# Quenched limits and fluctuations of the empirical measure for plane rotators in random media

Eric Luçon[*]eric.lucon@etu.upmc.fr

### Abstract

The Kuramoto model has been introduced to describe synchronization phenomena observed in groups of cells, individuals, circuits, etc. The model consists of $N$ interacting oscillators on the one dimensional sphere $\mathbf{S}^1$, driven by independent Brownian Motions with constant drift chosen at random. This quenched disorder is chosen independently for each oscillator according to the same law $\mu$. The behaviour of the system for large $N$ can be understood via its empirical measure: we prove here the convergence as $N \to \infty$ of the quenched empirical measure to the unique solution of coupled McKean-Vlasov equations, under weak assumptions on the disorder $\mu$ and general hypotheses on the interaction. The main purpose of this work is to address the issue of quenched fluctuations around this limit, motivated by the dynamical properties of the disordered system for large but fixed $N$: hence, the main result of this paper is a quenched Central Limit Theorem for the empirical measure. Whereas we observe a self-averaging for the law of large numbers, this no longer holds for the corresponding central limit theorem: the trajectories of the fluctuations process are sample-dependent.

[*]Laboratoire de Probabilités et Modèles Aléatoires (CNRS U.M.R. 7599) and Université Paris 6 – Pierre et Marie Curie, U.F.R. Mathematiques, Case 188, 4 place Jussieu, 75252 Paris cedex 05, France

# 1 Introduction

In this work, we study the fluctuations in the Kuramoto model, which is a particular case of interacting diffusions with a mean field Hamiltonian that depends on a random disorder. The Kuramoto model was first introduced in the 70's by Yoshiki Kuramoto ([17], see also [1] and references therein) to describe the phenomenon of synchronization in biological or physical systems. More precisely, the Kuramoto model is a particular case of a system of $N$ oscillators (considered as elements of the one-dimensional sphere $\mathbf{S}^1 := \mathbf{R}/2\pi\mathbf{Z}$) solutions to the following SDE:

$$dx_t^{i,N} = \frac{1}{N}\sum_{j=1}^N b(x^{i,N}, x^{j,N}, \omega_j)\,dt + c(x^{i,N}, \omega_i)\,dt + dB_t^i, \quad t \in [0,T], \ i = 1\ldots N, \qquad (1)$$

where $T > 0$ is a fixed (but arbitrary) time, $b$ and $c$ are smooth periodic functions. The Kuramoto case corresponds to a sine interaction ($b(x,y,\omega) = K\sin(y-x)$ and $c(x,\omega) = \omega$). This case which has the particularity of being rotationally invariant (namely, if $(x^{i,N})_i$ is a solution of the Kuramoto model, $(x^{i,N} + c)_i$, $c$ a constant, is also a solution), will be referred to in this work as the *sine-model*.

The parameter $K > 0$ is the coupling strength and $(\omega_j)$ is a sequence of randomly chosen reals (being i.i.d. realizations of a law $\mu$). The sequence $(\omega_j)_j$ is called *disorder* and represents the fact that the behaviour of each rotator $x^{j,N}$ depends on its own local frequency $\omega_j$.

Due to the mean field character of (1), the behaviour of the system can be understood via its empirical measure $\nu^N$, process with values in $\mathcal{M}_1(\mathbf{S}^1 \times \mathbf{R})$, that is the set of probability measures on oscillators and disorder:

$$\forall(\omega) \in \mathbf{R^N}, \ \forall t \in [0,T], \quad \nu_t^{N,(\omega)} := \frac{1}{N}\sum_{j=1}^N \delta_{(x_t^{j,N}, \omega_j)},$$

where $(\omega) = (\omega_j)_{j\geq 1}$ is a fixed sequence of disorder in $\mathbf{R^N}$.

The purpose of this paper is to address the issue of both convergence and fluctuations of the empirical measure, as $N \to \infty$; thus the main theorem of this paper (Theorem 2.10) concerns a Central Limit Theorem in a quenched set-up (namely the quenched fluctuations of $\nu^N$ around its limit).

Some heuristic results have been obtained in the physical literature ([1] and references therein) concerning the convergence of the empirical measure, as $N \to \infty$, to a time-dependent measure $(P_t(dx, d\omega))_{t\in[0,T]}$, whose density w.r.t. Lebesgue measure at time $t$, $q_t(x,\omega)$ is the solution of a deterministic non-linear McKean-Vlasov equation (see Eq.(5)). It is well understood ([1], [11]) that crucial features of this equation are captured in the sine-model by order parameters $r_t$ and $\psi_t$ defined by:

$$r_t e^{i\psi_t} = \int_{\mathbf{S}^1 \times \mathbf{R}} e^{ix} q_t(x,\omega)\,dx\mu(d\omega).$$

The quantity $r_t$ captures in fact the degree of synchronization of a solution (the profile $q_t \equiv \frac{1}{2\pi}$ for example corresponds to $r = 0$ and represents a total lack of synchronization) and $\psi_t$ identifies the center of synchronization: this is true and rather intuitive for unimodal profiles. Moreover ([22], see also [11], p.118) it turns out that if $\mu$ is symmetric, all the stationary solutions can be parameterized (up to rotation) by the stationary version $r$ of $r_t$ which must satisfy a fixed point

relation $r = \Psi_{K,\mu}(r)$, with $\Psi_{K,\mu}(\cdot)$ an explicit function such that $\Psi_{K,\mu}(0) = 0$. For $K$ small, $r = 0$ is the only solution of such an equation and the system is not synchronized, but for $K$ large, non-trivial solutions appear (synchronization). In the easiest instances, such a non-trivial solution is unique (in the sense that $r = \Psi_{K,\mu}(r)$ admits a unique non-zero solution but of course one obtains an infinite number of solutions by rotation invariance that is $\psi$ can be chosen arbitrarily).

In [9], Dai Pra and den Hollander have rigorously shown the convergence of the averaged empirical measure $L^N \in \mathscr{M}_1(\mathscr{C}([0,T], \mathbf{S}^1) \times \mathbf{R})$ (probability measure on the whole trajectories and the disorder):

$$L^N = \frac{1}{N} \sum_{j=1}^{N} \delta_{(x^{j,N}, \omega_j)}.$$

This convergence of the law of $L^N$ *under the joint law of the oscillators and the disorder* is shown via an averaged large deviations principle in the case where $b(x, y, \omega) = K \cdot f(y - x)$ and $c(x, \omega) = g(x, \omega)$ for $f$ and $g$ smooth and *bounded* functions. As a corollary, it is deduced in [9] the convergence of $L_N$ and of $\nu^N$, via a contraction principle, in the averaged set-up. In the case of unbounded disorder, the same proof can be generalized (thesis in preparation) under the following assumption:

$$\forall t > 0, \int_{\mathbf{R}} e^{t|\omega|} \mu(d\omega) < \infty. \tag{$H_\mu^A$}$$

One aim of this paper is to obtain the limit of $\nu^{N,(\omega)}$ in the *quenched* model, namely for a *fixed* realization of the disorder ($\omega$). This result can be deduced from the large deviations estimates in [9], via a Borel-Cantelli argument, but our result is more direct and works under the much weaker assumption on $\mu$, $\int_{\mathbf{R}} |\omega| \mu(d\omega) < \infty$.

A crucial aspect of the quenched convergence result, which is a law of large numbers, is that it shows the *self-averaging* character of this limit: every typical disorder configuration leads as $N \to \infty$ to the same evolution equation.

However, it seems quite clear even at a superficial level that if we consider the central limit theorem associated to this convergence, self-averaging does not hold since the fluctuations of the disorder compete with the dynamical fluctuations. This leads for example to a remarkable phenomenon (pointed out e.g. in [2] on the basis of numerical simulations): even if the distribution $\mu$ is symmetric, the fluctuations of a fixed chosen sample of the disorder makes it *not symmetric* and thus the center of the synchronization of the system *slowly* (i.e. with a speed of order $1/\sqrt{N}$) rotates in one direction and with a speed that depends on the sample of the disorder (Fig. 1 and 2). This non-self averaging phenomenon can be tackled in the sine-model by computing the finite-size order parameters (Fig. 2):

$$r_t^{N,(\omega)} e^{i\psi_t^{N,(\omega)}} = \frac{1}{N} \sum_{j=1}^{N} e^{ix_t^{j,N}} = \left\langle \nu_t^{N,(\omega)}, e^{ix} \right\rangle. \tag{2}$$

As a step toward understanding this non self-averaging phenomenon, the second and main goal of this paper is to establish a fluctuations result around the McKean-Vlasov limit in the quenched set-up (see Theorem 2.10). A central limit theorem for the averaged model is addressed in [9], applying techniques introduced by Bolthausen [6]. A fluctuations theorem may also be found in [8] for a
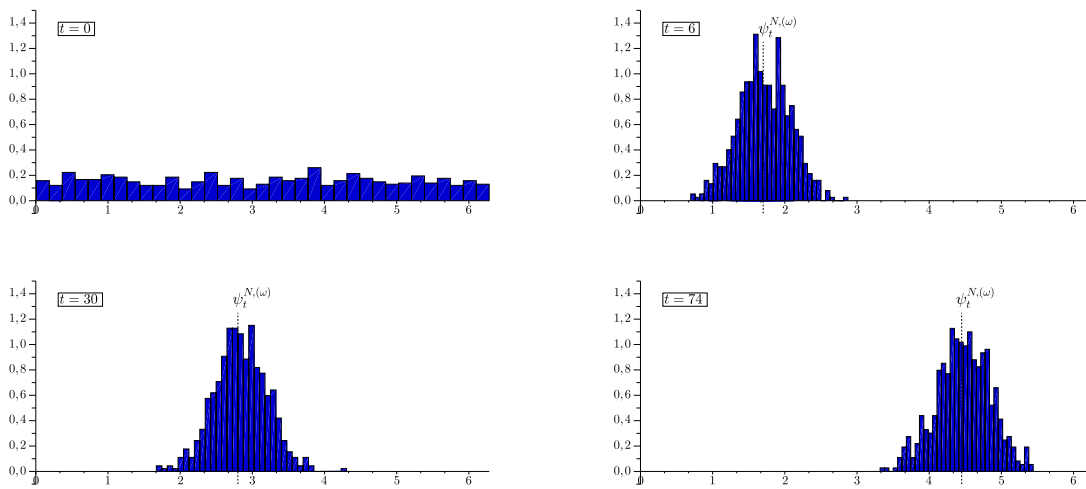
Figure 1: We plot here the evolution of the marginal on $\mathbf{S}^1$ of $\nu^{N,(\omega)}$ for $N = 600$ oscillators in the sine-model ($\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$, $K = 6$). The oscillators are initially chosen independently and uniformly on $[0, 2\pi]$ independently of the disorder. First the dynamics leads to synchronization of the oscillators ($t = 6$) to a profile which is close to a non-trivial stationary solution of McKean-Vlasov equation. We then observe that the center $\psi_t^{N,(\omega)}$ of this density moves to the right with an approximately constant speed; what is more, this speed of fluctuation turns out to be sample-dependent (see Fig. 2).

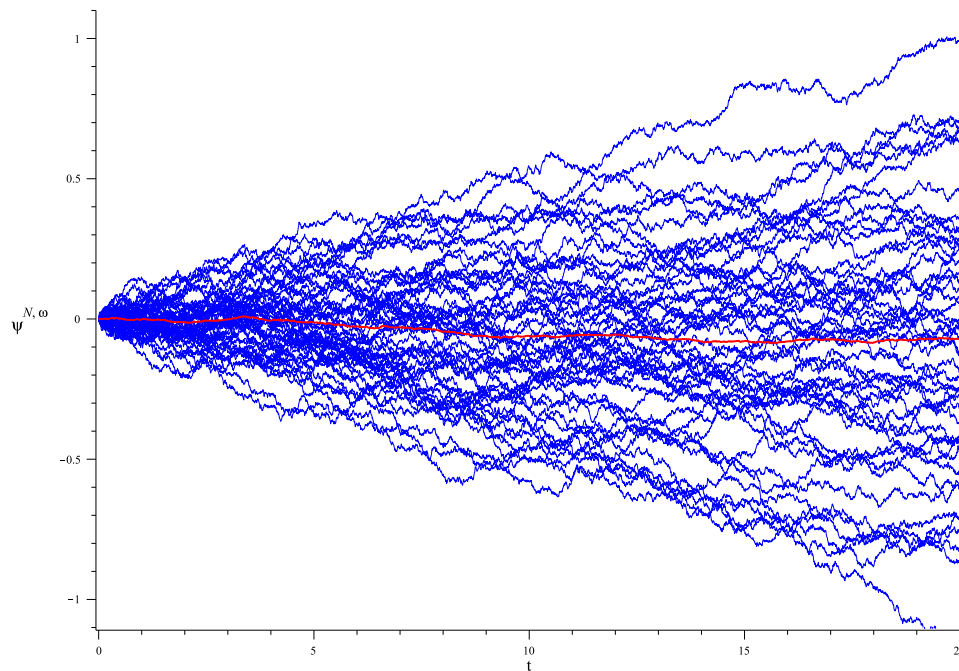Figure 2: Trajectories of the center of synchronization $\psi^{N,(\omega)}$ in the sine-model for different realizations of the disorder ($\mu = \frac{1}{2}(\delta_{-0.5} + \delta_{0.5})$, $K = 4$, $N = 400$). We observe here the non self-averaging phenomenon: direction and speed of the center depend on the choice of the initial $N$-sample of the disorder. Moreover, these simulations are compatible with speeds of order $1/\sqrt{N}$.

model of social interaction in an averaged set-up. Here we prove convergence in law of the *quenched fluctuations process*

$$\eta^{N,(\omega)} := \sqrt{N}\left(\nu^{N,(\omega)} - P\right),$$

seen as a continuous process in the Schwartz space $\mathscr{S}'$ of tempered distributions on $\mathbf{S}^1 \times \mathbf{R}$ to the solution of an Ornstein-Uhlenbeck process. The quenched convergence is here understood as a weak convergence *in law w.r.t. the disorder* and is more technically involved than the convergence in the averaged system. The main techniques we exploit have been introduced by Fernandez and Méléard [12] and Hitsuda and Mitoma [15], who studied similar fluctuations in the case without disorder. In [7], A Large Deviation Principle is also proved. We refer to Section 4 for detailed definitions.

