Electronic Journal of Probability

Electron. J. Probab. **20** (2015), no. 59, 1–19.
ISSN: 1083-6489 DOI: 10.1214/EJP.v20-3077

Poisson-Dirichlet statistics for the extremes of the two-dimensional discrete Gaussian free field*

Louis-Pierre Arguin[†] Olivier Zindy[‡]

Abstract

In a previous paper, the authors introduced an approach to prove that the statistics of the extremes of a log-correlated Gaussian field converge to a Poisson-Dirichlet variable at the level of the Gibbs measure at low temperature and under suitable test functions. The method is based on showing that the model admits a one-step replica symmetry breaking in spin glass terminology. This implies Poisson-Dirichlet statistics by general spin glass arguments. In this note, this approach is used to prove Poisson-Dirichlet statistics for the two-dimensional discrete Gaussian free field, where boundary effects demand a more delicate analysis.

Keywords: Gaussian free field; Gibbs measure; Poisson-Dirichlet variable; Spin glasses.
AMS MSC 2010: Primary 60G15; 60F05, Secondary 82B44; 60G70; 82B26.
Submitted to EJP on October 17, 2013, final version accepted on May 31, 2015.
Supersedes arXiv:1310.2159.

1 Introduction

1.1 The model

Consider a finite box A of $\mathbb{Z}^2$. The Gaussian free field (GFF) on A with Dirichlet boundary condition is the centered Gaussian field $(\phi_v, v \in A)$ with the covariance matrix

$$G_A(v,v') := E_v\left[\sum_{k=0}^{\tau_A} 1_{v'}(S_k)\right] , \tag{1.1}$$

where $(S_k, k \geq 0)$ is a simple random walk with $S_0 = v$ of law P_v killed at the first exit time of A, τ_A, i.e. the first time where the walk reaches the boundary ∂A. Throughout the paper, for any $A \subset \mathbb{Z}^2$, ∂A will denote the set of vertices in A^c that share an edge with a vertex of A. We will write $\mathbb{P}$ for the law of the Gaussian field and $\mathbb{E}$ for the expectation. For $B \subset A$, we denote the σ-algebra generated by $\{\phi_v, v \in B\}$ by $\mathcal{F}_B$.

*L.-P. A. is supported by a NSERC discovery grant and a grant FQRNT *Nouveaux chercheurs*. O.Z. is partially supported by the french ANR project MEMEMO2 2010 BLAN 0125.

[†]Université de Montréal, Canada; City University of New York (Baruch College and Graduate Center), United States. E-mail: louis-pierre.arguin@baruch.cuny.edu

[‡]Université Pierre et Marie Curie, France. E-mail: olivier.zindy@upmc.fr

We are interested in the case where $A = V_N := \{1, \dots, N\}^2$ in the limit $N \to \infty$. For $0 \le \delta < 1/2$, we denote by V_N^δ the set of the points of V_N whose distance to the boundary ∂V_N is greater than δN. In this set, the variance of the field diverges logarithmically with N (cf. Lemma 2.1 in [13])

$$\mathbb{E}[\phi_v^2] = G_{V_N}(v,v) = \frac{1}{\pi}\log N^2 + O_N(1), \qquad \forall v \in V_N^\delta, \tag{1.2}$$

where $O_N(1)$ will always be a term which is uniformly bounded in N and in $v \in V_N^\delta$. (The term $o_N(1)$ will denote a term which goes to 0 as $N \to \infty$ uniformly in all other parameters.) In addition, there exists $c_0 > 0$ independent of N such that (see Lemma 1 in [7])

$$\sup_{v \in V_N} \mathbb{E}[\phi_v^2] \le \frac{1}{\pi}\log N^2 + c_0. \tag{1.3}$$

In V_N^δ, the covariances can be estimated by (cf. Lemma 2.1 in [13])

$$\mathbb{E}[\phi_v \phi_{v'}] = G_{V_N}(v,v') = \frac{1}{\pi}\log\frac{N^2}{\|v - v'\|^2} + O_N(1), \qquad \forall v, v' \in V_N^\delta, \tag{1.4}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{Z}^2$. In view of (1.2) and (1.4), the Gaussian field $(\phi_v, v \in V_N)$ is said to be *log-correlated*. On the other hand, there are many points that are outside V_N^δ (of the order of N^2 points) for which the estimates (1.2) and (1.4) are not correct. Essentially, the closer the points are to the boundary the lesser are the variance and covariance as the simple random walk in (1.1) has a higher probability of exiting V_N early. This decoupling effect close to the boundary complicates the analysis of the extrema of the GFF by comparison with log-correlated Gaussian fields with stationary distribution.

1.2 Main results

It was shown by Bolthausen, Deuschel, and Giacomin [7] that the maximum of the GFF in V_N satisfies

$$\lim_{N\to\infty} \frac{\max_{v \in V_N} \phi_v}{\log N^2} = \sqrt{\frac{2}{\pi}}, \qquad \text{in probability.} \tag{1.5}$$

Their technique was later refined by Daviaud [13] who computed the *log-number of high points* in V_N^δ: for $0 < \lambda < 1$,

$$\lim_{N\to\infty} \frac{1}{\log N^2} \log \#\{v \in V_N^\delta : \phi_v \ \ge\ \lambda\sqrt{\frac{2}{\pi}}\log N^2\} = 1 - \lambda^2, \qquad \text{in probability.} \tag{1.6}$$

It is a simple exercise using Laplace's method to show that the *free energy* in V_N of the model is given by

$$f(\beta) := \lim_{N\to\infty} \frac{1}{\log N^2} \log \sum_{v \in V_N} e^{\beta\phi_v} = \begin{cases} 1 + \frac{\beta^2}{2\pi}, & \text{if } \beta \le \sqrt{2\pi}, \\ \sqrt{\frac{2}{\pi}}\beta, & \text{if } \beta \ge \sqrt{2\pi}, \end{cases} \tag{1.7}$$

in probability and in L^1. Indeed, there is the clear lower bound $\log \sum_{v \in V_N} e^{\beta\phi_v} \ge \log \sum_{v \in V_N^\delta} e^{\beta\phi_v}$, which can be evaluated using the estimate on the log-number of high points (1.6). The upper bound follows from a first moment calculation on the number of high points.

A striking fact is that the three above results correspond to the expressions for N^2 independent Gaussian variables of variance $\frac{1}{\pi}\log N^2$. In other words, correlations have no effects on the above observables of the extremes. The purpose of the paper is to extend this correspondence to observables related to the Gibbs measure.

To this aim, consider the *normalized Gibbs weights* or *Gibbs measure*

$$\mathcal{G}_{\beta,N}(\{v\}) := \frac{e^{\beta\phi_v}}{Z_N(\beta)}, \qquad v \in V_N,$$

where $Z_N(\beta) := \sum_{v\in V_N} e^{\beta\phi_v}$. We consider the normalized covariance or *overlap*

$$q(v,v') := \frac{\mathbb{E}[\phi_v\phi_{v'}]}{\frac{1}{\pi}\log N^2 + c_0}, \qquad \forall v,v' \in V_N, \tag{1.8}$$

where c_0 is introduced in Equation (1.3). This is the covariance divided by the uniform upper bound of the variance. Note that $\frac{1}{\pi}\log N^2$ is the dominant term of the variance in the bulk. It is clear that $q(v,v') \in [0,1]$ for any $v,v' \in V_N$.

In spin-glass theory, the relevant object to classify the extreme value statistics of strongly correlated variables is the *two-overlap distribution function*

$$x_{\beta,N}(q) := \mathbb{E}\left[\mathcal{G}_{\beta,N}^{\times 2}\left\{q(v,v') \le q\right\}\right], \qquad 0 \le q \le 1. \tag{1.9}$$

The main result shows that the 2D GFF falls within the class of models that exhibit a *one-step replica symmetry breaking* at low temperature.

Theorem 1.1. *For $\beta > \beta_c = \sqrt{2\pi}$,*

$$\lim_{N\to\infty} x_{\beta,N}(r) = \lim_{N\to\infty} \mathbb{E}\left[\mathcal{G}_{\beta,N}^{\times 2}\left\{q(v,v') \le r\right\}\right] = \begin{cases} \frac{\beta_c}{\beta} & \text{for } 0 \le r < 1, \\ 1 & \text{for } r = 1. \end{cases}$$

Note that for $\beta \le \beta_c$, it follows from (1.7) that the overlap is 0 almost surely. The result is the analogue for the 2D GFF of the results obtained by Derrida & Spohn [15] and Bovier & Kurkova [9, 10] for the branching Brownian motion and for Derrida's *Generalized Random Energy models* (GREM) [14]. In [4], such a result was proved for a non-hierarchical log-correlated Gaussian field constructed from the multifractal random measure of Bacry & Muzy [5], see also [20] for a closely related model. This type of result was conjectured by Carpentier & Le Doussal [12]. We also remark that Theorem 1.1 shows that at low temperature two points sampled with the Gibbs measure have overlaps 0 or 1 in the limit $N \to \infty$. More precisely, Theorem 1.1 implies that, for all $\varepsilon > 0$,

$$\mathcal{G}_{\beta,N}^{\times 2}\left\{\varepsilon < q(v,v') < 1-\varepsilon\right\} \longrightarrow 0, \qquad N \to \infty,$$

and

$$\mathcal{G}_{\beta,N}^{\times 2}\left\{q(v,v') < \varepsilon\right\} \longrightarrow \frac{\beta_c}{\beta}, \qquad N \to \infty,$$

in probability, which is consistent with the result of Ding & Zeitouni [17] who showed that the extremal values of GFF are at distance from each other of order one or of order N.

A general method to prove Poisson-Dirichlet statistics for the distribution of the overlaps from the one-step replica symmetry breaking was laid out in [4]. This connection is done via the Ghirlanda-Guerra identities. Another equivalent approach would be using *stochastic stability* as developed in [1, 2, 3]. The reader is referred to Section 2.3 of [4] where the connection is explained in details for general Gaussian fields. For the sake of conciseness, we simply state the consequence for the 2D GFF.

Consider the product measure $\mathcal{G}_{\beta,N}^{\times s}$ on s *replicas* $(v_1,\dots,v_s)\in V_N^{\times s}$. Let $F:[0,1]^{\frac{s(s-1)}{2}}\to\mathbb{R}$ be a continuous function. Write $F((q_{ll'})_{l<l'})$ for the function evaluated at $q_{ll'}:=q(v_l,v_{l'})$, $l<l'$, for $(v_1,\dots,v_s)\in V_N^{\times s}$. We write $\mathbb{E}\mathcal{G}_{\beta,N}^{\times s}\left[F((q_{ll'})_{l<l'})\right]$ for the averaged expectation. Recall that a *Poisson-Dirichlet variable* ξ of parameter $\kappa\in(0,1)$ is a random variable on the space of decreasing weights $\boldsymbol{s}=(s_1,s_2,\dots)$ with $1\geq s_1\geq s_2\geq\dots\geq 0$ and $\sum_i s_i\leq 1$ which has the same law as $\left(\eta_i/\sum_j\eta_j, i\in\mathbb{N}\right)_{\downarrow}$ where $\downarrow$ stands for the decreasing rearrangement and $\eta=(\eta_i,i\in\mathbb{N})$ are the atoms of a Poisson random measure on $(0,\infty)$ of intensity measure $s^{-\kappa-1}\,ds$.

