Diffusion limits at small times for Λ-coalescents with a Kingman component[*]

Vlada Limic[†]   Anna Talarczyk[‡]

Abstract

We consider standard Λ-coalescents (or coalescents with multiple collisions) with a non-trivial "Kingman part". Equivalently, the driving measure Λ has an atom at 0; $\Lambda(\{0\}) = c > 0$. It is known that all such coalescents come down from infinity. Moreover, the number of blocks N_t is asymptotic to $v(t) = 2/(ct)$ as $t \to 0$. In the present paper we investigate the second-order asymptotics of N_t in the functional sense at small times. This complements our earlier results on the fluctuations of the number of blocks for a class of regular Λ-coalescents without the Kingman part. In the present setting it turns out that the Kingman part dominates, and the limit process is a Gaussian diffusion, as opposed to the stable limit in our previous work.

1 Introduction and main results

1.1 Background

The Kingman coalescent, introduced in [14, 15], is one of the pillar processes of mathematical population genetics. The research reported here is linked to some of the classical results on the Kingman coalescent. In particular, Griffiths in [11] derives the Gaussian behavior of the number of blocks (one dimensional distributions only). Similar limits are discussed by Aldous in [1] in the absence of mutations, with general acknowledgement (as folk theorem), but no specific reference provided. More precisely, let K_t be the number of blocks in the standard Kingman coalescent at time t. Then [1] outlines the argument for

$$\sqrt{\frac{3t}{2}} \left(K_t - \frac{2}{t} \right) \Rightarrow N(0,1), \text{ as } t \to 0.$$

[†]CNRS UMR 8628, Laboratoire de Mathématiques, Université Paris-Sud, Orsay, France.
 E-mail: vlada.limic@math.u-psud.fr
[‡]Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland.
 E-mail: annatal@mimuw.edu.pl

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The Λ-coalescents form the simplest class of processes with exchangeable dynamics that generalize the Kingman coalescent. They were introduced and first studied independently by Pitman [18] and Sagitov [19], and were also considered in a contemporaneous work of Donnelly and Kurtz [9]. For recent overviews of the literature we refer the reader to [5, 4].

Let Λ be an arbitrary finite measure on $[0, 1]$. We denote by $(\Pi_t, t \geq 0)$ the associated Λ-coalescent. This Markov jump process $(\Pi_t, t \geq 0)$ takes values in the set of partitions of $\{1, 2, \ldots\}$. Its law is specified by the requirement that, for any $n \in \mathbb{N}$, the restriction Π^n of Π to $\{1, \ldots, n\}$ is a continuous-time Markov chain with the following transitions: whenever Π^n has $b \in \{2, \ldots, n\}$ blocks, any given k-tuple of blocks coalesces at rate $\lambda_{b,k} := \int_{[0,1]} r^{k-2}(1-r)^{b-k}\Lambda(dr)$. The case $\Lambda(dx) = \delta_0(dx)$ corresponds to the classical Kingman coalescent, where each pairwise collision occurs at rate 1, and no multiple collision is possible. The total mass of Λ can be scaled to 1. This is convenient for the analysis, and corresponds to a constant time rescaling of the process. Henceforth we assume that Λ is a probability measure. One of our main current assumptions is that $\Lambda(\{0\}) = c > 0$. We distinguish two cases: if $c = 1$ we call the corresponding coalescent the *pure Kingman* coalescent, while if $c \in (0, 1)$ we call it the *mixed (with) Kingman* coalescent.

The *standard* Λ-coalescent starts from the trivial configuration $\{\{i\} : i \in \mathbb{N}\}$. We shall denote by N_t the number of blocks of $\Pi(t)$ at time t. Note that the law of N depends on Λ, but it will be clear from the context which Λ (and therefore which N) we currently consider. If $\mathbb{P}(N_t < \infty, \forall t > 0) = 1$ the coalescent is said to *come down from infinity* (CDI). It is well known that the Kingman coalescent has this property. Necessary and sufficient conditions for CDI for general Λ-coalescents were derived in [20] and [6]. In [3] and [2] the small time behavior of Λ coalescents was studied. In [2], for a general Λ-coalescent that comes down from infinity, the authors found a non-random function $t \mapsto v_t$, dependent on Λ, such that as $t \to 0$

$$\frac{N_t}{v_t} \to 1 \qquad \text{a.s. and in } L^p, \tag{1.1}$$

for any $p \geq 1$. Any function v satisfying (1.1) is referred to as the *speed of coming down from infinity*. There are many functions with this property, but clearly they have the same asymptotic behavior near 0. In our previous work [16] we investigated the second order asymptotics near 0 for the number of blocks in a Λ-coalescent that comes down from infinity, assuming that Λ has no atom at 0 (no Kingman part) and that $\Lambda(\{1\}) = 0$. We studied the asymptotic behavior in a functional sense. More precisely, we were interested in the processes

$$r(\varepsilon)\left(\frac{N_{\varepsilon t}}{v_{\varepsilon t}} - 1\right), \quad t \geq 0 \tag{1.2}$$

where $\varepsilon > 0$, and $r(\varepsilon)$ is an appropriate norming, such that these processes converge in law in the Skorokhod space $D([0, \infty))$ as $\varepsilon \to 0$. We have shown that if Λ is sufficiently regular near 0, that is, if Λ has a density in a neighborhood of 0 that behaves as $Ay^{-\beta}$ with $0 < \beta < 1$ for some $A \in (0, \infty)$, then (for an appropriate speed v) the correct norming is $r(\varepsilon) = \varepsilon^{-1/(1+\beta)}$. Furthermore, the limit process is a $(1+\beta)$-stable process of the form

$$K\frac{1}{t}\int_{[0,t]} u\, dL_u, \tag{1.3}$$

where L is a $(1+\beta)$-stable Lévy process, totally skewed to the left (it has no positive jumps), and K is a positive constant.

The object of the present paper is to present a complementary result, concerning the second order asymptotics of the number of blocks at small times for Λ-coalescents that

have non-zero Kingman part: $\Lambda(\{0\}) = c > 0$. The presence of an atom at 0 introduces some essential differences and the results of [16] cannot be applied to this case. However, as we will see, the main idea can be adapted to cover this case as well.

For other second-order fluctuation limits in the setting of exchangeable coalescents, we refer the reader to the works of Schweinsberg [21], Kersting [12], Dahmer, Kersting and Wakolbinger [8], Kersting, Schweinsberg and Wakolbinger [13].

1.2 Main results

Let $N = (N_t, t \geq 0)$ be the block counting process in a Λ-coalescent. In the mixed with Kingman case, where $\Lambda(\{0\}) = c > 0$, it is easy to see (by comparing with the Kingman coalescent slowed down by a factor c, which corresponds to the driving measure $\Lambda(dx) = c\delta_0(dx)$) that $\mathbb{P}(N_t < \infty) = 1$ for all $t > 0$, without any additional assumptions on Λ. Furthermore, from the results of Berestycki et al. [2] it follows that in this case (1.1) is satisfied with the function $t \mapsto \frac{2}{ct}$, which therefore is a speed of CDI for the corresponding Λ-coalescent. Note that this expression for the speed depends only on the atom at 0. In particular, the pure Kingman coalescent slowed down by a factor c will have exactly the same speed of CDI.

The object of interest is the process (1.2) as $\varepsilon \to 0$. We now set $r(\varepsilon) = \varepsilon^{-\frac{1}{2}}$.

We first consider the pure Kingman case. As already mentioned, a similar study had already been undertaken in [11, 1] in the setting of pure Kingman coalescent, but only for the marginal distributions (i.e. for fixed $t = 1$, an analogue of the classical CLT). In the present paper we study these fluctuations in a functional sense. We then proceed to the general result for the mixed Kingman coalescent, which is novel even for the one dimensional distributions.

Let $D([0, \infty))$ denote the Skorokhod space of càdlàg functions equipped with the usual J_1 topology.

Our main result in the pure Kingman setting is as follows:

Theorem 1.1. *Let $(N_t)_{t \geq 0}$ be the block counting process in a standard Kingman coalescent. Then the process X_ε defined by*

$$X_\varepsilon(t) = \varepsilon^{-\frac{1}{2}} \left(\frac{\varepsilon t}{2} N_{\varepsilon t} - 1 \right), \quad t \geq 0, \qquad X_\varepsilon(0) = 0 \tag{1.4}$$

converges in law in $D([0, \infty))$ as $\varepsilon \to 0$ to a Gaussian process

$$Z_t = \frac{1}{\sqrt{2}\, t} \int_0^t u\, dW_u, \quad t > 0, \qquad Z_0 = 0, \tag{1.5}$$

where W is a standard Brownian motion.

Remark 1.2. (a) The limit process Z has the same form, as the one in [16] (in the case where $c = 0$, and where Λ has a density near zero, which behaves as $Cy^{-\beta}$; cf. (1.3)), if one formally sets $\beta = 1$.
(b) It is easy to see that the process Z satisfies the equation

$$Z_t = -\int_0^t Z_s \frac{1}{s} ds + \frac{1}{\sqrt{2}} W_t, \qquad Z_0 = 0. \tag{1.6}$$

(c) It is worth pointing out that the limit process (1.5) also appeared in the context of scaling limits related to hierarchical random walks (see [7], Proposition 2.11).

To state our result in the general setting of mixed Kingman coalescents we first need to recall the speeds of CDI used in [2] and [16], while introducing some additional notation.