While numerical computations of the trajectories of the limit process of fluctuations clearly show a non self-averaging phenomenon, the dynamical properties of the fluctuations process that we find are not completely understood so far. Progress in this direction requires a good understanding of the spectral properties of the linearized operator of McKean-Vlasov equation around its non-trivial stationary solution; the stability of the non-synchronized solution $q \equiv \frac{1}{2\pi}$ has been treated by Strogatz and Mirollo in [24]. In the particular case without disorder, spectral properties of the evolution operator linearized around the non-trivial stationary solution are obtained in [3], but the case with a general distribution $\mu$ needs further investigations.

This work is organized as follows: Section 2 introduces the model and the main results. Section 3 focuses on the quenched convergence of $\nu_N$. In Section 4, the quenched Central Limit Theorem is proved. The last section 5 applies the fluctuations result to the behaviour of the order parameters in the sine-model.

## 2 Notations and main results

### 2.1 Notations

- if $X$ is a metric space, $\mathscr{B}_X$ will be its Borel $\sigma$-field,

- $\mathscr{C}_b(X)$ (resp. $\mathscr{C}_b^p(X)$, $p = 1,\ldots,\infty$), the set of bounded continuous functions (resp. bounded continuous with bounded continuous derivatives up to order $p$) on $X$, ($X$ will be often $\mathbf{S}^1 \times \mathbf{R}$),

- $\mathscr{C}_c(X)$ (resp. $\mathscr{C}_c^p(X)$, $p = 1,\ldots,\infty$), the set of continuous functions with compact support (resp. continuous with compact support with continuous derivatives up to order $p$) on $X$,

- $\mathbf{D}([0,T],X)$, the set of right-continuous with left limits functions with values on $X$, endowed with the Skorokhod topology,

- $\mathscr{M}_1(Y)$, the set of probability measures on $Y$ ($Y$ topological space, with a regular $\sigma$-field $\mathscr{B}$),

- $\mathscr{M}_F(Y)$, the set of finite measures on $Y$,

- $(\mathscr{M}_1(Y),w)$: $\mathscr{M}_1(Y)$ endowed with the topology of weak convergence, namely the coarsest topology on $\mathscr{M}_1(Y)$ such that the evaluations $\nu \mapsto \int f \, d\nu$ are continuous, where $f$ are bounded continuous,

- $(\mathcal{M}_1(Y), v)$: $\mathcal{M}_1(Y)$ endowed with the topology of vague convergence, namely the coarsest topology on $\mathcal{M}_1(Y)$ such that the evaluations $v \mapsto \int f \, dv$ are continuous, where $f$ are continuous with compact support.

We will use $C$ as a constant which may change from a line to another.

## 2.2 The model

We consider the solutions of the following system of SDEs:

for $i = 1, \ldots, N$, for $T > 0$, for all $t \leq T$,

$$x_t^{i,N} = \xi^i + \frac{1}{N} \sum_{j=1}^{N} \int_0^t b(x_s^{i,N}, x_s^{j,N}, \omega_j) \, ds + \int_0^t c(x_s^{i,N}, \omega_i) \, ds + B_t^i, \tag{3}$$

where the initial conditions $\xi^i$ are independent and identically distributed with law $\lambda$, and independent of the Brownian motion $(B) = (B^i)_{i \geq 1}$, and where $b$ (resp. $c$) is a smooth function, $2\pi$-periodic w.r.t. the two first (resp. first) variables. The disorder $(\omega) = (\omega_i)_{i \geq 1}$ is a realization of i.i.d. random variables with law $\mu$.

*Remark 2.1.*
The assumption that the random variables $(\omega_i)$ are independent will not always be necessary and will be weakened when possible.

Instead of considering $x^{i,N}$ as elements of $\mathbf{R}$, we will consider their projection on $\mathbf{S}^1$. For simplicity, we will keep the same notation $x^{i,N}$ for this projection[1].

We introduce the empirical measure $v^N$ (on the trajectories and disorder):

**Definition 2.2.**
*For all $t \leq T$, for a fixed trajectory $(x^1, \ldots, x^N) \in \mathscr{C}([0, T], (\mathbf{S}^1)^N)$ and a fixed sequence of disorder $(\omega)$, we define an element of $\mathcal{M}_1(\mathbf{S}^1 \times \mathbf{R})$ by:*

$$v_t^{N,(\omega)} = \frac{1}{N} \sum_{i=1}^{N} \delta_{(x_t^{i,N}, \omega_i)}.$$

Finally, we introduce the fluctuations process $\eta^{N,(\omega)}$ of $v^{N,(\omega)}$ around its limit $P$ (see Th. 2.5):

**Definition 2.3.**
*For all $t \leq T$, for fixed $(\omega) \in \mathbf{R}^{\mathbf{N}}$, we define:*

$$\eta_t^{N,(\omega)} = \sqrt{N} \left( v_t^{N,(\omega)} - P_t \right).$$

Throughout this article, we will denote as $\mathbf{P}$ the law of the sequence of Brownian Motions and as $\mathbb{P}$ the law of the sequence of the disorder. The corresponding expectations will be denoted as $\mathbf{E}$ and $\mathbb{E}$ respectively.

---

[1]See Remarks 2.7 and 2.12 for possible generalizations to the non-compact case.

### 2.3 Main results

#### 2.3.1 Quenched convergence of the empirical measure

In [9], Dai Pra and den Hollander are interested in the *averaged* model, i.e. in the convergence in law of the empirical measure *under the joint law of both oscillators and disorder*. The model studied here, which is more interesting as far as the biological applications are concerned is *quenched*: for a fixed realization of the disorder ($\omega$), do we have the convergence of the empirical measure? Moreover the convergence is shown under weaker assumptions on the moments of the disorder.

We consider here the general case where $b(x, y, \omega)$ is bounded, Lipschitz-continuous, and $2\pi$-periodic w.r.t. the two first variables. $c$ is assumed to be Lipschitz-continuous w.r.t. its first variable, but not necessarily bounded (see the sine-model, where $c(x, \omega) = \omega$). We also suppose that the function $\omega \mapsto S(\omega) := \sup_{x \in \mathbf{S}^1} |c(x, \omega)|$ is continuous (this is in particular true if $c$ is uniformly continuous w.r.t. to both variables $(x, \omega)$, and obvious in the sine-model where $S(\omega) = |\omega|$). The Lipschitz bounds for $b$ and $c$ are supposed to be uniform in $\omega$.

The disorder ($\omega$), is assumed to be a sequence of identically distributed random variables (but not necessarily independent), such that the law of each $\omega_i$ is $\mu$. We suppose that the sequence ($\omega$) satisfies the following property: for $\mathbb{P}$-almost every sequence ($\omega$),

$$\frac{1}{N} \sum_{i=1}^{N} \sup_{x \in \mathbf{S}^1} |c(x, \omega_i)| \to_{N \to \infty} \int \sup_{x \in \mathbf{S}^1} |c(x, \omega)| \mu(\mathrm{d}\omega) < \infty. \tag{$H_\mu^Q$}$$

We make the following hypothesis on the initial empirical measure:

$$\nu_0^{N,(\omega)} \xrightarrow{N \to \infty} \nu_0, \quad \text{in law, in } (\mathcal{M}_1(\mathbf{S}^1 \times \mathbf{R}), w). \tag{$H_0$}$$

*Remark 2.4.* 1. The required hypotheses about the disorder and the initial conditions are weaker than for the large deviation principle:

- the (identically distributed) variables ($\omega_i$) need not be independent: we simply need a convergence (similar to a law of large numbers) only concerning the function $S$,
- Condition ($H_\mu^Q$) is weaker than ($H_\mu^A$) on page 796; for the sine-model, ($H_\mu^Q$) reduces to $\int |\omega| \mu(\mathrm{d}\omega) < \infty$,
- the initial values need not be independent, we only assume a convergence of the empirical measure.

2. The hypothesis ($H_\mu^Q$) is verified, for example, in the case of i.i.d. random variables, or in the case of an ergodic stationary Markov process.

3. Under ($H_0$), the second marginal of $\nu_0$ is $\mu$.

In Section 3, we show the following:

**Theorem 2.5.**
*Under the hypothesis ($H_0$) and ($H_\mu^Q$), for $\mathbb{P}$-almost every sequence ($\omega_i$), the random variable $\nu^{N,(\omega)}$ converges in law to $P$, in the space $\mathbf{D}([0, T], (\mathcal{M}_1(\mathbf{S}^1 \times \mathbf{R}), w))$, where $P$ is the only solution of the*

*following weak equation (for every $f$ continuous bounded on $\mathbf{S}^1 \times \mathbf{R}$, twice differentiable, with bounded derivatives):*

$$\langle P_t, f \rangle = \langle v_0, f \rangle + \frac{1}{2} \int_0^t ds \, \langle P_s, f'' \rangle + \int_0^t ds \, \langle P_s, f'(b[\cdot, P_s] + c) \rangle, \tag{4}$$

*where*

$$b[x, m] = \int b(x, y, \pi) m(dy, d\pi).$$

*Moreover, with the same hypotheses, the law of $v^N$ under the joint law of the oscillators and disorder (averaged model) converges weakly to $P$ as well.*

*Remark 2.6.*
An easy calculation shows that $P$ can be considered as a weak solution to the family of coupled McKean-Vlasov equations (see [9]):

1. $P$ can be written as $P(dx, d\omega) = \mu(\omega) P^\omega(dx)$,

2. if we define $q_t^\omega$ through $P_t(dx, d\omega) = \mu(\omega) q_t^\omega(dx)$, $q_t^\omega$ is the unique weak solution of the McKean-Vlasov equation:

$$\frac{d}{dt} q_t^\omega = \mathscr{L}^\omega q_t^\omega, \quad q_0^\omega = \lambda. \tag{5}$$

where, $\mathscr{L}^\omega$ is the following differential operator:

$$\mathscr{L}^\omega q_t^\omega = -\frac{\partial}{\partial x} \left[ \left( \int_{\mathbf{R}} b(x, y, \pi) q_t^\pi(dy) \mu(d\pi) + c(x, \omega) \right) q_t^\omega \right] + \frac{1}{2} \frac{\partial^2}{\partial x^2} q_t^\omega. \tag{6}$$

We insist on the fact that Eq. (5) is indeed a (possibly) infinite system of coupled non-linear PDEs. To fix ideas, one may consider the simple case where $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$. Then (5) reduces to two equations (one for $+1$, the other for $-1$) which are coupled via the averaged measure $\frac{1}{2}(q_t^{+1} + q_t^{-1})$. But for more general situations ($\mu = \mathscr{N}(0, 1)$ say) this would consist of an infinite number of coupled equations.

*Remark 2.7 (Generalization to the non compact case).*
The assumption that the state variables are in $\mathbf{S}^1$, although motivated by the Kuramoto model, is not absolutely essential: Theorem 2.5 still holds in the non-compact case (e.g. when $\mathbf{S}^1$ is replaced by $\mathbf{R}^d$), under the additional assumptions of boundedness of $x \mapsto |c(x, \omega)|$ and the first finite moment of the initial condition: $\int |x| \lambda(dx) < \infty$.

We know turn to the statement of the main result of the paper: Theorem 2.10.