The theorem below is a direct consequence of the Theorem 1.1, the differentiability of the free energy (1.7) as well as Corollary 2.5 and Theorem 2.6 of [4].

Theorem 1.2. *Let $\beta>\beta_c$ and $\xi=(\xi_k,k\in\mathbb{N})$ be a Poisson-Dirichlet variable of parameter $\beta_c/\beta\in(0,1)$. Denote by E the expectation with respect to ξ. For any continuous function $F:[0,1]^{\frac{s(s-1)}{2}}\to\mathbb{R}$ of the overlaps of s replicas:*

$$\lim_{N\to\infty}\mathbb{E}\mathcal{G}_{\beta,N}^{\times s}\left[F((q_{ll'})_{l<l'})\right]=E\left[\sum_{k_1\in\mathbb{N},\dots,k_s\in\mathbb{N}}\xi_{k_1}\dots\xi_{k_s}\;F((\delta_{k_lk_{l'}})_{l<l'})\right].$$

The above is one of the few rigorous results known on the Gibbs measure of log-correlated fields at low temperature. Theorem 1.2 is a step closer to the conjecture of Duplantier, Rhodes, Sheffield & Vargas (see Conjecture 11 in [18] and Conjecture 6.3 in [25]) that the Gibbs measure, as a random probability measure on V_N, should be atomic in the limit with the size of the atoms being Poisson-Dirichlet. Theorem 1.2 falls short of the full conjecture because only test-functions of the overlaps are considered. Finally, it is expected that the Poisson-Dirichlet statistics emerging here is related to the Poissonian statistics of the thinned extrema of the 2D GFF proved by Biskup & Louidor in [6] based on the convergence of the maximum established by Bramson, Ding & Zeitouni [11]. To recover the Gibbs measure from the extremal process, some properties of the cluster of points near the maxima must be known.

1.3 Outline of the proof

The proof of Theorem 1.1 relies on a technique introduced by Bovier & Kurkova in [10] for GREM's. The idea of the method is to relate the overlap distribution of a given model to the free energy of a perturbed version of the model. The main advantage of the approach is that computing free energies is in general a much simpler task than a direct computation of overlaps.

With this in mind, we define in Section 2 a perturbed version of the GFF that we call the $(\alpha,\boldsymbol{\sigma})$-generalized GFF. The generalization is close in spirit to the GREM and also to a non-hierarchical version introduced by Fyodorov & Bouchaud in [21]. In essence, the parameter $\boldsymbol{\sigma}=(\sigma_1,\sigma_2)$ controls the strength of the perturbation, whereas the parameter α specifies at what scale the perturbation is applied. The pivotal equation in this approach is the following identity relating the overlap distribution x_β of the original model to the limiting free energy $f^{(\alpha,\boldsymbol{\sigma})}(\beta)$ of the perturbed model for $\boldsymbol{\sigma}=\boldsymbol{\sigma}(u)$ depending on a small parameter u:

$$\int_\alpha^1 x_\beta(r)dr=\frac{\pi}{\beta^2}\frac{\partial}{\partial u}f^{(\alpha,\boldsymbol{\sigma})}(\beta)\Big|_{u=0}\,. \tag{1.10}$$

In the case of the 2D GFF, the identity (1.10) is approximate due to the dependence of the covariances on the relative position of the points in the box. To control the effect of the boundary, we need to limit the analysis to a box $A_{N,\rho}$ in V_N containing the points at distance greater than $N^{1-\rho}$ from the boundary for some $\rho > 0$. We have a good control of the overlap as a function of the distance between the points in that box, cf. Lemma 3.4. In Section 3.1, we show that the Gibbs measure samples in $A_{N,\rho}$ with large probability so that the overlap distribution is well approximated by the overlap distribution in $A_{N,\rho}$. In Section 3.2, we derive the relation corresponding to (1.10) for the overlap distribution in $A_{N,\rho}$, cf. Proposition 3.2. The proof of Theorem 1.1 follows from this and the explicit formula for the free energy of the generalized model given in Theorem 2.1. The derivation of this formula is given in Section 4 and is the same as the one of a GREM with two levels.

2 The multiscale decomposition and a generalized GFF

In this section, we construct a Gaussian field from the GFF whose variance is scale-dependent. The construction uses a multiscale decomposition along each vertex. The construction is analogous to a GREM, but where correlations are non-hierarchical. Here, only two different values of the variance will be needed though the construction can be directly generalized to any finite number of values.

Consider $0 < \alpha < 1$. We assume to simplify the notation that $N^{1-\alpha}$ is an even integer and that N^{α} divides N. The case of general α's can also be done by making trivial corrections along the construction.

For $v \in V_N$, we write $[v]_\alpha$ for the unique box with $N^{1-\alpha}$ vertices in $\mathbb{Z}^2$ on each side and centered at v. If $[v]_\alpha$ is not entirely contained in V_N, we take the convention that $[v]_\alpha$ is the intersection of the square box with V_N. For $\alpha = 1$, take $[v]_1 = v$. The σ-algebra $\mathcal{F}_{[v]_\alpha^c}$ is the σ-algebra generated by the field outside $[v]_\alpha$. We define

$$\phi_{[v]_\alpha} := \mathbb{E}\left[\phi_v \;\middle|\; \mathcal{F}_{[v]_\alpha^c}\right] = \mathbb{E}\left[\phi_v \;\middle|\; \mathcal{F}_{\partial[v]_\alpha}\right] \ ,$$

where the second equality holds by the Markov property of the Gaussian free field, see Lemma 5.1. Clearly, for any $v \in V_N$, the random variable $\phi_{[v]_\alpha}$ is Gaussian. Moreover, by Lemma 5.1,

$$\phi_{[v]_\alpha} = \sum_{u \in \partial[v]_\alpha} p_{\alpha,v}(u)\phi_u \ , \tag{2.1}$$

where $p_{\alpha,v}(u) = P_v(S_{\tau_{[v]_\alpha}} = u)$ is the probability that a simple random walk starting at v hits u at the first exit time of $[v]_\alpha$.

The following *multiscale decomposition* holds trivially

$$\phi_v = \phi_{[v]_\alpha} + \left(\phi_v - \phi_{[v]_\alpha}\right) \ , \tag{2.2}$$

where $\phi_{[v]_\alpha}$ and $\phi_v - \phi_{[v]_\alpha}$ are independent. The decomposition suggests the following scale-dependent perturbation of the field. For $0 < \alpha < 1$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2) \in \mathbb{R}_+^2$, consider for $v \in V_N$,

$$\psi_v := \sigma_1 \phi_{[v]_\alpha} + \sigma_2 \left(\phi_v - \phi_{[v]_\alpha}\right) \ . \tag{2.3}$$

The Gaussian field $(\psi_v, v \in V_N)$ will be called the $(\alpha, \boldsymbol{\sigma})$-GFF on V_N.

To control the boundary effects, it is necessary to consider the field in a box slightly smaller than V_N. For $\rho \in (0,1)$, let

$$A_{N,\rho} := \{v \in V_N : d_1(v, \partial V_N) \geq N^{1-\rho}\} \ , \tag{2.4}$$

where $d_1(v, B) := \inf\{\|v - u\| \,;\, u \in B\}$ for any set $B \subset \mathbb{Z}^2$. We always take $\rho < \alpha$ so that $[v]_\alpha$ is a square of side-length $N^{1-\alpha}$ for any $v \in A_{N,\rho}$. We write $\mathcal{G}^{(\alpha,\boldsymbol{\sigma})}_{\beta,N,\rho}(\cdot)$ for the Gibbs measure of $(\alpha, \boldsymbol{\sigma})$-GFF restricted to $A_{N,\rho}$

$$\mathcal{G}^{(\alpha,\boldsymbol{\sigma})}_{\beta,N,\rho}(\{v\}) := \frac{e^{\beta\psi_v}}{Z^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta)}, \qquad v \in A_{N,\rho},$$

where $Z^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta) := \sum_{v\in A_{N,\rho}} e^{\beta\psi_v}$.

The associated free energy is given by

$$f^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta) := \frac{1}{\log N^2} \log Z^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta), \qquad \forall \beta > 0.$$

(Note that $\log \# A_{N,\rho} = (1 + o_N(1)) \log N^2$.) As it will be explained in Section 3.2, the limit of its expectation is central to obtain the overlap distribution of the original model. This limit is better expressed in terms of the free energy of the REM model consisting of N^2 i.i.d. Gaussian variables of variance $\frac{\theta^2}{\pi} \log N^2$ with $\theta > 0$:

$$f(\beta;\theta^2) := \begin{cases} 1 + \frac{\beta^2\theta^2}{2\pi}, & \text{if } \beta \le \beta_c(\theta^2) := \frac{\sqrt{2\pi}}{\theta}, \\ \sqrt{\frac{2}{\pi}}\theta\beta, & \text{if } \beta \ge \beta_c(\theta^2). \end{cases} \tag{2.5}$$

Theorem 2.1. *Fix $\alpha \in (0,1)$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2) \in \mathbb{R}^2_+$ and let $\Gamma_{12} := \sigma_1^2\alpha + \sigma_2^2(1-\alpha)$. Then, for any $0 < \rho < \alpha$, and for all $\beta > 0$*

$$\lim_{N\to\infty} \mathbb{E}\left[f^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta)\right] = f^{(\alpha,\boldsymbol{\sigma})}(\beta) := \begin{cases} f(\beta;\Gamma_{12}), & \text{if } \sigma_1 \le \sigma_2, \\ \alpha f(\beta;\sigma_1^2) + (1-\alpha) f(\beta;\sigma_2^2), & \text{if } \sigma_1 \ge \sigma_2. \end{cases} \tag{2.6}$$

The theorem is proved in Section 4. Note that the limit does not depend on ρ. We expect that Theorem 2.1 also holds for the free energy in the whole box V_N. However, this is not straightforward from our analysis. To do so, one would need to find an upper bound for the field in V_N including vertices close to the boundary. Again, this is complicated by the sensitivity of the Green's function on the boundary. Since this case is not necessary for the result on the overlaps, we have decided to omit it from our treatment.

We observe that the right-hand side of (2.6) is exactly equal to the limiting free energy of a 2-level GREM with the same parameters α and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$. Precisely, let $(X^{(1)}_{v_1}, v_1 \le N^{2\alpha})$ and $(X^{(2)}_{v_1,v_2}; v_1 \le N^{2\alpha}, v_2 \le N^{2(1-\alpha)})$ be i.i.d. centered Gaussian random variables with variance $\log N$. The corresponding 2-level GREM on N^2 points is the Gaussian field of the form

$$X_v = \sigma_1 X^{(1)}_{v_1} + \sigma_2 X^{(2)}_{v_1,v_2}, \qquad v = (v_1, v_2), \tag{2.7}$$

where $\sigma_1, \sigma_2 > 0$. The correlations of this Gaussian field have an exact underlying tree structure with two levels. This is not the case for the $(\alpha, \boldsymbol{\sigma})$-GFF for finite N. However, a tree structure emerges, at least at the level of the free energy, in the limit $N \to \infty$. We refer to [8] for a nice introduction to the 2-level GREM, the study of its maximum, high-points, free energy and Gibbs measure.