Assume that $\Lambda(\{0\}) = c > 0$ (clearly $c \le 1$), hence Λ has the form $\Lambda = c\delta_0 + (1-c)\Lambda_1$, where for $0 < c < 1$, Λ_1 is the uniquely determined probability measure on $[0,1]$ such that $\Lambda_1(\{0\}) = 0$. For $c = 1$ set $\Lambda_1 \equiv 0$. Denote

$$\Psi(q) = \int_{[0,1]} (qy - 1 + (1-y)^q) \frac{\Lambda(dy)}{y^2}, \qquad q \ge 1, \tag{1.7}$$

where the function $y \mapsto \frac{qy - 1 + (1-y)^q}{y^2}$ is continuously extended on $[0,1]$, so that its value at $y = 0$ is $\frac{q(q-1)}{2}$. Let Ψ_1 denote the function given by (1.7), with Λ replaced by Λ_1. In particular we have

$$\Psi(q) = c\frac{q(q-1)}{2} + (1-c)\Psi_1(q), \qquad \forall q \ge 1. \tag{1.8}$$

Similarly let

$$\Psi^*(q) = \int_{[0,1]} (qy - 1 + e^{-qy}) \frac{\Lambda(dy)}{y^2}, \qquad q \ge 0. \tag{1.9}$$

These functions have already appeared in earlier papers (see e.g. [2] and [16] for some of their properties and a discussion on relation between Ψ and Ψ^*). In particular, we know that the functions Ψ and $q \mapsto \frac{\Psi(q)}{q}$ are increasing, and the same holds for Ψ^* in place of Ψ.

From the assumption $c > 0$ it follows that, for any $a > 1$, the integral $\int_a^\infty \frac{1}{\Psi(q)} dq$ is finite and the same is true for Ψ^* (which is also a condition for CDI, see [6]). As in [16], we define the function $v : (0, \infty) \mapsto (0, \infty)$ by

$$t = \int_{v_t}^\infty \frac{1}{\Psi(q)} dq. \tag{1.10}$$

By (1.8) and Lemma 2.1 in [16] it follows that $\int_1^\infty \frac{1}{\Psi(q)} dq = \infty$, hence $v_t > 1$ for all $t > 0$. (Note that the assumption $\Lambda(\{1\}) = 0$ in the formulation of Lemma 2.1 in [16] was not used in the proof.)

Analogously to v, one can define v^*, by substituting Ψ^* for Ψ in (1.10). From the results of [2] it follows that $N_t/v_t^* \to 1$ as $t \to 0$, almost surely and in L^p for any $p \ge 1$, hence v_t^* is a speed of CDI. The same is true for v in place of v^*. Moreover, both v and v^* are asymptotic to w near zero, where

$$w_t = \frac{2}{ct}, \qquad t \ge 0.$$

Let us denote

$$X_\varepsilon^v(t) = \varepsilon^{-\frac{1}{2}} \left(\frac{N_{\varepsilon t}}{v_{\varepsilon t}} - 1 \right), \quad X_\varepsilon^v(0) = 0. \tag{1.11}$$

Similarly, let $X_\varepsilon^{v^*}$ and X_ε^w be the processes defined as in (1.11) with v replaced by v^* and w, respectively.

The convergence result for the number of blocks of the mixed Kingman coalescent, when normalized by the speed v or v^*, is analogous to Theorem 1.1. The only assumption made on the measure Λ is that $\Lambda(\{0\}) > 0$. However, if one wishes to replace the speed v by the simpler function w, given above, then additional assumptions on the measure Λ are necessary:

Theorem 1.3. *Assume that $\Lambda(\{0\}) = c > 0$ and write $\Lambda = c\delta_0 + (1-c)\Lambda_1$, as above, $0 < c \le 1$, $\Lambda_1 \equiv 0$ if $c = 1$.*
(i) The processes X_ε^v and $X_\varepsilon^{v^*}$ converge in law in $D([0,\infty))$ as $\varepsilon \to 0$ to the process $\sqrt{c}Z$, where Z is defined in (1.5).

(ii) Suppose additionally that the function Ψ_1 defined by (1.7) with Λ replaced by Λ_1 satisfies

$$\lim_{q \to \infty} \frac{\Psi_1(q)}{q^{3/2}} = 0. \tag{1.12}$$

Then X_ε^w converges in law in $D([0, \infty))$ as $\varepsilon \to 0$ to the process $\sqrt{c}Z$.

Remark 1.4. (a) If $c = 1$, then part *(ii)* clearly restates Theorem 1.1.

(b) In part *(i)* the measure Λ_1 can be completely arbitrary, the limit only depends on $\Lambda(\{0\})$, which shows that the Kingman part dominates.

(c) Here we see the same phenomenon as in [16], that the speed of CDI has to be carefully chosen, and that we cannot always replace v by w. In fact, condition (1.12) is sharp. This is discussed in more detail in Remark 3.9 at the end of Section 3.4 after the proof of part (ii) of the theorem.

It is known (and easy to see) that the asymptotic behavior of $\Psi_1(q)$ as $q \to \infty$ depends quite strongly on the behavior of Λ_1 near 0 (see for example [16], Lemma 2.5). However, to ensure (1.12) one does not need to assume much about the regularity of Λ_1 near 0.

A simple sufficient condition for is now given.

Proposition 1.5. *If $\int_{[0,1]} \frac{1}{\sqrt{y}}\Lambda_1(dy) < \infty$, then (1.12) holds.*

Due to Remark 3.9, it is easy to give examples of Λ_1 where the hypothesis of the proposition is not satisfied and neither (1.12) nor the conclusion of Theorem 1.3 (ii) hold (e.g. Λ_1 is Beta$(2 - \alpha, \alpha)$ distribution for any $\alpha \geq 3/2$). The intuition is that for such Λ_1 the mass near 0 is not sufficiently strong (when compared to the atom at 0) to change the class of speeds, but it is sufficiently strong to perturb the second-order asymptotics of N_t.

The proofs of Theorems 1.1 and 1.3 use some of the main ideas of our recent paper [16], where we studied the case $\Lambda(\{0\}) = 0$, and where Λ was sufficiently regular at 0. These techniques relied heavily on a representation of Λ-coalescents satisfying $\Lambda(\{0\}) = 0$ via Poisson random measures. This representation has been already observed by Pitman (see [18]), but we reformulated it in [16] to better suit our needs.

At first sight the case of the (mixed) Kingman coalescent is different, since the same representation cannot be used. However, it turns out that if, somewhat artificially, one writes the effect of the Kingman part with the help of a different Poisson random measure, then many of the arguments used in [16] may be adapted to this case as well.

In particular, we begin by explicitly writing out an integral equation for the number of blocks N_t. This equation involves an integral with respect to a certain Poisson random measure. In the setting where $\Lambda(\{0\}) \in (0, 1)$, this measure consists of two essentially different pieces: the first corresponding to the Kingman part (the atom at 0), and the second to multiple collisions (the measure Λ_1). The latter piece, which accounts for the individual block coloring, was introduced and thoroughly studied in [16]. We shall rely on the results of that analysis.

However, some of the technical estimates need to be done differently. In a certain sense, the case $\Lambda(\{0\}) > 0$ is simpler, since the Kingman part dominates, and the limits are Gaussian. As in [16], one has to consider terms resulting from the non-Kingman part, but now one can use less precise estimates of these terms. We also make use of a standard result (found e.g. in [10]), a version of a martingale central limit theorem, which ensures convergence in law of martingales whose jumps are well controlled and whose skew brackets converge to a deterministic function.

The remainder of the paper is organized as follows. Section 2 contains the proof of Theorem 1.1. Section 3 is dedicated to the proof of Theorem 1.3, and it also contains a proof of Proposition 1.5 and Remark 3.9.

Throughout the paper $C, C_1, C_2, \ldots$ denote positive constants, which may differ from line to line.

The symbol $\Rightarrow$ denotes convergence in law in the Skorokhod space $D([0, \infty))$ equipped with J_1 topology.

2 The pure Kingman case

In this section we prove Theorem 1.1.

Let us recall first the following deterministic easy lemma, that will be used frequently in the proofs.

Lemma 2.1. *([2], Lemma 10) Suppose $f, g : [a, b] \mapsto \mathbb{R}$ are càdlàg functions such that $\sup_{x \in [a,b]} \left| f(x) + \int_a^x g(u) \, du \right| \leq K$, for some $K < \infty$. If in addition $f(x)g(x) > 0$, $x \in [a, b]$ whenever $f(x) \neq 0$, then*

$$\sup_{x \in [a,b]} \left| \int_a^x g(u) \, du \right| \leq K \quad and \quad \sup_{x \in [a,b]} |f(x)| \leq 2K.$$

Recall that now $\Lambda = \delta_0$, so that N_t denotes the number of blocks of the Kingman coalescent at time t.

The process $(N_t)_{t \geq 0}$ is a pure death continuous time Markov chain, and as such has a simple description. If it is at state n, $n \geq 2$, then it jumps to $n - 1$ with intensity $\binom{n}{2}$. For our purpose, it will be convenient to express this process with the help of a Poisson random measure. This will facilitate the study of fine asymptotic behavior of N near zero and it will allow us to use some of the standard techniques of the theory of integration with respect to Poisson random measures. We refer to Chapter 8 of [17] for a summary of the main properties of such integrals. We always take càdlàg versions of martingales expressed as integrals with respect to a compensated Poisson random measure.