### 2.3.2 Quenched fluctuations of the empirical measure

Theorem 2.5 says that for $\mathbb{P}$-almost every realization $(\omega)$ of the disorder, we have the convergence of $\nu^{N,(\omega)}$ towards $P$, which is a law of large numbers. We are now interested in the corresponding Central Limit Theorem associated to this convergence, namely, for a *fixed* realization of the disorder $(\omega)$, in the asymptotic behaviour, as $N \to \infty$ of the fluctuations field $\eta^{N,(\omega)}$ taking values in the set of signed measures:

$$\forall t \in [0, T], \ \eta_t^{N,(\omega)} := \sqrt{N} \left( \nu_t^{N,(\omega)} - P_t \right).$$

In the case with no disorder, such fluctuations have already been studied by numerous authors (eg. Sznitman [25], Fernandez-Méléard [12], Hitsuda-Mitoma [15]). More particularly, Fernandez and Méléard show the convergence of the fluctuations field in an appropriate Sobolev space to an Ornstein-Uhlenbeck process.

Here, we are interested in the *quenched* fluctuations, in the sense that the fluctuations are studied for fixed realizations of the disorder. We will prove a weak convergence of the law of the process $\eta^{N,(\omega)}$, *in law w.r.t. the disorder*.

In addition to the hypothesis made in §2.3.1, we make the following assumptions about $b$ and $c$ (where $\mathscr{D}_p$ is the set of all differential operators of the form $\partial_{u^k} \partial_{\pi^l}$ with $k + l \leq p$):

$$\begin{cases} b \in \mathscr{C}_b^\infty(\mathbf{S}^1 \times \mathbf{R}), \quad c \in \mathscr{C}^\infty(\mathbf{S}^1 \times \mathbf{R}), \\[2mm] \exists \alpha > 0, \ \sup_{D \in \mathscr{D}_6} \int_{\mathbf{R}} \dfrac{\sup_{u \in \mathbf{S}^1} |Dc(u, \pi)|^2}{1 + |\pi|^{2\alpha}} \, d\pi < \infty, \end{cases} \tag{$H_{b,c}$}$$

Furthermore, we make the following assumption about the law of the disorder ($\alpha$ is defined in $(H_{b,c})$):

$$\text{the } (\omega_j) \text{ are i.i.d. and } \int_{\mathbf{R}} |\omega|^{4\alpha} \mu(d\omega) < \infty. \tag{$H_\mu^{\mathscr{F}}$}$$

*Remark 2.8.*
The regularity hypothesis about $b$ and $c$ can be weakened (namely $b \in \mathscr{C}_b^n(\mathbf{S}^1 \times \mathbf{R})$ and $c \in \mathscr{C}^m(\mathbf{S}^1 \times \mathbf{R})$ for sufficiently large $n$ and $m$) but we have kept $m = n = \infty$ for the sake of clarity.

*Remark 2.9.*
In the case of the sine-model, Hypothesis $(H_{b,c})$ is satisfied with $\alpha = 2$ for example.

In order to state the fluctuations theorem, we need some further notations: for all $s \leq T$, let $\mathscr{L}_s$ be the second order differential operator defined by

$$\mathscr{L}_s(\varphi)(y, \pi) := \frac{1}{2} \varphi''(y, \pi) + \varphi'(y, \pi)(b[y, P_s] + c(y, \pi)) + \langle P_s, \varphi'(\cdot, \cdot) b(\cdot, y, \pi) \rangle.$$

Let $W$ the Gaussian process with covariance:

$$\mathbf{E}(W_t(\varphi_1) W_s(\varphi_2)) = \int_0^{s \wedge t} \langle P_u, \varphi_1' \varphi_2' \rangle \, du. \tag{7}$$

For all $\varphi_1, \varphi_2$ bounded and continuous on $\mathbf{S}^1 \times \mathbf{R}$, let

$$\Gamma_1(\varphi_1, \varphi_2) = \int_{\mathbf{R}} \mathrm{Cov}_\lambda \left( \varphi_1(\cdot, \omega), \varphi_2(\cdot, \omega) \right) \mu(\mathrm{d}\omega), \tag{8}$$

$$= \int_{\mathbf{S}^1 \times \mathbf{R}} \left( \varphi_1 - \int_{\mathbf{S}^1} \varphi_1(\cdot, \omega) \mathrm{d}\lambda \right) \left( \varphi_2 - \int_{\mathbf{S}^1} \varphi_2(\cdot, \omega) \mathrm{d}\lambda \right) \lambda(\mathrm{d}x)\mu(\mathrm{d}\omega),$$

and

$$\Gamma_2(\varphi_1, \varphi_2) = \mathrm{Cov}_\mu \left( \int_{\mathbf{S}^1} \varphi_1 \mathrm{d}\lambda, \int_{\mathbf{S}^1} \varphi_2 \mathrm{d}\lambda \right), \tag{9}$$

$$= \int_{\mathbf{R}} \left( \int_{\mathbf{S}^1} \varphi_1 \mathrm{d}\lambda - \int_{\mathbf{S}^1 \times \mathbf{R}} \varphi_1 \mathrm{d}\lambda \, \mathrm{d}\mu \right) \left( \int_{\mathbf{S}^1} \varphi_2 \mathrm{d}\lambda - \int_{\mathbf{S}^1 \times \mathbf{R}} \varphi_2 \mathrm{d}\lambda \, \mathrm{d}\mu \right) \mathrm{d}\mu.$$

For fixed $(\omega)$, we may consider $\mathcal{H}_N(\omega)$, the law of the process $\eta^{N,(\omega)}$; $\mathcal{H}_N(\omega)$ belongs to $\mathcal{M}_1(\mathcal{C}([0,T], \mathcal{S}'))$, where $\mathcal{S}'$ is the usual Schwartz space of tempered distributions on $\mathbf{S}^1 \times \mathbf{R}$. We are here interested in the law of the random variable $(\omega) \mapsto \mathcal{H}_N(\omega)$ which is hence an element of $\mathcal{M}_1(\mathcal{M}_1(\mathcal{C}([0,T], \mathcal{S}')))$.

The main theorem (which is proved in Section 4) is the following:

**Theorem 2.10 (Fluctuations in the quenched model).**
*Under $(H_\mu^{\mathcal{F}})$, $(H_{b,c})$, the sequence $(\omega) \mapsto \mathcal{H}_N(\omega)$ converges in law to the random variable $\omega \mapsto \mathcal{H}(\omega)$, where $\mathcal{H}(\omega)$ is the law of the solution to the Ornstein-Uhlenbeck process $\eta^\omega$ solution in $\mathcal{S}'$ of the following equation:*

$$\eta_t^\omega = X(\omega) + \int_0^t \mathcal{L}_s^* \eta_s^\omega \, \mathrm{d}s + W_t, \tag{10}$$

*where, $\mathcal{L}_s^*$ is the formal adjoint operator of $\mathcal{L}_s$ and for all fixed $\omega$, $X(\omega)$ is a non-centered Gaussian process with covariance $\Gamma_1$ and with mean value $C(\omega)$. As a random variable in $\omega$, $\omega \mapsto C(\omega)$ is a Gaussian process with covariance $\Gamma_2$. Moreover, $W$ is independent on the initial value $X$.*

*Remark 2.11.*
In the evolution (10), the linear operator $\mathcal{L}_s^*$ is deterministic ; the only dependence in $\omega$ lies in the initial condition $X(\omega)$, through its non trivial means $C(\omega)$. However, numerical simulations of trajectories of $\eta^\omega$ (see Fig. 3) clearly show a non self-averaging phenomenon, analogous to the one observed in Fig 2: $\eta_t^\omega$ not only depends on $\omega$ through its initial condition $X(\omega)$, but also for all positive time $t > 0$.

Understanding how the deterministic operator $\mathcal{L}_s^*$ propagates the initial dependence in $\omega$ on the whole trajectory is an intriguing question which requires further investigations (work in progress). In that sense, one would like to have a precise understanding of the spectral properties of $\mathcal{L}_s^*$, which appears to be deeply linked to the differential operator in McKean-Vlasov equation (6) linearized around its non-trivial stationary solution.

*Remark 2.12 (Generalization to the non-compact case).*
As in Remark 2.7, it is possible to extend Theorem 2.10 to the (analogous but more technical) case

where $\mathbf{S}^1$ is replaced by $\mathbf{R}^d$. To this purpose, one has to introduce an additional weight $(1+|x|^\alpha)^{-1}$ in the definition of the Sobolev norms in Section 4 and to suppose appropriate hypothesis concerning the first moments of the initial condition $\lambda$ ($\int |x|^\beta \lambda(\mathrm{d}x) < \infty$ for a sufficiently large $\beta$).

### 2.3.3 Fluctuations of the order parameters in the Kuramoto model

For given $N \geq 1$, $t \in [0, T]$ and disorder $(\omega) \in \mathbf{R}^N$, let $r_t^{N,(\omega)} > 0$ and $\zeta_t^{N,(\omega)} \in \mathbf{S}^1$ such that

$$r_t^{N,(\omega)} \zeta_t^{N,(\omega)} = \frac{1}{N} \sum_{j=1}^N e^{ix_t^{j,N}} = \left\langle \nu_t^{N,(\omega)}, e^{ix} \right\rangle.$$

**Proposition 2.13 (Convergence and fluctuations for $r_t^{N,(\omega)}$).**
*We have the following:*

1.  *Convergence of $r_t^{N,(\omega)}$: For $\mathbb{P}$-almost every realization of the disorder, $r^{N,(\omega)}$ converges in law in $\mathscr{C}([0,T], \mathbf{R})$, to $r$ defined by*

    $$t \in [0, T] \mapsto r_t := \left( \langle P_t, \cos(\cdot) \rangle^2 + \langle P_t, \sin(\cdot) \rangle^2 \right)^{\frac{1}{2}}.$$

2.  *If $r_0 > 0$ then*
    $$\forall t \in [0, T], \quad r_t > 0. \tag{$H_r$}$$

3.  *Fluctuations of $r_t^{N,(\omega)}$ around its limit: Let*

    $$t \mapsto \mathscr{R}_t^{N,(\omega)} := \sqrt{N} \left( r_t^{N,(\omega)} - r_t \right)$$

    *be the fluctuations process. For fixed disorder $(\omega)$, let $\mathfrak{R}^{N,(\omega)} \in \mathscr{M}_1(\mathscr{C}([0,T], \mathbf{R}))$ be the law of $\mathscr{R}^{N,(\omega)}$. Then, under $(H_r)$, the random variable $(\omega) \mapsto \mathfrak{R}^{N,(\omega)}$ converges in law to the random variable $\omega \mapsto \mathfrak{R}^\omega$, where $\mathfrak{R}^\omega$ is the law of $\mathscr{R}^\omega :=$ $\frac{1}{r} \left( \langle P, \cos(\cdot) \rangle \cdot \langle \eta^\omega, \cos(\cdot) \rangle + \langle P, \sin(\cdot) \rangle \cdot \langle \eta^\omega, \sin(\cdot) \rangle \right)$.*

*Remark 2.14.*
In simpler terms, this double convergence in law corresponds for example to the convergence in law of the corresponding characteristic functions (since the tightness is a direct consequence of the tightness of the process $\eta$); i.e. for $t_1, \dots, t_p \in [0, T]$ ($p \geq 1$) the characteristic function of $(\mathscr{R}_{t_1}^{N,(\omega)}, \dots, \mathscr{R}_{t_p}^{N,(\omega)})$ for fixed $(\omega)$ converges in law, as a random variable in $(\omega)$, to the random characteristic function of $(\mathscr{R}_{t_1}^\omega, \dots, \mathscr{R}_{t_p}^\omega)$.

**Proposition 2.15 (Convergence and fluctuations for $\zeta^{N,(\omega)}$).**
*We have the following:*

1.  *Convergence of $\zeta^{N,(\omega)}$: Under $(H_r)$, for $\mathbb{P}$-almost every realization of the disorder $(\omega)$, $\zeta^{N,(\omega)}$ converges in law to $\zeta : t \in [0, T] \mapsto \zeta_t := \frac{\langle P_t, e^{ix} \rangle}{r_t}$,*
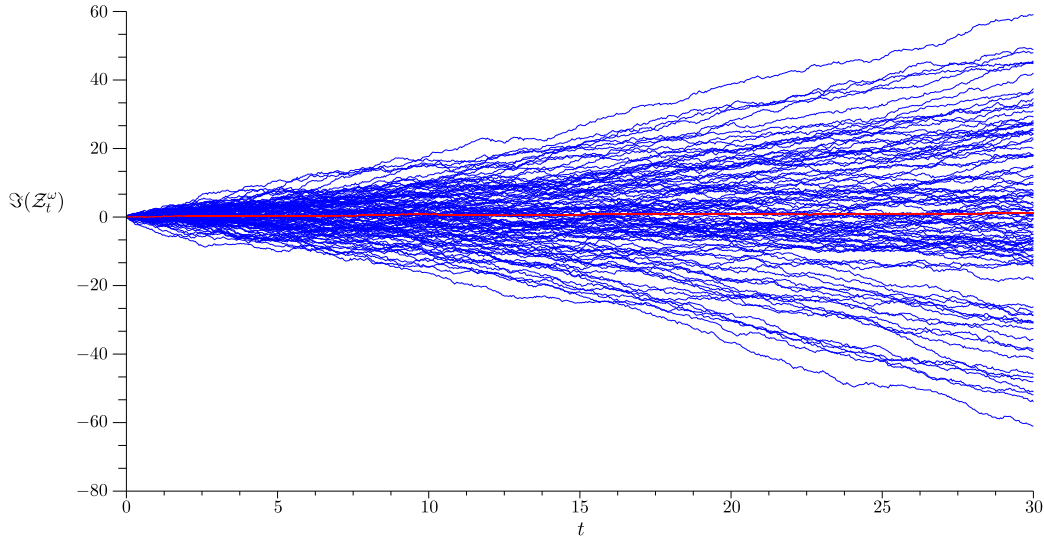
Figure 3: We plot here the evolution of the imaginary part of the process $\mathscr{Z}^\omega$, for different realizations of $\omega$; the trajectories are sample-dependent, as in Fig 2.

