Electronic Journal of Probability

Electron. J. Probab. **20** (2015), no. 6, 1–20.
ISSN: 1083-6489 DOI: 10.1214/EJP.v20-3379

Directed polymers in a random environment with a defect line*

Kenneth S. Alexander† Gökhan Yıldırım†

Abstract

We study the depinning transition of the $1+1$ dimensional directed polymer in a random environment with a defect line. The random environment consists of i.i.d. potential values assigned to each site of $\mathbb{Z}^2$; sites on the positive axis have the potential enhanced by a deterministic value u. We show that for small inverse temperature β the quenched and annealed free energies differ significantly at most in a small neighborhood (of size of order β) of the annealed critical point $u_c^a = 0$. For the case $u = 0$, we show that the difference between quenched and annealed free energies is of order β^4 as $\beta \to 0$, assuming only finiteness of exponential moments of the potential values, improving existing results which required stronger assumptions.

Keywords: random walk; depinning transition; pinning; Lipschitz percolation.
AMS MSC 2010: Primary 82B44, Secondary 82D60; 60K35.
Submitted to EJP on March 12, 2014, final version accepted on December 16, 2014.
Supersedes arXiv:1402.6660v3.

1 Introduction.

1.1 Physical Motivation

The directed polymer in a random environment (DPRE) models a one-dimensional object interacting with disorder. The $1+1$ dimensional version of the model first appeared in the physics literature in [21] as a model for the interface in two-dimensional Ising models with random exchange interaction. Since then it has been used in models of various growth phenomena: formation of magnetic domains in spin-glasses [21], vortex lines in superconductors [30], turbulence in viscous incompressible fluids (Burger turbulence) [8], roughness of crack interfaces [20], and the KPZ equation [25].

A related problem is the competition between extended and point defects as reflected in pinning phenomena, arising for example in the context of high-temperature superconductors [7, 10]. On a lattice this can be described by a random potential, typically i.i.d. at each lattice site, representing the point defects, with an additional fixed potential u added for sites along some line, representing the extended defect. The polymer must choose between roughly following the extended defect, or finding the best path(s)

*This research was supported by NSF grant DMS-0804934.
†University of Southern California, USA. E-mail: alexandr@usc.edu,gyildiri@usc.edu

through the point defects. As u is decreased, one expects a depinning transition at some critical u_c where the polymer ceases to follow the extended defect.

In the (nonrigorous) physics literature, there have been disagreeing predictions as to whether $u_c = 0$. Kardar [24] examined this problem numerically and found that $u_c > 0$ for the $1+1$ dimensional DPRE with defect line. On the other hand, Tang and Lyuksyutov in [31] argued that the same model satisfies $u_c = 0$, and claimed that $u_c > 0$ only above $1+1$ dimensions. Their conclusion was supported by Balents and Kardar [2], numerically and via a functional renormalization group analysis, and later by Hwa and Natterman [22] in another renormalization group analysis. It is hoped that a mathematically rigorous analysis can eventually resolve the question.

1.2 Mathematical Formulation of the Problem

The DPRE in $1+d$ dimensions is formulated as follows. Let P_ν be the distribution of the symmetric simple random walk (SSRW) $S = \{S_j, j \geq 0\}$ on $\mathbb{Z}^d$ with initial distribution ν, and let E_ν be the corresponding expectation. We write P_x, E_x when $\nu = \delta_x$, and P, E for P_0, E_0. The polymer configuration is represented by the path $\{(j, S_j)\}_{j=1}^n$ in $\mathbb{N} \times \mathbb{Z}^d$. The random environment, or bulk disorder, is given by mean zero, variance one i.i.d. random variables $V = \{v(i,x) : i \geq 1, x \in \mathbb{Z}^d\}$ with law denoted Q satisfying

$$\Lambda(\beta) = \log E^Q[e^{\beta v(i,x)}] < \infty \quad \text{for all} \quad \beta \in \mathbb{R}. \tag{1.1}$$

The Hamiltonian for paths s is

$$H_N(s) = \sum_{j=1}^{N} v(j, s_j),$$

and the *quenched polymer measure* $\mu_N^{\beta,q}$ is defined in the usual Boltzmann-Gibbs way:

$$\frac{d\mu_N^{\beta,q}}{dP}(s) = \frac{1}{Z_N^{\beta,q}} e^{\beta H_N(s)}, \tag{1.2}$$

where $\beta > 0$ is the inverse temperature and $Z_N^{\beta,q} = E_0\left[e^{\beta H_N(S)}\right]$ is the *quenched partition function.*

The first rigorous mathematical work on directed polymers in $1+d$ dimensions was done by Imbrie and Spencer [23], proving that in dimension $d \geq 3$ with Bernoulli disorder and small enough β, the end point of the polymer scales as $n^{1/2}$, i.e. the polymer is diffusive. Bolthausen [6] considered the nonnegative martingale $W_n^{\beta,q} = Z_n^{\beta,q}/E^Q[Z_n^{\beta,q}]$ and observed that the almost sure limit $W_\infty = \lim_{n\to\infty} W_n^{\beta,q}$ is subject to a dichotomy: there are only two possibilities for the positivity of the limit, $Q(W_\infty > 0) = 1$ (known as *weak disorder*) or $Q(W_\infty = 0) = 1$ (known as *strong disorder*), because the event $\{W_\infty = 0\}$ is a tail event. Bolthausen also improved the result of Imbrie and Spencer to a central limit theorem for the end point of the walk, which means that in $d \geq 3$ entropy dominates at high enough temperature, in that the polymer behaves almost as if the disorder were absent. Comets and Yoshida [12, 11], showed that there exists a critical value $\beta_c = \beta_c(d, v) \in [0, \infty]$ with $\beta_c = 0$, for $d = 1, 2$ and $0 < \beta_c \leq \infty$ for $d \geq 3$, such that $Q(W_\infty > 0) = 1$ if $\beta \in \{0\} \cup (0, \beta_c)$, and $Q(W_\infty = 0) = 1$ if $\beta > \beta_c$. In particular, for the $1+1$ dimensional case we consider here, disorder is always strong. See [13] for a survey.

There has been substantial investigation of pinning models in which disorder is present only in the defect line $\{0\} \times \mathbb{N}$; see ([17, 18] and [32]) for surveys. In such models (which we call *pinning models with defect-line potential*), the energy gains from pinning compete only with the entropy loss inherent in the class of pinned paths. Here, by contrast, we enhance the potential in the DPRE by a fixed amount u at each site of the defect line, so that energy gains from the enhancement for pinned paths also compete with the possibility of better energy gains from the potential $v(i,x)$ along depinned paths compared to pinned ones. Specifically, we define the Hamiltonian and the *quenched polymer measure* by

$$H_N^u(s) \quad = \quad \sum_{j=1}^{N}(v(j,s_j)+u1_{s_j=0}) = H_N(s)+uL_N(s), \tag{1.3}$$

$$\frac{d\mu_N^{\beta,u,q}}{dP}(s) = \frac{1}{Z_N^{\beta,u,q}}e^{\beta H_N^u(s)}, \tag{1.4}$$

where

$$L_N(s)=\sum_{j=1}^{N}1_{s_j=0}, \quad Z_N^{\beta,u,q}=E_0\left[e^{\beta H_N^u(S)}\right]$$

are the *local time* and the *quenched partition function*, respectively. Here P is the distribution of the SSRW with $S_0=0$.

In general for a partition function Z, the restriction to a set Ω of SSRW paths will be denoted $Z(\Omega)$; we add a subscript ν when the SSRW has initial distribution ν, and include V as an argument of Z when we wish to emphasize the dependence on the disorder configuration V. Thus for example,

$$Z_{N,\nu}^{\beta,u,q}(\Omega,V) := E_\nu\left(e^{\beta H_N^u(S)}1_\Omega(S)\right).$$

When $\nu=\delta_x$ we write x in place of ν.

Our results concern only $d=1$ so we restrict to that case henceforth. Our first result is on the existence of the quenched free energy of the model:

Theorem 1.1. *For every $\beta>0$ and $u\in\mathbb{R}$,*

$$f^q(\beta,u)=\lim_{N\to\infty}\frac{1}{N}\log Z_N^{\beta,u,q}=\lim_{N\to\infty}\frac{1}{N}E^Q[\log Z_N^{\beta,u,q}] \tag{1.5}$$

exists Q-a.s. and in Q-mean.

The *annealed polymer measure* $\mu_N^{\beta u}$ is obtained by taking the expected value over the disorder of the quenched Boltzmann-Gibbs weight, yielding

$$\frac{d\mu_N^{\beta u}}{dP}(s)=\frac{1}{Z_N^{\beta,u}}e^{\beta u L_N(s)+\Lambda(\beta)N}, \tag{1.6}$$

where

$$Z_N^\gamma=E_0(e^{\gamma L_N(S)}), \quad Z_N^{\beta,u}=Z_N^{\beta u}e^{\Lambda(\beta)N}=E_0(e^{\beta u L_N(S)+\Lambda(\beta)N})$$

is the *annealed partition function*. Note that $\mu_N^{\beta u}$ depends only on the product βu. Letting

$$\mathrm{F}(\gamma)=\lim_{N\to\infty}\frac{1}{N}\log Z_N^\gamma,$$

the *annealed free energy* is

$$f^a(\beta,u)=\lim_{N\to\infty}\frac{1}{N}\log Z_N^{\beta,u}=\mathrm{F}(\beta u)+\Lambda(\beta).$$

Here $\mathrm{F}(\cdot)$ is the free energy of the pinning model with homogeneous defect-line potential, that is, with disorder $v\equiv 0$.