3 Proof of Theorem 1.1

3.1 The Gibbs measure close to the boundary

The first step in the proof of Theorem 1.1 is to show that points close to the boundary do not carry any weight in the Gibbs measure of the GFF in V_N. The result would not necessarily hold if we considered instead the complement of V_N^δ which is much larger than the complement of $A_{N,\rho}$.

Lemma 3.1. *For any $\rho > 0$,*

$$\lim_{N\to\infty} \mathcal{G}_{\beta,N}(A^{c}_{N,\rho}) = 0, \qquad \textit{in } \mathbb{P}\textit{-probability.} \tag{3.1}$$

Before turning to the proof, we claim that the lemma implies that, for any $r \in [0,1]$ and $\rho \in (0,1)$,

$$\lim_{N\to\infty} \big|x_{\beta,N}(r) - x_{\beta,N,\rho}(r)\big| = 0 \ , \tag{3.2}$$

where

$$x_{\beta,N,\rho}(r) := \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\{q(v,v') \le r\}, \qquad r \in [0,1] \ . \tag{3.3}$$

is the two-overlap distribution of the Gibbs measure of the GFF $(\phi_v, v \in V_N)$ restricted to $A_{N,\rho}$

$$\mathcal{G}_{\beta,N,\rho}(\{v\}) := \frac{\mathrm{e}^{\beta\phi_v}}{Z_{N,\rho}(\beta)}, \qquad v \in A_{N,\rho},$$

for $Z_{N,\rho}(\beta) := \sum_{v\in A_{N,\rho}} \mathrm{e}^{\beta\phi_v}$. Indeed, introducing an auxiliary term

$$\begin{aligned}\big|x_{\beta,N}(r) - x_{\beta,N,\rho}(r)\big| \le{}& \big|\mathbb{E}\mathcal{G}^{\times 2}_{\beta,N}\big\{q(v,v') \le r\big\} - \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N}\big\{q(v,v') \le r; v,v' \in A_{N,\rho}\big\}\big| \\ &+ \big|\mathbb{E}\mathcal{G}^{\times 2}_{\beta,N}\big\{q(v,v') \le r; v,v' \in A_{N,\rho}\big\} - \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\big\{q(v,v') \le r\big\}\big| \ .\end{aligned}$$

The first term is smaller than $2\ \mathbb{E}\mathcal{G}_{\beta,N}(A^{c}_{N,\rho})$. The second term equals

$$\begin{aligned}&\mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\big\{q(v,v') \le r\big\} - \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N}\big\{q(v,v') \le r; v,v' \in A_{N,\rho}\big\} \\ &\qquad = \mathbb{E}\left[\frac{\mathcal{G}^{\times 2}_{\beta,N}\big\{q(v,v') \le r; v,v' \in A_{N,\rho}\big\}}{\mathcal{G}^{\times 2}_{\beta,N}\big\{v,v' \in A_{N,\rho}\big\}}\Big(1 - \mathcal{G}^{\times 2}_{\beta,N}\big\{v,v' \in A_{N,\rho}\big\}\Big)\right],\end{aligned}$$

which is also smaller than $2\ \mathbb{E}\mathcal{G}_{\beta,N}(A^{c}_{N,\rho})$. Lemma 3.1 then implies (3.2) as claimed.

Proof of Lemma 3.1. Let $\epsilon > 0$ and $\lambda > 0$. The probability can be split as follows

$$\begin{aligned}\mathbb{P}\left(\mathcal{G}_{\beta,N}(A^{c}_{N,\rho}) > \epsilon\right) \le{}& \mathbb{P}\left(\mathcal{G}_{\beta,N}(A^{c}_{N,\rho}) > \epsilon, \left|\frac{1}{\log N^2}\log Z_N(\beta) - f(\beta)\right| \le \lambda\right) \\ &+ \mathbb{P}\left(\left|\frac{1}{\log N^2}\log Z_N(\beta) - f(\beta)\right| > \lambda\right) \ ,\end{aligned}$$

where $f(\beta)$ is defined in (1.7). The second term converges to zero by (1.7). The first term is smaller than

$$\mathbb{P}\left(\frac{1}{\log N^2}\log \sum_{v\in A^{c}_{N,\rho}} \exp\beta\phi_v > f(\beta) - \lambda + \frac{\log\epsilon}{\log N^2}\right) . \tag{3.4}$$

Since the free energy is a Lipschitz function of the variables ϕ_v, see e.g. Theorem 2.2.4 in [26], the free energy self-averages, that is for any $t > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\frac{1}{\log N^2}\log \sum_{v\in A^{c}_{N,\rho}} \exp\beta\phi_v - \frac{1}{\log N^2}\mathbb{E}\left[\log \sum_{v\in A^{c}_{N,\rho}} \exp\beta\phi_v\right]\right| \ge t\right) = 0 \ .$$

To conclude the proof, it remains to show that for some $C < 1$ (independent of N but dependent on ρ)

$$\limsup_{N\to\infty} \frac{1}{\log N^2}\mathbb{E}\left[\log \sum_{v\in A^{c}_{N,\rho}} \exp\beta\phi_v\right] < Cf(\beta). \tag{3.5}$$

Note that by Lemma 5.1, the maximal variance of ϕ_v in V_N is $\frac{1}{\pi}\log N^2 + O_N(1)$. Pick $(g_v, v \in A^c_{N,\rho})$ independent centered Gaussians (and independent of $(\phi_v)_{v\in A^c_{N,\rho}}$) with variance given by $\mathbb{E}[g_v^2] = \frac{1}{\pi}\log N^2 + O_N(1) - \mathbb{E}[\phi_v^2]$. Jensen's inequality applied to the Gibbs measure $(\exp\beta\phi_v/\sum_{u\in A^c_{N,\rho}}\exp\beta\phi_u)_{v\in A^c_{N,\rho}}$ implies that $\mathbb{E}[\log\sum_{v\in A^c_{N,\rho}}\exp\beta(\phi_v + g_v)] \geq \mathbb{E}[\log\sum_{v\in A^c_{N,\rho}}\exp\beta\phi_v]$. Moreover, by a standard comparison argument (see Lemma 5.3 in the Appendix), $\mathbb{E}[\log\sum_{v\in A^c_{N,\rho}}\exp\beta(\phi_v+g_v)]$ is smaller than the expectation for i.i.d. variables with identical variances. The two last observations imply that

$$\frac{1}{\log N^2}\mathbb{E}\left[\log\sum_{v\in A^c_{N,\rho}}\exp\beta\phi_v\right] \leq \frac{1}{\log N^2}\mathbb{E}\left[\log\sum_{v\in A^c_{N,\rho}}\exp\beta\widetilde{\phi}_v\right],$$

where $(\widetilde{\phi}_v, v\in A^c_{N,\rho})$ are i.i.d. centered Gaussians of variance $\frac{1}{\pi}\log N^2 + O_N(1)$. Since $\#A^c_{N,\rho} = N^2 - |A_{N,\rho}| = 4N^{2-\rho}(1+o_N(1))$, the free energy of these i.i.d. Gaussians in the limit $N\to\infty$ is given by (2.5)

$$\lim_{N\to\infty}\frac{1}{\log 4N^{2-\rho}}\mathbb{E}\left[\log\sum_{v\in A^c_{N,\rho}}\exp\beta\widetilde{\phi}_v\right] = \begin{cases} 1+\frac{\beta^2}{2\pi}\left(1-\frac{\rho}{2}\right)^{-1}, & \beta < \sqrt{2\pi}\left(1-\frac{\rho}{2}\right)^{1/2}, \\ \sqrt{\frac{2}{\pi}}\left(1-\frac{\rho}{2}\right)^{-1/2}\beta, & \beta \geq \sqrt{2\pi}\left(1-\frac{\rho}{2}\right)^{1/2}. \end{cases}$$

The last two equations then imply

$$\limsup_{N\to\infty}\frac{1}{\log N^2}\mathbb{E}\left[\log\sum_{v\in A^c_{N,\rho}}\exp\beta\phi_v\right] \leq \begin{cases} \left(1-\frac{\rho}{2}\right)+\frac{\beta^2}{2\pi}, & \beta < \sqrt{2\pi}\left(1-\frac{\rho}{2}\right)^{1/2}, \\ \sqrt{\frac{2}{\pi}}\left(1-\frac{\rho}{2}\right)^{1/2}\beta, & \beta \geq \sqrt{2\pi}\left(1-\frac{\rho}{2}\right)^{1/2}. \end{cases}$$

It is then straightforward to check that, for every β, the right side is strictly smaller than $f(\beta)$ as claimed. □

3.2 An adaptation of the Bovier-Kurkova technique

The Bovier-Kurkova technique is a way to compute the overlap distribution of a model in terms of the free energy of a perturbed version of that model. In the context of this paper, this connection is established by Proposition 3.2 below. One difficulty in the present case is the fact that the Green's function depends on the relative position to the boundary. The restriction to the set $A_{N,\rho}$ is a way to control this, cf. Lemma 3.4.