Denote $\Delta = \{(i, j) \in \mathbb{Z}_+^2 : 1 \leq i < j\}$. We will often denote a typical element of Δ by $\mathbf{k}$. Let π^0 be a Poisson random measure on $\mathbb{R}_+ \times \Delta$ with the intensity measure $\nu^0 = \ell \otimes \sum_{(i,j) \in \Delta} \delta_{(i,j)}$, where ℓ is the Lebesgue measure on $\mathbb{R}_+$ and $\delta_{(i,j)}$ is the Dirac delta measure. In other words, $((\pi^0([0, t] \times \{\mathbf{k}\}))_{t \geq 0})_{\mathbf{k} \in \Delta}$ are independent Poisson processes with intensity 1.

The standard Kingman coalescent may be constructed from π^0 as follows: Arrivals in the process indexed by $\mathbf{k} = (i, j)$ correspond to potential times of coalescence of blocks currently labeled by i and by j, but coalescence occurs only if there are at least j blocks in the current configuration. More precisely, initially we have trivial configuration $\{\{1\}, \{2\}, \ldots\}$ consisting of singleton blocks. After each coalescence event, the blocks are reordered according to their smallest element. The ith and jth block in the current ordering coalesce into one block at the next arrival time of $\pi^0(\cdot \times \{(i, j)\})$. This construction is very much related to the Donnelly-Kurtz modified lookdown process, see [9].

By $\hat{\pi}^0$ we denote the compensated Poisson random measure

$$\hat{\pi}^0 = \pi^0 - \nu^0.$$

Let us also denote $\Delta_k = \{(i, j) \in \Delta : 1 \leq i < j \leq k\}$ for $k \in \mathbb{Z}_+$. Note that $\#\Delta_k = \binom{k}{2}$.

The following lemma is important for our analysis.

Lemma 2.2. *Under the assumptions of Theorem 1.1 we have*

$$\frac{t}{2} N_t = 1 - \int_0^t \left(\frac{s}{2} N_s - 1 \right) \frac{1}{s} ds - M_t + R_t, \qquad t \geq 0, \tag{2.1}$$

where

$$M_t = \frac{1}{2} \int_{[0,t]} \int_{\Delta} s \mathbb{1}_{\Delta_{N_{s-}}} (\mathbf{k}) \hat{\pi}^0(ds d\mathbf{k}), \qquad t \geq 0. \tag{2.2}$$

and R is a continuous process such that for any $T > 0$ there exists $C > 0$ such that

$$E \sup_{s \leq t} |R_s| \leq Ct \qquad t \leq T. \tag{2.3}$$

Proof. As already mentioned, the Kingman coalescent comes down from infinity, hence for any $0 < r \leq t$ we have $N_t \leq N_r < \infty$, almost surely. We may and will assume that the coalescent is constructed using the procedure described before Lemma 2.2.

Due to this construction, we have $N_t = N_r - \int_{(r,t] \times \Delta} \mathbb{1}_{\Delta_{N_{s-}}} (\mathbf{k}) \pi^0(ds, d\mathbf{k})$. Therefore, introducing the compensated π^0,

$$N_t = N_r - \int_{(r,t]} \int_{\Delta} \mathbb{1}_{\Delta_{N_{s-}}} (\mathbf{k}) \nu^0(ds d\mathbf{k}) - \int_{(r,t]} \int_{\Delta} \mathbb{1}_{\Delta_{N_{s-}}} (\mathbf{k}) \hat{\pi}^0(ds d\mathbf{k})$$

$$= N_r - \int_r^t \frac{N_s(N_s - 1)}{2} ds - \int_{(r,t]} \int_{\Delta} \mathbb{1}_{\Delta_{N_{s-}}} (\mathbf{k}) \hat{\pi}^0(ds d\mathbf{k}). \tag{2.4}$$

This is permissible, since the jumps of N on $[r,t]$ are discrete (isolated). Clearly we have

$$tN_t = rN_r + \int_r^t N_s \, ds + \int_r^t s \, dN_s.$$

Hence, using (2.4) we obtain,

$$\frac{t}{2} N_t = \frac{r}{2} N_r + \int_r^t \left(\frac{N_s}{2} - s \frac{N_s(N_s - 1)}{4} \right) ds - \int_{(r,t]} \int_{\Delta} \frac{s}{2} \mathbb{1}_{\Delta_{N_{s-}}} (\mathbf{k}) \hat{\pi}^0(ds d\mathbf{k}), \tag{2.5}$$

for any $t \geq r$. If one formally plugs in $r = 0$ in this final expression, one readily sees that the final term equals M_t from (2.2), and that the drift term can be written as

$$-\int_0^t \frac{sN_s}{2} \frac{1}{s} \left(\frac{sN_s}{2} - 1 \right) ds + \int_0^t \frac{sN_s}{4} ds.$$

The point is that for $s \approx 0$ we have $\frac{sN_s}{2} \approx 1$, and this explains the form of the drift in (2.1), provided we can argue that the errors are small. We will in fact show that both M and the integral in (2.1) are well defined, and that for any fixed $t > 0$, as $r \to 0$, the left and the right hand side of (2.5) converge in probability to the corresponding left and right hand side of (2.1). Due to the càdlàg property of all the processes under consideration, (2.1) holds for all $t \geq 0$ simultaneously.

First we show that M is well defined. Due to [2], Theorem 2 we have

$$\lim_{t \to 0} E \sup_{s \leq t} \left(\frac{s}{2} N_s - 1 \right)^2 = 0. \tag{2.6}$$

Hence $\frac{r}{2} N_r \to 1$ in L^2 (this convergence also holds a.s. and in any L^p, $p \geq 1$). Now (2.6) and the fact that N is non-increasing immediately imply that for any $T > 0$

$$E \sup_{s \leq T} \left(\frac{s}{2} N_s \right)^2 < \infty. \tag{2.7}$$

Using the definition of ν^0 and (2.7) we hence obtain

$$E \int_{[0,t]} \int_{\Delta} s^2 \mathbb{1}_{\Delta_{N_{s-}}} (\mathbf{k}) \nu^0(ds d\mathbf{k}) = E \int_0^t s^2 \frac{N_s(N_s - 1)}{2} ds \leq Ct \quad \text{if } t \leq T. \tag{2.8}$$

Due to Theorem 8.23 in [17] and (2.8) we obtain that M given by (2.2) is a well defined square integrable martingale. Moreover, it satisfies $EM_t^2 \le Ct$ for $t \le T$ Hence, by Doob's L^2 maximal inequality

$$E \sup_{s \le t} M_s^2 \le 4Ct \qquad for \ t \le T. \tag{2.9}$$

We observe that the last term on the right hand side of (2.5) is equal to $M_t - M_r$, and from (2.9) it follows that $M_r \to 0$ in L^2 as $r \to 0$.

Let us now examine the drift term in (2.5). It can be written as

$$A_r(t) := \frac{1}{2} \int_r^t N_s ds - \frac{1}{2} \int_r^t s \frac{N_s(N_s - 1)}{2} ds$$
$$= -\int_r^t \frac{N_s}{2} \left(\frac{s}{2} N_s - 1 \right) ds + \int_r^t \frac{s}{4} N_s ds. \tag{2.10}$$

This allows us to improve (2.6) in a similar way as it was done in [16], Lemma 3.7 for Λ-coalescents without the Kingman part. More precisely, using (2.5), (2.10) and Lemma 2.1 (with $g(s) = \frac{N_s}{2}(\frac{s N_s}{2} - 1)$), for $r \le t \le T$ we have

$$\sup_{r \le s \le t} \left| \frac{s}{2} N_s - 1 \right| \le 2 \left(\left| \frac{r}{2} N_r - 1 \right| + |M_r| + \sup_{r \le s \le t} |M_s| + \int_r^t \frac{s}{4} N_s ds \right).$$

Squaring both sides of the last expression, applying expectation and using (2.6), (2.7) and (2.9), we obtain that for any $T > 0$ there exists $C > 0$ such that

$$E \sup_{s \le t} \left(\frac{s}{2} N_s - 1 \right)^2 \le Ct \qquad t \le T. \tag{2.11}$$

Estimate (2.11), together with Jensen's inequality readily implies that the integral with respect to ds in (2.1) is well defined for all t simultaneously, almost surely.

Moreover, we can express the drift term A_r of (2.10) as

$$A_r(t) = -\int_r^t \left(\frac{s}{2} N_s - 1 \right)^2 \frac{1}{s} ds - \int_r^t \left(\frac{s}{2} N_s - 1 \right) \frac{1}{s} ds + \frac{1}{2} \int_r^t \frac{s}{2} N_s ds.$$

By (2.11) and (2.7)

$$E \left| A_r(t) + \int_r^t \left(\frac{s}{2} N_s - 1 \right) \frac{1}{s} ds \right| \le C_1 t \qquad for \ all \ t \le T,$$

where C_1 does not depend on r.