2. *Fluctuations of $\zeta^{N,(\omega)}$ around its limit:* Let

$$t \mapsto \mathscr{Z}_t^{N,(\omega)} := \sqrt{N}\left(\zeta_t^{N,(\omega)} - \zeta_t\right)$$

*be the fluctuations process. For fixed disorder ($\omega$), let $\mathfrak{Z}^{N,(\omega)} \in \mathscr{M}_1(\mathscr{C}([0,T],\mathbf{R}))$ be the law of $\mathscr{Z}^{N,(\omega)}$. Then, under ($H_r$), the random variable ($\omega$) $\mapsto \mathfrak{Z}^{N,(\omega)}$ converges in law to the random variable $\omega \mapsto \mathfrak{Z}^\omega$, where $\mathfrak{Z}^\omega$ is the law of $\mathscr{Z}^\omega := \frac{1}{r^2}\left(r\left\langle \eta^\omega, \cos(\cdot)\right\rangle + \left\langle P, e^{ix}\right\rangle \mathscr{R}^\omega\right)$.*

In the sine-model, we have $\zeta^{N,(\omega)} = e^{i\psi^{N,(\omega)}}$ where $\psi^{N,(\omega)}$ is defined in (2) and is plotted in Fig. 2. Some trajectories of the process $\mathscr{Z}^\omega$ are plotted in Fig. 3.

This fluctuations result is proved in Section 5.

## 3  Proof of the quenched convergence result

In this section we prove Theorem 2.5. Reformulating (3) in terms of $\nu^{N,(\omega)}$, we have:

$$\forall i = 1\ldots N, \ \forall t \in [0,T], \ x_t^{i,N} = \xi^i + \int_0^t b[x_s^{i,N}, \nu_s^N]\,\mathrm{d}s + \int_0^t c(x_s^{i,N}, \omega_i)\,\mathrm{d}s + B_t^i, \qquad (11)$$

where we recall that $b[x,m] := \int b(x,y,\pi)m(\mathrm{d}y,\mathrm{d}\pi)$.

The idea of the proof of Theorem 2.5 is the following: we show the tightness of the sequence $(\nu^{N,(\omega)})$ firstly in $\mathbf{D}([0,T],(\mathscr{M}_F,\nu))$ (recall Notations in §2.1), which is quite simple since $\mathscr{C}_c(\mathbf{S}^1 \times \mathbf{R})$ is

separable and by an argument of boundedness of the second marginal of any accumulation point, thanks to $(H^Q_\mu)$, we show the tightness in $\mathbf{D}([0,T],(\mathcal{M}_F,w))$. The proof is complete when we prove the uniqueness of any accumulation point.

## 3.1 Proof of the tightness result

We want to show successively:

1. Tightness of $\mathcal{L}(v^{N,(\omega)})$ in $\mathbf{D}([0,T],(\mathcal{M}_F,v))$,

2. Equation verified by any accumulation point,

3. Characterization of the marginals of any limit,

4. Convergence in $\mathbf{D}([0,T],(\mathcal{M}_F,w))$.

### 3.1.1 Equation verified by $v^{N,(\omega)}$

For $f \in \mathcal{C}^2_b(\mathbf{S}^1 \times \mathbf{R})$, we denote by $f'$, $f''$ the first and second derivative of $f$ with respect to the first variable. Moreover, if $m \in \mathcal{M}_1(\mathbf{S}^1 \times \mathbf{R})$, then $\langle m, f \rangle$ stands for $\int_{\mathbf{S}^1 \times \mathbf{R}} f(x, \pi) m(\mathrm{d}x, \mathrm{d}\pi)$.

Applying Ito's formula to (11), we get, for all $f \in \mathcal{C}^2_b(\mathbf{S}^1 \times \mathbf{R})$,

$$
\left\langle v^{N,(\omega)}_t, f \right\rangle = \left\langle v^{N,(\omega)}_0, f \right\rangle + \frac{1}{2} \int_0^t \mathrm{d}s \left\langle v^{N,(\omega)}_s, f'' \right\rangle
$$

$$
+ \int_0^t \mathrm{d}s \left\langle v^{N,(\omega)}_s, f' \cdot (b[\cdot, v^{N,(\omega)}_s] + c) \right\rangle + M_{N,f}(t),
$$

where $M_{N,f}(t) := \frac{1}{N} \sum_{j=1}^N \int_0^t f'(x^{N,(\omega)}_j, \omega_j) \mathrm{d}B_j(s)$ is a martingale ($f'$ bounded).

### 3.1.2 Tightness of $\mathcal{L}(v^{N,(\omega)})$ in $\mathbf{D}([0,T],(\mathcal{M}_F,v))$

$\mathcal{C}_c(\mathbf{S}^1 \times \mathbf{R})$ is separable: let $(f_k)_{k \geq 1}$ (elements of $\mathcal{C}^\infty(\mathbf{S}^1 \times \mathbf{R})$) a dense sequence in $\mathcal{C}_c(\mathbf{S}^1 \times \mathbf{R})$, and let $f_0 \equiv 1$. We define $\Omega := \mathbf{D}([0,T],(\mathcal{M}_1,v))$ and the applications $\Pi_f$, $f \in \mathcal{C}_c(\mathbf{S}^1 \times \mathbf{R})$ by:

$$
\begin{array}{rcl}
\Pi_f : \quad \Omega & \to & \mathbf{D}([0,T],\mathbf{R}) \\
m & \mapsto & \langle m, f \rangle.
\end{array}
$$

Let $(P_n)_n$ a sequence of probabilities on $\Omega$ and $(\Pi_f P_n) = P_n \circ \Pi_f^{-1} \in \mathbf{D}([0,T],\mathbf{R})$. We recall the following result:

**Lemma 3.1.**
*If for all $k \geq 0$, the sequence $(\Pi_{f_k} P_n)_n$ is tight in $\mathcal{M}_1(\mathbf{D}([0,T],\mathbf{R}))$, then the sequence $(P_n)_n$ is tight in $\mathcal{M}_1(\mathbf{D}([0,T],(\mathcal{M}_1,v)))$.*

Hence, it suffices to have a criterion for tightness in $\mathbf{D}([0,T],\mathbf{R})$. Let $X_t^n$ be a sequence of processes in $\mathbf{D}([0,T],\mathbf{R})$ and $\mathscr{F}_t^n$ a sequence of filtrations such that $X^n$ is $\mathscr{F}^n$-adapted. Let $\phi^n = \{$stopping times for $\mathscr{F}^n\}$. We have (cf. Billingsley [4]):

**Lemma 3.2 (Aldous' criterion).**
*If the following holds,*

1. *$\mathscr{L}\left(\sup_{t\leq T}|X_t^n|\right)_n$ is tight,*

2. *For all $\varepsilon > 0$ and $\eta > 0$, there exists $\delta > 0$ such that*
$$\limsup_n \sup_{S,S'\in\phi^n; S\leq S'\leq (S+\delta)\wedge T} \mathbf{P}\left(\left|X_S^n - X_{S'}^n\right| > \eta\right) \leq \varepsilon,$$

*then $\mathscr{L}(X^n)$ is tight.*

**Proposition 3.3.**
*The sequence $\mathscr{L}(v^{N,(\omega)})$ is tight in $\mathbf{D}([0,T],(\mathcal{M}_F,v))$.*

*Proof.* For all $\varepsilon > 0$, for all $k \geq 1$ (the case $k = 0$ is straightforward),
$$\mathbf{P}\left(\sup_{t\leq T}\left|\left\langle v_t^{N,(\omega)}, f_k\right\rangle\right| > \frac{1}{\varepsilon}\right) \leq \varepsilon \left\|f_k\right\|_\infty \mathbf{E}\left[\sup_{t\leq T}\underbrace{\left|\left\langle v_t^{N,(\omega)}, 1\right\rangle\right|}_{=1}\right], \qquad \text{(Markov Inequality)}.$$

The tightness follows.
For all $k \geq 1$, we have the following decomposition:
$$\left\langle v_t^{N,(\omega)}, f_k\right\rangle = \left\langle v_0^{N,(\omega)}, f_k\right\rangle + A_t^{N,(\omega)}(f_k) + M_t^{N,(\omega)}(f_k),$$

where $A_t^{N,(\omega)}(f_k)$ is a process of bounded variations, and $M_t^{N,(\omega)}(f_k)$ is a square-integrable martingale. Then it suffices to verify Lemma 3.2, (2) for $A$ and $M$ separately. For all $\varepsilon > 0$ and $\eta > 0$, for all stopping times $S,S' \in \phi^N; S \leq S' \leq (S+\delta)\wedge T$, we have:
$$a_N := \mathbf{P}\left(\left|A_{S'}^{N,(\omega)}(f_k) - A_S^{N,(\omega)}(f_k)\right| > \eta\right),$$
$$\leq \frac{1}{\eta}\mathbf{E}\left[\int_S^{S'} ds \left|\left\langle v_s^{N,(\omega)}, f_k' \cdot (b[\cdot, v_s^{N,(\omega)}] + c)\right\rangle\right|\right] + \frac{1}{\eta}\mathbf{E}\left[\frac{1}{2}\int_S^{S'} ds \left|\left\langle v_s^{N,(\omega)}, f_k''\right\rangle\right|\right],$$
$$\leq \frac{C}{\eta}\mathbf{E}\left[S' - S\right] \leq \varepsilon, \quad \text{for } \delta \text{ sufficiently small.}$$

(we use here that $f_k$ are of compact support for $k \geq 1$; in particular the function $(x,\pi) \mapsto f_k'(x,\pi)c(x,\pi)$ is bounded). Furthermore,
$$\mathbf{P}\left(\left|M_{S'}^{N,(\omega)}(f_k) - M_S^{N,(\omega)}(f_k)\right| > \eta\right) = \mathbf{P}\left(\left|M_{S'}^{N,(\omega)}(f_k) - M_S^{N,(\omega)}(f_k)\right|^2 > \eta^2\right),$$
$$\leq \frac{1}{\eta^2}\mathbf{E}\left[\left|M_{S'}^{N,(\omega)}(f_k) - M_S^{N,(\omega)}(f_k)\right|^2\right],$$
$$\leq \frac{1}{(N\eta)^2}\mathbf{E}\left[\sum_{i=1}^N \int_S^{S'} f_k'^2(x_s^i, \omega_i)\,ds\right] \leq \frac{C}{N\eta^2}\delta. \qquad \square$$

At this point, $\mathscr{L}(v^{N,(\omega)})$ is tight in $\mathbf{D}([0,T],(\mathscr{M}_F,v))$.