Proposition 3.2. *Let $\boldsymbol{\sigma} = (1, 1+u)$ where $|u| \leq 1$ and consider the $(\alpha,\boldsymbol{\sigma})$-GFF as in Theorem 2.1. Then for every $0<\rho<\alpha$, every $\varepsilon > \rho$, and N large enough,*

$$\left|\int_\alpha^1 x_{\beta,N,\rho}(r)dr - \frac{\pi}{\beta^2}\frac{\partial}{\partial u}\mathbb{E}f^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta)\Big|_{u=0}\right| \leq c\big(x_{\beta,N,\rho}(\alpha+\varepsilon) - x_{\beta,N,\rho}(\alpha-\varepsilon)\big) + O(\rho) + o_N(1), \quad (3.6)$$

where the two-overlap distribution $x_{\beta,N,\rho}$ is defined in (3.3)*, $O(\rho)$ is uniform in N, and c is an absolute constant.*

The proof of Proposition 3.2 is based on combining the following identities. It is convenient for the statement and the proof to define

$$q_\alpha(v,v') := \left(\frac{1}{\pi}\log N^2\right)^{-1}\mathbb{E}[\phi_{v'}(\phi_v - \phi_{[v]_\alpha})] \text{ for } v, v' \in V_N.$$

Lemma 3.3. *For every $\rho < \alpha < 1$ and $N\in\mathbb{N}$,*

$$\int_\alpha^1 x_{\beta,N,\rho}(r)dr = 1-\alpha-\mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\big[q(v,v')-\alpha; q(v,v')\geq\alpha\big], \quad (3.7)$$

$$\frac{\pi}{\beta^2}\frac{\partial}{\partial u}\mathbb{E}f^{(\alpha,\boldsymbol{\sigma})}_{N,\rho}(\beta)\Big|_{u=0} = 1-\alpha-\mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}\left[q_\alpha(v,v')\,; v'\in[v]_\alpha\right] + o_N(1), \quad (3.8)$$

where $\mathbb{E}[X;A]$ *stands for the expectation of a random variable* X *on the event* A*.*

Proof. The identity (3.7) holds since by Fubini's theorem

$$\begin{aligned}\int_\alpha^1 x_{\beta,N,\rho}(r)dr &= \mathbb{E}\mathcal{G}_{\beta,N,\rho}^{\times 2}\left[\int_\alpha^1 1_{\{r\geq q(v,v')\}}dr\right] \\ &= \mathbb{E}\mathcal{G}_{\beta,N,\rho}^{\times 2}\big[1-\alpha; q(v,v')<\alpha\big] + \mathbb{E}\mathcal{G}_{\beta,N,\rho}^{\times 2}\big[1-q(v,v'); q(v,v')\geq\alpha\big] \\ &= (1-\alpha) - \mathbb{E}\mathcal{G}_{\beta,N,\rho}^{\times 2}\big[q(v,v')-\alpha; q(v,v')\geq\alpha\big] \ .\end{aligned}$$

For the identity (3.8), direct differentiation gives

$$\frac{\pi}{\beta^2}\frac{\partial}{\partial u}\mathbb{E}f_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta)\Big|_{u=0} = \left(\frac{\beta}{\pi}\log N^2\right)^{-1}\sum_{v\in A_{N,\rho}}\mathbb{E}\left[\frac{(\phi_v-\phi_{[v]_\alpha})e^{\beta\phi_v}}{\sum_{v'\in A_{N,\rho}}\exp\beta\phi_{v'}}\right] . \tag{3.9}$$

The identity is then obtained using Gaussian integration by parts. Precisely, for a centered Gaussian vector $\boldsymbol{X}=(X_1,\ldots,X_n)$ and a twice-continuously differentiable function F on $\mathbb{R}^n$, of moderate growth at infinity, we have the formula $\mathbb{E}[X_iF(\boldsymbol{X})] = \sum_{j=1}^n \mathbb{E}[X_iX_j]\ \mathbb{E}[\partial_{X_j}F(\boldsymbol{X})]$. Here the relevant Gaussian vector for a given $v\in A_{N,\rho}$ is $(\phi_v-\phi_{[v]_\alpha};\phi_{[v]_\alpha};\phi_{v'},v'\neq v)$, and the function F is $\exp\beta\phi_v/\sum_{v'\in A_{N,\rho}}\exp\beta\phi_{v'}$. Note that $\mathbb{E}[\phi_{v'}(\phi_v-\phi_{[v]_\alpha})]=0$ for $v'\notin[v]_\alpha$. Applying the formula to the right-hand side of (3.9) yields

$$\left(\frac{1}{\pi}\log N^2\right)^{-1}\Big(\mathbb{E}\mathcal{G}_{\beta,N,\rho}\big[\,\mathbb{E}[(\phi_v-\phi_{[v]_\alpha})^2]\big]-\mathbb{E}\mathcal{G}_{\beta,N,\rho}^{\times 2}\big[\,\mathbb{E}[\phi_{v'}(\phi_v-\phi_{[v]_\alpha})]\,;v'\in[v]_\alpha\big]\Big) . \tag{3.10}$$

The field $(\phi_u-\mathbb{E}[\phi_u|\mathcal{F}_{[v]_\alpha^c}],u\in[v]_\alpha)$ has the law of a GFF in $[v]_\alpha$ by Lemma 5.1. Since $\rho<\alpha$ implies that $[v]_\alpha$ is a square of side-length $N^{1-\alpha}$ for any $v\in A_{N,\rho}$, it follows from Lemma 5.2 that

$$\left(\frac{1}{\pi}\log N^2\right)^{-1}\mathbb{E}\big[(\phi_v-\phi_{[v]_\alpha})^2\big] = 1-\alpha+o_N(1) \ .$$

This proves (3.8). □

To prove (3.6) from Lemma 3.3, we need to relate the overlap $q(v,v')-\alpha$ to the covariance $\mathbb{E}[\phi_{v'}(\phi_v-\phi_{[v]_\alpha})]$ as well as relate the event $\{q(v,v')\geq\alpha\}$ to the event $\{v'\in[v]_\alpha\}$. One obstacle is to establish an approximate correspondence between the Green's function for two points v and v' and their relative distance. The problem is that Green's function is also sensitive to the relative position to the boundary. The restriction to $A_{N,\rho}$ allows for a sufficient control of the Green's function.

Lemma 3.4. *Let* $v,v'\in A_{N,\rho}$*.*

(i) If $q(v,v')\geq\alpha+\varepsilon$ *for some* $\varepsilon>0$*, then* $\|v-v'\|^2\leq cN^{2(1-\alpha)-\varepsilon}$ *for some constant* c *independent of* N *and* ρ*. In particular,* $v'\in[v]_\alpha$ *for* N *large enough.*

(ii) If $v'\in[v]_\alpha$*, then* $q(v,v')\geq\alpha-\rho+o_N(1)$*.*

Proof. For $v,v'\in A_{N,\rho}$, Lemma 5.2 gives

$$1-\rho-\frac{\log\|v-v'\|^2}{\log N^2}+O_N((\log N)^{-1})\leq q(v,v')\leq 1-\frac{\log\|v-v'\|^2}{\log N^2}+O_N((\log N)^{-1}) \ . \tag{3.11}$$

The assertion (i) is direct from the right inequality. The claim (ii) follows from the left inequality since $v'\in[v]_\alpha$ implies $\|v-v'\|^2\leq cN^{2(1-\alpha)}$ for some constant c. □

Proof of Proposition 3.2. Note that $q_\alpha(v,v')$ is bounded by a constant uniformly in N. Indeed, by the Cauchy-Schwarz inequality, we get

$$|q_\alpha(v,v')|^2 \le \left(\frac{1}{\pi}\log N^2\right)^{-2} \mathbb{E}[\phi_{v'}^2]\ \mathbb{E}[(\phi_v - \phi_{[v]_\alpha})^2]\ ,$$

and the boundedness follows from the upper bound (1.3) and Lemma 5.1. Fix $\varepsilon > \rho$. Then, Lemma 3.4 (ii) implies that $\{v' \in [v]_\alpha\} \subset \{q(v,v') > \alpha - \varepsilon\}$ for N large enough. Since Lemma 3.4 (i) yields $\{q(v,v') \ge \alpha + \varepsilon\} \subset \{v' \in [v]_\alpha\}$, one obtains, for N large enough,

$$\{v' \in [v]_\alpha\} \setminus \{q(v,v') \ge \alpha+\varepsilon\} \subset \{\alpha - \varepsilon < q(v,v') < \alpha + \varepsilon\}.$$

Because $q_\alpha(v,v')$ is bounded, we get for all $\varepsilon > \rho$ and some constant $c > 0$,

$$\begin{aligned}\Big|\mathbb{E}G^{\times 2}_{\beta,N,\rho}\left[q_\alpha(v,v'); v' \in [v]_\alpha\right] - \mathbb{E}G^{\times 2}_{\beta,N,\rho}\left[q_\alpha(v,v'); q(v,v') \ge \alpha+\varepsilon\right]\Big| \\ \le c\big(x_{\beta,N,\rho}(\alpha+\varepsilon) - x_{\beta,N,\rho}(\alpha-\varepsilon)\big)\ .\end{aligned} \tag{3.12}$$

If $q(v,v') \ge \alpha + \varepsilon$, then $\|v-v'\|^2 \le cN^{2(1-\alpha)-\varepsilon}$ by Lemma 3.4 (i), and, in particular, $v' \in [v]_\alpha$ for N large enough. For such v and v', we have by conditioning,

$$q_\alpha(v,v') = \left(\frac{1}{\pi}\log N^2\right)^{-1} \mathbb{E}[(\phi_{v'} - \mathbb{E}[\phi_{v'}|\mathcal{F}_{[v]^c_\alpha}])(\phi_v - \phi_{[v]_\alpha})]\ .$$

Since $(\phi_u - \mathbb{E}[\phi_u|\mathcal{F}_{[v]^c_\alpha}], u \in [v]_\alpha)$ has the law of a GFF in $[v]_\alpha$ by Lemma 5.1 and since $\|v-v'\|^2 \le cN^{2(1-\alpha)-\varepsilon}$, Lemma 5.2 gives

$$q_\alpha(v,v') = 1 - \alpha - \frac{\log\|v-v'\|^2}{\log N^2} + o_N(1) \ \text{ if } q(v,v') \ge \alpha+\varepsilon.$$

Moreover, by (3.11), $q(v,v') = 1 - \frac{\log\|v-v'\|^2}{\log N^2} + o_N(1) + O(\rho)$, where the term $O(\rho)$ is uniform in N. This shows that

$$\mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}[q_\alpha(v,v'); q(v,v') \ge \alpha+\varepsilon] = \mathbb{E}\mathcal{G}^{\times 2}_{\beta,N,\rho}[q(v,v') - \alpha; q(v,v') \ge \alpha+\varepsilon] + o_N(1) + O(\rho)\ . \tag{3.13}$$

Finally, because $q(v,v') - \alpha$ is bounded by 1, we have

$$\begin{aligned}0 \le \mathbb{E}G^{\times 2}_{\beta,N,\rho}\left[q(v,v') - \alpha; q(v,v') \ge \alpha\right] - \mathbb{E}G^{\times 2}_{\beta,N,\rho}\left[q(v,v') - \alpha; q(v,v') \ge \alpha+\varepsilon\right] \\ \le x_{\beta,N,\rho}(\alpha+\varepsilon) - x_{\beta,N,\rho}(\alpha-\varepsilon)\ .\end{aligned} \tag{3.14}$$

The conclusion follows from Equations (3.7) and (3.8) by combining Equations (3.12), (3.13), and (3.14).