This shows that, as $r \to 0$, $A_r(t)$ converges in L^1 to $-\int_0^t \left(\frac{s}{2} N_s - 1 \right) \frac{1}{s} ds + R_t$, where

$$R_t = -\int_0^t \left(\frac{s}{2} N_s - 1 \right)^2 \frac{1}{s} ds + \frac{1}{2} \int_0^t \frac{s}{2} N_s ds.$$

Again (2.11) and (2.7) yield (2.3). □

Recall (1.4), let M be the martingale defined by (2.2), and define

$$Y_t = -\frac{1}{t} \int_{[0,t]} u \, dM_u, \quad t > 0, \qquad Y_0 = 0. \tag{2.12}$$

and

$$Y_\varepsilon(t) = \varepsilon^{-\frac{1}{2}} Y(\varepsilon t). \tag{2.13}$$

Lemma 2.3. *The process* $(Y_t)_{t \in \mathbb{R}_+}$ *satisfies the equation*

$$Y_t = -\int_0^t Y_s \frac{1}{s} ds - M_t. \tag{2.14}$$

Moreover, for any $T > 0$ *there exists* $C > 0$ *such that for all* $t \leq T$

$$E \sup_{s \leq t} Y_s^2 \leq Ct \tag{2.15}$$

and

$$\lim_{\varepsilon \to 0} E \sup_{t \leq T} |X_\varepsilon(t) - Y_\varepsilon(t)| = 0. \tag{2.16}$$

Proof. It is clear that Y is well defined. Moreover, if we denote $H_t = tY_t$, then by the definition of Y and M we have

$$H_t = -\int_{[0,t]} u \, dM_u = -\frac{1}{2} \int_{[0,t]} \int_\Delta s^2 \mathbb{1}_{\Delta_{N_{s-}}}(\mathbf{k}) \hat{\pi}^0(ds d\mathbf{k}), \qquad t \geq 0.$$

Thus H is a martingale with quadratic variation

$$[H]_t = \frac{1}{4} \int_0^t \int_\Delta s^4 \mathbb{1}_{\Delta_{N_{s-}}}(\mathbf{k}) \pi^0(ds d\mathbf{k}), \qquad t \geq 0,$$

(cf. Theorem 8.23 in [17]). Consequently, using (2.7) we obtain

$$\begin{aligned}
E[H]_t &= \frac{1}{4} E \int_0^t \int_\Delta s^4 \mathbb{1}_{\Delta_{N_{s-}}}(\mathbf{k}) \nu^0(ds d\mathbf{k}) \\
&= \frac{1}{4} E \int_0^t s^4 \frac{N_s(N_s - 1)}{2} ds \\
&\leq C \int_0^t s^2 ds = \frac{C}{3} t^3.
\end{aligned}$$

In particular,

$$EY_t^2 = \frac{1}{t^2} E H_t^2 = \frac{1}{t^2} E[H]_t \leq \frac{C}{3} t. \tag{2.17}$$

The identity (2.14) follows by simple integration by parts (note that $t \mapsto \frac{1}{t}$ is continuous and of finite variation on any interval $[a, b]$, $0 < a < b$). The only subtle point is the lack of regularity of $t \mapsto \frac{1}{t}$ at 0. This difficulty is easily overcome, by writing first the formula for $Y_t - Y_r$, for any $0 < r < t$,

$$Y_t - Y_r = \int_r^t \frac{1}{s^2} \int_0^s u \, dM_u \, ds - \int_{(r,t]} dM_s = -\int_r^t \frac{1}{s} Y_s ds - M_t + M_r, \tag{2.18}$$

and then letting $r \to 0$. Here we use (2.17) to bound $\int_r^t \frac{1}{s} |Y_s| \, ds$ uniformly in $r > 0$, implying that $\int_0^t \frac{1}{s} Y_s \, ds$ exists in the absolute sense, almost surely.

Estimate (2.15) follows from (2.18), Lemma 2.1 and (2.9).

To prove (2.16) we set $X_t = \frac{t}{2} N_t - 1$ and observe that by (2.1) and (2.14) we have

$$X_t - Y_t = -\int_0^t (X_s - Y_s) \frac{1}{s} ds + R_t.$$

Another application of Lemma 2.1 yields

$$\sup_{s \leq t} |X_s - Y_s| \leq 2 \sup_{s \leq t} |R_s|,$$

so (2.3) implies $E \sup_{t \leq T} |X_\varepsilon(t) - Y_\varepsilon(t)| \leq 2C\sqrt{\varepsilon} T$, and hence (2.16). $\qquad \square$

We are now ready to proceed to the proof of the second order asymptotics of the number of blocks of the pure Kingman coalescent.

Proof of Theorem 1.1. Due to (2.16) and the symmetry of the law of W it suffices to show that the process $-Y_\varepsilon$, given by (2.12)–(2.13), converges in law in $D([0,\infty))$ to the process Z given by (1.5).

We start by showing that for H_ε, where $H_\varepsilon(t) := -tY_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}\,\varepsilon}\int_0^{t\varepsilon} u\,dM_u$, $t \geq 0$ we have

$$(H_\varepsilon(t))_{t\geq 0} \Rightarrow \left(\frac{1}{\sqrt{2}}\int_0^t u\,dW_u\right)_{t\geq 0}. \tag{2.19}$$

For this we use Theorem 1.4 in Chapter 7.1 of [10], as noted in the introduction. Observe that $H_\varepsilon(t) = -\varepsilon^{-\frac{3}{2}}H(\varepsilon t)$, where H is taken from the proof of Lemma 2.3. Therefore $H_\varepsilon(t)$ is again an L^2-martingale and it has the form

$$H_\varepsilon(t) = \varepsilon^{-\frac{3}{2}}\frac{1}{2}\int_0^{\varepsilon t}\int_\Delta s^2 \mathbb{1}_{\Delta_{N_{s-}}}(\mathbf{k})\hat{\pi}^0(ds\,d\mathbf{k}). \tag{2.20}$$

By the properties of the compensated Poisson integral we have

$$\langle H_\varepsilon\rangle(t) = \frac{1}{4\varepsilon^3}\int_0^{\varepsilon t}\int_\Delta s^4 \mathbb{1}_{\Delta_{N_{s-}}}(\mathbf{k})\nu^0(ds\,d\mathbf{k}) \tag{2.21}$$

$$= \frac{1}{4\varepsilon^3}\int_0^{\varepsilon t} s^4 \frac{N_s(N_s-1)}{2}ds$$

$$= \frac{1}{2}\int_0^t s^2(\varepsilon s)^2 \frac{N_{\varepsilon s}(N_{\varepsilon s}-1)}{4}ds. \tag{2.22}$$

We next verify the assumptions (b) of [10], Theorem 1.4 in Chapter 7.1, with $c(t) = c_{11}(t) = \frac{1}{2}\int_0^t u^2\,du$, and $\langle H_\varepsilon\rangle$ corresponding to $A^{(n)}$. Since $\langle H_\varepsilon\rangle$ is continuous, we only need to prove that $\langle H_\varepsilon\rangle(t)$ converges to $\frac{1}{2}\int_0^t u^2\,du$ in probability, for each fixed $t > 0$, and that for any $T > 0$

$$\lim_{\varepsilon\to 0} E\sup_{t\leq T}|H_\varepsilon(t) - H_\varepsilon(t-)|^2 = 0. \tag{2.23}$$

The first claim follows readily from (2.22) and (2.11). Equality (2.23) is true due to (2.20), since from this representation of H_ε it follows that the jumps of H_ε on $[0,T]$ are uniformly bounded by $\frac{1}{2}\varepsilon^{-\frac{3}{2}}(\varepsilon T)^2$.

This finishes the proof of (2.19). To see that convergence of H_ε implies the required convergence of $-Y_\varepsilon$, one can apply the argument from [16]: use the continuity of $t \mapsto 1/t$ away from 0, and near 0 use the estimate (2.15) together with an analogous bound $E\sup_{s\leq t}|Z_s|^2 \leq Ct$ for $t \leq T$, where Z is the limit process . Due to Lemma 2.1 the latter bound follows from (1.6) in the same way that (2.15) followed from (2.9) . See Steps 2-4 in the proof of Lemma 4.8 in [16] for more details. Note that here Step 3 simplifies due to (2.15). $\qquad\square$

3 The mixed with Kingman case

In this section we prove Theorem 1.3. In Section 3.1 we present an outline of the proof of Theorem 1.3 for X_ε^v, in Section 3.2 we prove the key technical lemmas needed to fill in this outline, and in Section 3.3 we discuss the convergence of the processes $X_\varepsilon^{v^*}$ and X_ε^w. The final subsection contains the proof of Proposition 1.5.

3.1 Outline of the proof of Theorem 1.3 for X_ε^v.

The proof combines the ideas from [16] and the proof of Theorem 1.1, therefore we only briefly sketch it, omitting the details and concentrating on the differences. Proofs of the technical lemmas that require some new calculations (Lemmas 3.3, 3.5 and 3.6) will be given in Section 3.2.

We will use the Poissonian construction of Λ-coalescents (cf. [18]). More precisely, we make use of the notation introduced in the pure Kingman case, as well as the "enriched" Poisson random measure taken directly from [16]. Recall (1.7)–(1.8) and (1.11). Suppose that we are given Λ as in the introduction, and recall how c and Λ_1 were defined there. Let π_c^0 be a Poisson random measure on $\mathbb{R}_+ \times \Delta$ with intensity measure $c\nu^0$. It is defined analogously to π^0, introduced before Lemma 2.2. Let $\pi_{1-c}^{1,E}$ be a Poisson random measure on $\mathbb{R}_+ \times [0,1] \times [0,1]^{\mathbb{N}}$ with intensity measure $(1-c)ds\frac{\Lambda(dy)}{y^2}\mu(d\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \ldots)$ and where μ is a countable product of Lebesgue measures on $[0,1]$ (or equivalently, the law of an i.i.d. sequence of random variables, distributed uniformly on $[0,1]$) . It corresponds to the non-Kingman part of the coalescent, and it is directly related to π^E from [16]. Since μ is a product of Lebesgue measures, we will usually abbreviate $\mu(d\mathbf{x})$ as $d\mathbf{x}$. See [16] for more details and an interpretation.