### 3.1.3 Equation satisfied by any accumulation point in $\mathbf{D}([0,T],(\mathscr{M}_F,v))$

Using hypothesis $(H_0)$, it is easy to show that the following equation is satisfied for every accumulation point $v$, for every $f \in \mathscr{C}_c^2(\mathbf{S}^1 \times \mathbf{R})$ (we use here that $\mathbf{S}^1$ is compact):

$$\langle v_t, f \rangle = \langle v_0, f \rangle + \int_0^t \mathrm{d}s \, \langle v_s, f' \cdot (b[\cdot, v_s] + c) \rangle + \frac{1}{2} \int_0^t \mathrm{d}s \, \langle v_s, f'' \rangle. \tag{12}$$

For any accumulation point $v$, the following lemma gives a uniform bound for the second marginal of $v$:

**Lemma 3.4.**
*Let $Q$ be an accumulation point of $\mathscr{L}(v^{N,(\omega)})_N$ in $\mathbf{D}([0,T],(\mathscr{M}_1,v))$ and let be $v \sim Q$. For all $t \in [0,T]$, we define by $(v_{t,2})$ the second marginal of $v_t$:*

$$\forall A \in \mathscr{B}(\mathbf{R}), \quad (v_{t,2})(A) = \int_{\mathbf{S}^1 \times A} v_t(\mathrm{d}x, \mathrm{d}\pi).$$

*Then, for all $t \in [0,T]$,*

$$\int_{\mathbf{R}} \sup_{x \in \mathbf{S}^1} |c(x,\pi)| (v_{t,2})(\mathrm{d}\pi) \le \int_{\mathbf{R}} \sup_{x \in \mathbf{S}^1} |c(x,\pi)| \mu(\mathrm{d}\pi).$$

*Proof.* Let $\phi$ be a $\mathscr{C}^2$ positive function such that $\phi \equiv 1$ on $[-1,1]$, $\phi \equiv 0$ on $[-2,2]$ and $\|\phi\|_\infty \le 1$. Let,
$$\forall k \ge 1, \quad \phi_k := \pi \mapsto \phi\left(\frac{\pi}{k}\right).$$

Then $\phi_k \in \mathscr{C}_c^2(\mathbf{R})$ and $\phi_k(\pi) \to_{k \to \infty} 1$, for all $\pi$. We have also for all $\pi \in \mathbf{R}$, $|\phi_k(\pi)| \le \|\phi\|_\infty$, $|\phi_k'(\pi)| \le \|\phi'\|_\infty$, $|\phi_k''(\pi)| \le \|\phi''\|_\infty$.
We have successively, denoting $S(\pi) := \sup_{x \in \mathbf{S}^1} |c(x,\pi)|$,

$$\int_{\mathbf{S}^1 \times \mathbf{R}} S(\pi) v_t(\mathrm{d}x, \mathrm{d}\pi) = \int_{\mathbf{S}^1 \times \mathbf{R}} \liminf_{k \to \infty} \phi_k(\pi) S(\pi) v_t(\mathrm{d}x, \mathrm{d}\pi),$$

$$\le \liminf_{k \to \infty} \int_{\mathbf{S}^1 \times \mathbf{R}} \phi_k(\pi) S(\pi) v_t(\mathrm{d}x, \mathrm{d}\pi), \ \text{(Fatou's lemma)},$$

$$= \liminf_{k \to \infty} \lim_{N \to \infty} \int_{\mathbf{S}^1 \times \mathbf{R}} \phi_k(\pi) S(\pi) v_t^{N,(\omega)}(\mathrm{d}x, \mathrm{d}\pi), \tag{13}$$

$$\le \lim_{N \to \infty} \int_{\mathbf{S}^1 \times \mathbf{R}} S(\pi) v_t^{N,(\omega)}(\mathrm{d}x, \mathrm{d}\pi), \ \text{(since } \|\phi\|_\infty \le 1\text{)}.$$

The equality (13) is true since $(x,\pi) \mapsto \phi_k(\pi) S(\pi)$ is of compact support in $\mathbf{S}^1 \times \mathbf{R}$ (recall that $S$ is supposed to be continuous by hypothesis).

But, by definition of $v_t^{N,(\omega)}$, and using the hypothesis $(H_\mu^Q)$ concerning $\mu$, we have,

$$\lim_{N\to\infty} \int_{\mathbf{S}^1\times\mathbf{R}} S(\pi)v_t^{N,(\omega)}(\,\mathrm{d}x,\,\mathrm{d}\pi) = \int_{\mathbf{R}} S(\pi)\mu(\,\mathrm{d}\pi). \qquad (14)$$

The result follows. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

*Remark 3.5.*
(14) is only true for $\mathbb{P}$-almost every sequence $(\omega)$. We assume that the sequence $(\omega)$ given at the beginning satisfies this property.

### 3.1.4 Tightness in $\mathbf{D}([0,T],(\mathcal{M}_F,w))$

We have the following lemma (cf. [21]):

**Lemma 3.6.**
*Let $(X_n)$ be a sequence of processes in $\mathbf{D}([0,T],(\mathcal{M}_F,w))$ and $X$ a process belonging to $\mathscr{C}([0,T],(\mathcal{M}_F,w))$. Then,*

$$X_n \xrightarrow{\mathscr{L}} X \Longleftrightarrow \begin{cases} X_n \xrightarrow{\mathscr{L}} X & \text{in } \mathbf{D}([0,T],(\mathcal{M}_F,v)), \\ \langle X_n,1\rangle \xrightarrow{\mathscr{L}} \langle X,1\rangle & \text{in } \mathbf{D}([0,T],\mathbf{R}). \end{cases}$$

So, it suffices to show, for any accumulation point $v$:

1. $\langle v,1\rangle = 1$: Eq. (12) is true for all $f \in \mathscr{C}_c^2(\mathbf{S}^1\times\mathbf{R})$, so in particular for $f_k(x,\pi) := \phi_k(\pi)$. Using the boundedness shown in lemma 3.4, we can apply dominated convergence theorem in Eq. (12). We then have $\langle v_t,1\rangle = 1$, for all $t\in[0,T]$. The fact that Eq. (12) is verified for all $f \in \mathscr{C}_b^2(\mathbf{S}^1\times\mathbf{R})$ can be shown in the same way.

2. Continuity of the limit: For all $0\le s\le t\le T$, for all $f\in\mathscr{C}_b^2(\mathbf{S}^1\times\mathbf{R})$,

$$\left|\langle v_t,f\rangle - \langle v_s,f\rangle\right| \le K\int_s^t \left|\langle v_u,f'\cdot b[\cdot,v_u]\rangle\right|\,\mathrm{d}u + \frac{1}{2}\int_s^t \left|\langle v_u,f''\rangle\right|\,\mathrm{d}u$$
$$+ \int_s^t \left|\langle v_u,f'\cdot c\rangle\right|\,\mathrm{d}u \le C\times|t-s|, \quad \text{for some constant } C.$$

Noticing that we used again Lemma 3.4 to bound the last term, we have the result.

*Remark 3.7.*
It is then easy to see that the second marginal (on the disorder) of any accumulation point is $\mu$.

At this point $\mathscr{L}(v^{N,(\omega)})$ is tight in $\mathbf{D}([0,T],(\mathcal{M}_F,w))$. It remains to show the uniqueness of any accumulation point: it shows firstly that the sequence effectively converges and that the limit does not depend on the given sequence $(\omega)$.

## 3.2 Uniqueness of the limit

**Proposition 3.8.**
*There exists a unique element P of* $\mathbf{D}([0,T], \mathscr{M}_F(\mathbf{S}^1 \times \mathbf{R}))$ *which satisfies equation* (12)*,* $P_0 \in \mathscr{M}_1$ *and* $P_{0,2} = \mu.$

*Remark 3.9.*
Since this proof is an adaptation of Oelschläger [20] Lemma 10, p.474, we only sketch the proof:

We can rewrite Eq. (12) in a more compact way:

$$\langle P_t, f \rangle = \langle P_0, f \rangle + \int_0^t \langle P_s, L(P_s)(f) \rangle \, ds, \quad \forall f \in \mathscr{C}_b^2(\mathbf{S}^1 \times \mathbf{R}), 0 \le t \le T, \tag{15}$$

where,

$$L(P)(f)(x, \omega) := \frac{1}{2} f''(x, \omega) + h(x, \omega, P) f'(x, \omega), \quad \forall x \in \mathbf{S}^1, \forall \omega \in \mathbf{R},$$

and,

$$h(x, \pi, P) = b[x, P] + c(x, \pi).$$

Let $t \mapsto P_t$ be any solution of Eq. (15). One can then introduce the following SDE (where $\xi \in \mathbf{R}$ and $\tilde{\xi} \in \mathbf{S}^1$ is its projection on $\mathbf{S}^1$):

$$\begin{cases} \mathrm{d}\xi_t &= h(\tilde{\xi}_t, \omega_t, P_t) \mathrm{d}t + \mathrm{d}W_t, \quad \xi_0 = \tilde{\xi}_0, \; (\tilde{\xi}_0, \omega_0) \sim P_0, \\ \mathrm{d}\omega_t &= 0. \end{cases} \tag{16}$$

Eq. (16) has a unique (strong) solution $(\xi_t, \omega_t)_{t \in [0,T]} = (\xi_t, \omega_0)_{t \in [0,T]}$. The proof of uniqueness in Eq. (12) consists in two steps:

1. $P_t = \mathscr{L}(\tilde{\xi}_t, \omega_t)$, for all $t \in [0, T]$,

2. Uniqueness of $\tilde{\xi}$.

We refer to Oelschläger [20] for the details. Another proof of uniqueness can be found in [9] or in [13] via a martingale argument.

# 4 Proof of the fluctuations result

Now we turn to the proof of Theorem 2.10. To that purpose, we need to introduce some distribution spaces:

## 4.1 Distribution spaces

Let $\mathscr{S} := \mathscr{S}(\mathbf{S}^1 \times \mathbf{R})$ be the usual Schwartz space of rapidly decreasing infinitely differentiable functions. Let $\mathscr{D}_p$ be the set of all differential operators of the form $\partial_{u^k} \partial_{\pi^l}$ with $k + l \leq p$. We know from Gelfand and Vilenkin [14] p. 82-84, that we can introduce on $\mathscr{S}$ a nuclear Fréchet topology by the system of seminorms $\|\cdot\|_p$, $p = 1, 2, \ldots$, defined by

$$\left\| \phi \right\|_p^2 = \sum_{k=0}^{p} \int_{\mathbf{S}^1 \times \mathbf{R}} (1 + |\pi|^2)^{2p} \sum_{D \in \mathscr{D}_k} |D\phi(y, \pi)|^2 \, \mathrm{d}y \, \mathrm{d}\pi.$$

Let $\mathscr{S}'$ be the corresponding dual space of tempered distributions. Although, for the sake of simplicity, we will mainly consider $\eta^{N,(\omega)}$ as a process in $\mathscr{C}([0, T], \mathscr{S}')$, we need some more precise estimations to prove tightness and convergence. We need here the following norms:

For every integer $j$, $\alpha \in \mathbf{R}^+$, we consider the space of all real functions $\varphi$ defined on $\mathbf{S}^1 \times \mathbf{R}$ with derivative up to order $j$ such that

$$\left\| \varphi \right\|_{j,\alpha} := \left( \sum_{k_1 + k_2 \leq j} \int_{\mathbf{S}^1 \times \mathbf{R}} \frac{|\partial_{x^{k_1}} \partial_{\pi^{k_2}} \varphi(x, \pi)|^2}{1 + |\pi|^{2\alpha}} \, \mathrm{d}x \, \mathrm{d}\pi \right)^{1/2} < \infty.$$

Let $W_0^{j,\alpha}$ be the completion of $\mathscr{C}_c^\infty(\mathbf{S}^1 \times \mathbf{R})$ for this norm; $(W_0^{j,\alpha}, \|\cdot\|_{j,\alpha})$ is a Hilbert space. Let $W_0^{-j,\alpha}$ be its dual space.

Let $C^{j,\alpha}$ be the space of functions $\varphi$ with continuous partial derivatives up to order $j$ such that

$$\lim_{|\pi| \to \infty} \sup_{x \in \mathbf{S}^1} \frac{|\partial_{x^{k_1}} \partial_{\pi^{k_2}} \varphi(x, \pi)|}{1 + |\pi|^\alpha} = 0, \text{ for all } k_1 + k_2 \leq j,$$

with norm

$$\left\| \varphi \right\|_{C^{j,\alpha}} = \sum_{k_1 + k_2 \leq j} \sup_{x \in \mathbf{S}^1} \sup_{\pi \in \mathbf{R}} \frac{|\partial_{x^{k_1}} \partial_{\pi^{k_2}} \varphi(x, \pi)|}{1 + |\pi|^\alpha}.$$

We have the following embeddings:

$$W_0^{m+j,\alpha} \hookrightarrow C^{j,\alpha}, m > 1, j \geq 0, \alpha \geq 0,$$

i.e. there exists some constant $C$ such that

$$\left\| \varphi \right\|_{C^{j,\alpha}} \leq C \left\| \varphi \right\|_{m+j,\alpha}. \tag{17}$$

Moreover,

$$W_0^{m+j,\alpha} \hookrightarrow W_0^{j,\alpha+\beta}, m > 1, j \geq 0, \alpha \geq 0, \beta > 1.$$

Thus there exists some constant $C$ such that

$$\left\| \varphi \right\|_{j,\alpha+\beta} \leq C \left\| \varphi \right\|_{m+j,\alpha}.$$

We then have the following dual continuous embedding:

$$W_0^{-j,\alpha+\beta} \hookrightarrow W_0^{-(m+j),\alpha}, \ m > 1, \alpha \geq 0, \beta > 1. \tag{18}$$

It is quite clear that $\mathscr{S} \hookrightarrow W_0^{j,\alpha}$ for any $j$ and $\alpha$, with a continuous injection.