The *quenched* and *annealed critical points* are

$$u_c^q(\beta) = \inf\{u : f^q(\beta,u) > f^q(\beta,0)\}, \quad u_c^a(\beta) = \inf\{u : f^a(\beta,u) > f^a(\beta,0)\}.$$

Note that the last inequality is equivalent to $\mathrm{F}(\beta u) > 0$, so $\beta u_c^a(\beta)$ does not depend on β. In fact, it is standard (see [17]) that in the present situation $u_c^a(\beta) = 0$ for all β because the random walk on $\mathbb{Z}$ with distribution P is recurrent. When $u > u_c^q(\beta)$ the quenched polymer is said to be *pinned*. Note also that $f^q(\beta,u) \le f^a(\beta,u)$ by Jensen's inequality.

As mentioned above, physicists have differed on the question of whether $u_c^q(\beta) = 0$, for $d = 1$. One approach which at least provides a bound for $u_c^q(\beta)$ is to find a value $\Delta_0(\beta)$ such that for $u > \Delta_0(\beta)$, the quenched and annealed free energies are approximately the same; in particular this means the quenched free energy is strictly greater than $\Lambda(\beta)$ and thus also strictly greater than $f^q(\beta,0)$, meaning that $u > u_c^q(\beta)$. We thereby obtain that $u_c^q(\beta) \le \Delta_0(\beta)$. This is the approach taken in [1] for the pinning model with defect-line potential; in the case where the underlying process is 1-dimensional SSRW one has $\Delta_0(\beta)$ of order at most e^{-K/β^2} for some constant K, for small β. Here our main result has a similar form, but with bound $\Delta_0(\beta)$ of order β. This larger size of $\Delta_0(\beta)$ is rooted in the larger overlap present in the DPRE—overlap is counted throughout the bulk of $\mathbb{Z}^2$, as opposed to just on the axis. (Here by overlap we mean intersections between two independent copies of the path—see (3.7).) We do not know whether $\Delta_0(\beta)$ of order β is optimal; the physicists' predictions in ([2],[22],[31]) point toward $u_c(\beta) = 0$. Analogs of $u_c(\beta) = 0$ were in fact proved for the randomized polynuclear growth model [5] (see also [4]) and recently also for the longest increasing subsequence problem and last passage percolation [3]. At any rate, the theorem says in effect that the disorder alters the free energy significantly at most for u in a neighborhood of size $O(\beta)$ of the annealed critical point $u_c^a(\beta) = 0$.

We can now state our main results.

Theorem 1.2. *Consider the $1+1$ dimensional DPRE with defect line, with Hamiltonian as in (1.3). Suppose that the disorder variables $V = \{v(i,x) : i \ge 1, x \in \mathbb{Z}^d\}$ are i.i.d. mean zero variance one random variables which satisfy the condition* (1.1).

Then given $0 < \epsilon < 1$, there exists a $K = K(\epsilon)$ as follows. Provided that β and βu are sufficiently small and $u \ge K\beta$, we have

$$\Lambda(\beta) + \mathrm{F}(\beta u) \ge f^q(\beta,u) \ge \Lambda(\beta) + (1-\epsilon)\mathrm{F}(\beta u). \tag{1.7}$$

Further, for small β,

$$0 \le u_c^q(\beta) \le K(\epsilon)\beta. \tag{1.8}$$

With minor modifications, the proof of Theorem 1.2 also proves the following.

Theorem 1.3. *Under the hypotheses of Theorem 1.2, there exist constants C_1, C_2 such that for sufficiently small β,*

$$C_1\beta^4 \le f^a(\beta,0) - f^q(\beta,0) \le C_2\beta^4. \tag{1.9}$$

Lacoin [27] proved (1.9) in the case of Gaussian disorder, and proved a similar statement with an upper bound of $C_2\beta^4(1 + (\log\beta)^2)$ for the general disorder we consider here. Watbled [34] extended (1.9) to infinitely divisible disorder.

The full strength of assumption (1.1) is used only to establish the existence of the free energy for all $\beta > 0$, in Theorem 1.1. For Theorems 1.2 and 1.3 we need only that $\Lambda(\beta) < \infty$ for small β.

In the following sections, the $K_i's$ are universal constants, except where they depend on a parameter, which is shown in parentheses.

2 Proof of Theorem 1.1:Existence of the Free Energy

In the case $u = 0$, the existence of the quenched free energy is a consequence of the concentration of $\log Z_N^{\beta,0,q}$ around its mean, together with superadditivity of $E^Q(\log Z_N^{\beta,0,q})$ in N, which yields a limit for $N^{-1}E^Q(\log Z_N^{\beta,0,q})$; see [9], [12]. For $u \neq 0$, though, the superadditivity fails because $E^Q(\log Z_N^{\beta,u,q})$ is inhomogeneous, in the sense that if we start paths at some (j,x) instead of $(0,0)$, the distribution depends on x. Let us write $Z_N(x)$ or $Z_N(x,V)$ for $Z_{N,x}^{\beta,u,q}$, and Z_N for $Z_N^{\beta,u,q}$ (suppressing the β,u,q for notational convenience), and define

$$Z_N(x,y) = Z_N(x,y;V) := E_x\left[e^{\sum_{j=1}^N \beta(v(j,S_j)+u1_{S_j=0})}1_{S_N=y}\right],$$

where P_x is the SSRW measure when $S_0 = x$. As we will see below, for general u one can easily obtain superadditivity of $E^Q(\log Z_N(0,0))$, and the proof of concentration of $\log Z_N$ around its mean requires little change; the main task is to bound the difference between $E^Q(\log Z_N)$ and $\frac{1}{2}E^Q(\log Z_{2N}(0,0))$.

2.1 The Constrained Model

In the *constrained model* (quenched or annealed), we restrict to paths ending at $s_N = 0$, so the quenched partition function is $Z_N(0,0)$.

Due to the periodicity of SSRW, we assume that N, M are even integers for this section. Let $\theta_{n,y}$ be the space-time shift operator on the environment V:

$$(\theta_{n,y}v)(k,x) = v(k+n, x+y).$$

From the Markov property of SSRW, we have

$$Z_{N+M}(0,0;V) \geq Z_N(0,x;V)Z_M(x,0;\theta_{N,0}V) \quad \text{for all } N,M,x. \tag{2.1}$$

For $N = M$, after taking logs and expectations this yields

$$E^Q[\log Z_N(0,x;V)] \leq \frac{1}{2}E^Q[\log Z_{2N}(0,0;V)] \quad \text{for all } N,x. \tag{2.2}$$

Similarly we obtain

$$E^Q[\log Z_{N+M}(0,0;V)] \geq E^Q[\log Z_N(0,0;V)] + E^Q[\log Z_M(0,0;V)]. \tag{2.3}$$

This superadditivity establishes the existence of the limit

$$\lim_{N\to\infty} \frac{1}{N}E^Q[\log Z_N(0,0;V)] = \sup_{N\geq 1} \frac{1}{N}E^Q[\log Z_N(0,0;V)].$$

It follows from (2.1), with $x = 0$, and the subadditive ergodic theorem [26] that the constrained free energy exists and Q-a.s. constant:

$$f^{q,c}(\beta,u) = \lim_{N\to\infty} \frac{1}{N}\log Z_N(0,0;V) = \lim_{N\to\infty} \frac{1}{N}E^Q[\log Z_N(0,0;V)].$$

The non-randomness (a.s.) of $f^{q,c}(\beta,u)$ is called the *self-averaging property* of the quenched free energy.

2.2 The Unconstrained Model

Since $Z_N(0,0) \leq Z_N$, if we show

$$E^Q[\log Z_N] \leq \frac{1}{2}E^Q[\log Z_{2N}(0,0)] + o(N), \tag{2.4}$$

it follows that

$$\lim_{N\to\infty} \frac{1}{N} E^Q[\log Z_N(0,0)] = \lim_{N\to\infty} \frac{1}{N} E^Q[\log Z_N]. \tag{2.5}$$

Inside the proof of ([12], Proposition 2.5), the following is established for the case $u = 0$: the deviation from the mean can be expressed as a sum of martingale differences,

$$\log Z_N - E^Q(\log Z_N) = \sum_{j=1}^{N} W_{N,j},$$

satisfying

$$E^Q\left(e^{|W_{N,j}|}\right) \leq K_0(\beta) < \infty \quad \text{for all } N, j.$$

This proof extends to $Z_N(0,x)$ simply by restricting to paths ending at x, and it extends to general u by adding $\beta u L_N$ to the exponent in the definition of $\hat{e}_{N,j}$ in the proof in [12]. Then by ([28] Theorem 3.6), there exists $K_1(\beta,p)$ such that for all $t > 0$ and all N, x,

$$Q\Big(\Big|\frac{1}{N}\log Z_N(0,x) - \frac{1}{N}E^Q[\log Z_N(0,x)]\Big| \geq t\Big) \leq \frac{K_1(\beta,p)}{t^p N^{p/2}}. \tag{2.6}$$

We can now establish (2.4). Let $\Lambda_N = \{(i,x) : 1 \leq i \leq N, |x| \leq i, x - i \text{ even}\}$. Then using (2.2)

$$\begin{aligned}
E^Q[\log Z_N] &= E^Q\left[\log\left(\sum_{x:(N,x)\in\Lambda_N} e^{\log Z_N(0,x)}\right)\right] \\
&= E^Q\left[\log\left(\sum_{x:(N,x)\in\Lambda_N} e^{E^Q[\log Z_N(0,x)]} e^{(\log Z_N(0,x) - E^Q[\log Z_N(0,x)])}\right)\right] \\
&\leq \frac{1}{2}E^Q[\log Z_{2N}(0,0)] + E^Q\left[\log\left(\sum_{x:(N,x)\in\Lambda_N} e^{(\log Z_N(0,x) - E^Q[\log Z_N(0,x)])}\right)\right] \\
&\leq \frac{1}{2}E^Q[\log Z_{2N}(0,0)] \\
&\qquad + E^Q\Big[\log\Big((2N+1)\max_{x:(N,x)\in\Lambda_N} e^{(\log Z_N(0,x) - E^Q[\log Z_N(0,x)])}\Big)\Big] \\
&\leq \frac{1}{2}E^Q[\log Z_{2N}(0,0)] + \log(2N+1) \\
&\qquad + E^Q\Big[\max_{x:(N,x)\in\Lambda_N}(\log Z_N(0,x) - E^Q[\log Z_N(0,x)])\Big] \\
&\leq \frac{1}{2}E^Q[\log Z_{2N}(0,0)] + \log(2N+1) \\
&\qquad + \int_0^\infty Q\Big(\max_{x:(N,x)\in\Lambda_N} |\log Z_N(0,x) - E^Q[\log Z_N(0,x)]| \geq s\Big)\, ds.
\end{aligned} \tag{2.7}$$