□

3.3 Proof of Theorem 1.1

By Equation (3.2), it suffices to prove that

$$\lim_{\rho\to 0}\lim_{N\to\infty} x_{\beta,N,\rho}(r) = \begin{cases} \beta_c/\beta, & \text{if } 0 \le r < 1, \\ 1, & \text{if } r = 1\ . \end{cases} \tag{3.15}$$

Recall that the space of probability measures on $[0,1]$ is compact under the topology induced by weak convergence. Consider a converging subsequence of the probability measures on $[0,1]$ corresponding to the cumulative distribution functions $x_{\beta,N,\rho}$ (when $N \to \infty$ and then $\rho \to 0$). Write x_β for the cumulative distribution function of the limiting probability measure. The proof is reduced to show that x_β is given by the right-hand side of (3.15). In fact, since $x_\beta(1) = 1$, it remains to show that $x_\beta(r) = \beta_c/\beta$ for $0 \leq r < 1$. The points of continuity of x_β are dense in $[0,1]$ (because it is monotone) so it suffices to show that $x_\beta(r) = \beta_c/\beta$ for all points of continuity of x_β in $[0,1)$. Let α be such a point of continuity. Proposition 3.2 implies that, after taking the limits $N \to \infty$, $\rho \to 0$ and then $\varepsilon \to 0$,

$$\int_\alpha^1 x_\beta(r)dr = \lim_{\rho\to 0}\lim_{N\to\infty} \frac{\pi}{\beta^2}\frac{\partial}{\partial u}\mathbb{E}f_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta)\Big|_{u=0} ,$$

for $\boldsymbol{\sigma} = (1, 1+u)$. Recall Theorem 2.1. It is straightforward to check that $u \mapsto f_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta)$ is a convex function. In particular,

$$\lim_{\rho\to 0}\lim_{N\to\infty} \frac{\partial}{\partial u}\mathbb{E}f_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta) = \frac{\partial}{\partial u} f^{(\alpha,\boldsymbol{\sigma})}(\beta) , \tag{3.16}$$

at any points u where $f^{(\alpha,\boldsymbol{\sigma})}(\beta)$ is differentiable. We calculate the derivative at $u = 0$ by computing the right and left derivatives at 0. In our case $\sigma_1 = 1$, $\sigma_2 = 1+u$ and $\Gamma_{12} = \alpha + (1+u)^2(1-\alpha)$. If $u > 0$, we are in the case $\sigma_1 < \sigma_2$. Since $\beta > \beta_c = \sqrt{2\pi}$, we can pick u small enough such that $\beta > \sqrt{2\pi}/\sqrt{\Gamma_{12}}$. Therefore Theorem 2.1 gives

$$\frac{\partial}{\partial u^+} f^{(\alpha,\boldsymbol{\sigma})}(\beta)\Big|_{u=0} = \sqrt{\frac{2}{\pi}}\beta\, \frac{\partial}{\partial u^+}\big(\alpha + (1+u)^2(1-\alpha)\big)^{1/2}\Big|_{u=0} = \sqrt{\frac{2}{\pi}}\beta(1-\alpha) . \tag{3.17}$$

Similarly, for the left derivative, pick $u < 0$ so we are in the case $\sigma_1 > \sigma_2$. Since $\beta > \beta_c = \sqrt{2\pi}$, we can pick $|u|$ small enough such that $\beta > \sqrt{2\pi}/(1+u)$. Therefore Theorem 2.1 gives

$$\frac{\partial}{\partial u^-} f^{(\alpha,\boldsymbol{\sigma})}(\beta)\Big|_{u=0} = \sqrt{\frac{2}{\pi}}\beta\, \frac{\partial}{\partial u^-}\big(\alpha + (1+u)(1-\alpha)\big)\Big|_{u=0} = \sqrt{\frac{2}{\pi}}\beta(1-\alpha) . \tag{3.18}$$

Equations (3.16), (3.17) and (3.18) imply

$$\int_\alpha^1 x_\beta(r)dr = \frac{\beta_c}{\beta}(1-\alpha) . \tag{3.19}$$

Since x_β must be non-decreasing, this gives $x_\beta(\alpha) \leq \beta_c/\beta$. If $x_\beta(\alpha) < \beta_c/\beta$, there would exist another point of continuity $\alpha' > \alpha$ such that $x_\beta(\alpha') < \beta_c/\beta$ by the right-continuity of x_β. Therefore $\int_\alpha^{\alpha'} x_\beta(r)dr < \beta_c/\beta(\alpha'-\alpha)$ contradicting (3.19). This means that $x_\beta(\alpha) = \beta_c/\beta$ at all points of continuity α and concludes the proof of the theorem.

4 The free energy of the $(\alpha, \boldsymbol{\sigma})$-GFF: proof of Theorem 2.1

The computation of the expectation of the free energy of the $(\alpha,\boldsymbol{\sigma})$-GFF is divided in two steps. First, an upper bound is found by comparing the field ψ in $A_{N,\rho}$ to a standard 2-level GREM as in (2.7). Second, we get a matching lower bound using the trivial inequality $f_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta) \geq \frac{1}{\log N^2}\log\sum_{v\in V_N^\delta} \mathrm{e}^{\beta\psi_v}$. The limit of the expectation of the right term is computed following the method of Daviaud [13].

4.1 Proof of the upper bound

For conciseness, we only prove the case $\sigma_1 \geq \sigma_2$. This is done by comparing the free energy of the field ψ in $A_{N,\rho}$ with a 2-level GREM as in (2.7). To account for the boundary effect, the comparison is done via two intermediate Gaussian fields $\widetilde{\psi}$ and $\overline{\psi}$ defined below. The field $\widetilde{\psi}$ will be a "non-homogeneous" GREM in the sense that σ_1 in (2.7) will depend on v. On the other hand, the field $\overline{\psi}$ will differ from (2.7) only by a factor $O_N(1)$ in the variance. The case $\sigma_1 \leq \sigma_2$ is done similarly by comparing with a REM.

Divide the set $A_{N,\rho}$ into square boxes of side-length $N^{1-\alpha}/100$. (The factor $1/100$ is a convenient choice. We simply need these boxes to be smaller than the neighborhoods $[v]_\alpha$, yet of the same order of length in N.) Pick the boxes in such a way that each $v \in A_{N,\rho}$ belongs to one and only one of these boxes. The collection of boxes is denoted by $\mathcal{B}_\alpha$ and $\partial\mathcal{B}_\alpha$ denotes $\bigcup_{B\in\mathcal{B}_\alpha} \partial B$. For $v \in A_{N,\rho}$, we write $B(v)$ for the box of $\mathcal{B}_\alpha$ to which v belongs. For $B \in \mathcal{B}_\alpha$, denote by $\widetilde{B} \supset B$ the square box given by the intersections of all $[u]_\alpha$, $u \in B$, see figure 1. Remark that the side-length of $\widetilde{B}$ is $cN^{1-\alpha}$, for some constant c. For short, write $\phi_{\widetilde{B}} := \mathbb{E}[\phi_{v_B} | \mathcal{F}_{\widetilde{B}^c}]$ where v_B is the center of the box B. The idea in constructing the "non-homogeneous" GREM is to associate to each point $v \in B$ the same contribution at scale α, namely $\phi_{\widetilde{B}}$. One problem is that $\phi_{\widetilde{B}}$ will not have the same variance for every B since it depends on the distance to the boundary. This is the reason why the comparison with a 2-level GREM needs to be done using intermediate fields.

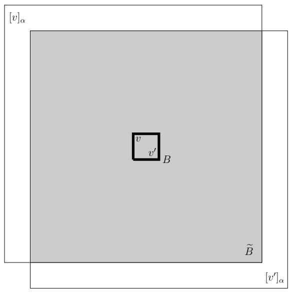

Figure 1: The box $B \in \mathcal{B}_\alpha$ and the corresponding box $\widetilde{B}$ which is the intersection of all the neighborhoods $[v]_\alpha$, $v \in B$.

First, consider the hierarchical Gaussian field $(\widetilde{\psi}_v, v \in A_{N,\rho})$:

$$\widetilde{\psi}_v = g^{(1)}_{B(v)} + g^{(2)}_v, \tag{4.1}$$

where the variables $(g^{(1)}_B, B \in \mathcal{B}_\alpha)$ are independent centered Gaussians with variance chosen to be $\sigma_1^2 \mathbb{E}[\phi^2_{\widetilde{B}}] + C$ (for some constant $C \in \mathbb{R}$ independent of B in $\mathcal{B}_\alpha$ and independent of N) and $(g^{(2)}_v, v \in A_{N,\rho})$ are independent centered Gaussians (also independent from $(g^{(1)}_B, B \in \mathcal{B}_\alpha)$) with variance

$$\mathbb{E}[(g^{(2)}_v)^2] = \mathbb{E}[\psi_v^2] - \mathbb{E}[(g^{(1)}_{B(v)})^2] \ .$$

(Equations (4.4) and (4.5) below will guarantee that the right-hand term is non-negative.) Note that with this definition $\mathbb{E}[\psi_v^2] = \mathbb{E}[\widetilde{\psi}_v^2]$ for all $v \in A_{N,\rho}$. The next lemma ensures that

$$\mathbb{E}[\psi_v \psi_{v'}] \geq \mathbb{E}[\widetilde{\psi}_v \widetilde{\psi}_{v'}] \ . \tag{4.2}$$

Lemma 4.1. *Consider the field $(\psi_v, v \in A_{N,\rho})$ as in (2.3). Then $\mathbb{E}[\psi_v \psi_{v'}] \geq 0$. Moreover, if v and v' both belong to $B \in \mathcal{B}_\alpha$, then*

$$\mathbb{E}[\psi_v \psi_{v'}] \geq \sigma_1^2 \mathbb{E}[\phi^2_{\widetilde{B}}] + C \ ,$$

for some constant $C \in \mathbb{R}$ independent of N.

Proof. For the first assertion, write

$$\psi_v = (\sigma_1 - \sigma_2)\phi_{[v]_\alpha} + \sigma_2 \phi_v \ .$$

The representation $\phi_{[v]_\alpha} = \sum_{u\in\partial[v]_\alpha} p_{\alpha,v}(u)\ \phi_u$ of Lemma 5.1 and the fact that $\sigma_1 > \sigma_2$ imply that $\mathbb{E}[\psi_v\psi'_v] \geq 0$ since the field ϕ is positively correlated by (1.1).

Suppose now that $v, v' \in B$ where $B \in \mathcal{B}_\alpha$. The covariance can be written as

$$\begin{aligned}\mathbb{E}[\psi_v\psi_{v'}] &= \sigma_1^2\mathbb{E}\left[\phi_{[v]_\alpha}\phi_{[v']_\alpha}\right] + \sigma_2^2\mathbb{E}\left[(\phi_v - \phi_{[v]_\alpha})(\phi_{v'} - \phi_{[v']_\alpha})\right] \\ &\quad + \sigma_1\sigma_2\mathbb{E}\left[\phi_{[v]_\alpha}(\phi_{v'} - \phi_{[v']_\alpha})\right] + \sigma_1\sigma_2\mathbb{E}\left[\phi_{[v']_\alpha}(\phi_v - \phi_{[v]_\alpha})\right] \ .\end{aligned} \tag{4.3}$$

We first prove that the last two terms of (4.3) are positive. By Lemma 5.1, we can write $\phi_{[v]_\alpha} = \sum_{u\in\partial[v]_\alpha} p_{\alpha,v}(u)\ \phi_u$. Note that the vertices u that are in $[v']^c_\alpha$ will not contribute to the covariance $\mathbb{E}\left[\phi_{[v]_\alpha}(\phi_{v'} - \phi_{[v']_\alpha})\right]$ by conditioning. Thus

$$\begin{aligned}\mathbb{E}\left[\phi_{[v]_\alpha}(\phi_{v'} - \phi_{[v']_\alpha})\right] &= \sum_{u\in\partial[v]_\alpha\cap[v']_\alpha} p_{\alpha,v}(u)\ \mathbb{E}\left[\phi_u(\phi_{v'} - \phi_{[v']_\alpha})\right] \\ &= \sum_{u\in\partial[v]_\alpha\cap[v']_\alpha} p_{\alpha,v}(u)\ \mathbb{E}\left[(\phi_u - \mathbb{E}[\phi_u|\mathcal{F}_{[v']^c_\alpha}])(\phi_{v'} - \mathbb{E}[\phi_{v'}|\mathcal{F}_{[v']^c_\alpha}])\right] \ .\end{aligned}$$

Lemma 5.1 ensures that the covariances in the sum are positive (the field $(\phi_u - \mathbb{E}[\phi_u|\mathcal{F}_{[v']^c_\alpha}])_{u\in[v']_\alpha}$ has the law of a GFF on $[v']_\alpha$).