We now prove some continuity of linear mappings in the corresponding spaces:

**Lemma 4.1.**
*For every $x, y \in \mathbf{S}^1$, $\omega \in \mathbf{R}$, for all $\alpha$, the linear mappings $W_0^{3,\alpha} \to \mathbf{R}$ defined by*

$$D_{x,y,\omega}(\varphi) := \varphi(x,\omega) - \varphi(y,\omega); D_{x,\omega} := \varphi(x,\omega); H_{x,\omega} = \varphi'(x,\omega),$$

*are continuous and*

$$\left\| D_{x,y,\omega} \right\|_{-3,\alpha} \leq C|x-y|(1+|\omega|^\alpha), \tag{19}$$

$$\left\| D_{x,\omega} \right\|_{-3,\alpha} \leq C(1+|\omega|^\alpha), \tag{20}$$

$$\left\| H_{x,\omega} \right\|_{-3,\alpha} \leq C(1+|\omega|^\alpha). \tag{21}$$

*Proof.* Let $\varphi$ be a function of class $\mathscr{C}^\infty$ with compact support on $\mathbf{S}^1 \times \mathbf{R}$, then,

$$|\varphi(x,\omega) - \varphi(y,\omega)| \leq |x-y| \sup_u \left| \varphi'(u,\omega) \right|,$$

$$\leq |x-y|(1+|\omega|^\alpha) \sup_{u,\omega} \left( \frac{\left| \varphi'(u,\omega) \right|}{1+|\omega|^\alpha} \right),$$

$$\leq |x-y|(1+|\omega|^\alpha) \left\| \varphi \right\|_{1,\alpha},$$

$$\leq C|x-y|(1+|\omega|^\alpha) \left\| \varphi \right\|_{3,\alpha},$$

following (17) with $j = 1$ and $m = 2 > 1$. Then, (19) follows from a density argument. (20) and (21) are proved in the same way. $\qquad \square$

## 4.2 The non-linear process

The proof of convergence is based on the existence of the non-linear process associated to McKean-Vlasov equation. Such existence has been studied by numerous authors (eg. Dawson [10], Jourdain-Méléard [16], Malrieu [18], Shiga-Tanaka [23], Sznitman [25], [26]) mostly in order to prove some propagation of chaos properties in systems without disorder. We consider the present similar case where disorder is present. Let us give some intuition of this process. One can replace the non-linearity in Eq. (5) by an arbitrary measure $m(dx, d\omega)$:

$$\partial_t q_t^\omega = \frac{1}{2} \partial_{xx} q_t^\omega - \partial_x \left[ (b[x,m] + c(x,\omega)) q_t^\omega \right].$$

In this particular case, it is usual to interpret $q_t^\omega$ as the time marginals of the following diffusion:

$$dx_t^\omega = dB_t + b[x_t^\omega, m]dt + c(x_t^\omega, \omega)dt, \ \omega \sim \mu.$$

It is then natural to consider the following problem, where $m$ is replaced by the proper measure $P$: on a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, B, x_0, Q)$, endowed with a Brownian motion $B$ and with a $\mathscr{F}_0$ measurable random variable $x_0$, we introduce the following system:

$$\begin{cases} x_t^\omega & = x_0 + \int_0^t b[x_s^\omega, P_s] \, ds + \int_0^t c(x_s^\omega, \omega) \, ds + B_t, \\ \omega & \sim \mu, \\ P_t & = \mathscr{L}(x_t, \omega), \forall t \in [0,T]. \end{cases} \tag{22}$$

**Proposition 4.2.**
*There is pathwise existence and uniqueness for Equation* (22).

*Proof.* The proof is the same as given in Sznitman [26], Th 1.1, p.172, up to minor modifications. The main idea consists in using a Picard iteration in the space of probabilities on $\mathscr{C}([0,T], \mathbf{S}^1 \times \mathbf{R})$ endowed with an appropriate Wasserstein metric. We refer to it for details. $\qquad\square$

### 4.3  Fluctuations in the quenched model

The key argument of the proof is to explicit the speed of convergence as $N \to \infty$ for the rotators to the non-linear process (see Prop. 4.3).

A major difference between this work and [12] is that, since in our quenched model, we only integrate w.r.t. oscillators *and not* w.r.t. the disorder, one has to deal with remaining terms, see $Z_N$ in Proposition 4.3, to compare with [12], Lemma 3.2, that would have disappeared in the *averaged model*. The main technical difficulty of Proposition 4.3 is to control the asymptotic behaviour of such terms, see (24). As in [12], having proved Prop. 4.3, the key argument of the proof is a uniform estimation of the norm of the process $\eta^{N,(\omega)}$, see Propositions 4.4 and 4.8, based on the generalized stochastic differential equation verified by $\eta^{N(\omega)}$, see (30).

#### 4.3.1  Preliminary results

We consider here a fixed realization of the disorder $(\omega) = (\omega_1, \omega_2, \dots)$. On a common filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, (B^i)_{i \geq 1}, Q)$, endowed with a sequence of i.i.d. $\mathscr{F}_t$-adapted Brownian motions $(B_i)$ and with a sequence of i.i.d. $\mathscr{F}_0$ measurable random variables $(\xi^i)$ with law $\lambda$, we define as $x^{i,N}$ the solution of (11), and as $x^{\omega_i}$ the solution of (22), with the same Brownian motion $B^i$ and with the same initial value $\xi^i$.

The main technical proposition, from which every norm estimation of $\eta^{N,(\omega)}$ follows is the following:

**Proposition 4.3.**

$$\mathbf{E}\left[\sup_{t \leq T}\left|x_t^{i,N} - x_t^{\omega_i}\right|^2\right] \leq C/N + Z_N(\omega_1, \dots, \omega_N), \tag{23}$$

*where the random variable* $(\omega) \mapsto Z_N(\omega)$ *is such that:*

$$\lim_{A \to \infty} \limsup_{N \to \infty} \mathbb{P}\left(N Z_N(\omega) > A\right) = 0. \tag{24}$$

The (rather technical) proof of Proposition 4.3 is postponed to the end of the document (see §A). Once again, we stress the fact that the term $Z_N$ would have disappeared in the averaged model.

The first norm estimation of the process $\eta^{N,(\omega)}$ (which will be used to prove tightness) is a direct consequence of Proposition 4.3 and of a Hilbertian argument:

**Proposition 4.4.**
*Under the hypothesis* $(H_\mu^{\mathscr{F}})$ *on $\mu$, the process* $\eta^{N,(\omega)}$ *satisfies the following property: for all $T > 0$,*

$$\sup_{t \leq T} \mathbf{E}\left[\left\|\eta_t^{N,(\omega)}\right\|_{-3,2\alpha}^2\right] \leq A_N(\omega_1, \dots, \omega_N), \tag{25}$$

*where*

$$\lim_{A\to\infty} \limsup_{N\to\infty} \mathbb{P}\left(A_N > A\right) = 0.$$

*Proof.* For all $\varphi \in W_0^{3,2\alpha}$, writing

$$\left\langle \eta_t^{N,(\omega)}, \varphi \right\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ \varphi(x_t^{i,N}, \omega_i) - \varphi(x_t^{\omega_i}, \omega_i) \right\} + \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left\{ \varphi(x_t^{\omega_i}, \omega_i) - \langle P_s, \varphi \rangle \right\},$$

$$=: S_t^{N,(\omega)}(\varphi) + T_t^{N,(\omega)}(\varphi),$$

we have:

$$\left\langle \eta_t^{N,(\omega)}, \varphi \right\rangle^2 \leq 2 \left( S_t^{N,(\omega)}(\varphi)^2 + T_t^{N,(\omega)}(\varphi)^2 \right). \tag{26}$$

But, by convexity,

$$S_t^{N,(\omega)}(\varphi)^2 \leq \sum_{i=1}^{N} D^2_{x_t^{i,N}, x_t^{\omega_i}, \omega_i}(\varphi).$$

Then, applying the latter equation to an orthonormal system $(\varphi_p)_{p\geq 1}$ in the Hilbert space $W_0^{3,2\alpha}$, summing on $p$, we have by Parseval's identity on the continuous functional $D_{x_t^{i,N}, x_t^{\omega_i}}$,

$$\mathbf{E}\left[\left\| S_t^{N,(\omega)} \right\|^2_{-3,2\alpha}\right] \leq \mathbf{E}\left[\sum_{i=1}^{N} \left\| D_{x_t^{i,N}, x_t^{\omega_i}, \omega_i} \right\|^2_{-3,2\alpha}\right],$$

$$\leq C \sum_{i=1}^{N} (1 + |\omega_i|^{4\alpha}) \mathbf{E}\left[\left| x_t^{i,N} - x_t^{\omega_i} \right|^2\right], \tag{27}$$

$$\leq C \sum_{i=1}^{N} \left(1 + |\omega_i|^{4\alpha}\right) \left(C/N + Z_N(\omega_1, \ldots, \omega_N)\right), \tag{28}$$

where we used (19) in (27), and (23) in (28).

On the other hand,

$$\mathbf{E}\left[T_t^{N,(\omega)}(\varphi)^2\right] = \frac{1}{N} \mathbf{E}\left[\left\{\sum_{i=1}^{N} (\varphi(x_t^{\omega_i}, \omega_i) - \langle P_t, \varphi \rangle)\right\}^2\right],$$

$$= \frac{1}{N} \mathbf{E}\left[\sum_{i=1}^{N} (\varphi(x_t^{\omega_i}, \omega_i) - \langle P_t, \varphi \rangle)^2\right]$$

$$+ \frac{1}{N} \mathbf{E}\left[\sum_{i\neq j} (\varphi(x_t^{\omega_i}, \omega_i) - \langle P_t, \varphi \rangle)(\varphi(x_t^{\omega_j}, \omega_j) - \langle P_t, \varphi \rangle)\right],$$

$$\leq \frac{2}{N} \mathbf{E}\left[\sum_{i=1}^{N} (\varphi(x_t^{\omega_i}, \omega_i)^2 + \langle P_t, \varphi \rangle^2)\right] + \frac{1}{N} \sum_{i\neq j} G(\varphi)(\omega_i) G(\varphi)(\omega_j),$$

$$\leq \frac{2}{N} \mathbf{E}\left[\sum_{i=1}^{N} \varphi(x_t^{\omega_i}, \omega_i)^2\right] + 2 \langle P_t, \varphi \rangle^2 + \left(\frac{1}{\sqrt{N}} \sum_{i=1}^{N} G(\varphi)(\omega_i)\right)^2,$$

where $G(\varphi)(\omega) := \int \varphi(y, \omega_i) P_t^{\omega_i}(\mathrm{d}y) - \langle P_t, \varphi \rangle$. If we apply the same Hilbertian argument as for $S^{N,(\omega)}$, we see

$$\mathbf{E}\left[\left\|T_t^{N,(\omega)}\right\|_{-3,2\alpha}^2\right] \le \frac{2C}{N}\mathbf{E}\left[\sum_{i=1}^{N}(1+|\omega_i|^{4\alpha})\right] + C + \left\|\phi \mapsto \left(\frac{1}{\sqrt{N}}\sum_{i=1}^{N}G(\phi)(\omega_i)\right)\right\|_{-3,2\alpha}^2, \quad (29)$$

It is easy to see that the last term in (29) can be reformulated as $B_N(\omega_1, \ldots, \omega_N)$, with the property that $\lim_{A\to\infty}\lim\sup_{N\to\infty}\mathbb{P}(B_N > A) = 0$. Combining (24), (26), (28) and (29), Proposition 4.4 is proved. $\qquad\square$

### 4.3.2 Tightness of the fluctuations process

Applying Ito's formula to (11), we obtain, for all $\varphi$ bounded function on $\mathbf{S}^1 \times \mathbf{R}$, with two bounded derivatives w.r.t. $x$, for every sequence $(\omega)$, for all $t \le T$:

$$\left\langle \eta_t^{N,(\omega)}, \varphi \right\rangle = \left\langle \eta_0^{N,(\omega)}, \varphi \right\rangle + \int_0^t \left\langle \eta_s^{N,(\omega)}, \mathscr{L}_s^{v^N}(\varphi) \right\rangle \mathrm{d}s + M_t^{N,(\omega)}(\varphi), \quad (30)$$

where, for all $y \in \mathbf{S}^1$, $\pi \in \mathbf{R}$,

$$\mathscr{L}_s^{v^N}(\varphi)(y, \pi) = \frac{1}{2}\varphi''(y, \pi) + \varphi'(y, \pi)\left(b[y, v_s^N] + c(y, \pi)\right) + \left\langle P_s, \varphi'(\cdot, \cdot)b(\cdot, y, \pi)\right\rangle,$$

and $M_t^{N,(\omega)}(\varphi)$ is a real continuous martingale with quadratic variation process

$$\left\langle M^{N,(\omega)}(\varphi)\right\rangle_t = \int_0^t \left\langle v_s^{N,(\omega)}, \varphi'(y, \pi)^2 \right\rangle \mathrm{d}s.$$

