\gamma_1) = h_n(\gamma_2)) = \frac{1}{2\pi\sigma_n}\int_{-\pi\sigma_n}^{\pi\sigma_n} dt\, |\hat{f}_n(t)|^2. \tag{B.8}$$

The desired upper bound now follows at once from (a) and (b) of Lemma B.3.

Next we lower bound $\mathcal{Z}_{0,L}$ by $\mathcal{Z}_{0,L}^{\mathrm{reg}}$. Any $\gamma \in \Omega_{0,L}^{\mathrm{reg}}$ can be seen as formed by two paths γ_i, $i = 1, 2$, the first one starting from the origin and running forward and the second one starting from $(L, 0)$ and running backward, until they hit $\mathcal{L}_n$ at the same height. These two paths are i.i.d with law $\hat{\mathcal{P}}_n(\gamma)$. Thus

$$\frac{\mathcal{Z}_{0,L}^{\mathrm{reg}}}{\Xi_n^2} = \hat{\mathcal{P}}_n \otimes \hat{\mathcal{P}}_n(h_n(\gamma_1) = h_n(\gamma_2))\left(\frac{\hat{\Xi}_n}{\Xi_n}\right)^2.$$

Using (b) of Lemma B.3, (B.8) is bounded from below by $1/C_1 \max\left(1, \sqrt{\sigma_n^2}\right)$ for some large C_1 independent of β as in Proposition B.1. The ratio $\hat{\Xi}_n/\Xi_n$ is greater than e.g. $1/2$ using [8]*Equations 4.8.5, 4.8.6. The proof of part (a) is complete.

As for part (b), using Proposition A.2, $\lim_{\beta\to\infty} \mathbb{P}_{0,L}(\Omega_{0,L}^{\mathrm{reg}}) = 1$ uniformly in L. Hence it is enough to prove the statement of the lemma for $\mathbb{P}_{0,L}^{\mathrm{reg}}(\cdot) := \mathbb{P}_{0,L}(\cdot \mid \Omega_{0,L}^{\mathrm{reg}})$, with $3/4$ replaced by e.g. $3/5$. By writing as before $\gamma \in \Omega_{0,L}^{\mathrm{reg}}$ as the concatenation of two independent paths γ_1, γ_2 we get that

$$\mathbb{P}_{0,L}^{\mathrm{reg}}(\gamma_{\max}(L/2) \geqslant -j) = \hat{\mathcal{P}}_n \otimes \hat{\mathcal{P}}_n(h_n(\gamma_1) \geqslant -j \mid h_n(\gamma_1) = h_n(\gamma_2)).$$

At this stage we proceed exactly as in part (b) of Proposition B.1 (cf. (B.4)). Using again Lemma B.3, we get that, for any small δ and any $L \geqslant Ce^\beta$ with C large enough depending on δ,

$$\frac{\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |\hat{f}_n(t)|}{\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |\hat{f}_n(t)|^2} \leqslant \sqrt{2} + \delta.$$

As far as the quantity $\hat{\mathcal{P}}_n(h(\gamma_1) \geqslant -j)$ is concerned we can appeal to the following formula (cf. e.g. [10]*Ch. XVI formula 3.13):

$$\max_h |\hat{\mathcal{P}}_n(h(\gamma) \geqslant h\sigma_n) - \Phi(-h)| \leqslant \frac{1}{\pi} \int_{-T}^{T} dt \, \frac{1}{t} |\hat{f}_n(t) - e^{-t^2/2}| + \frac{24}{\sqrt{2\pi^3}T}, \quad \forall T > 0,$$

valid for all h. Choose now $T = \frac{48}{\sqrt{2\pi^3}\delta}$ and a constant C large enough depending on δ in a such a way that $\int_{-T}^{T} dt \, \frac{1}{t} |\hat{f}_n(t) - e^{-t^2/2}| \leqslant \delta/2$ for all $L \geqslant Ce^\beta$. Thus the r.h.s. above is smaller than δ. Taking $h = -j/\sigma_n$ concludes the proof. $\qquad\square$

We end with a simple lemma that allows one to bound canonical probabilities with their grand canonical counterpart. We use the notation introduced so far.

Lemma B.4. *Fix $k \in \mathbb{N}$, $L \geqslant k$ and let $A \subset \Omega_{0,L}$ depend only on the first k edges of the path γ. If $\lim_{\beta\to\infty} \sup_{L \geqslant k} \mathcal{P}_L(A) = 0$ then the same holds for the canonical measure $\mathbb{P}_{0,L}$.*

Proof. Fix a large constant C. If $L \leqslant Ce^\beta$ then it is immediate to check that, as $\beta \to \infty$, the measure $\mathbb{P}_{0,L}$ is concentrated over paths which are flat for the first k steps as is the case for the grand canonical measure, so the two marginals on the first k steps are essentially identical. If instead $L \geqslant Ce^\beta$ we argue as in part (b) of Proposition B.2 to write (with $n = \lfloor L \rfloor$)

$$\lim_{\beta\to\infty} \sup_{L \geqslant k} \mathbb{P}_{0,L}(A) = \lim_{\beta\to\infty} \sup_{L \geqslant k} \mathbb{P}_{0,L}^{\mathrm{reg}}(A) = \lim_{\beta\to\infty} \sup_{2n \geqslant k} \hat{\mathcal{P}}_n \otimes \hat{\mathcal{P}}_n \left(A \mid h_n(\gamma_1) = h_n(\gamma_2)\right)$$

$$\leqslant \lim_{\beta\to\infty} \sup_{2n \geqslant k} \frac{\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |\hat{f}_n(t)|}{\int_{-\pi\sigma_n}^{\pi\sigma_n} dt \, |\hat{f}_n(t)|^2} \hat{\mathcal{P}}_n(A) \leqslant (\sqrt{2} + \delta) \lim_{\beta\to\infty} \sup_{L \geqslant k} \hat{\mathcal{P}}_L(A),$$

where $\delta = \delta(C)$ tends to zero as $C \to +\infty$. Finally we observe that

$$\lim_{\beta\to\infty} \sup_{L \geqslant k} \hat{\mathcal{P}}_L(A) = \lim_{\beta\to\infty} \sup_{L \geqslant k} \mathcal{P}_L(A) = 0.$$

$\qquad\square$

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