For the first term of (4.3), the idea is to show that $\phi_{[v]_\alpha}$ and $\phi_{\widetilde{B}}$ are close in the L^2-sense. Write

$$\phi_v - \phi_{[v]_\alpha} = (\phi_v - \mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}]) + (\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}] - \phi_{[v]_\alpha})\ ,$$

and observe that, since $\phi_v - \mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}]$ is independent of $\mathcal{F}_{\widetilde{B}^c}$ and $\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}] - \phi_{[v]_\alpha}$ is $\mathcal{F}_{\widetilde{B}^c}$-measurable (indeed $\mathcal{F}_{[v]^c_\alpha} \subset \mathcal{F}_{\widetilde{B}^c}$), this implies

$$\mathbb{E}[(\phi_v - \phi_{[v]_\alpha})^2] = \mathbb{E}[(\phi_v - \mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}])^2] + \mathbb{E}[(\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}] - \phi_{[v]_\alpha})^2]\ .$$

Moreover, $\mathbb{E}[(\phi_v - \mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}])^2]$ and $\mathbb{E}[(\phi_v - \phi_{[v]_\alpha})^2]$ are both equal to $\frac{1-\alpha}{\pi}\log N^2 + O_N(1)$ by Lemma 5.1, Lemma 5.2 and the fact that distances of v to vertices in $\partial\widetilde{B}$ and $\partial[v]_\alpha$ are both proportional to $N^{1-\alpha}$ (here the condition $\rho < \alpha$ and the fact that the boxes in $\mathcal{B}_\alpha$ have side-length $N^{1-\alpha}/100$ are used). Therefore

$$\mathbb{E}\left[\left(\phi_{[v]_\alpha} - \mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}]\right)^2\right] = O_N(1)\ . \tag{4.4}$$

Moreover, since v and v_B are also at a distance smaller than $N^{1-\alpha}/100$ from each other, Lemma 12 in [7] implies that

$$\mathbb{E}\left[\left(\phi_{\widetilde{B}} - \mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}]\right)^2\right] = O_N(1)\ . \tag{4.5}$$

Equations (4.4) and (4.5) give $\mathbb{E}[(\phi_{\widetilde{B}} - \phi_{[v]_\alpha})^2] = O_N(1)$ and similarly for v'. All the above sum up to

$$\sigma_1^2\mathbb{E}\left[\phi_{[v]_\alpha}\phi_{[v']_\alpha}\right] = \sigma_1^2\mathbb{E}[\phi^2_{\widetilde{B}}] + O_N(1)\ . \tag{4.6}$$

It remains to show that the second term of (4.3) is greater than $O_N(1)$. Since $\phi_{[v]_\alpha}$ and $\phi_{[v']_\alpha}$ are $\mathcal{F}_{\widetilde{B}^c}$-measurable by definition of the box $\widetilde{B}$, we have the decomposition

$$\begin{aligned}\mathbb{E}\left[(\phi_v - \phi_{[v]_\alpha})(\phi_{v'} - \phi_{[v']_\alpha})\right] &= \mathbb{E}[(\phi_v - \mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}])(\phi_{v'} - \mathbb{E}[\phi_{v'}|\mathcal{F}_{\widetilde{B}^c}])] \\ &\quad + \mathbb{E}[(\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}] - \phi_{[v]_\alpha})(\mathbb{E}[\phi_{v'}|\mathcal{F}_{\widetilde{B}^c}] - \phi_{[v']_\alpha})]\ .\end{aligned}$$

The first term is positive by Lemma 5.1. As for the second, Equation (4.4) shows that

$$\mathbb{E}\Big[\left(\mathbb{E}[\phi_v|\mathcal{F}_{\widetilde{B}^c}] - \phi_{[v]_\alpha}\right)\left(\mathbb{E}[\phi_{v'}|\mathcal{F}_{\widetilde{B}^c}] - \phi_{[v']_\alpha}\right)\Big] = O_N(1)\ .$$

This concludes the proof of the lemma. □

Equation (4.2) implies that the expectation of the free energy of ψ is smaller than the one of $\widetilde{\psi}$ by a standard comparison lemma, see Lemma 5.3 in the Appendix. It remains to prove an upper bound for the expectation of the free energy of $\widetilde{\psi}$.

The field $\widetilde{\psi}$ is not a 2-level GREM as in (2.7) because the variances of $g_B^{(1)}$, $B \in \mathcal{B}_\alpha$, are not the same for every B, as it depends on the relative position of B to the boundary. However, the variances of $\phi_{\widetilde{B}}$, $B \in \mathcal{B}_\alpha$, are uniformly bounded above by $\frac{\alpha}{\pi}\log N^2 + O_N(1)$; indeed

$$\begin{aligned}\mathbb{E}\left[\phi_{\widetilde{B}}^2\right] &= \mathbb{E}\left[\phi_{v_B}^2\right] - \mathbb{E}\left[(\phi_{v_B} - \phi_{\widetilde{B}})^2\right]\\ &= \mathbb{E}\left[\phi_{v_B}^2\right] - \frac{1-\alpha}{\pi}\log N^2 + O_N(1)\\ &\leq \frac{1}{\pi}\log N^2 - \frac{1-\alpha}{\pi}\log N^2 + O_N(1) = \frac{\alpha}{\pi}\log N^2 + O_N(1),\end{aligned}$$

where, in the second equality, Lemmas 5.1 and 5.2 imply that $\mathbb{E}[(\phi_{v_B} - \phi_{\widetilde{B}})^2]$ is equal to $\frac{1-\alpha}{\pi}\log N^2 + O_N(1)$ (again the condition $\alpha > \rho$ guarantees that $\widetilde{B}$ is a square of side-length $cN^{1-\alpha}$), and the inequality comes from Equation (1.3).

Moreover, note that for $v \in B$,

$$\mathbb{E}[(g_v^{(2)})^2] = \mathbb{E}[\psi_v^2] - \mathbb{E}[(g_B^{(1)})^2] = \sigma_1^2\big(\mathbb{E}[\phi_{[v]_\alpha}^2] - \mathbb{E}[\phi_{\widetilde{B}}^2]\big) + \sigma_2^2\frac{1-\alpha}{\pi}\log N^2 - C\sigma_1^2\ .$$

The first term is of order $O_N(1)$ by Equations (4.4) and (4.5). Thus one has

$$\mathbb{E}[(g_v^{(2)})^2] = \sigma_2^2\frac{1-\alpha}{\pi}\log N^2 + O_N(1)\ .$$

The important point is that the variance of $g_v^{(2)}$ of $\widetilde{\psi}$ is uniform in v, up to lower order terms. Now consider the Gaussian field $(\overline{\psi}_v, v \in A_{N,\rho})$

$$\overline{\psi}_v = \overline{g}_B^{(1)} + g_v^{(2)} \tag{4.7}$$

where $(\overline{g}_B^{(1)}, B \in \mathcal{B}_\alpha)$ are i.i.d. centered Gaussians of variance $\sigma_1^2\frac{\alpha}{\pi}\log N^2 + O_N(1)$ and $(g_v^{(2)}, v \in A_{N,\rho})$ are as before. This field is not exactly a 2-level GREM as in (2.7) since the Gaussians at each level have an additional $O_N(1)$ term in their variances. It differs from $\widetilde{\psi}$ only from the fact that the variance of $\overline{g}_B^{(1)}$ is the same for all B and is equal to the maximal variance of $(g_B^{(1)}, B \in \mathcal{B}_\alpha)$. The calculation of the free energy of $(\overline{\psi}_v, v \in A_{N,\rho})$ is a standard computation (the $O_N(1)$ does not affect the free energy) and gives the correct upper bound in the statement of Theorem 2.1. (We refer to [8] for the detailed computation of the free energy of the GREM.) The fact that the expectation of the free energy of $\overline{\psi}$ is larger than the one of $\widetilde{\psi}$ follows from the next lemma showing that the free energy of a hierarchical field is an increasing function of the variance of each point at the first level.

Lemma 4.2. *Consider $N_1, N_2 \in \mathbb{N}$. Let $(X_{v_1}^{(1)}, v_1 \leq N_1)$ and $(X_{v_1,v_2}^{(2)}; v_1 \leq N_1, v_2 \leq N_2)$ be i.i.d. standard Gaussian random variables. Consider the Gaussian field of the form*

$$X_v = \sigma_1(v_1)X_{v_1}^{(1)} + \sigma_2 X_{v_1,v_2}^{(2)}\ ,\ \ v = (v_1, v_2)\ ,$$

where $\sigma_2 > 0$ and $\sigma_1(v_1) > 0$, for all $v_1 \leq N_1$. Then $\mathbb{E}\left[\log\sum_v e^{\beta X_v}\right]$ is an increasing function in each variable $\sigma_1(v_1)$.

Proof. Direct differentiation gives

$$\frac{\partial}{\partial\sigma_1(v_1)}\mathbb{E}\left[\log\sum_v e^{\beta X_v}\right] = \beta\mathbb{E}\left[\frac{\sum_{v_2} X_{v_1} e^{\beta X_{v_1,v_2}}}{Z_N(\beta)}\right]\ ,$$

where $Z_N(\beta) = \sum_v e^{\beta X_v}$. Gaussian integration by part then yields

$$\beta \mathbb{E}\left[\frac{\sum_{v_2} X_{v_1} e^{\beta X_{v_1,v_2}}}{Z_N(\beta)}\right] = \beta^2 \sigma_1(v_1) \mathbb{E}\left[\frac{\sum_{v_2} e^{\beta X_{v_1,v_2}}}{Z_N(\beta)} - \frac{\sum_{v_2,v_2'} e^{\beta X_{v_1,v_2}} e^{\beta X_{v_1,v_2'}}}{Z_N(\beta)^2}\right] .$$

The right side is clearly positive, hence proving the lemma. □

4.2 Proof of the lower bound

Recall the definition of V_N^δ given in the introduction. The two following propositions are used to compute the log-number of high points of the field ψ in V_N^δ. The treatment follows the treatment of Daviaud [13] for the standard GFF. The lower bound for the free energy is then computed using Laplace's method. Recall that $\Gamma_{12} = \sigma_1^2 \alpha + \sigma_2^2(1-\alpha)$.