**Lemma 4.5.**
*For every $N$, the operator $\mathscr{L}_s^{v^N}$ defines a linear mapping from $\mathscr{S}$ into $\mathscr{S}$ and for all $\varphi \in \mathscr{S}$,*

$$\left\|\mathscr{L}_s^{v^N}(\varphi)\right\|_{3,2\alpha}^2 \le C\left\|\varphi\right\|_{6,\alpha}^2.$$

*Proof.* The terms $\frac{1}{2}\varphi''(y, \pi)$ and $\varphi'(y, \pi)b[y, v_s^N]$ clearly satisfy the lemma. We study the two remaining terms:

$$\left\|\langle P_s, \varphi'b(\cdot, y, \pi)\rangle\right\|_{3,2\alpha}^2 = \sum_{k_1+k_2\le 3}\int_{\mathbf{S}^1\times\mathbf{R}}\frac{\left\langle P_s, \varphi'\partial_{y^{k_1}}\partial_{\pi^{k_2}}b(\cdot, y, \pi)\right\rangle^2}{1+|\pi|^{4\alpha}}\,\mathrm{d}y\,\mathrm{d}\pi,$$

$$\le C\int_{\mathbf{R}}\frac{1}{1+|\pi|^{4\alpha}}\,\mathrm{d}\pi\int_{\mathbf{S}^1\times\mathbf{R}}\varphi'(y, \pi)^2 P_s(\mathrm{d}y, \mathrm{d}\pi),$$

$$\le C\left\|\varphi\right\|_{C^{3,\alpha}}^2\int_{\mathbf{R}}\frac{1}{1+|\pi|^{4\alpha}}\,\mathrm{d}\pi\int_{\mathbf{S}^1\times\mathbf{R}}(1+|\pi|^{\alpha})^2 P_s(\mathrm{d}y, \mathrm{d}\pi),$$

$$\le C\left\|\varphi\right\|_{6,\alpha}^2\int_{\mathbf{R}}\frac{1}{1+|\pi|^{4\alpha}}\,\mathrm{d}\pi\int_{\mathbf{R}}(1+|\pi|^{\alpha})^2\mu(\mathrm{d}\pi).$$

And,

$$\left\| \varphi'(y,\pi)c(y,\pi) \right\|_{3,2\alpha}^2 = \sum_{k_1+k_2\leq 3} \int_{\mathbf{S}^1\times\mathbf{R}} \frac{\left( \partial_{y^{k_1}} \partial_{\pi^{k_2}} \{\varphi'(y,\pi)c(y,\pi)\} \right)^2}{1+|\pi|^{4\alpha}} \, dy \, d\pi.$$

It suffices to estimate, for every differential operator $D_i = \partial_{y^{u_i}} \partial_{\pi^{v_i}}$, $i = 1, 2$ with $u_1 + u_2 + v_1 + v_2 \leq 3$, the following term:

$$\int_{\mathbf{S}^1\times\mathbf{R}} \frac{|D_1\varphi'(y,\pi)D_2c(y,\pi)|^2}{1+|\pi|^{4\alpha}} \, dy \, d\pi \leq \int_{\mathbf{S}^1\times\mathbf{R}} \frac{|D_1\varphi'(y,\pi)|^2}{(1+|\pi|^\alpha)^2} \frac{|D_2c(y,\pi)|^2(1+|\pi|^\alpha)^2}{1+|\pi|^{4\alpha}} \, dy \, d\pi,$$

$$\leq C \left\| \varphi \right\|_{6,\alpha}^2 \int_{\mathbf{R}} \frac{\sup_{y\in\mathbf{S}^1} |D_2c(y,\pi)|^2}{1+|\pi|^{2\alpha}} \, d\pi.$$

The result follows from the assumptions made on $c$. $\qquad\square$

For the tightness criterion used below, we need to ensure that the trajectories of the fluctuations process are almost surely continuous: in that purpose, we need some more precise evaluations than in Prop. 4.4.

**Proposition 4.6.**
*The process $(M_t^{N,(\omega)})$ satisfies, for every $(\omega)$, and for every $T > 0$,*

$$\mathbf{E}\left[ \sup_{t\leq T} \left\| M_t^{N,(\omega)} \right\|_{-3,2\alpha}^2 \right] \leq \frac{C}{N} \sum_{i=1}^N \left( 1 + |\omega_i|^{4\alpha} \right).$$

*Remark 4.7.*
In particular, a consequence of $(H_\mu^{\mathscr{F}})$ is that, for $\mathbb{P}$-almost every sequence $(\omega)$,

$$\sup_N \mathbf{E}\left[ \sup_{t\leq T} \left\| M_t^{N,(\omega)} \right\|_{-3,2\alpha}^2 \right] \leq \sup_N \frac{C}{N} \sum_{i=1}^N \left( 1 + |\omega_i|^{4\alpha} \right) < \infty. \qquad (31)$$

*Proof.* Let $(\varphi_p)_{p\geq 1}$ a complete orthonormal system in $W_0^{3,2\alpha}$. For fixed $N$, by Doob's inequality, $\sum_{p\geq 1} \mathbf{E}\left[ \sup_{t\leq T} (M_t^{N,(\omega)}(\varphi_p))^2 \right]$ is bounded by

$$C\sum_{p\geq 1} \mathbf{E}\left[ M_T^{N,(\omega)}(\varphi_p)^2 \right] = C\sum_{p\geq 1} \mathbf{E}\left[ \int_0^T \left\langle v_s^{N,(\omega)}, \varphi_p'(y,\pi)^2 \right\rangle \, ds \right],$$

$$= \frac{1}{N} \sum_{i=1}^N \mathbf{E}\left[ \int_0^T \sum_{p\geq 1} \varphi_p'(x_s^{i,N}, \omega_i)^2 \, ds \right],$$

$$= \frac{1}{N} \sum_{i=1}^N \mathbf{E}\left[ \int_0^T \left\| H_{x_s^{i,N}, \omega_i} \right\|_{3,2\alpha}^2 \, ds \right],$$

$$\leq \frac{C}{N} \sum_{i=1}^N \left( 1 + |\omega_i|^{4\alpha} \right), \text{(using (21))}. \qquad\square$$

**Proposition 4.8.**
*For every $N$, every $(\omega)$,*

$$\mathbf{E}\left[\sup_{t \leq T}\left\|\eta_t^{N,(\omega)}\right\|_{-6,\alpha}^2\right] < C_N(\omega_1,\ldots,\omega_N), \tag{32}$$

*with*

$$\lim_{A \to \infty} \limsup_{N \to \infty} \mathbb{P}\left(C_N > A\right) = 0.$$

*Proof.* Let $(\psi_p)$ be a complete orthonormal system in $W_0^{6,\alpha}$ of $\mathscr{C}^\infty$ function on $\mathbf{S}^1 \times \mathbf{R}$ with compact support. We prove the stronger result:

$$\mathbf{E}\left[\sum_{p \geq 1} \sup_{t \leq T}\left\langle \eta_t^{N,(\omega)}, \psi_p \right\rangle^2\right] < \infty.$$

Indeed,

$$\left\langle \eta_t^{N,(\omega)}, \psi_p \right\rangle^2 \leq C\left(\left\langle \eta_0^{N,(\omega)}, \psi_p \right\rangle^2 + T\int_0^t \left\langle \eta_s^{N,(\omega)}, \mathscr{L}_s^{\nu^N}(\psi_p)\right\rangle^2 \, ds + M_t^{N,(\omega)}(\psi_p)^2\right).$$

By Doob's inequality,

$$\mathbf{E}\left[\sum_{p \geq 1} \sup_{t \leq T}\left\langle \eta_t^{N,(\omega)}, \psi_p \right\rangle^2\right] \leq C\left(\mathbf{E}\left[\left\|\eta_0^{N,(\omega)}\right\|_{-6,\alpha}^2\right] + \mathbf{E}\int_0^T \sum_{p \geq 1}\left\langle \eta_s^{N,(\omega)}, \mathscr{L}_s^{\nu^N}(\psi_p)\right\rangle^2 \, ds\right.$$

$$\left. + \sum_{p \geq 1} \mathbf{E}\left[M_T^{N,(\omega)}(\psi_p)^2\right]\right).$$

By Lemma 4.5, we have:

$$\left|\left\langle \eta_s^{N,(\omega)}, \mathscr{L}_s^{\nu^N}(\psi)\right\rangle\right| \leq C\left\|\eta_s^{N,(\omega)}\right\|_{-3,2\alpha}\left\|\psi\right\|_{6,\alpha}.$$

Then,

$$\mathbf{E}\left[\int_0^T \sum_{p \geq 1}\left\langle \eta_s^{N,(\omega)}, \mathscr{L}_s^{\nu^N}(\psi_p)\right\rangle^2 \, ds\right] \leq C^2 \int_0^T \mathbf{E}\left[\left\|\eta_s^{N,(\omega)}\right\|_{-3,2\alpha}^2\right] \, ds,$$

$$\leq C^2 T \sup_{s \leq T}\mathbf{E}\left[\left\|\eta_s^{N,(\omega)}\right\|_{-3,2\alpha}^2\right] \leq C^2 T A_N,$$

where $A_N$ is defined in Proposition 4.4. The result follows. $\qquad\square$

**Proposition 4.9.** *1. For every $N$, for $\mathbb{P}$-almost every $(\omega)$, the trajectories of the fluctuations process $\eta^{N,(\omega)}$ are almost surely continuous in $\mathscr{S}'$,*

*2. For every $N$, for $\mathbb{P}$-almost every $(\omega)$, the trajectories of $M^{N,(\omega)}$ are almost surely continuous in $\mathscr{S}'$.*

*Proof.* We only prove for $M^{N,(\omega)}$, since, using Proposition 4.8, the proof is the same for $\eta^{N,(\omega)}$. Let $(\varphi_p)$ be a complete orthonormal system in $W_0^{-3,2\alpha}$, then for every fixed $N$ and $(\omega)$, we know from the proof of Proposition 4.6, that for all $\varepsilon > 0$, there exists some $M_0 > 0$ such that

$$\sum_{p \geq M_0} \sup_{t \leq T} (M_t^{N,(\omega)}(\varphi_p))^2 < \frac{\varepsilon}{3}, a.s.$$

Let $(t_m)$ be a sequence in $[0, T]$ such that $t_m \to_{m \to \infty} t$.

$$\left\| M_{t_m}^{N,(\omega)} - M_t^{N,(\omega)} \right\|_{-3,2\alpha}^2 = \sum_{p \geq 1} \left( M_{t_m}^{N,(\omega)} - M_t^{N,(\omega)} \right)^2 (\varphi_p),$$

$$\leq \sum_{p=1}^{M_0} \left( M_{t_m}^{N,(\omega)} - M_t^{N,(\omega)} \right)^2 (\varphi_p) + \frac{2\varepsilon}{3} \leq \varepsilon,$$

if $t_m$ is sufficiently large. $\qquad\qquad\square$

We are now in position to prove the tightness of the fluctuations process. Let us recall some notations: for fixed $N$ and $(\omega)$ $\mathscr{H}^{N,(\omega)}$ is the law of the process $\eta^{N,(\omega)}$. Hence, $\mathscr{H}^{N,(\omega)}$ is an element of $\mathscr{M}_1(\mathscr{C}([0,T], \mathscr{S}'))$, endowed with the topology of weak convergence and with $\mathscr{B}^*$, the smallest $\sigma$-algebra such that the evaluations $Q \mapsto \langle Q, f \rangle$ are measurable, $f$ being measurable and bounded.

We will denote by $\Theta_N$ the law of the random variable $(\omega) \mapsto \mathscr{H}^{N,(\omega)}$. The main result of this part is the following:

**Theorem 4.10.** *1. for $\mathbb{P}$-almost every sequence $(\omega)$, the law of the process $M^{N,(\omega)}$ is tight in $\mathscr{M}_1(\mathscr{C}([0,T], \mathscr{S}'))$,*

*2. The law of the sequence $(\omega) \mapsto \mathscr{H}^{N,(\omega)}$ is tight on $\mathscr{M}_1(\mathscr{M}_1(\mathscr{C}([0,T], \mathscr{S}')))$.*

Before proving Theorem 4.10, we recall the following result and notations (cf. Mitoma [19], Th 3.1, p. 993):

**Proposition 4.11 (Mitoma's criterion).**
*Let $(P_N)$ be a sequence of probability measures on $(\mathscr{C}_{\mathscr{S}'} := \mathscr{C}([0,T], \mathscr{S}'), \mathscr{B}_{\mathscr{C}_{\mathscr{S}'}})$. For each $\varphi \in \mathscr{S}$, we denote by $\Pi_\varphi$ the mapping of $\mathscr{C}_{\mathscr{S}'}$ to $\mathscr{C} := \mathscr{C}([0,T], \mathbf{R})$ defined by*

$$\Pi_\varphi : \psi(\cdot) \in \mathscr{C}_{\mathscr{S}'} \mapsto \langle \psi(\cdot), \varphi \rangle \in \mathscr{C}.$$

*Then, if for all $\varphi \in \mathscr{S}$, the sequence $(P_N \Pi_\varphi^{-1})$ is tight in $\mathscr{C}$, the sequence $(P_N)$ is tight in $\mathscr{C}_{\mathscr{S}'}$.*

*Remark 4.12.*
A closer look to the proof of Mitoma shows that it suffices to verify the tightness of $(P_N \Pi_\varphi^{-1})$ for $\varphi$ in a countable dense subset of the nuclear Fréchet space $(\mathscr{S}, \| \cdot \|_p, p \geq 1)$.