It is important to assume that π_c^0 and $\pi_{1-c}^{1,E}$ are independent. Then one can construct a version of the Λ-coalescent by the following procedure (blocks are again ordered according to their smallest element): (a) upon arrival of an atom $(t, \mathbf{k})$ of π_c^0, perform the collapsing of blocks as described above Lemma 2.2; (b) upon arrival of an atom $(t, y, \mathbf{x})$ of $\pi_{1-c}^{1,E}$, the j-th block present in the configuration at time $t-$ is colored if and only if $x_j \leq y$. Once the colors are assigned, in order to form the configuration at time t, merge all the colored blocks into a single block, and leave the other (uncolored) blocks intact.

Following [16], we define a function

$$f(k, y, \mathbf{x}) = \left(\sum_{j=1}^{k} \mathbb{1}_{\{x_j \leq y\}} - 1\right)^+ = \sum_{j=1}^{k} \mathbb{1}_{\{x_j \leq y\}} - 1 + \mathbb{1}_{\bigcap_{j=1}^{k}\{x_j > y\}}, \tag{3.1}$$

which quantifies the decrease in the number of blocks during one coalescent event induced by $\pi_{1-c}^{1,E}$, given that k blocks are present just before this event.

Observe that

$$\Psi(k) = c\frac{k(k-1)}{2} + (1-c)\Psi_1(k) \tag{3.2}$$

$$= c\nu^0(\Delta_k) + (1-c)\int_{[0,1]}\int_{[0,1]^{\mathbb{N}}} f(k, y, \mathbf{x})d\mathbf{x}\frac{\Lambda_1(dy)}{y^2}, \tag{3.3}$$

since $\int_{[0,1]^{\mathbb{N}}} f(k, y, \mathbf{x})d\mathbf{x} = E(\xi - 1)^+$, where ξ has the Binomial(k, y) distribution.

From Corollary 15 in [2] the following lemma can be derived:

Lemma 3.1.

$$\lim_{q\to\infty} \frac{\Psi_1(q)}{q^2} = 0, \tag{3.4}$$

$$\lim_{q\to\infty} \frac{\Psi(q)}{q^2} = \frac{c}{2}, \tag{3.5}$$

$$\lim_{t\to0+} \frac{ct}{2}v_t = 1. \tag{3.6}$$

More precisely, in [2] this was formulated for Ψ^* and v^*, but their behavior at ∞ and $0+$ is the same as that of Ψ and v, respectively. For completeness we include a short argument in Section 3.2.

We keep the convention that $\hat{\pi}$ denotes the compensated Poisson random measure π. Using the Poissonian construction of the Λ-coalescent described above, then compensating, and applying (3.3) we have

$$
\begin{aligned}
N_t = N_r &- \int_r^t \Psi(N_s)\, ds - c \int_{(r,t]\times\Delta} \mathbb{1}_{\Delta_{N_{s-}}}(\mathbf{k})\hat{\pi}_c^0(ds, d\mathbf{k}) \\
&- (1-c) \int_{(r,t]\times[0,1]\times[0,1]^{\mathbb{N}}} f(N_{s-}, y, \mathbf{x})\hat{\pi}_{1-c}^{1,E}(ds, dy, d\mathbf{x}),\ t \geq r.
\end{aligned}
$$

Next, realizing that (1.10) implies $v_t' = -\Psi(v_t)$, for all $t > 0$, one can obtain the following lemma in the same way as (2.5) or [16], Lemma 3.3.

Lemma 3.2. *For any $r > 0$ and all $t \geq r$ we have*

$$
\begin{aligned}
\frac{N_t}{v_t} = \frac{N_r}{v_r} &- \int_r^t \frac{N_s}{v_s}\left(\frac{\Psi(N_s)}{N_s} - \frac{\Psi(v_s)}{v_s}\right) ds \\
&- \int_{(r,t]}\int_\Delta \frac{\mathbb{1}_{\Delta_{N_{s-}}}(\mathbf{k})}{v_s}\hat{\pi}_c^0(dsd\mathbf{k}) - \int_{(r,t]}\int_{[0,1]}\int_{[0,1]^{\mathbb{N}}} \frac{f(N_{s-}, y, \mathbf{x})}{v_s}\hat{\pi}_{1-c}^{1,E}(dsdyd\mathbf{x}).
\end{aligned}
$$

As in the pure Kingman case, we wish to write the above equation starting from $r = 0$. In particular, we need to show that

$$
M_t^0 := \int_{(0,t]}\int_\Delta \frac{\mathbb{1}_{\Delta_{N_{s-}}}(\mathbf{k})}{v_s}\hat{\pi}_c^0(dsd\mathbf{k}), \tag{3.7}
$$

$$
M_t^1 := \int_{(0,t]}\int_{[0,1]}\int_{[0,1]^{\mathbb{N}}} \frac{f(N_{s-}, y, \mathbf{x})}{v_s}\hat{\pi}_{1-c}^{1,E}(dsdyd\mathbf{x}), \tag{3.8}
$$

and

$$
A(t) := \int_0^t \frac{N_s}{v_s}\left(\frac{\Psi(N_s)}{N_s} - \frac{\Psi(v_s)}{v_s}\right) ds \tag{3.9}
$$

are all well defined. The integrals in (3.7) and (3.8) are to be understood in the sense of the usual (compensated) Poisson integration.

As before, from Theorem 2 in [2] it follows that

$$
\lim_{t\to 0} E\sup_{s\leq t}\left(\frac{N_s}{v_s} - 1\right)^2 = 0. \tag{3.10}
$$

We have already observed that $v_t \geq 1$ and N_t is non-increasing, hence similarly to (2.7) we find that there exists $C > 0$ such that

$$
E\sup_{s\geq 0}\left(\frac{N_s}{v_s}\right)^2 \leq C. \tag{3.11}
$$

In Section 3.2 we will prove the following lemma.

Lemma 3.3. *The processes M^0 and M^1 given by (3.7) and (3.8) are well defined square integrable martingales. Moreover, for any $T > 0$ there exists $C > 0$ such that*

$$
E\sup_{s\leq t}(M_s^i)^2 \leq Ct, \qquad t \leq T, \quad i = 0, 1. \tag{3.12}
$$

The function $q \mapsto \frac{\Psi(q)}{q}$ is increasing (see e.g. Lemma 2.1(iv) in [16]), therefore, Lemma 3.2, (3.11), (3.12) and Lemma 2.1 imply

Lemma 3.4. *For any $T > 0$ there exists $C > 0$ such that*

$$E \sup_{s \leq t} \left(\frac{N_s}{v_s} - 1 \right)^2 \leq Ct, \qquad t \in [0, T]. \tag{3.13}$$

The proof is very similar to that of (2.11).

Regarding the drift, we will prove the following in Section 3.2:

Lemma 3.5. *For each $t > 0$, the integral in (3.9) is a well defined Lebesgue integral, almost surely.*

Moreover,

$$A(t) = \int_0^t \left(\frac{N_s}{v_s} - 1 \right) \frac{1}{s} ds + U_t, \qquad t \geq 0, \tag{3.14}$$

where the process U satisfies

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} E \sup_{s \leq t} |U_{\varepsilon s}| = 0. \tag{3.15}$$

Note that this is simpler than the corresponding Lemma 4.4 in [16], where at this point in the analogue of (3.14) we had to use a more complicated function instead of $\frac{1}{s}$.

We will also show in Section 3.2 that the effect of M^1 is negligeable in the limit:

Lemma 3.6. *For any $1 \leq r < 2$ and $T > 0$ we have*

$$\lim_{\varepsilon \to 0} E \sup_{s \leq T} \left| \frac{1}{\sqrt{\varepsilon}} M^1_{\varepsilon t} \right|^r = 0.$$

Using the above lemmas and Lemma 2.1 again, it is easy to deduce the following analogue of Lemma 2.2:

Lemma 3.7.

$$\frac{N_t}{v_t} - 1 = - \int_0^t \left(\frac{N_s}{v_s} - 1 \right) \frac{1}{s} ds - M^0_t + R_t,$$

where, for any $T > 0$, R satisfies

$$\lim_{\varepsilon \to 0} E \sup_{s \leq T} \left| \frac{1}{\sqrt{\varepsilon}} R_{\varepsilon t} \right| = 0. \tag{3.16}$$

The rest of the proof is the same as in the pure Kingman case. Setting $R = 0$ we study the process (Y_t) satisfying

$$Y_t = - \int_0^t Y_s \frac{1}{s} ds - M^0_t, \qquad Y_0 = 0,$$

which can be written explicitly as

$$Y_t = -\frac{1}{t} \int_0^t s \, dM^0_s ds, \qquad t > 0, \qquad Y_0 = 0.$$

From (3.16) and Lemma 2.1 it follows that the convergence in law in $D([0, \infty))$ of X^v_ε is equivalent to the same convergence of

$$Y_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} Y_{\varepsilon t}.$$

This convergence is shown in exactly the same way as in the proof of Theorem 1.1, using (3.10) and the fact that $v_t \sim \frac{2}{ct}$ as $t \to 0$. The constant c appearing in the limit comes from the intensity of the Poisson random measure in the definition of M^0, which is cv^0 in this case, compared to ν^0 in the definition of M in Section 2. In particular, the effect of this change is visible in (2.21).