For $q_N > 0$ we can bound the last integral using (2.6) with $p = 3$:

$$\int_0^\infty Q\Big(\max_{x:(N,x)\in\Lambda_N} |\log Z_N(0,x) - E^Q[\log Z_N(0,x)]| \geq s\Big)\, ds$$
$$\leq q_N + (2N+1) \int_{q_N}^\infty \max_{x:(N,x)\in\Lambda_N} Q\Big(|\log Z_N(0,x) - E^Q[\log Z_N(0,x)]| \geq s\Big)\, ds$$
$$\leq q_N + (2N+1)N \int_{q_N/N}^\infty \max_{x:(N,x)\in\Lambda_N} Q\Big(|\log Z_N(0,x) - E^Q[\log Z_N(0,x)]| \geq Nt\Big)\, dt$$
$$\leq q_N + 3K_1(\beta,3)N^{1/2} \int_{q_N/N}^\infty t^{-3}\, dt$$
$$\leq q_N + \frac{3}{2}K_1(\beta,3)N^{5/2}q_N^{-2}.$$

Choosing $q_N = N^{5/6}$ we see that the integral on the right side of (2.7) is $O(N^{5/6})$, and hence (2.4) holds. Therefore so does (2.5).

The Borel-Cantelli lemma, and (2.6) with $p > 2$, then establish the equality of the free energies in the original and constrained models.

3 Proof of Theorem 1.2

3.1 Proof Outline

We take a block length N which is a multiple (of order ϵ^{-2}) of the annealed correlation length, so that the associated finite-volume annealed free energy is large. We use the second moment method to show that on scale N, the quenched partition function is with high probability within a constant of the annealed one; here the condition $u \geq K(\epsilon)\beta$ allows necessary control of the overlap. This remains true if we restrict the partition functions to a set Ω_N of paths which stay inside an $N \times 4\sqrt{N}$ box centered on the axis, and end within $\sqrt{N}/4$ of the axis. Having paths end close to the axis facilitates concatenating a large number L of the boxes together to make a length-LN corridor in such a way that the corresponding partition function is approximately the product of the L single-box partition functions.

Certain boxes in this corridor, though, may have very small values for the associated quenched partition function, making this product of single-box partition functions unacceptably small relative to the annealed one. This requires re-routing the corridor through off-axis boxes in places, to avoid "bad" on-axis boxes; bad off-axis boxes must also be avoided in this process. The result is a dependent percolation problem on coarse-grained scale; one needs an infinite directed path of "good" boxes, with most of these boxes being on-axis, where the extra potential u is relevant. We use results of [14], [19] and [29] to establish the existence of such a path. The restriction of the quenched partition function to length-LN paths following the corresponding (non-coarse-grained) corridor then provides a lower bound for the full quenched partition function at length LN, and taking a limit as $L \to \infty$ yields the desired result.

3.2 Further Preliminaries

Recall that $\mathrm{F}(\gamma)$ denotes the free energy of the homogeneous (or annealed) model with defect-line potential. As observed in ([1], equation (2.7)), $\gamma + \log E_0(e^{\gamma L_n})$ is subadditive in n for all $\gamma \geq 0$. It follows that

$$E_0(e^{\gamma L_N}) \geq e^{-\gamma} e^{N\mathrm{F}(\gamma)} \quad \text{for all } N \geq 1. \tag{3.1}$$

In what follows, in service of clean notation, we omit (but implicitly assume) integer part notation for large quantities which in fact must be integers, such as M in the next lemma.

The following is essentially the same as ([1], equation (2.22)).

Lemma 3.1. *There exist $K_2, K_3 > 0$ such that*

$$\forall j \geq 1, \gamma > 0, \quad E_0(e^{\gamma L_{jM}}) \leq K_2 j e^{K_3 j},$$

where $M = 1/\mathrm{F}(\gamma)$ is the correlation length.

For the proof of the following see [17] or [18].

Proposition 3.2. *The free energy $\mathrm{F}(\gamma)$ has the following properties:*

a) $\mathrm{F}(\gamma)$ *is* 0 *on* $(-\infty, 0]$ *and strictly increasing and positive on* $(0, \infty)$.
b) *for some* $K_4 > 0$, $\mathrm{F}(\gamma) \sim K_4 \gamma^2$, *as* $\gamma \to 0^+$.

For any $x \in \mathbb{Z}$, $\gamma \geq 0$, conditioning on the hitting time of 0 yields

$$E_x e^{\gamma L_N} \leq E_0 e^{\gamma(L_N+1)}. \tag{3.2}$$

For $k > 1$, conditioning on $S_{(k-1)N}$, applying (3.2) and iterating we obtain

$$E_x e^{\gamma L_{kN}} \leq \left(E_0 e^{\gamma(L_N+1)}\right)^k. \tag{3.3}$$

The following is a straightforward consequence of Donsker's invariance principle.

Lemma 3.3. *For one dimensional SSRW, we have*

$$A^{\mathrm{forward}} := \liminf_{N\to\infty} \inf_{|x|\leq \frac{\sqrt{N}}{4}} P_x\Big(\max_{1\leq i\leq N} |S_i| \leq 2\sqrt{N}, |S_N| \leq \frac{\sqrt{N}}{4}\Big) > 0,$$

$$A^{\mathrm{up}} := \liminf_{N\to\infty} \inf_{|x|\leq \frac{\sqrt{N}}{4}} P_x\Big(\max_{1\leq i\leq N} |S_i| \leq 2\sqrt{N}, |S_N - \sqrt{N}| \leq \frac{\sqrt{N}}{4}\Big) > 0,$$

$$A^{\mathrm{down}} := \liminf_{N\to\infty} \inf_{|x|\leq \frac{\sqrt{N}}{4}} P_x\Big(\max_{1\leq i\leq N} |S_i| \leq 2\sqrt{N}, |S_N + \sqrt{N}| \leq \frac{\sqrt{N}}{4}\Big) > 0.$$

The proof of the following is due to S.R.S. Varadhan [33].

Lemma 3.4. *There exists a constant $0 < \epsilon_0 < 1$, such that for $\gamma > 0$, for all sufficiently large N and $|x| \leq \frac{\sqrt{N}}{4}$,*

$$E_x\Big(e^{\gamma L_N} 1_{\Omega_N}\Big) \geq \epsilon_0 E_x\Big(e^{\gamma L_N}\Big),$$

where

$$\Omega_N = \{s : \max_{1\leq i\leq N} |s_i| \leq 2\sqrt{N}, |s_N| \leq \frac{\sqrt{N}}{4}\}.$$

Proof. We define a polymer measure on the space of SSRW paths:

$$\mu^{\gamma}_{N,x}(A) := \frac{E_x[e^{\gamma L_N} 1_A]}{E_x[e^{\gamma L_N}]}.$$

Let $W(n,x) = E_x[e^{\gamma L_n}]$.

Under $\mu^{\gamma}_{N,x}(\cdot)$ we have a non-stationary Markov process with transition probabilities from z to $y = z \pm 1$ at time $k < N$ given by

$$\begin{aligned}
\pi(z,y,k,N,\gamma) &= \frac{E_x[e^{\gamma L_N} 1_{S_k=z} 1_{S_{k+1}=y}]}{E_x[e^{\gamma L_N} 1_{S_k=z}]} \\
&= \frac{E_x[e^{\gamma L_k} 1_{S_k=z}] E_z[e^{\gamma L_{N-k}} 1_{S_1=y}]}{E_x[e^{\gamma L_k} 1_{S_k=z}] E_z[e^{\gamma L_{N-k}}]} \\
&= \frac{1}{2} \frac{e^{\gamma\delta_0(y)} E_y[e^{\gamma L_{N-k-1}}]}{E_z[e^{\gamma L_{N-k}}]} \\
&= \frac{e^{\gamma\delta_0(y)}}{2} \frac{W(N-k-1,y)}{W(N-k,z)}.
\end{aligned} \tag{3.4}$$

For all z,

$$W(N-k,z)=\frac{1}{2}e^{\gamma\delta_0(z+1)}W(N-k-1,z+1)+\frac{1}{2}e^{\gamma\delta_0(z-1)}W(N-k-1,z-1) \quad (3.5)$$

while for $z \geq 1$ we have monotonicity in z:

$$W(N-k-1,z+1)\leq W(N-k-1,z) \text{ and } W(N-k-1,1)\leq e^{\gamma}W(N-k-1,0),$$

which follows from the fact that the hitting time of 0 is stochastically smaller when starting from a lower height $z \geq 0$. Similarly for $z \leq -1$,

$$W(N-k-1,z)\leq W(N-k-1,z+1) \text{ and } W(N-k-1,-1)\leq e^{\gamma}W(N-k-1,0).$$

Therefore for $z \geq 1$, the second term on the right in (3.5) is the larger one, and by (3.4) we thus have

$$\pi(z,z-1,k,N,\gamma)\geq\frac{1}{2},$$

while for $z \leq -1$, similarly,

$$\pi(z,z+1,k,N,\gamma)\geq\frac{1}{2}.$$

Hence, the $\mu_{N,x}^{\gamma}$ chain can be coupled to the P_x chain (i.e. SSRW) in a such a way that the $\mu_{N,x}^{\gamma}$ chain is always smaller or equal in magnitude. Therefore

$$\mu_{N,x}^{\gamma}(\Omega_N)\geq P_x(\Omega_N),$$

and the result then follows from Lemma 3.3. □

Let

$$\tau_x=\inf\{n\geq 1: S_n=x\}.$$

Lemma 3.5. *Let $0<\epsilon<1$ be given. Then, for sufficiently large N and $|x|\leq\frac{\sqrt{N}}{4}$, for all $\gamma>0$,*

$$E_x\left(e^{\gamma L_N}\right)\geq\frac{1}{2}\mathbf{P}(\xi\geq\frac{1}{4\sqrt{\epsilon}})e^{(1-\epsilon)N\mathrm{F}(\gamma)},$$

where ξ denotes a standard normal random variable.