Proposition 4.3.

$$\lim_{N\to\infty} \mathbb{P}\left(\max_{v\in V_N^\delta} \psi_v \geq \sqrt{\frac{2}{\pi}} \gamma_{max} \log N^2\right) = 0,$$

where

$$\gamma_{max} = \gamma_{max}(\alpha, \boldsymbol{\sigma}) := \begin{cases} \sqrt{\Gamma_{12}}, & \text{if } \sigma_1 \leq \sigma_2, \\ \sigma_1\alpha + \sigma_2(1-\alpha), & \text{if } \sigma_1 \geq \sigma_2. \end{cases}$$

Proof. The case $\sigma_1 \leq \sigma_2$ is direct by a union bound. In the case $\sigma_1 \geq \sigma_2$, note that the field $\widetilde{\psi}$ defined in (4.1) but restricted to V_N^δ is of the form (2.7) (up to $O_N(1)$ terms in the variance) with $cN^{2\alpha}$ (for some $c > 0$) Gaussian variables of variance $\frac{\sigma_1^2\alpha}{\pi} \log N^2 + O_N(1)$ at the first level. Indeed, for the field restricted to V_N^δ, the variance of $\mathbb{E}[\phi^2_{\widetilde{B}}]$ is $\frac{\sigma_1^2\alpha}{\pi} \log N^2 + O_N(1)$ by Lemma 5.2 since the distance to the boundary is a constant times N. Therefore, by Lemma 5.3 and Equation (4.2), we have

$$\mathbb{P}\left(\max_{v\in V_N^\delta} \psi_v \geq \sqrt{\frac{2}{\pi}} \gamma_{max} \log N^2\right) \leq \mathbb{P}\left(\max_{v\in V_N^\delta} \widetilde{\psi}_v \geq \sqrt{\frac{2}{\pi}} \gamma_{max} \log N^2\right) .$$

Then, the study of the first order of the maximum of $\widetilde{\psi}$ restricted to V_N^δ is a standard GREM result (indeed the additional $O_N(1)$ in the variance does not affect the first order of the maximum). The proof is not hard and omitted for conciseness. The reader is referred to Theorem 1.1 in [9] where a stronger result on the maximum is given. □

Proposition 4.4. *Let* $\mathcal{H}_N^{\psi,\delta}(\gamma) := \left\{v \in V_N^\delta : \psi_v \geq \sqrt{\frac{2}{\pi}}\gamma \log N^2\right\}$ *be the set of* γ*-high points within* V_N^δ *and define*

$$\text{if } \sigma_1 \leq \sigma_2 \qquad \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma) := 1 - \frac{\gamma^2}{\Gamma_{12}};$$

$$\text{if } \sigma_1 \geq \sigma_2 \qquad \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma) := \begin{cases} 1 - \frac{\gamma^2}{\Gamma_{12}}, & \text{if } \gamma < \frac{\Gamma_{12}}{\sigma_1}, \\ (1-\alpha) - \frac{(\gamma - \sigma_1\alpha)^2}{\sigma_2^2(1-\alpha)}, & \text{if } \gamma \geq \frac{\Gamma_{12}}{\sigma_1}. \end{cases}$$

Then, for all $0 < \gamma < \gamma_{max}$, *and for any* $\mathcal{E} < \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma)$, *there exists* c *such that*

$$\mathbb{P}\left(|\mathcal{H}_N^{\psi,\delta}(\gamma)| \leq N^{2\mathcal{E}}\right) \leq \exp\{-c(\log N)^2\}. \tag{4.8}$$

Proposition 4.4 is obtained by a two-step recursion. Two lemmas are needed. The first is a straightforward generalization of the lower bound in Daviaud's theorem (see Theorem 1.2 in [13] and its proof). For all $0 < \alpha < 1$, denote by Π_α the centers of the square boxes in $\mathcal{B}_\alpha$ (as defined in Section 4.1) which also belong to V_N^δ.

Lemma 4.5. *Let $\alpha', \alpha'' \in (0,1]$ such that $0 < \alpha' < \alpha'' \leq \alpha$ or $\alpha \leq \alpha' < \alpha'' \leq 1$. Denote by σ the parameter σ_1 if $0 < \alpha' < \alpha'' \leq \alpha$ and by σ the parameter σ_2 if $\alpha \leq \alpha' < \alpha'' \leq 1$. Assume that the event*

$$\Xi := \left\{ \#\{v \in \Pi_{\alpha'} : \psi_v(\alpha') \geq \gamma' \sqrt{\frac{2}{\pi}} \log N^2\} \geq N^{\mathcal{E}'} \right\},$$

is such that

$$\mathbb{P}(\Xi^c) \leq \exp\{-c'(\log N)^2\},$$

for some $\gamma' \geq 0$, $\mathcal{E}' > 0$ and $c' > 0$.

Let

$$\mathcal{E}(\gamma) := \mathcal{E}' + (\alpha'' - \alpha') - \frac{(\gamma - \gamma')^2}{\sigma^2(\alpha'' - \alpha')} > 0.$$

Then, for any γ'' such that $\mathcal{E}(\gamma'') > 0$ and any $\mathcal{E} < \mathcal{E}(\gamma'')$, there exists c such that

$$\mathbb{P}\left(\#\{v \in \Pi_{\alpha''} : \psi_v(\alpha'') \geq \gamma'' \sqrt{\frac{2}{\pi}} \log N^2\} \leq N^{2\mathcal{E}} \right) \leq \exp\{-c(\log N)^2\}.$$

We stress that γ'' may be such that $\mathcal{E}(\gamma'') < \mathcal{E}'$. The second lemma, which follows, serves as the starting point of the recursion and is proved in [7] (see Lemma 8 in [7]).

Lemma 4.6. *For any α_0 such that $0 < \alpha_0 < \alpha$, there exists $\mathcal{E}_0 = \mathcal{E}_0(\alpha_0) > 0$ and $c = c(\alpha_0)$ such that*

$$\mathbb{P}\left(\#\{v \in \Pi_{\alpha_0} : \psi_v(\alpha_0) \geq 0\} \leq N^{\mathcal{E}_0}\right) \leq \exp\{-c(\log N)^2\}.$$

Proof of Proposition 4.4. Let γ such that $0 < \gamma < \gamma_{max}$ and choose $\mathcal{E}$ such that $\mathcal{E} < \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma)$. By Lemma 4.6, for $\alpha_0 < \alpha$ arbitrarily close to 0, there exists $\mathcal{E}_0 = \mathcal{E}_0(\alpha_0) > 0$ and $c_0 = c_0(\alpha_0) > 0$, such that

$$\mathbb{P}\left(\#\{v \in \Pi_{\alpha_0} : \psi_v(\alpha_0) \geq 0\} \leq N^{2\mathcal{E}_0}\right) \leq \exp\{-c_0(\log N)^2\}. \tag{4.9}$$

Moreover, let

$$\mathcal{E}_1(\gamma_1) := \mathcal{E}_0 + (\alpha - \alpha_0) - \frac{\gamma_1^2}{\sigma_1^2(\alpha - \alpha_0)}. \tag{4.10}$$

Lemma 4.5 is applied from α_0 to α. For any γ_1 with $\mathcal{E}_1(\gamma_1) > 0$ and any $\mathcal{E}_1 < \mathcal{E}_1(\gamma_1)$, there exists $c_1 > 0$ such that

$$\mathbb{P}\left(\#\{v \in \Pi_{\alpha} : \psi_v(\alpha) \geq \gamma_1 \sqrt{\frac{2}{\pi}} \log N^2\} \leq N^{2\mathcal{E}_1} \right) \leq \exp\{-c_1(\log N)^2\}.$$

Therefore, Lemma 4.5 can be applied again from α to 1 for any γ_1 with $\mathcal{E}_1(\gamma_1) > 0$. Define similarly $\mathcal{E}_2(\gamma_1, \gamma_2) := \mathcal{E}_1(\gamma_1) + (1 - \alpha) - (\gamma_2 - \gamma_1)^2/\sigma_2^2(1 - \alpha)$. Then, for any γ_2 with $\mathcal{E}_2(\gamma_1, \gamma_2) > 0$, and $\mathcal{E}_2 < \mathcal{E}_2(\gamma_1, \gamma_2)$, there exists $c_2 > 0$ such that

$$\mathbb{P}\left(\#\{v \in V_N^\delta : \psi_v \geq \gamma_2 \sqrt{\frac{2}{\pi}} \log N^2\} \leq N^{2\mathcal{E}_2} \right) \leq \exp\{-c_2(\log N)^2\}. \tag{4.11}$$

Observing that $0 \leq \mathcal{E}_0 \leq \alpha_0$, Equation (4.8) follows from (4.11) if it is proved that $\lim_{\alpha_0 \to 0} \mathcal{E}_2(\gamma_1, \gamma) = \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma)$ for an appropriate choice of γ_1 (in particular such that $\mathcal{E}_1(\gamma_1) > 0$). It is easily verified that, for a given γ, the quantity $\mathcal{E}_2(\gamma_1, \gamma)$ is maximized at $\gamma_1^* = \gamma\sigma_1^2(\alpha - \alpha_0)/(\Gamma_{12} - \sigma_1^2\alpha_0)$. Plugging these back in (4.10) shows that $\mathcal{E}_1(\gamma_1^*) > 0$ provided that $\gamma < \Gamma_{12}/\sigma_1 =: \gamma_{crit}$, with α_0 small enough (depending on γ). Furthermore, since $\mathcal{E}_2(\gamma_1^*,\gamma) = \mathcal{E}_0 + (1-\alpha_0) - \gamma^2/(\Gamma_{12} - \sigma_1^2\alpha_0)$, we obtain $\lim_{\alpha_0\to 0}\mathcal{E}_2(\gamma_1^*,\gamma) = \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma)$, which concludes the proof in the case $0<\gamma<\gamma_{crit}$.