3.2 Proofs of the auxiliary lemmas stated in Section 3.1

Proof of Lemma 3.1. For $q \geq 1$ we can rewrite $\Psi_1(q)$ as

$$\Psi_1(q) = q(q-1) \int_0^1 \int_0^1 \int_0^u (1-ry)^{q-2} dr du \Lambda_1(dy). \tag{3.17}$$

Hence (3.4) follows since $(1 - ry)^{q-2} \to 0$ as $q \to \infty$, and it is bounded by 1 if $q \geq 2$. Convergence (3.5) is a direct consequence of (3.2) and (3.4).

From the definition of v_t we have that $tv_t = v_t \int_{v_t}^\infty \frac{1}{\Psi(q)} dq$, so that the l'Hospital rule and (3.5) imply

$$\lim_{t \to 0+} \frac{ct}{2} v_t = \frac{c}{2} \lim_{y \to \infty} y \int_y^\infty \frac{1}{\Psi(q)} dq = \frac{c}{2} \lim_{y \to \infty} \frac{-\frac{1}{\Psi(y)}}{-\frac{1}{y^2}} = 1.$$

$\square$

Proof of Lemma 3.3. The argument for M^0 is the same as in Lemma 2.2. Using (3.11) we obtain the following analogue of (2.8):

$$E \int_{(0,t]} \int_\Delta \left(\frac{\mathbb{1}_{\Delta_{N_{s-}}}(\mathbf{k})}{v_s} \right)^2 c\nu^0(d\mathbf{k}) ds = c \int_0^t E \left(\frac{N_s(N_s - 1)}{2v_s^2} \right) ds \leq C_1 t. \tag{3.18}$$

From the standard theory of Poisson integration (see e.g. [17]) it follows that M^0 is well defined. Moreover, it is a square integrable martingale with the following skew bracket

$$\langle M^0 \rangle_t = c \int_0^t \int_\Delta \left(\frac{\mathbb{1}_{\Delta_{N_{s-}}}(\mathbf{k})}{v_s} \right)^2 \nu^0(d\mathbf{k}) ds = c \int_0^t \frac{N_s(N_s - 1)}{2v_s^2} ds.$$

The bound (3.12) for M^0 now follows from Doob's L^2 maximal inequality and (3.18).

The argument for M^1 is similar. We need to get a bound on

$$I(t) := E \int_0^t \int_{[0,1]} \int_{[0,1]^{\mathbb{N}}} \frac{f^2(N_{s-}, y, \mathbf{x})}{v_s^2} d\mathbf{x} \frac{\Lambda_1(dy)}{y^2} dy ds.$$

As $E \int_{[0,1]^{\mathbb{N}}} f^2(k, y, \mathbf{x}) d\mathbf{x} = E\left((\xi - 1)^+\right)^2$, where ξ is a Binomial(k, y) r.v., it is elementary to check (see e.g Lemma 17 in [2] or (3.11) in [16]) that

$$\int_{[0,1]^{\mathbb{N}}} f^2(k, y, \mathbf{x}) d\mathbf{x} = k(k-1)y^2 - k(k-1) \int_0^y \int_0^r (1-u)^{k-2} du dr. \tag{3.19}$$

By (3.19) and (3.11), for each $t > 0$ we have

$$I(t) \leq E \int_0^t \int_{[0,1]} \frac{N_{s-}(N_{s-} - 1)}{v_s^2} \Lambda_1(dy) ds \leq Ct.$$

This implies that M^1 is well defined. Moreover, it is a square integrable martingale satisfying $E(M_t^1)^2 \leq Ct$. As before, an application of Doob's L^2 maximal inequality finishes the proof. $\square$

Proof of Lemma 3.5. Fix any $T > 0$, and suppose that $t \in [0, T]$. Observe that from (3.13) and Jensen's inequality for $s \leq t$ we have

$$E \left| \frac{N_s}{v_s} - 1 \right| \leq C_1 \sqrt{s}. \tag{3.20}$$

Hence

$$E \int_0^t \left| \frac{N_s - v_s}{v_s} \right| \frac{1}{s} ds \leq C\sqrt{t} < \infty, \tag{3.21}$$

so the integral in (3.14) is well defined.

Recalling (3.2), and formally rewriting $A(t)$ defined by (3.9), we have

$$
\begin{aligned}
A(t) =& c \int_0^t \frac{N_s}{v_s} \frac{N_s - v_s}{2} ds + (1 - c) \int_0^t \frac{N_s}{v_s} \left(\frac{\Psi_1(N_s)}{N_s} - \frac{\Psi_1(v_s)}{v_s} \right) ds \\
=& \int_0^t \frac{N_s - v_s}{v_s} \frac{1}{s} ds + \int_0^t \frac{N_s - v_s}{v_s} \left(\frac{cs}{2} v_s - 1 \right) \frac{1}{s} ds \\
& + \frac{c}{2} \int_0^t \left(\frac{N_s - v_s}{v_s} \right)^2 v_s ds + (1 - c) \int_0^t \frac{N_s}{v_s} \left(\frac{\Psi_1(N_s)}{N_s} - \frac{\Psi_1(v_s)}{v_s} \right) ds. \tag{3.22}
\end{aligned}
$$

It suffices to show that each of the terms in (3.22) is a well defined Lebesgue integral for all t simultaneously, almost surely. For this it is enough to show finiteness of $E \int_0^t |\cdots| ds$ in each of the cases. The first term has already been estimated.

Considering the remaining terms in (3.22), we denote

$$I_1(t) = \int_0^t \left| \frac{N_s - v_s}{v_s} \left(\frac{cs}{2} v_s - 1 \right) \right| \frac{1}{s} ds \tag{3.23}$$

$$I_2(t) = \int_0^t \left(\frac{N_s - v_s}{v_s} \right)^2 v_s ds \tag{3.24}$$

$$I_3(t) = \int_0^t \left| \frac{N_s}{v_s} \left(\frac{\Psi_1(N_s)}{N_s} - \frac{\Psi_1(v_s)}{v_s} \right) \right| ds \tag{3.25}$$

By (3.22) and (3.21), the proof of the lemma will be completed once we show that

$$EI_i(t) < \infty, \qquad i = 1, 2, 3 \tag{3.26}$$

and

$$\lim_{\varepsilon \to 0} \varepsilon^{-\frac{1}{2}} EI_i(\varepsilon T) = 0, \qquad i = 1, 2, 3. \tag{3.27}$$

As already observed, v_t is decreasing and $v_t \geq 1$, so from (3.6) it follows that there exists $C > 0$ such that

$$v_s \leq C \left(\frac{1}{s} \vee 1 \right). \tag{3.28}$$

In particular, this implies that sv_s is bounded on $[0, T]$.

Estimates (3.28), (3.13) and (3.20) easily imply (3.26) for $i = 1, 2$ and (3.27) for $i = 2$. To show (3.27) for $i = 1$ we additionally make an appropriate substitution and use the dominated convergence theorem.

It remains to consider I_3. Let us denote $h_1(q) = \frac{\Psi_1(q)}{q}$. We can rewrite h_1 as

$$h_1(q) = \int_{[0,1]} \int_0^y (1 - (1 - r)^{q-1}) dr \frac{\Lambda_1(dy)}{y^2}.$$

Using this representation, it is easy to see (cf. [16]) that

$$\sup_{q>1} h_1'(q) \equiv \sup_{q>1} |h_1'(q)| < \infty \text{ and } \lim_{q \to \infty} (h_1)'(q) = 0. \tag{3.29}$$

By (3.25) and the mean value theorem we have

$$EI_3(t) \leq E \int_0^t \frac{N_s}{v_s} |N_s - v_s| \sup_{q \geq N_s \wedge v_s} h_1'(q) ds.$$

Writing $\frac{N_s}{v_s} = \left(\frac{N_s}{v_s} - 1\right) + 1$, and estimating further, we arrive at

$$EI_3(t) \leq CE \int_0^t \left(\frac{N_s}{v_s} - 1\right)^2 v_s ds + \int_0^t v_s E \left(\frac{|N_s - v_s|}{v_s} \sup_{q \geq N_s \wedge v_s} h_1'(q)\right) ds.$$

Note that the first term on the right hand side is just $CI_2(t)$, which has already been estimated. The second term is finite by (3.20) and (3.29). To obtain (3.27) for $i = 3$, we apply the Cauchy-Schwarz inequality for the expectation inside the second integral, and then the dominated convergence theorem. Here we use (3.13), and the fact that $E(\sup_{q \geq N_{\varepsilon s} \wedge v_{\varepsilon s}} h_1'(q))^2$ is bounded and tends to 0 as $\varepsilon \to 0$, due to (3.29), together with $N_{\varepsilon s} \wedge v_{\varepsilon s} \to_{\varepsilon \to 0} \infty$, a.s. This finishes the proof of (3.24) and (3.25) for $i = 3$. $\qquad \square$

Proof of Lemma 3.6. Fix $T > 0$, and let $0 \leq t \leq T$. We write

$$M_t^1 = L_t + U_t, \tag{3.30}$$

where

$$L_t = \int_0^t \int_{[0,1]} \int_{[0,1]^{\mathbb{N}}} y \hat{\pi}_{1-c}^{1,E}(dsdyd\mathbf{x})$$

$$U_t = \int_0^t \int_{[0,1]} \int_{[0,1]^{\mathbb{N}}} \left(\frac{f(N_{s-}, y, \mathbf{x})}{v_s} - y\right) \hat{\pi}_{1-c}^{1,E}(dsdyd\mathbf{x}).$$

Note that the process $(L_t)_{t \geq 0}$ is simply a Lévy process with Lévy measure $(1 - c)\frac{\Lambda_1(dy)}{y^2}$. Both U and L are square integrable martingales.