Proof. For a given $0<\epsilon<1$, there exists an $N_0=N_0(\epsilon)$ such that for all $N\geq N_0$ and for $0<x\leq\frac{\sqrt{N}}{4}$,

$$\begin{aligned} P_x(\tau_0\leq\epsilon N) &= P_0(\tau_x\leq\epsilon N)\\ &\geq P_0(S_{\epsilon N}\geq\frac{\sqrt{N}}{4})\\ &\geq \frac{1}{2}\mathbf{P}(\xi\geq\frac{1}{4\sqrt{\epsilon}}). \end{aligned} \quad (3.6)$$

The right side of (3.6) is also a lower bound for the left side for $-\frac{\sqrt{N}}{4}\leq x<0$ by symmetry, and for $x=0$ after increasing N_0 if necessary. Therefore, for sufficiently large N and $|x|\leq\frac{\sqrt{N}}{4}$, using (3.1) and (3.6),

$$\begin{aligned} E_x\left(e^{\gamma L_N}\right) &\geq \sum_{k=x}^{\epsilon N}e^{\gamma}E_0\left(e^{\gamma L_{N-k}}\right)P_x(\tau_0=k)\\ &\geq \sum_{k=x}^{\epsilon N}e^{(1-\epsilon)N\mathrm{F}(\gamma)}P_x(\tau_0=k)\\ &= e^{(1-\epsilon)N\mathrm{F}(\gamma)}P_x(\tau_0\leq\epsilon N)\\ &\geq \frac{1}{2}\mathbf{P}(\xi\geq\frac{1}{4\sqrt{\epsilon}})e^{(1-\epsilon)N\mathrm{F}(\gamma)}. \end{aligned}$$

□

For SSRW paths s^1, s^2, define the overlap

$$B_N(s^1, s^2) = \sum_{i=1}^{N} 1_{s_i^1 = s_i^2} \tag{3.7}$$

For independent copies S^1, S^2 of the Markov chain S, (S^1, S^2) is also a Markov chain, so as a special case of (3.2),

$$E^{\otimes 2}_{(x,x')} e^{\gamma B_N} \le E^{\otimes 2}_{(0,0)} e^{\gamma (B_N + 1)}, \tag{3.8}$$

and as a special case of (3.3), for $k \ge 1, \gamma \ge 0$, and $x, x' \in \mathbb{Z}$, we have

$$E^{\otimes 2}_{(x,x')} e^{\gamma B_{kN}} \le \left(E^{\otimes 2}_{(0,0)} e^{\gamma (B_N + 1)} \right)^k. \tag{3.9}$$

We need information about the excursion length distribution of (p, q)-walks. First, a definition:

Definition 3.6. *A (p, q)-walk is a random walk in which the steps X_i have distribution $\mathbf{P}(X_1 = b) = \mathbf{P}(X_1 = -b) = p/2 \in (0, 1/2)$ and $\mathbf{P}(X_1 = 0) = q > 0$, where $p + q = 1$ and b is a positive integer.*

Let $\bar{S}_N = S_N^1 - S_N^2$, where S_N^1, S_N^2 are independent SSRWs. Then $(\bar{S}_N)_{N \ge 1}$ is a $(1/2, 1/2)$-walk with $b = 2$, and $B_N(S^1, S^2) = L_N(\bar{S})$.

For the proof of the following, see [16] and [17].

Proposition 3.7. *For any (p, q)-walk, $p \in (0, 1)$, we have*

$$\mathbf{P}(\tau_0 = n) \sim \sqrt{\frac{p}{2\pi}} n^{-3/2} \text{ as } n \to \infty.$$

For $(1, 0)$-walk,

$$\mathbf{P}(\tau_0 = 2n) \sim \sqrt{\frac{1}{4\pi}} n^{-3/2} \text{ as } n \to \infty.$$

Let us define

$$\Phi(\beta) = \Lambda(2\beta) - 2\Lambda(\beta) \tag{3.10}$$

where $\Lambda(\beta) = \log E^Q[e^{\beta v(i,x)}]$.

The next result is similar to ([1] equation (2.40)), but specialized to the present situation.

Proposition 3.8. *Let $0 < a < 1$ be given. Then there exists a constant $K_5 = K_5(a) > 0$ such that for sufficiently small β and $R \le K_5 \beta^{-4}$ we have*

$$E^{\otimes 2}_{(0,0)} \left(e^{2\Phi(\beta)(B_R(S^1, S^2) + 1)} - 1 \right) \le a. \tag{3.11}$$

Proof. Let E_i denote the length of the i^{th} excursion of $\bar{S} = S^1 - S^2$ from 0 (that is, the time from the $(i-1)$st to the ith visit to 0.) Then

$$P(B_R + 1 > k) \quad \le \quad P(\max_{1 \le i \le k} E_i \le R) = (1 - P(E_1 > R))^k \quad \text{for all } k \ge 1.$$

By Proposition 3.7, $P(E_1 > R) \sim (\pi R)^{-1/2}$ as $R \to \infty$, so for sufficiently large R,

$$P(B_R + 1 > k) \le \left(1 - \frac{1}{\sqrt{2\pi R}} \right)^k \quad \text{for all } k \ge 1. \tag{3.12}$$

Therefore $B_R + 1$ is stochastically dominated by a geometric random variable with parameter

$$p_R = (2\pi R)^{-1/2} \ge \frac{\beta^2}{\sqrt{2\pi K_5}} \tag{3.13}$$

Therefore for R large and β small,

$$E^{\otimes 2}_{(0,0)}\left(e^{2\Phi(\beta)(B_R(S^1,S^2)+1)}-1\right) \le \frac{p_R e^{2\Phi(\beta)}}{1-(1-p_R)e^{2\Phi(\beta)}}-1, \tag{3.14}$$

provided that

$$p_R > 1-e^{-2\Phi(\beta)}. \tag{3.15}$$

To bound (3.14) by the given a, we need

$$p_R \ge \frac{a+1}{a}(1-e^{-2\Phi(\beta)}). \tag{3.16}$$

Since $\Lambda(\beta)\sim\beta^2/2$, and hence $\Phi(\beta)\sim\beta^2$, as $\beta\to 0$, if $K_5(a)$ is taken sufficiently small, then (3.15) and (3.16) follow from (3.13). This proves (3.11) for $R \le K_5\beta^{-4}$ with R large. Since the left side of (3.11) is monotone in R, $R\le K_5\beta^{-4}$ alone is sufficient.

□

3.3 The Coarse Grained Lattice $\mathbb{L}_{CG}$

In this section, we introduce a coarse grained lattice

$$\mathbb{L}_{CG} := \{(I,J)\in\mathbb{Z}^2 : I\ge 0, 0\le J\le I\}.$$

Note this is really a "half lattice" since we only consider $J\ge 0$.

Recall that the annealed correlation length is $M=1/\mathrm{F}(\beta u)$. Let $N=k_0 M$, with k_0 to be specified. For notational convenience we assume that N and $\sqrt{N}$ are integers. We use capital letters (I,J) for a site in the coarse grained lattice which corresponds to the vertical window

$$R(I,J) := \{(k,l)\in\mathbb{Z}^2 : k=IN,\ (J-\frac{1}{4})\sqrt{N}\le l\le (J+\frac{1}{4})\sqrt{N}\}$$

in the original lattice $\mathbb{Z}^2$.

The *box* starting from the window $R(I,J)$ is the following region in $\mathbb{Z}^2$:

$$B(I,J) := [IN,(I+1)N]\times[(J-2)\sqrt{N},(J+2)\sqrt{N}].$$

We say that there is a *link* between sites (I,J) and $(I+1,L)$ if $|L-J|\le 1$. The link is *down, forward* or *up* according as $L=J-1, J$ or $J+1$. A *path* $\Gamma=\Gamma_{(I,J)\to(K,L)}$ from site (I,J) to site (K,L) in $\mathbb{L}_{CG}$ is a sequence of sites $(I,J)=(I_0,J_0),(I_1,J_1),\cdots,(I_N,J_N)=(K,L)$ such that there is a link between (I_i,J_i) and (I_{i+1},J_{i+1}) for all $i<N$. $\Gamma(I_i)$ will denote the second coordinate J_i of the unique site (I_i,J_i) in the path Γ. We will use the alternate notation $\Gamma_{(I,J)}$ for $\Gamma_{(0,0)\to(I,J)}$. Given paths Γ^1,Γ^2 from some (I,J) to (K,L), we say that Γ^1 is *closer to the x-axis than* Γ^2 if

$$\Gamma^1(I_i)\le\Gamma^2(I_i) \text{ for each } I\le I_i\le K.$$

Suppose each site $(I,J)\in\mathbb{L}_{CG}$ is designated as *open* or *closed*. We then say a path $\Gamma_{(I,J)\to(K,L)}$ is

(i) *open* if its all sites are open;

(ii) *maximal* if it has the maximum number of open sites among all paths from site (I,J) to site (K,L);

(iii) *optimal* if it is the maximal path which is closest to the x-axis.

$\Gamma^{\infty}_{(I,J)}$ denotes a generic infinite open path from the site (I,J). There is exactly one optimal path for given sites (I,J) and (K,L) and we denote it by $\Gamma^{\text{opt}}_{(I,J)\to(K,L)}$.