If $\gamma_{crit} \le \gamma < \gamma_{max}$, the condition $\mathcal{E}_1(\gamma_1^*) > 0$ is violated as α_0 goes to zero. However, the previous arguments can easily be adapted and we refer to subsection 3.1.2 in [4] for more details. □

Proof of the lower bound of Theorem 2.1. Define $\gamma_i := i\gamma_{\max}/M$ for $0 \le i \le M$ (M will be chosen large enough). Notice that Proposition 4.3, Proposition 4.4 and the symmetry property of centered Gaussian random variables imply that the event

$$B_{N,M,\nu} \quad := \quad \bigcap_{i=0}^{M-1}\left\{|\mathcal{H}_N^{\psi,\delta}(\gamma_i)| \ge N^{2\mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma_i)-\nu/3}\right\}\bigcap\left\{\max_{v\in V_N^\delta}|\psi_v| \le \sqrt{\frac{2}{\pi}}\gamma_{\max}\log N^2\right\}$$

satisfies

$$\mathbb{P}(B_{N,M,\nu}) \longrightarrow 1, \qquad N\to\infty,$$

for all $M\in\mathbb{N}^*$ and all $\nu>0$. Since $|\mathcal{H}_N^{\psi,\delta}(\gamma_M)| = 0$ on $B_{N,M,\nu}$, we have

$$\begin{aligned}
Z_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta) &\ge \sum_{v\in V_N^\delta} \mathrm{e}^{\beta\psi_v} \ge \sum_{i=1}^{M}(|\mathcal{H}_N^{\psi,\delta}(\gamma_{i-1})| - |\mathcal{H}_N^{\psi,\delta}(\gamma_i)|)N^{2\sqrt{\frac{2}{\pi}}\gamma_{i-1}\beta}\\
&= \sum_{i=1}^{M-1}\left(N^{2\sqrt{\frac{2}{\pi}}\gamma_i\beta} - N^{2\sqrt{\frac{2}{\pi}}\gamma_{i-1}\beta}\right)|\mathcal{H}_N^{\psi,\delta}(\gamma_i)| + |\mathcal{H}_N^{\psi,\delta}(\gamma_0)|N^{2\sqrt{\frac{2}{\pi}}\gamma_0\beta} - |\mathcal{H}_N^{\psi,\delta}(\gamma_M)|N^{2\sqrt{\frac{2}{\pi}}\gamma_{M-1}\beta}\\
&\ge \frac{1}{2}\sum_{i=1}^{M-1} N^{2\sqrt{\frac{2}{\pi}}\gamma_i\beta}|\mathcal{H}_N^{\psi,\delta}(\gamma_i)|,
\end{aligned}$$

where the last inequality holds for N large enough. Define $P_\beta(\gamma) := \mathcal{E}^{(\alpha,\boldsymbol{\sigma})}(\gamma) + \sqrt{\frac{2}{\pi}}\beta\gamma$. On the event $B_{N,M,\nu}$, this estimate for the logarithm becomes

$$\begin{aligned}
f_{N,\rho}^{(\alpha,\boldsymbol{\sigma})}(\beta) &\ge \frac{1}{\log N^2}\log\left(\sum_{i=1}^{M-1} N^{2P_\beta(\gamma_i)}\right) - \frac{\nu}{6} + o_N(1)\\
&\ge \max_{1\le i\le M-1} P_\beta(\gamma_i) - \frac{\nu}{6} + o_N(1)\ .
\end{aligned}$$

Using the expression of $\mathcal{E}^{(\alpha,\boldsymbol{\sigma})}$ in Proposition 4.4 on the different intervals, it is easily checked by differentiation that $\max_{\gamma\in[0,\gamma_{\max}]} P_\beta(\gamma) = f^{(\alpha,\boldsymbol{\sigma})}(\beta)$. Furthermore, the continuity of $\gamma\mapsto P_\beta(\gamma)$ on $[0,\gamma_{\max}]$ yields

$$\max_{1\le i\le M-1} P_\beta(\gamma_i) \longrightarrow \max_{\gamma\in[0,\gamma_{\max}]} P_\beta(\gamma) = f^{(\alpha,\boldsymbol{\sigma})}(\beta), \qquad M\to\infty.$$

Therefore, choosing M large enough yields the result. □

5 Appendix

The conditional expectation of the GFF has nice features such as the Markov property, see e.g. Theorems 1.2.1 and 1.2.2 in [19] for a general statement on Markov fields constructed from symmetric Markov processes.

Lemma 5.1. *Let $B\subset A$ be subsets of $\mathbb{Z}^2$. Let $(\phi_v, v\in A)$ be a GFF on A. Then*

$$\mathbb{E}[\phi_v|\mathcal{F}_{B^c}] = \mathbb{E}[\phi_v|\mathcal{F}_{\partial B}], \qquad \forall v\in B,$$

and

$$(\phi_v - \mathbb{E}[\phi_v|\mathcal{F}_{\partial B}], v\in B)$$

has the law of a GFF on B. Moreover, if P_v is the law of a simple random walk starting at v and τ_B is the first exit time of B, we have

$$\mathbb{E}[\phi_v|\mathcal{F}_{\partial B}] = \sum_{u\in\partial B} P_v(S_{\tau_B} = u)\ \phi_u\ .$$

The following estimate on the Green function can be found as Lemma 2.2 in [16] and is a combination of Proposition 4.6.2 and Theorem 4.4.4 in [23].

Lemma 5.2. *There exists a function $a : \mathbb{Z}^2 \times \mathbb{Z}^2 \mapsto [0,\infty)$ of the form*

$$a(v,v') = \frac{2}{\pi}\log\|v-v'\| + \frac{2\gamma_0 \log 8}{\pi} + O(\|v-v'\|^{-2})$$

(where γ_0 denotes the Euler's constant) such that $a(v,v) = 0$ and

$$G_A(v,v') = E_v\left[a(v', S_{\tau_A})\right] - a(v,v')\ .$$

The following Slepian's comparison lemma for the tail of the maximum and the expectation of the log-partition function of two Gaussian fields can be found in [24] and in [22].

Lemma 5.3. *Let $(X_1,\cdots,X_N)$ and $(Y_1,\cdots,Y_N)$ be two centered Gaussian vectors in N variables such that*

$$\mathbb{E}[X_i^2] = \mathbb{E}[Y_i^2]\ \forall i, \qquad \mathbb{E}[X_iX_j] \geq \mathbb{E}[Y_iY_j]\ \forall i\neq j\ .$$

Then for all $\beta > 0$

$$\mathbb{E}\left[\log\sum_{i=1}^N e^{\beta X_i}\right] \leq \mathbb{E}\left[\log\sum_{i=1}^N e^{\beta Y_i}\right]\ ,$$

and for all $\lambda > 0$,

$$\mathbb{P}\left(\max_{i=1,\dots,N} X_i > \lambda\right) \leq \mathbb{P}\left(\max_{i=1,\dots,N} Y_i > \lambda\right)\ .$$

References

[1] M. Aizenman and P. Contucci, *On the stability of the quenched state in mean-field spin-glass models*, J. Statist. Phys. **92** (1998), no. 5-6, 765–783. MR-1657840

[2] L.-P. Arguin, *A dynamical characterization of Poisson-Dirichlet distributions*, Electron. Comm. Probab. **12** (2007), 283–290 (electronic). MR-2342707

[3] L.-P. Arguin and S. Chatterjee, *Random overlap structures: properties and applications to spin glasses*, Probab. Theory Related Fields **156** (2013), no. 1-2, 375–413. MR-3055263

[4] L.-P. Arguin and O. Zindy, *Poisson-Dirichlet statistics for the extremes of a log-correlated Gaussian field*, Ann. Appl. Probab. **24** (2014), no. 4, 1446–1481. MR-3211001

[5] E. Bacry and J. F. Muzy, *Log-infinitely divisible multifractal processes*, Comm. Math. Phys. **236** (2003), no. 3, 449–475. MR-2021198

[6] M. Biskup and O. Louidor, *Extreme local extrema of two-dimensional discrete gaussian free field*, Preprint, arxiv:1306.2602 (2013).

[7] E. Bolthausen, J.-D. Deuschel, and G. Giacomin, *Entropic repulsion and the maximum of the two-dimensional harmonic crystal*, Ann. Probab. **29** (2001), no. 4, 1670–1692. MR-1880237

[8] E. Bolthausen and A.-S. Sznitman, *Ten lectures on random media*, DMV Seminar, vol. 32, Birkhäuser Verlag, Basel, 2002. MR-1890289

[9] A. Bovier and I. Kurkova, *Derrida's generalised random energy models. I. Models with finitely many hierarchies*, Ann. Inst. H. Poincaré Probab. Statist. **40** (2004), no. 4, 439–480. MR-2070334

[10] ______, *Derrida's generalized random energy models. II. Models with continuous hierarchies*, Ann. Inst. H. Poincaré Probab. Statist. **40** (2004), no. 4, 481–495. MR-2070335

[11] M. Bramson, J. Ding, and O. Zeitouni, *Convergence in law of the maximum of the two-dimensional discrete gaussian free field*, Preprint, arxiv:1301.6669v2 (2013).

[12] D. Carpentier and P. Le Doussal, *Glass transition of a particle in a random potential, front selection in nonlinear renormalization group, and entropic phenomena in liouville and sinh-gordon models*, Phys. Rev. E **63** (2001), 026110.

[13] O. Daviaud, *Extremes of the discrete two-dimensional Gaussian free field*, Ann. Probab. **34** (2006), no. 3, 962–986. MR-2243875

[14] B. Derrida, *A generalisation of the random energy model that includes correlations between the energies*, J. Phys. Lett. **46** (1985), no. 3, 401–407.

[15] B. Derrida and H. Spohn, *Polymers on disordered trees, spin glasses, and traveling waves*, J. Statist. Phys. **51** (1988), no. 5-6, 817–840, New directions in statistical mechanics (Santa Barbara, CA, 1987). MR-971033

[16] J. Ding, *Exponential and double exponential tails for maximum of two-dimensional discrete Gaussian free field*, Probab. Theory Related Fields **157** (2013), no. 1-2, 285–299. MR-3101848

[17] J. Ding and O. Zeitouni, *Extreme values for two-dimensional discrete Gaussian free field*, Ann. Probab. **42** (2014), no. 4, 1480–1515. MR-3262484

[18] B. Duplantier, R. Rhodes, S. Sheffield, and V. Vargas, *Critical Gaussian multiplicative chaos: convergence of the derivative martingale*, Ann. Probab. **42** (2014), no. 5, 1769–1808. MR-3262492

[19] E. B. Dynkin, *Markov processes and random fields*, Bull. Amer. Math. Soc. (N.S.) **3** (1980), no. 3, 975–999. MR-585179

[20] Y. V. Fyodorov and J.-P. Bouchaud, *Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential*, J. Phys. A **41** (2008), no. 37, 372001, 12. MR-2430565

[21] ______, *Statistical mechanics of a single particle in a multiscale random potential: Parisi landscapes in finite-dimensional Euclidean spaces*, J. Phys. A **41** (2008), no. 32, 324009, 25. MR-2425780

[22] J.-P. Kahane, *Sur le chaos multiplicatif*, Ann. Sci. Math. Québec **9** (1985), no. 2, 105–150. MR-829798

[23] G. F. Lawler and V. Limic, *Random walk: a modern introduction*, Cambridge Studies in Advanced Mathematics, vol. 123, Cambridge University Press, Cambridge, 2010. MR-2677157

[24] M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 23, Springer-Verlag, Berlin, 1991, Isoperimetry and processes. MR-1102015

[25] R. Rhodes and V. Vargas, *Gaussian multiplicative chaos and applications: a review*, Probab. Surv. **11** (2014), 315–392. MR-3274356

[26] M. Talagrand, *Spin glasses: a challenge for mathematicians*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 46, Springer-Verlag, Berlin, 2003, Cavity and mean field models. MR-1993891

Acknowledgments. The authors would like to thank the Centre International de Rencontres Mathématiques in Luminy for hospitality and financial support during part of this work. The authors also wish to thank the referees for useful comments that improved the presentation of the paper.