By the standard properties of Poisson integrals (see e.g. [17], Theorem 8.23) we have

$$EU_t^2 = E \int_0^t \int_{[0,1]} \int_{[0,1]^{\mathbb{N}}} \left(\frac{f(N_{s-}, y, \mathbf{x})}{v_s} - y\right)^2 d\mathbf{x} \frac{\Lambda_1(dy)}{y^2} ds \leq 2\left(J_1(t) + J_2(t)\right), \tag{3.31}$$

where (using $N_s = N_{s-}$ for almost all s)

$$J_1(t) = E \int_0^t \int_{[0,1]} \int_{[0,1]^{\mathbb{N}}} \left(\frac{N_s}{v_s}\right)^2 \left(\frac{f(N_s, y, \mathbf{x})}{N_s} - y\right)^2 d\mathbf{x} \frac{\Lambda_1(dy)}{y^2} ds, \text{ and}$$

$$J_2(t) = E \int_0^t \int_{[0,1]} \left(\frac{N_s}{v_s} - 1\right)^2 y^2 \frac{\Lambda_1(dy)}{y^2} ds.$$

By (3.13) we have

$$J_2(t) \leq Ct^2, \qquad \text{for } t \leq T. \tag{3.32}$$

To estimate $J_1(t)$, we recall that for $k \geq 1$, $k \in \mathbb{N}$

$$\int_{[0,1]^{\mathbb{N}}} f(k, y, \mathbf{x}) d\mathbf{x} = k \int_0^y (1 - (1 - r)^{k-1}) dr.$$

This, together with (3.19) gives

$$\int_{[0,1]^{\mathbb{N}}} (f(k, y, \mathbf{x}) - ky)^2 d\mathbf{x} \leq 2k^2 y \int_0^y (1 - r)^{k-1} dr.$$

Hence

$$J_1(t) \leq 2E \int_0^t \int_{[0,1]} \left(\frac{N_s}{v_s}\right)^2 \int_0^y (1 - r)^{N_s - 1} dr \frac{\Lambda_1(dy)}{y} ds.$$

Due to (3.11) and $(1-r)^{N_s-1} \leq 1$, we get $J_1(t) \leq Ct$.

Moreover,

$$\frac{1}{\varepsilon} J_1(\varepsilon T) \leq 2 \int_0^T \int_{[0,1]} \int_0^1 E \left(\frac{N_{\varepsilon s}}{v_{\varepsilon s}} \right)^2 (1-ry)^{N_{\varepsilon s}-1} dr \Lambda_1(dy) ds. \qquad (3.33)$$

Estimating $\left(\frac{N_{\varepsilon s}}{v_{\varepsilon s}} \right)^2 \leq 2 \left(\frac{N_{\varepsilon s}}{v_{\varepsilon s}} - 1 \right)^2 + 2$, using $N_{\varepsilon s} \to \infty$ a.s. and $0 \leq (1-ry) < 1$ a.e. on the domain of integration, we have

$$E \left(\frac{N_{\varepsilon s}}{v_{\varepsilon s}} \right)^2 (1-ry)^{N_{\varepsilon s}-1} \leq 2E \left(\frac{N_{\varepsilon s}}{v_{\varepsilon s}} - 1 \right)^2 + 2E(1-ry)^{N_{\varepsilon s}-1} \to 0 \quad \text{as } \varepsilon \to 0.$$

Moreover, by (3.11), the integrand in (3.33) is bounded for $s \leq T$, $\varepsilon \leq 1$, hence the dominated convergence theorem applies. We obtain that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} J_1(\varepsilon T) = 0. \qquad (3.34)$$

By (3.31), (3.32) and (3.34) we have

$$\lim_{\varepsilon \to 0} E \left(\frac{1}{\sqrt{\varepsilon}} U_{\varepsilon T} \right)^2 = 0. \qquad (3.35)$$

Since U is a càdlàg martingale, (3.35) and Doob's L^2 maximal inequality imply that also

$$\lim_{\varepsilon \to 0} E \sup_{t \leq T} \left(\frac{1}{\sqrt{\varepsilon}} U_{\varepsilon t} \right)^2 = 0. \qquad (3.36)$$

Let us now consider the process L. Define

$$L_\varepsilon(t) = \frac{1}{\sqrt{\varepsilon}} L(\varepsilon t).$$

We will show that for any $0 < r < 2$ we have

$$\lim_{\varepsilon \to 0} E \sup_{t \leq T} |L_\varepsilon(t)|^r = 0. \qquad (3.37)$$

By Jensen's inequality it is clearly enough to consider $1 < r < 2$. Moreover, if $r \in (1,2)$ then the Doob L^r maximal inequality applied to the càdlàg martingale L_ε implies that to obtain (3.37) it suffices to show that

$$\lim_{\varepsilon \to 0} E |L_\varepsilon(T)|^r = 0. \qquad (3.38)$$

Fix any $r \in (1,2)$. In order to prove (3.38), we will show that $(|L_\varepsilon(T)|^r)_{\varepsilon>0}$ is uniformly integrable, and that $L_\varepsilon(T)$ converges in law to 0, and hence in probability.

The first assertion above follows from the simple estimate

$$\sup_{\varepsilon>0} E(L_\varepsilon(T))^2 \leq \sup_{\varepsilon>0} \frac{1}{\varepsilon} (1-c) \int_0^{\varepsilon T} \int_{[0,1]} y^2 \frac{\Lambda_1(dy)}{y^2} ds = (1-c)T.$$

To show convergence in law of $L_\varepsilon(T)$, we write out its characteristic function:

$$\begin{aligned} E e^{iuL_\varepsilon(T)} &= \exp \left\{ \varepsilon T \int_{[0,1]} \left(e^{iu\frac{1}{\sqrt{\varepsilon}}y} - 1 - iu\frac{1}{\sqrt{\varepsilon}}y \right) \frac{\Lambda_1(dy)}{y^2} \right\} \\ &= \exp \left\{ Tu^2 \int_{[0,1]} \frac{e^{iu\frac{1}{\sqrt{\varepsilon}}y} - 1 - iu\frac{1}{\sqrt{\varepsilon}}y}{(\frac{y}{\sqrt{\varepsilon}})^2 u^2} \Lambda_1(dy) \right\}. \end{aligned}$$

Since $\sup_{x \in \mathbb{R}, x \neq 0} \left| \frac{e^{ix} - 1 - ix}{x^2} \right|$ is finite, and since $\lim_{|x| \to \infty} \frac{e^{ix} - 1 - ix}{x^2} = 0$, the Lebesgue dominated convergence theorem implies that the right hand side above converges to 1, as $\varepsilon \to 0$, yielding the needed claim.

Due to the previous discussion, (3.38) and therefore (3.37) holds .

The assertion of the Lemma now follows by (3.30), (3.36), (3.37) and Jensen's inequality. □

3.3 Convergence of $X_\varepsilon^{v^*}$.

The proof is almost exactly the same as in the case of Λ-coalescents without an atom at 0, studied in [16] (see Theorem 1.4 and Lemma 2.2 therein). Recalling the definitions of $X_\varepsilon^{v^*}$ and X_ε^v we can write

$$X_\varepsilon^{v^*}(t) = X_\varepsilon^v(t) + \frac{1}{\sqrt{\varepsilon}} \frac{N_{\varepsilon t}}{v_{\varepsilon t}} \left(\frac{v_{\varepsilon t}}{v_{\varepsilon t}^*} - 1 \right). \tag{3.39}$$

Therefore, to prove the desired convergence of $X_\varepsilon^{v^*}$, it suffices to show that for any $T > 0$ we have

$$\lim_{\varepsilon \to 0} E \sup_{t \leq T} \frac{1}{\sqrt{\varepsilon}} \frac{N_{\varepsilon t}}{v_{\varepsilon t}} \left| \frac{v_{\varepsilon t}}{v_{\varepsilon t}^*} - 1 \right| = 0. \tag{3.40}$$

From (3.11) it follows that (3.40) will hold provided that

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} \left(\frac{v_\varepsilon}{v_\varepsilon^*} - 1 \right) = 0. \tag{3.41}$$

As in the proof of Theorem 1.4 in [16] (see Section 5.1 therein), the proof is thus reduced to a purely deterministic problem of showing (3.41).