When an infinite open path from a site (I,J) exists, the one which is closest to the x-axis among all such paths is called the *infinite good path from the site* (I,J), and we denote it by $\Gamma^{\text{G},\infty}_{(I,J)}$. $\Gamma^{\text{G},\infty}$ denotes the infinite good path from the site $(0,0)$, when it exists. For $0 \le I \le K$, $\Gamma^{\text{G},\infty}_{I\to K}$ will denote the segment of the path $\Gamma^{\text{G},\infty}$ between the sites with first coordinates I and K. Note that if the site (I_0,J_0) is on the infinite good path from $(0,0)$, then

$$\Gamma^{\text{opt}}_{(0,0)\to(I_0,J_0)} = \Gamma^{\text{G},\infty}_{0\to I_0}. \tag{3.17}$$

Given a path $\Gamma = \Gamma_{(0,0)\to(I,J)} = \{(L,J_L) : L \le I\}$ in $\mathbb{L}_{CG}$, we identify a subset $\Omega^{(I,J)}$ of the SSRW paths of length IN in the following way:

$$\begin{aligned}\Omega^{(I,J)} &:= \Omega^{(I,J)}(\Gamma)\\ &:= \Big\{ s = \{(n,s_n)\}_{n\le IN} : s_0 = 0, s_{LN} \in R(L,J_L)\ \forall L \le I, s \subset \cup_{L<I} B(L,J_L) \Big\}.\end{aligned}$$

When $\Gamma^{\text{G},\infty} = \{(L,J^G_L) : L \ge 0\}$ exists, for $0 \le I \le K$ we define

$$\Omega^{\text{G},\infty}_{I\to K} := \Big\{ s = \{(n,s_n)\}_{IN\le n\le KN} : s_{LN} \in R(L,J^G_L)\ \forall I \le L \le K, s \subset \cup_{I\le L<K} B(L,J^G_L) \Big\},$$

otherwise we define $\Omega^{\text{G},\infty}_{I\to K} := \phi$. We define quenched probability measures on the windows $R(I,J)$, using SSRW paths associated to the optimal coarse-grained path to that window, as follows: for $I \ge 1$ and $x \in R(I,J)$, let

$$\nu^q_{(I,J)}(x) := \frac{Z^{\beta,u,q}_{IN}\left(\Omega^{(I,J)}(\Gamma^{\text{opt}}_{(0,0)\to(I,J)}) \cap \{s_{IN} = x\}\right)}{Z^{\beta,u,q}_{IN}\left(\Omega^{(I,J)}(\Gamma^{\text{opt}}_{(0,0)\to(I,J)})\right)}, \quad x \in R(I,J), \tag{3.18}$$

and let $\nu^q_{(0,0)} := \delta_0$. The measure

$$\tilde{\nu}^q_{(I,J)}(x) = \nu^q_{(I,J)}((IN,JN)+x), \quad x \in R(0,0),$$

is the translate of $\nu^q_{(I,J)}$ to $R(0,0)$.

Define the following sets of SSRW paths, corresponding to up, forward and down links in a coarse-grained path:

$$\Omega^{\text{up}}_N := \{(s_0,\cdots,s_N) : |s_0| \le \frac{\sqrt{N}}{4}, |s_N - \sqrt{N}| \le \frac{\sqrt{N}}{4}, |s_i| \le 2\sqrt{N}, 1 \le i \le N\},$$

$$\Omega^{\text{forward}}_N := \{(s_0,\cdots,s_N) : |s_0| \le \frac{\sqrt{N}}{4}, |s_N| \le \frac{\sqrt{N}}{4}, |s_i| \le 2\sqrt{N}, 1 \le i \le N\},$$

and

$$\Omega^{\text{down}}_N := \{(s_0,\cdots,s_N) : |s_0| \le \frac{\sqrt{N}}{4}, |s_N + \sqrt{N}| \le \frac{\sqrt{N}}{4}, |s_i| \le 2\sqrt{N}, 1 \le i \le N\}.$$

Note that the up, forward and down sets of SSRW paths start at the window $R(I,J)$, stay in the box $B(I,J)$, and end at the window $R(I+1,J+l)$, $l = +1,0,-1$, respectively.

Of particular interest are the *link partition functions*

$$Z^{\beta,u,q}_{N,\tilde{\nu}^q_{(I,J)}}(\Omega^{\text{g}}_N, \theta_{IN,JN}(V)), \quad g = \text{up, forward, down},$$

corresponding to SSRW paths in the box $B(I,J)$ from the window $R(I,J)$ to $R(I+1,J+l)$, with $l = 1,0,-1$ according to the value of g. When $J = 0$ and $g =$ forward, we refer to the link or partition function as *on-axis*, otherwise it is *off-axis*.

3.4 Open and Closed Sites in the Coarse Grained Lattice.

Define the filtrations

$$\mathcal{F}_I := \sigma(\{v(i,x) : 1 \le i \le IN, x \in \mathbb{Z}\}),\ I \ge 1,$$

and note that the measures $\nu^q_{(I,J)}$ are $\mathcal{F}_I$-measurable for all $J \ge 0$. One expects on-axis link partition functions to be larger than off-axis ones in general, and we will specify constants $U_{\text{on}} \ge U_{\text{off}}$ which will serve as lower bounds for these partition functions, satisfying

$$U_{\text{on}} \le \frac{1}{2} E^Q \Big(Z^{\beta,u,q}_{N,\tilde{\nu}^q_{(I,0)}} \left(\Omega^{\text{forward}}_N, \theta_{IN,0}(V)\right) \ \Big|\ \mathcal{F}_I \Big) \quad Q - a.s. \text{ for each } I \ge 0,$$

and for $I > 0, J \le I$ and g = forward, up, down,

$$U_{\text{off}} \le \frac{1}{2} E^Q \Big(Z^{\beta,u,q}_{N,\tilde{\nu}^q_{(I,J)}} \left(\Omega^{\text{g}}_N, \theta_{IN,JN}(V)\right) \ \Big|\ \mathcal{F}_I \Big) \quad Q - a.s.$$

For $I \ge 1$, by Lemma 3.5 and 3.4, for sufficiently small βu, Q-a.s.

$$\begin{aligned}
&E^Q \Big(Z^{\beta,u,q}_{N,\tilde{\nu}^q_{(I,0)}} \left(\Omega^{\text{forward}}_N, \theta_{IN,0}(V)\right) \ \Big|\ \mathcal{F}_I \Big) \\
&= \sum_{x \in R(0,0)} \tilde{\nu}^q_{(I,0)}(x) E^Q \left(E_x \left[e^{\beta \sum_{k=1}^N \left(v(IN+k, S_k) + u 1_{S_k=0} \right)} 1_{\Omega^{\text{forward}}_N} \right] \right) \\
&= \sum_{x \in R(0,0)} \tilde{\nu}^q_{(I,0)}(x) e^{\Lambda(\beta)N} E_x \left[e^{\sum_{k=1}^N \beta u 1_{S_k=0}} 1_{\Omega^{\text{forward}}_N} \right] \\
&\ge \sum_{x \in R(0,0)} \tilde{\nu}^q_{(I,0)}(x) e^{\Lambda(\beta)N} \frac{\epsilon_0}{2} \mathbf{P}(\xi \ge \frac{1}{4\sqrt{\epsilon}}) e^{(1-\epsilon)N\mathrm{F}(\beta u)} \\
&\ge \frac{\epsilon_0}{2} \mathbf{P}(\xi \ge \frac{1}{4\sqrt{\epsilon}}) e^{(\Lambda(\beta) + (1-\epsilon)\mathrm{F}(\beta u))N}.
\end{aligned}$$

Hence we define

$$\Theta_{\text{on}} := \Theta_{\text{on}}(\epsilon) := \frac{\epsilon_0}{4} \mathbf{P}(\xi \ge \frac{1}{4\sqrt{\epsilon}}), \quad U_{\text{on}} := \Theta_{\text{on}} e^{(\Lambda(\beta) + (1-\epsilon)\mathrm{F}(\beta u))N}. \tag{3.19}$$

For sufficiently small $\beta u > 0$, for all $I \ge 0, J \ge 1$ and for g = forward, up, down, by Lemma 3.3 we have Q-a.s.

$$\begin{aligned}
E^Q \Big(Z^{\beta,u,q}_{N,\tilde{\nu}^q_{(I,J)}} (\Omega^{\text{g}}_N, \theta_{IN,JN}(V)) | \mathcal{F}_I \Big) &\ge E^Q \Big(Z^{\beta,0,q}_{N,\tilde{\nu}^q_{(I,J)}} (\Omega^{\text{g}}_N, \theta_{IN,JN}(V)) | \mathcal{F}_I \Big) \\
&\ge e^{\Lambda(\beta)N} \sum_{x \in R(0,0)} \tilde{\nu}^q_{(I,J)}(x) P_x \left(\Omega^{\text{g}}_N \right) \\
&\ge \frac{1}{2} e^{\Lambda(\beta)N} \min(A^{\text{forward}}, A^{\text{up}}, A^{\text{down}}).
\end{aligned}$$

Hence we define

$$\Theta_{\text{off}} := \Theta_{\text{off}}(\epsilon) := \frac{1}{4} \min(A^{\text{forward}}, A^{\text{up}}, A^{\text{down}}, 4\Theta_{\text{on}}), \quad U_{\text{off}} := \Theta_{\text{off}} e^{\Lambda(\beta)N}. \tag{3.20}$$

We can then define open sites inductively on I. The site $(0,0)$ is called *open* if

$$Z^{\beta,u,q}_N(\Omega^{\text{up}}_N) \ge U_{\text{off}} \text{ and } Z^{\beta,u,q}_N(\Omega^{\text{forward}}_N) \ge U_{\text{on}},$$

otherwise $(0,0)$ is *closed*. Assume that all the sites (K,L), for $0 \le K < I$ and $0 \le L \le K$ have been defined as open or closed. Then the site $(I,0)$ is *open* if

$$Z^{\beta,u,q}_{N,\tilde{\nu}^q_{(I,0)}}(\Omega_N^{\text{up}}, \theta_{IN,0}(V)) \ge U_{\text{off}} \quad \text{and} \quad Z^{\beta,u,q}_{N,\tilde{\nu}^q_{(I,0)}}(\Omega_N^{\text{forward}}, \theta_{IN,0}(V)) \ge U_{\text{on}},$$

and the site $(I,J),\ 0 < J \le I$, is *open* if

$$Z^{\beta,u,q}_{N,\tilde{\nu}^q_{(I,J)}}(\Omega_N^{\text{g}}, \theta_{IN,JN}(V)) \ge U_{\text{off}}, \quad \text{g = up, forward, down,} \tag{3.21}$$

otherwise (I,J) is *closed*. Note the inductive definition is necessary because the previously defined open/closed values determine the optimal path from $(0,0)$ to (I,J), which determines $\tilde{\nu}^q_{(I,J)}$. Let $X_{(I,J)} = 1_{\{(I,J) \text{ is open}\}}$.

3.5 Second Moment Method and Probability of an Open Site.

We will use the second moment method to show the probability of a closed site is small. In general, for Y a random variable with finite mean and variance, and $\theta, \epsilon \in (0,1)$, by Chebychev's Inequality we have

$$P((1-\theta)EY \le Y \le (1+\theta)EY) \ge 1-\epsilon, \tag{3.22}$$

provided that

$$\frac{Var(Y)}{(EY)^2} \le \theta^2 \epsilon. \tag{3.23}$$

Hence for a site $(I,0)$ on the x-axis, applying (3.22) and (3.23) with $\theta = 1/2$ we see that, Q-a.s.,

$$Q(X_{(I,0)} = 1 | \mathcal{F}_I) \ge 1-\epsilon, \tag{3.24}$$

provided

$$\frac{Var_Q\Big(Z^{\beta,u,q}_{N,\tilde{\nu}^q_{(I,0)}}(\Omega_N^{\text{g}}, \theta_{IN,0}(V)) \;\big|\; \mathcal{F}_I\Big)}{\Big(E^Q\Big(Z^{\beta,u,q}_{N,\tilde{\nu}^q_{(I,0)}}(\Omega_N^{\text{g}}, \theta_{IN,0}(V)) \;\big|\; \mathcal{F}_I\Big)\Big)^2} \le \frac{\epsilon}{8}, \quad \text{g = forward, up.} \tag{3.25}$$

Similarly, for (I,J) with $J \ge 1$, we see that, Q-a.s.,

$$Q(X_{(I,J)} = 1 | \mathcal{F}_I) \ge 1-\epsilon, \tag{3.26}$$

provided

$$\frac{Var_Q\Big(Z^{\beta,u,q}_{N,\tilde{\nu}^q_{(I,J)}}(\Omega_N^{\text{g}}, \theta_{IN,JN}(V)) \;\big|\; \mathcal{F}_I\Big)}{\Big(E^Q\Big(Z^{\beta,u,q}_{N,\tilde{\nu}^q_{(I,J)}}(\Omega_N^{\text{g}}, \theta_{IN,JN}(V)) \;\big|\; \mathcal{F}_I\Big)\Big)^2} \le \frac{\epsilon}{12}, \quad \text{g = up, forward, down.} \tag{3.27}$$

For SSRW paths s^1 and s^2, we have

$$E^Q\left(e^{\beta H_N(s^1)+\beta u L_N(s^1)} e^{\beta H_N(s^2)+\beta u L_N(s^2)}\right) = e^{\beta u L_N(s^1)} e^{\beta u L_N(s^2)} e^{\Phi(\beta) B_N(s^1,s^2)} e^{2\Lambda(\beta) N}. \tag{3.28}$$

Recall $N = k_0 M$. Using (3.3), (3.9), the Cauchy-Schwartz inequality and the fact that $(t-1)^2 \le t^2 - 1$ for $t \ge 1$, for all (I,J) we get Q-a.s.

$$\begin{aligned}
&Var_Q\Big(Z^{\beta,u,q}_{N,\tilde\nu^q_{(I,J)}}(\Omega^{\mathrm{g}}_N,\theta_{IN,JN}(V))|\mathcal{F}_I\Big)\\
&= e^{2\Lambda(\beta)N}\sum_{x,x'\in R(I,J)}\tilde\nu^q_{(I,J)}(x)\tilde\nu^q_{(I,J)}(x')\\
&\qquad\qquad\cdot E^{\otimes 2}_{(x,x')}\Big(\Big(e^{\Phi(\beta)B_N(S^1,S^2)}-1\Big)e^{\beta u L_N(S^1)}e^{\beta u L_N(S^2)}1_{\Omega^{\mathrm{g}}_N\times\Omega^{\mathrm{g}}_N}\Big)\\
&\le e^{2\Lambda(\beta)N}\sum_{x,x'\in R(I,J)}\Big[\tilde\nu^q_{(I,J)}(x)\tilde\nu^q_{(I,J)}(x')\Big(E^{\otimes 2}_{(x,x')}\Big(e^{2\Phi(\beta)B_N(S^1,S^2)}-1\Big)\Big)^{1/2}\\
&\qquad\qquad\cdot\Big(E_x e^{2\beta u L_N(S^1)}\Big)^{1/2}\Big(E_{x'}e^{2\beta u L_N(S^2)}\Big)^{1/2}\Big]\\
&\le e^{2\Lambda(\beta)N}\bigg(\Big(E^{\otimes 2}_{(0,0)}e^{2\Phi(\beta)(B_M(S^1,S^2)+1)}\Big)^{k_0}-1\bigg)^{1/2}\Big(E_0 e^{2\beta u(L_M+1)}\Big)^{k_0}\\
&= e^{2\Lambda(\beta)N}\bigg(\Big(E^{\otimes 2}_{(0,0)}\Big(e^{2\Phi(\beta)(B_M(S^1,S^2)+1)}-1\Big)+1\Big)^{k_0}-1\bigg)^{1/2}\Big(E_0 e^{2\beta u(L_M+1)}\Big)^{k_0}.
\end{aligned}\tag{3.29}$$

For the denominator, by Lemma 3.3, for some $K_6 > 0$, Q-a.s.

$$\begin{aligned}
&E^Q\Big(Z^{\beta,u,q}_{N,\tilde\nu^q_{(I,J)}}(\Omega^{\mathrm{g}}_N,\theta_{IN,JN}(V))|\mathcal{F}_I\Big)\\
&= \sum_{x\in R(I,J)}\tilde\nu^q_{(I,J)}(x)E^Q\bigg(E_x\Big[e^{\beta\sum_{k=1}^N\big(v(IN+k,S_k)+u1_{S_k=0}\big)}1_{\Omega^{\mathrm{g}}_N}\Big]\bigg)\\
&\ge e^{\Lambda(\beta)N}\sum_{x\in R(I,J)}\tilde\nu^q_{(I,J)}(x)P_x\left(\Omega^{\mathrm{g}}_N\right)\\
&\ge e^{\Lambda(\beta)N}K_6.
\end{aligned}\tag{3.30}$$

By Proposition 3.2, we have $M = M(\beta u) \le 5M(2\beta u)$ for small βu. Therefore by Lemma 3.1, for K_2, K_3 from that lemma,

$$E_0 e^{2\beta u(L_M+1)} \le 6K_2 e^{5K_3} =: K_7.$$

Combining this with (3.29) and (3.30) we obtain that the left side of (3.27) is bounded by

$$K_6^{-2}K_7^{k_0}\bigg(\Big(E^{\otimes 2}_{(0,0)}\Big[e^{2\Phi(\beta)(B_M(S^1,S^2)+1)}-1\Big]+1\Big)^{k_0}-1\bigg)^{1/2}.\tag{3.31}$$

Hence for our given $0<\epsilon<1$, we wish to apply Proposition 3.8 with

$$R = M = \frac{1}{\mathrm{F}(\beta u)},\quad a = \Big(\frac{K_6^4\epsilon^2}{12^2K_7^{2k_0}}+1\Big)^{1/k_0}-1;\tag{3.32}$$

since $0 < K_6 < 1$ and $K_7 > 1$, we indeed have $a < 1$ as needed. From Proposition 3.2(b), for β small, provided $u \ge (2/K_4K_5(a))^{1/2}\beta$ we have $R \le K_5\beta^{-4}$, so Proposition 3.8 does apply. We then obtain from (3.31) that the left side of (3.27) (and also of (3.25)) is bounded by $\epsilon/12$. Thus (3.24) and (3.26) hold, for β and βu small.

3.6 Lipschitz Percolation

Lipschitz percolation, the existence of open Lipschitz surfaces, was first introduced and studied in [14] and [19]. In this section, we briefly summarize and adapt some of their results for dimension $d = 2$, to use in our context.

The independent site percolation model in $\mathbb{Z}^2$ is obtained by independently designating each site $x \in \mathbb{Z}^2$ *open* with probability p, otherwise *closed*. The corresponding probability measure on the sample space $\Omega = \{0,1\}^{\mathbb{Z}^2}$ will be denoted by $\mathbb{P}_p$, and expectation by $\mathbb{E}_p$.

Let $\mathbb{Z}_0^+ = \{0,1,2,3,\dots\}$. A function $\mathcal{L} : \mathbb{Z} \to \mathbb{Z}_0^+$ is called *Lipschitz* if for all $x, y \in \mathbb{Z}$ with $|x-y| = 1$, we have $|\mathcal{L}(x) - \mathcal{L}(y)| \leq 1$. $\mathcal{L}$ is called *open* if for each $x \in \mathbb{Z}$, the site $(x, \mathcal{L}(x)) \in \mathbb{Z}^2$ is open.