Observe that

$$0 \leq \Psi^*(q) - \Psi(q) = \frac{cq}{2} + (1 - c)(\Psi_1^*(q) - \Psi_1(q)).$$

Lemma 2.1 in [16] implies that

$$\Psi^*(q) - \Psi(q) \leq \frac{q}{2}. \tag{3.42}$$

This allows one to repeat the proof of Lemma 2.2 in [16]. (3.42) is used to obtain an estimate for the right hand side of (2.15) in [16].

From the analogue of (2.16) in [16] we obtain that there exists $t_0 > 0$ and $C > 0$ such that for all $\varepsilon \in [0, t_0]$

$$\left| \frac{v_\varepsilon}{v_\varepsilon^*} - 1 \right| \leq C\varepsilon.$$

Hence (3.41) follows.

3.4 Convergence of X_ε^w

Similarly to (3.39), we have

$$X_\varepsilon^w(t) = X_\varepsilon^v(t) + \frac{1}{\sqrt{\varepsilon}} \frac{N_{\varepsilon t}}{v_{\varepsilon t}} \left(\frac{v_{\varepsilon t}}{w_{\varepsilon t}} - 1 \right). \tag{3.43}$$

As before, the proof of convergence of X_ε^w reduces to showing that

$$\lim_{\varepsilon \to 0} \frac{1}{\sqrt{\varepsilon}} \left(\frac{v_\varepsilon}{w_\varepsilon} - 1 \right) = 0. \tag{3.44}$$

This can be solved using techniques from [16], which rely on writing an equation for $\log \frac{v_\varepsilon}{w_\varepsilon}$ and applying Lemma 2.1 (cf. (2.15)-(2.16) in [16]).

Below we will use a different approach, which is more direct and at the same time shows that condition (1.12) is sharp.

We have the following lemma:

Lemma 3.8. *Assume that*

$$\lim_{q \to \infty} \frac{\Psi_1(q)}{q^{3/2}} = A, \tag{3.45}$$

where $A \in [0, \infty]$. Then

$$\lim_{t \to 0+} \frac{1}{\sqrt{t}} \left(\frac{ct}{2} v_t - 1 \right) = -\frac{2\sqrt{2}}{3\sqrt{c}}(1-c)A. \tag{3.46}$$

Proof. Using (1.10) and an elementary integration we can write

$$\frac{1}{\sqrt{t}}\left(v_t \frac{ct}{2} - 1\right) = \frac{c}{2\sqrt{t}} v_t \left(t - \frac{2}{cv_t}\right) = \left(\frac{c}{2}\right)^{\frac{3}{2}} \frac{1}{\sqrt{\frac{c}{2}tv_t}} v_t^{\frac{3}{2}} \int_{v_t}^{\infty} \left(\frac{1}{\Psi(q)} - \frac{1}{\frac{c}{2}q^2}\right) dq. \tag{3.47}$$

Applying l'Hospital's rule, recalling (1.8) and then using (3.5) and (3.45), we obtain

$$\lim_{y \to \infty} \frac{\int_y^{\infty}\left(\frac{1}{\Psi(q)} - \frac{1}{\frac{c}{2}q^2}\right) dq}{y^{-3/2}} = \lim_{y \to \infty} \frac{\frac{\frac{c}{2}y - (1-c)\Psi_1(y)}{\frac{c}{2}y^2\Psi(y)}}{\frac{3}{2}y^{-5/2}} = -\frac{8}{3c^2}A(1-c). \tag{3.48}$$

Combining (3.6), (3.47) and (3.48) gives (3.46). $\qquad\square$

An application of Lemma 3.8 with $A = 0$ finishes the proof of convergence of X_ε^w.

Remark 3.9. Observe that if the limit in (3.45) exists and A is finite, then from (3.43), Theorem 1.3 (i), (3.10) and Lemma 3.8 it follows that X_ε^w converges in law in $D([0, \infty))$, as $\varepsilon \to 0$, to the process

$$\tilde{Z}_t = \sqrt{c}Z_t - \frac{2\sqrt{2}}{3\sqrt{c}}(1-c)A\sqrt{t}, \qquad t \geq 0,$$

where Z is given by (1.5). If (3.45) holds with $A = \infty$ then X_ε^w does not converge as $\varepsilon \to 0$.

It follows that the assumption (1.12) of Theorem 1.3 (ii) may not be relaxed. In particular, in the Beta$(2 - \alpha, \alpha)$ coalescent world, the parameter $\alpha = 3/2$ is critical for the convergence in Theorem 1.3 (ii).

3.5 Proof of Proposition 1.5

Recall that Ψ_1 can be expressed as in (3.17). Using the estimate $1 - x \leq e^{-x}$ and replacing the integral $\int_0^u \ldots dr$ by $\int_0^1 \ldots dr$ one obtains

$$\Psi_1(q) \leq q(q-1) \int_0^1 \int_0^1 e^{-(q-2)ry} dr\Lambda_1(dy).$$

Moreover, observe that for any $\delta > 0$ we have

$$\sqrt{q} \int_0^1 \int_0^1 e^{-qry} dr\Lambda_1(dy) \leq \int_{[0,\delta]} \frac{1}{\sqrt{y}} \int_0^1 \frac{1}{\sqrt{r}} \left(\sqrt{qry}e^{-qry}\right) dr\Lambda_1(dy) + \int_{(\delta,1]} \frac{1}{y\sqrt{q}}\Lambda_1(dy)$$

$$\leq C \int_{[0,\delta]} \frac{1}{\sqrt{y}}\Lambda_1(dy) + \frac{1}{\delta\sqrt{q}},$$

hence the proposition follows.

References

[1] D.J. Aldous, Deterministic and stochastic models for coalescence (aggregation and coagulation): a review of the mean-field theory for probabilists, *Bernoulli* **5** (1999), no. 1, 3–48. MR-1673235

[2] J. Berestycki, N. Berestycki and V. Limic, The Λ-coalescent speed of coming down from infinity, *Ann. Probab.* **38** (2010), no. 1, 207–233. MR-2599198

[3] J. Berestycki, N. Berestycki and J. Schweinsberg, Small-time behavior of beta coalescents, *Ann. Inst. Henri Poincaré Probab. Stat.* **44** (2008), no. 2, 214–238. MR-2446321

[4] N. Berestycki, *Recent progress in coalescent theory*, Ensaios Matemáticos [Mathematical Surveys], vol. 16, Sociedade Brasileira de Matemática, Rio de Janeiro, 2009. MR-2574323

[5] J. Bertoin, *Random fragmentation and coagulation processes*, Cambridge Studies in Advanced Mathematics, vol. 102, Cambridge University Press, Cambridge, 2006. MR-2253162

[6] J. Bertoin and J-F. Le Gall, Stochastic flows associated to coalescent processes. III. Limit theorems, *Illinois J. Math.* **50** (2006), no. 1-4, 147–181. MR-2247827

[7] T. Bojdecki, L. G. Gorostiza and A. Talarczyk, Number variance for hierarchical random walks and related fluctuations, *Electron. J. Probab.* **16** (2011), no. 75, 2059–2079. MR-2851056

[8] I. Dahmer, G. Kersting and A. Wakolbinger, The total external branch length of Beta-coalescents, *Combin. Probab. Comput.* **23** (2014), no. 6, 1010–1027. MR-3265836

[9] P. Donnelly and T. Kurtz, Particle representations for measure-valued population models, *Ann. Probab.* **27** (1999), no. 1, 166–205. MR-1681126

[10] S. N. Ethier and T. Kurtz, *Markov processes*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons, Inc., New York, 1986, Characterization and convergence. MR-838085

[11] R. C. Griffiths, Asymptotic line-of-descent distributions, *J. Math. Biol.* **21** (1984), no. 1, 67–75. MR-770713

[12] G. Kersting, The asymptotic distribution of the length of beta-coalescent trees, *Ann. Appl. Probab.* **22** (2012), no. 5, 2086–2107. MR-3025690

[13] G. Kersting, J. Schweinsberg and A. Wakolbinger, The evolving beta coalescent, *Electron. J. Probab.* **19** (2014), no. 64, 27. MR-3238784

[14] J. F. C. Kingman, The coalescent, *Stochastic Process. Appl.* **13** (1982), no. 3, 235–248. MR-671034

[15] J. F. C. Kingman, On the genealogy of large populations, *J. Appl. Probab.* (1982), no. Special Vol. 19A, 27–43, Essays in statistical science. MR-633178

[16] V. Limic and A. Talarczyk. Second-order asymptotics for the block counting process in a class of regularly varying Λ-coalescents, *Ann. Probab.* (to appear), arXiv:1304.5183

[17] S. Peszat and J. Zabczyk, *Stochastic partial differential equations with Lévy noise*, Encyclopedia of Mathematics and its Applications, vol. 113, Cambridge University Press, Cambridge, 2007. MR-2356959

[18] J. Pitman, Coalescents with multiple collisions, *Ann. Probab.* **27** (1999), no. 4, 1870–1902. MR-1742892

[19] S. Sagitov, The general coalescent with asynchronous mergers of ancestral lines, *J. Appl. Probab.* **36** (1999), no. 4, 1116–1125. MR-1742154

[20] J. Schweinsberg, A necessary and sufficient condition for the Λ-coalescent to come down from infinity, *Electron. Comm. Probab.* **5** (2000), 1–11. MR-1736720

[21] J. Schweinsberg, Dynamics of the evolving Bolthausen-Sznitman coalecent, *Electron. J. Probab.* **17** (2012), no. 91, 50. MR-2988406