Remark 3.9. In [14] and [19], it was assumed that $\mathcal{L} \geq 1$, but here it is more convenient to consider $\mathcal{L}(\cdot) \geq 0$, which of course does not change the results.

Let A_{Lip} be the event that there exists an open Lipschitz function $\mathcal{L} : \mathbb{Z} \to \mathbb{Z}_0^+$. Since A_{Lip} is invariant under horizontal translation, we have $\mathbb{P}_p(A_{Lip}) = 0$ or 1. Since A_{Lip} is also an increasing event, there exists a $p_L \in [0,1]$ such that

$$\mathbb{P}_p(A_{Lip}) = \begin{cases} 0 & \text{if } p < p_L, \\ 1 & \text{if } p > p_L. \end{cases}$$

It was proved in [14] that $0 < p_L < 1$ for general dimension, but for the present 2-dimensional case Lipschitz percolation is a special type of oriented percolation, so standard contour arguments similar to ([15] Section 10) suffice to show $p_L < 1$. For any family $\mathcal{F}$ of Lipschitz functions, the lowest function

$$\bar{\mathcal{L}}(x) = \inf\{\mathcal{L}(x) : \mathcal{L} \in \mathcal{F}\}$$

is also Lipschitz. Hence if there exists an open Lipschitz function, then there exists a *lowest open Lipschitz function*, and it will be again denoted by $\mathcal{L}$. From [14], $(\mathcal{L}(x) : x \in \mathbb{Z})$ is stationary and ergodic.

Let D be the set of all $x \in \mathbb{Z}$ for which $\mathcal{L}(x) > 0$. Let D_0 be the connected component of 0 in D, where connectedness is via adjacency in $\mathbb{Z}$. We define $D_0 = \emptyset$ if $0 \notin D$.

Theorem 3.10. *([14],[19]) Let $\mathcal{L}$ be the lowest open Lipschitz function. For $p > p_L$, there exists $\alpha = \alpha(p) > 0$ such that*

$$\mathbb{P}_p(\mathcal{L}(0) > n) \leq e^{-\alpha(n+1)},\ n > 0.$$

There exists $p'_L < 1$ such that for $p \geq p'_L$

$$\exp(-\lambda n) \leq \mathbb{P}_p(|D_0| \geq n) \leq \exp(-\gamma n),\ n \geq 1,$$

where $\lambda = \lambda(p)$ and $\gamma = \gamma(p)$ are positive and finite.

Remark 3.11. By Theorem 3.10, if the random field X stochastically dominates independent site percolation of a sufficiently high density, then with positive probability there exists an infinite good path starting from $(0,0)$ in $\mathbb{L}_{CG}$.

By Theorem 3.10, for $p \geq p'_L$ and $n \geq 1$ we have

$$\begin{aligned} 1 - \mathbb{P}_p(\mathcal{L}(0) = \mathcal{L}(1) = 0) &\leq \mathbb{P}_p(|D_0| > n) + \mathbb{P}_p((i,0) \text{ is closed for some } i \in (-n,n)) \\ &\leq e^{-\gamma(p)n} + (2n-1)(1-p). \end{aligned} \tag{3.33}$$

We may assume $\gamma(p)$ is nondecreasing in p. Then given $\epsilon > 0$, we can first apply (3.33) with $p = p'_L$, and choose n large enough so $e^{-\gamma(p'_L)n} < \epsilon/2$. Then for p sufficiently close to 1, both terms on the right side of (3.33) are bounded by $\epsilon/2$, so by the ergodic theorem,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} 1_{(\mathcal{L}(i-1)=\mathcal{L}(i)=0)} = \mathbb{P}_p(\mathcal{L}(0) = \mathcal{L}(1) = 0) > 1 - \epsilon, \quad \mathbb{P}_p - \text{a.s.} \tag{3.34}$$

3.7 Stochastic Domination

To obtain the domination referenced in Remark 3.11 we will need the following result of Liggett, Schonmann and Stacey [29].

Theorem 3.12. *Let $(X_s)_{s\in\mathbb{Z}}$ be a collection of 0-1 valued k-dependent random variables, and suppose that there exists a $p \in (0,1)$ such that for each $s \in \mathbb{Z}$*

$$\mathbf{P}(X_s = 1) \geq p.$$

Then if

$$p > 1 - \frac{k^k}{(k+1)^{k+1}},$$

then $(X_s)_{s\in\mathbb{Z}}$ is dominated from below by a product random field with density $0 < \rho(p) < 1$. Furthermore, $\rho(p) \to 1$ as $p \to 1$.

Fix $\epsilon > 0$ and choose $p < 1$ so that an open Lipschitz function exists a.s. and (3.34) holds. Then choose η with $\rho(1-\eta) > p$ (with $\rho(\cdot)$ from Theorem 3.12.) For fixed $I \geq 1$, the boxes $B(I,J), B(I,J')$ are disjoint for $|J - J'| > 4$, so conditionally on $\mathcal{F}_I$, $\{X_{(I,J)} : 0 \leq J \leq I\}$ is a 4-dependent collection of random variables. From (3.24) and (3.26), for sufficiently small $\beta u > 0$ and $\beta > 0$ with $u \geq K_8(\eta)\beta$,

$$Q(X_{(I,J)} = 1 | \mathcal{F}_I) \geq 1 - \eta \quad Q - a.s. \text{ for each } I \geq 1, J \geq 0.$$

We can apply Theorem 3.12 inductively on I to see that there exists a collection of i.i.d. 0-1 valued random variables $\{Y_{(I,J)} : (I,J) \in \mathbb{L}_{CG}\}$ with $Q(Y_{(I,J)} = 1) = \rho(1-\eta)$ and

$$Q(X_{(I,J)} \geq Y_{(I,J)} | \mathcal{F}_I) = 1 \;\; Q - a.s. \tag{3.35}$$

and therefore also unconditionally, $X(I,J) \geq Y(I,J)$ a.s. It follows that the configurations $\{X_{(I,J)} : (I,J) \in \mathbb{L}_{CG}\}$ and $\{Y_{(I,J)} : (I,J) \in \mathbb{L}_{CG}\}$ also a.s. have lowest open Lipschitz functions $\mathcal{L}_X \leq \mathcal{L}_Y$. With positive probability we have $\mathcal{L}_X(0) = 0$, in which case $\mathcal{L}_X = \Gamma^{G,\infty}$ is the infinite good path from $(0,0)$.

3.8 Final Steps

Let

$$R_L := \sum_{I=1}^{L} 1_{\{\Gamma^{G,\infty}(I-1) = \Gamma^{G,\infty}(I) = 0\}}.$$

Since $\Gamma^{G,\infty} \leq \mathcal{L}_X \leq \mathcal{L}_Y$ on $\mathbb{Z}_0^+$, it follows from (3.34) applied to $\mathcal{L}_Y$ that when $\Gamma^{G,\infty}$ exists,

$$\alpha = \alpha(\beta u) := \liminf_{L\to\infty} \frac{R_L}{L} > 1 - \epsilon. \tag{3.36}$$

Recall that

$$U_{\text{off}} = \Theta_{\text{off}} e^{\Lambda(\beta)N}, \quad U_{\text{on}} = \Theta_{\text{on}} e^{(\Lambda(\beta) + (1-\epsilon)\mathrm{F}(\beta u))N},$$

where

$$\Theta_{\text{on}} = \Theta_{\text{on}}(\epsilon) = \frac{\epsilon_0}{4}\mathbf{P}(\xi \geq \frac{1}{4\sqrt{\epsilon}}) \sim \frac{\epsilon_0\sqrt{\epsilon}}{\sqrt{2\pi}}\, e^{-1/32\epsilon} \text{ as } \epsilon \to 0 \tag{3.37}$$

and Θ_{off} is the minimum of Θ_{on} and a constant. Define $\Theta_0 = \Theta_0(\epsilon) = -(\alpha \log \Theta_{\text{on}} + (1-\alpha)\log \Theta_{\text{off}}) > 0$. For some $K_9 > 0$ we have

$$\Theta_0(\epsilon) \leq \frac{K_9}{\epsilon}, \quad \epsilon \in (0,1). \tag{3.38}$$

For $L \geq 1$ when an infinite good path from $(0,0)$ exists we have

$$\frac{1}{LN}\log Z_{LN}^{\beta,u,q} \geq \frac{1}{LN}\log Z_{LN}^{\beta,u,q}(\Omega_{0\to L}^{G,\infty})$$

and using (3.17),

$$Z_{LN}^{\beta,u,q}(\Omega_{0\to L}^{G,\infty}) = \prod_{I=1}^{L} \frac{Z_{IN}^{\beta,u,q}(\Omega_{0\to I}^{G,\infty})}{Z_{(I-1)N}^{\beta,u,q}(\Omega_{0\to I-1}^{G,\infty})} = \prod_{I=1}^{L} Z_{N,\tilde{\nu}^q_{(I-1,\Gamma^{G,\infty}(I-1))}}^{\beta,u,q}(\Omega_{I-1\to I}^{G,\infty}, \theta_{I-1,\Gamma^{G,\infty}(I-1)}V) \tag{3.39}$$

where $Z_0^{\beta,u,q} := 1$. Note that (3.17) also guarantees that the measures $\tilde{\nu}^q_{(I-1,\Gamma^{G,\infty}(I-1))}$ on the right side of (3.39) are the ones used in the definition of open/closed coarse-grained sites.

Let $p_{0,\infty} > 0$ be the probability that there is an infinite good path from $(0,0)$ in the configuration X. When such a path exists, by (3.39) we have for all $L \geq 1$

$$Z_{LN}^{\beta,u,q} \geq Z_{LN}^{\beta,u,q}(\Omega_{0\to L}^{G,\infty}) \geq U_{\text{on}}^{R_L} U_{\text{off}}^{L-R_L}. \tag{3.40}$$

Therefore

$$Q\Big(\frac{1}{LN}\log Z_{LN}^{\beta,u,q} \quad \geq \quad \frac{1}{LN}\log U_{\text{on}}^{R_L} U_{\text{off}}^{L-R_L} \text{ for all } L \geq 1\Big) \geq p_{0,\infty}.$$

Since the quenched free energy is self-averaging, recalling $N = k_0 M = k_0/\mathrm{F}(\beta u)$, $f^a(\beta,u) = \mathrm{F}(\beta u) + \Lambda(\beta)$ and $U_{\text{off}} \leq U_{\text{on}}$, using (3.38) we get

$$\begin{aligned} f^q(\beta,u) &\geq \alpha \frac{1}{N}\log U_{\text{on}} + (1-\alpha)\frac{1}{N}\log U_{\text{off}} \\ &= \alpha\left((1-\epsilon)\mathrm{F}(\beta u) + \Lambda(\beta)\right) - \frac{1}{N}\Theta_0 + (1-\alpha)\Lambda(\beta) \\ &\geq \Lambda(\beta) + \alpha(1-\epsilon)\mathrm{F}(\beta u) - \frac{K_9}{k_0\epsilon}\mathrm{F}(\beta u). \end{aligned} \tag{3.41}$$

By choosing $k_0 = \lfloor K_9\epsilon^{-2} + 1\rfloor$, we make the third term on the right side of (3.41) greater than $-\epsilon\mathrm{F}(\beta u)$. This and (3.36) show that

$$f^q(\beta,u) \geq \Lambda(\beta) + (1-3\epsilon)\mathrm{F}(\beta u) > \Lambda(\beta) = f^a(\beta,0) \geq f^q(\beta,0), \tag{3.42}$$

proving (1.7) and (1.8).

4 Proof of Theorem 1.3.

We describe here the necessary modifications to the proof of Theorem 1.2. We need only prove the upper bound, as the lower bound is proved in [27].

In place of separate "on" and "off" constants, we use simply (cf. (3.20))

$$\Theta = \frac{1}{4}\min(A^{\text{forward}}, A^{\text{up}}, A^{\text{down}}), \quad U := \Theta e^{\Lambda(\beta)N}.$$

A site (I,J) is now called *open* if (cf. (3.21))

$$Z_{N,\tilde{\nu}^q_{(I,J)}}^{\beta,u,q}(\Omega_N^{\mathrm{g}}, \theta_{IN,JN}(V)) \geq U, \quad \mathrm{g} = \text{up, forward, down}. \tag{4.1}$$

Given $\epsilon > 0$ we obtain a as in (3.32) and then $K_5(a)$ from Proposition 3.8, and take $\tilde{M} = K_5(a)\beta^{-4}$. We otherwise repeat the proof of Theorem 1.2 but with $u = 0$ and $\tilde{M}$ in place of M throughout, and $k_0 = 1$ so that $N = \tilde{M}$. The density of open sites can be made arbitrarily close to 1 by taking ϵ small, and then there is a positive probability that an infinite good path exists. In that case we have the lower bound (cf. (3.40))

$$Z_{LN}^{\beta,u,q} \geq U^L, \quad L \geq 1,$$

which as in (3.41) yields

$$f^q(\beta,0) \geq \frac{1}{N}\log U = f^a(\beta,0) - \frac{1}{N}\log\frac{1}{\Theta} \geq f^a(\beta,0) - C_2\beta^4,$$

concluding the proof.

References

[1] K. S. Alexander, *The effect of disorder on polymer depinning transitions*, Commun. Math. Phys **279** (2008), 117–146. MR-2377630

[2] L. Balents and M. Kardar, *Delocalization of flux lines from extended defects by bulk randomness*, Europhys. Lett. **23** (1993), 503–509.

[3] R. Basu, V. Sidoracicius, and A. Sly, Last passage percolation with a defect line and the solution of the slow bond problem. (2014) arXiv:1408.3464v2

[4] V. Beffara, V. Sidoravicius, H. Spohn, and M. E. Vares, *Polymer pinning in a random medium as influence percolation*, Dynamics and Stochastics, IMS Lecture Notes Monogr. Ser. 48, Inst. Math. Statist., Beachwood, OH, 2006. MR-2306183

[5] V. Beffara, V. Sidoravicius, and M. E. Vares, *Randomized polynuclear growth with a columnar defect*, Probab. Theory Rel. Fields **147** (2010), no. 3-4, 565–581. MR-2639715

[6] E. Bolthausen, *A note on the diffusion of directed polymer in a random environment.*, Commun. Math. Phys. **123** (1989), no. 4, 529–534. MR-1006293

[7] R. C. Budhani, M. Swenaga, and S. H. Liou, *Giant suppression of flux-flow resistivity in heavy-ion irradiated* $\mathrm{TL_2BA_2Ca_2Cu_3O_{10}}$ *films: Influence of linear defects on vortex transport*, Phys. Rev. Lett. **69** (1992), 3816–3819.

[8] J. M. Burgers, *Summation of series of fractions depending upon the roots of the airy function*, For Dirk Struik, Boston Stud. Philos. Sci. XV., 1974, pp. 15–19. MR-0774251

[9] P. Carmona and Y. Hu, *On the partition function of a directed polymer in a random environment*, Probab. Theory Rel. Fields **124** (2002), 431–457. MR-1939654

[10] L. Civale, A. D. Marwick, T. K. Worthington, M. A. Kirk, J. R. Thompson, L. Krusin-Elbaum, Y. Sum, J. R. Clem, and F. Holtzberg, *Vortex confinement by columnar defects in* $\mathrm{YBa_2Cu_3O_7}$ *crystals: Enhanced pinning at high fields and temperatures*, Phys. Rev. Lett. **67** (1991), 648–652.

[11] F. Comets and Yoshida N., *Directed polymers in a random environment are diffusive at weak disorder*, Ann. Probab. **34** (2006), no. 5, 1746–1770. MR-2271480

[12] F. Comets, T. Shiga, and N. Yoshida, *Directed polymers in random environment: Path localization and strong disorder*, Bernoulli **9** (2003), 705–723. MR-1996276

[13] F. Comets, T. Shiga, and N. Yoshida, *Probabilistic analysis of directed polymers in a random environment: A review*, Stochastic Analysis on Large Scale Interacting Systems, Math. Soc. Japan, Tokyo, 2004, pp. 115–142. MR-2073332

[14] N. Dirr, P. W. Dondl, G. R. Grimmett, A. E. Holroyd, and M. Scheutzow, *Lipschitz percolation*, Electron. Comm. Probab. **15** (2010), 14–21. MR-2581044

[15] R. Durrett, *Oriented percolation in two dimensions*, Ann. Probab. **12** (1984), 999–1040. MR-0757768

[16] W. Feller, *An introduction to probability theory and its applications*, 3rd ed., vol. 1, Wiley, New York, 1968. MR-0228020

[17] G. Giacomin, *Random polymer models*, Imperial College Press, London, 2007. MR-2380992

[18] G. Giacomin, *Disorder and critical phenomena through basic probability models, École d'Été de Probabilités de Saint-Flour XL*, Lecture Notes in Mathematics, vol. 2025, Springer, 2011. MR-2816225

[19] G. R. Grimmett and A. E. Holroyd, *Geometry of lipschitz percolation*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques **48** (2012), 309–326. MR-2954256

[20] A. Hansen, E. L. Hinrichsen, and S. Roux, *Roughness of crack interfaces*, Phys. Rev. Lett. **66** (1991), 2476–2479.

[21] D. A. Huse and C. L. Henley, *Pinning and roughening of domain wall in ising systems due to random impurities*, Phys. Rev. Lett. **54** (1985), 2708–2711.

[22] T. Hwa and T. Nattermann, *Disorder-induced depinning transition*, Phys. Rev. B **51** (1995), no. 1, 455–469.

[23] T. Imbrie, J. Z.; Spencer, *Diffusion of directed polymers in a random environment*, J. Statist. Phys. **52** (1988), no. 3-4, 609–626.

[24] M. Kardar, *Depinning by quenched randomness*, Phys. Rev. Lett **55** (1985), 2235–2238.

[25] M. Kardar, G. Parisi, and Y.-C. Zhang, *Dynamic scaling of growing interfaces.*, Phys. Rev. Lett. **56** (1986), no. 9, 889–892.

[26] J. F. C. Kingman, *The ergodic theory of subadditive processes*, J. Royal Stat. Soc. B **30** (1968), 499–510. MR-0254907

[27] H. Lacoin, *New bounds for the free energy of directed polymers in dimension* $1+1$ *and* $1+2$, Comm. Math. Phys. **294** (2010), no. 2, 471–503. MR-2579463

[28] E. Lesigne and D. Volný, *Large deviations for martingales*, Stoch. Proc. Appl. **96** (2001), 143–159. MR-1856684

[29] T. M. Liggett, R. H. Schonmann, and A. M. Stacey, *Domination by product measures*, Ann. Probab. **25** (1997), 71–95. MR-1428500

[30] D. R. Nelson, *Vortex entanglement in high-*$\mathrm{t_c}$ *superconductors*, Phys. Rev. Lett. **60** (1988), 1973–1976.

[31] L. H. Tang and I. F. Lyuksyutov, *Directed polymer localization in a disordered medium*, Phys. Rev. Lett. **71** (1993), 2745–2748.

[32] F. L. Toninelli, *Localization transition in disordered pinning models*, Methods of Contemporary Mathematical Statistical Physics, Lecture Notes in Mathematics 1970, Springer, Berlin, 2009, pp. 129–176. MR-2581605

[33] S. R. S. Varadhan, Personal communication. (2013)

[34] F. Watbled, *Sharp asymptotics for the free energy of* $1+1$ *dimensional directed polymers in an infinitely divisible environment*, Electron. Commun. Probab. **17** (2012), no. 53, 1–9. MR-2999981

Acknowledgments. The authors would like to thank S. R. S. Varadhan for the proof of Lemma 3.4, and an anonymous referee who pointed out that the proof of Theorem 1.2 also yielded Theorem 1.3.