Thanks to Mitoma's result, it suffices to have a tightness criterion in $\mathbf{R}$. We recall here the usual result (cf. Billingsley [4]): A sequence of $(\Omega^N, \mathscr{F}_t^N)$-adapted processes $(Y^N)$ with paths in $\mathscr{C}([0,T], \mathbf{R})$ is tight if both of the following conditions hold:

- Condition [T]: for all $t \leq T$ and $\delta > 0$, there exists $C > 0$ such that

$$\sup_N \mathbf{P}\left(|Y_t^N| > C\right) \leq \delta, \tag{$T_{t,\delta,C}$}$$

- Condition [A]: for all $\eta_1, \eta_2 > 0$, there exists $C > 0$ and $N_0$ such that for all $\mathscr{F}^N$-stopping times $\tau_N$,

$$\sup_{N \geq N_0} \sup_{\theta \leq C} \mathbf{P}\left(\left|Y_{\tau_N}^N - Y_{\tau_N+\theta}^N\right| \geq \eta_2\right) \leq \eta_1. \tag{$A_{\eta_1,\eta_2,C}$}$$

*Proof of Theorem 4.10.*     1. Tightness of $(M^{N,(\omega)})$: for a fixed realization of the disorder $(\omega)$, for fixed $\varphi \in \mathscr{S}$, we have:

- For all $t \in [0, T]$, for all $\delta > 0$, for all $C > 0$,

$$\mathbf{P}\left(\left|M_t^{N,(\omega)}(\varphi)\right| > C\right) \leq \frac{\mathbf{E}\left[\sup_{t \leq T}\left\{M_t^{N,(\omega)}(\varphi)^2\right\}\right]}{C^2},$$

$$\leq \frac{\mathbf{E}\left[\sup_{t \leq T}\left\|M_t^{N,(\omega)}\right\|_{-3,2\alpha}^2 \|\varphi\|_{3,2\alpha}^2\right]}{C^2},$$

$$\leq \frac{C\|\varphi\|_{3,2\alpha}^2}{a^2} \sup_N \frac{1}{N}\sum_{i=1}^N \left(1 + |\omega_i|^{4\alpha}\right), \text{(cf. (31))},$$

$$\leq \delta,$$

for a suitable $C > 0$ (depending on $(\omega)$). Condition [T] is proved.

- Let us verify Condition [A]: For every $\varphi \in \mathscr{S}$, for every $\delta, \theta, \eta_1, \eta_2 > 0, \theta \leq \delta$, for every stopping time $\tau_N$,

$$u_N := \mathbf{P}\left(\left|M_{\tau_N+\theta}^N(\varphi) - M_{\tau_N}^N(\varphi)\right| > \eta_2\right) \leq \frac{1}{\eta_2^2}\mathbf{E}\left[\left|M_{\tau_N+\theta}^N(\varphi) - M_{\tau_N}^N(\varphi)\right|^2\right],$$

$$\leq \frac{1}{\eta_2^2}\mathbf{E}\left[\int_{\tau_N}^{\tau_N+\theta}\left\langle v_s^N, \varphi'(y,\pi)^2\right\rangle ds\right],$$

$$\leq \|\varphi\|_{6,\alpha}^2 \frac{1}{\eta_2^2}\mathbf{E}\left[\int_{\tau_N}^{\tau_N+\theta}\int_{\mathbf{S}^1\times\mathbf{R}}\left\|H_{y,\pi}\right\|_{-6,\alpha}^2 dv_s^N ds\right],$$

$$\leq \|\varphi\|_{6,\alpha}^2 \frac{C}{\eta_2^2}\mathbf{E}\left[\int_{\tau_N}^{\tau_N+\theta}\frac{1}{N}\sum_{i=1}^N(1 + |\omega_i|^{4\alpha}) ds\right], \text{ (cf. (18) and (21))},$$

$$\leq \|\varphi\|_{6,\alpha}^2 \frac{C\delta}{\eta_2^2} \sup_N \left(\frac{1}{N}\sum_{i=1}^N(1 + |\omega_i|^{4\alpha})\right).$$

This last term is lower or equal than $\eta_1$ for $\delta$ sufficiently small (depending on $(\omega)$).

2. Tightness of $(\Theta_N)$: we need to be more careful here, since the tightness is *in law w.r.t. the disorder*. Let $(\varphi_j)_{j \geq 1}$ be a countable family in the nuclear Fréchet space $\mathscr{S}$. Without any

restriction, we can always suppose that $\left\| \phi_j \right\|_{6,\alpha} = 1$, for every $j \geq 1$. We define the following decreasing sequences (indexed by $J \geq 1$) of subsets of $\mathcal{M}_1\left(\mathscr{C}([0,T],\mathscr{S}')\right)$:

$$K_1^\varepsilon(\varphi_1,\ldots,\varphi_J) := \left\{ P \,;\, \forall t, \forall 1 \leq j \leq J, \, P\Pi_{\varphi_j}^{-1} \text{ satisfies } (T_{t,\delta,C_1}) \right\},$$

$$K_2^\varepsilon(\varphi_1,\ldots,\varphi_J) := \left\{ P \,;\, \forall 1 \leq j \leq J, \forall \eta_1, \eta_2 > 0, P\Pi_{\varphi_j}^{-1} \text{ satisfies } (A_{\eta_1,\eta_2,C_2}) \right\},$$

where $C_1 = C_1(\varepsilon,\delta)$, $C_2 = C_2(\varepsilon,\eta_1,\eta_2)$ will be precised later. By construction and by Mitoma's theorem (cf. Remark 4.12),

$$K^\varepsilon := \bigcap_J (K_1^\varepsilon(\varphi_1,\ldots,\varphi_J) \cap K_2^\varepsilon(\varphi_1,\ldots,\varphi_J))$$

is a relatively compact subset of $\mathcal{M}_1\left(\mathscr{C}([0,T],\mathscr{S}')\right)$. In order to prove tightness of $(\Theta_N)$, it is sufficient to prove that, for all $\varepsilon > 0$,

$$\forall i = 1,2, \; \limsup_N \Theta_N \left( \bigcup_J K_i^\varepsilon(\phi_1,\ldots,\phi_J)^c \right) \leq \varepsilon. \tag{33}$$

For $\varepsilon > 0$, let $A = A(\varepsilon)$ such that $\liminf_{N\to\infty} \mathbb{P}\left(A_N \leq A\right) \geq 1-\varepsilon$, and

$$\liminf_{N\to\infty} \mathbb{P}\left( \frac{1}{N}\sum_{i=1}^N (1+|\omega_i|^{4\alpha}) + A_N(\omega_1,\ldots,\omega_N) \leq A \right) \geq 1-\varepsilon,$$

where $A_N$ is the random variable defined in Proposition 4.4. We define the corresponding constants (for a sufficiently large constant $C$):

$$C_1(\varepsilon,\delta) := \sqrt{\frac{A(\varepsilon)}{\delta}}, \quad C_2(\varepsilon,\eta_1,\eta_2) := \frac{\eta_1 \eta_2^2}{CA(\varepsilon)}.$$

Then,

$$\Theta_N(K_1^\varepsilon(\phi_1,\ldots,\phi_J)) \geq \mathbb{P}\left( (\omega), \, \forall t, \, \forall 1 \leq j \leq J, \, \forall \delta, \, \frac{\mathbf{E}\left[\left|\left\langle \eta_t^{N,(\omega)}, \phi_j \right\rangle\right|^2\right]}{C_1(\delta,\varepsilon)^2} \leq \delta \right),$$

$$\geq \mathbb{P}\left( (\omega), \, \sup_{t\leq T} \mathbf{E}\left[\left\| \eta_t^{N,(\omega)} \right\|_{-6,\alpha}^2\right] \leq A \right), \text{ (by definition of } C_1),$$

$$\geq \mathbb{P}\left(A_N \leq A(\varepsilon)\right), \text{ (cf. (18) and (25))}.$$

Letting $J \to \infty$ in the latter inequality, we obtain: $\Theta_N(\bigcup_J K_1^\varepsilon(\phi_1,\ldots,\phi_J)^c) \leq \mathbb{P}(A_N > A)$. Taking on both sides $\limsup_{N\to\infty}$, we get the result.

Furthermore, for $\eta_2 > 0$, $0 < \theta \leq C_2$ and $\tau_N \leq T$ a stopping time, for all $1 \leq j \leq J$,

$$\mathbf{P}\left(\left|\int_{\tau_N}^{\tau_N+\theta}\left\langle \eta_s^{N,(\omega)}, \mathscr{L}_s^{\nu^N}(\varphi_j)\right\rangle \, ds\right| \geq \eta_2\right) \leq \frac{1}{\eta_2^2}\mathbf{E}\left[\left|\int_{\tau_N}^{\tau_N+\theta}\left\langle \eta_s^{N,(\omega)}, \mathscr{L}_s^{\nu^N}(\varphi_j)\right\rangle \, ds\right|^2\right],$$

$$\leq \frac{C_2}{\eta_2^2}\mathbf{E}\left[\int_{\tau_N}^{\tau_N+\theta}\left|\left\langle \eta_s^{N,(\omega)}, \mathscr{L}_s^{\nu^N}(\varphi_j)\right\rangle\right|^2 \, ds\right],$$

$$\leq \frac{C_2}{\eta_2^2}\int_0^T \mathbf{E}\left|\left\langle \eta_s^{N,(\omega)}, \mathscr{L}_s^{\nu^N}(\varphi_j)\right\rangle\right|^2 \, ds,$$

$$\leq \frac{CC_2}{\eta_2^2}\int_0^T \mathbf{E}\left[\left\|\eta_s^N\right\|_{-3,2\alpha}^2\right] \, ds,$$

$$\leq \frac{CTC_2}{\eta_2^2}A_N, \text{ (cf. (25))}.$$

And,

$$\mathbf{P}\left(\left|M_{\tau_N+\theta}^N(\varphi_j) - M_{\tau_N}^N(\varphi_j)\right| > \eta_2\right) \leq \frac{CC_2}{\eta_2^2}\left(\frac{1}{N}\sum_{i=1}^N (1+|\omega_i|^{4\alpha})\right).$$

So, for all $j \geq 1$, by definition of $C_2$,

$$\mathbf{P}\left(\left|\eta_{\tau_N+\theta}^{N,(\omega)}(\varphi_j) - \eta_{\tau_N}^{N,(\omega)}(\varphi_j)\right| \geq \eta_2\right) \leq \frac{\eta_1}{A(\varepsilon)}\left(A_N + \frac{1}{N}\sum_{i=1}^N (1+|\omega_i|^{4\alpha})\right).$$

Consequently,

$$\Theta_N(K_2^\varepsilon(\phi_1,\ldots,\varphi_J)) \geq \mathbb{P}\left(A_N + \frac{1}{N}\sum_{i=1}^N (1+|\omega_i|^{4\alpha}) > A(\varepsilon)\right).$$

Letting $J \to \infty$, we get $\limsup_N \Theta_N\left(\bigcup_J K_2^\varepsilon(\varphi_1,\ldots,\varphi_J)^c\right) \leq \varepsilon$. Eq. (33) is proved. $\qquad\square$

### 4.3.3   Identification of the limit

The proof of the fluctuations result will be complete when we identify any possible limit.

**Proposition 4.13 (Identification of the initial value).**
*The random variable $(\omega) \mapsto \mathscr{L}\left(\eta_0^{N,(\omega)}\right)$ converges in law to the random variable $\omega \mapsto \mathscr{L}(X(\omega))$, where for all $\omega$, $X(\omega) = C(\omega) + Y$, with $Y$ a centered Gaussian process with covariance $\Gamma_1$. Moreover $\omega \mapsto C(\omega)$ is a Gaussian process with covariance $\Gamma_2$, where $\Gamma_1$ and $\Gamma_2$ are defined in (8) and (9).*

*Proof.* For simplicity, we only identify here the law of $\left\langle \eta_0^{N,(\omega)}, \varphi\right\rangle$ for all $\varphi$. The same proof works for the law of finite-dimensional distributions $\left(\left\langle \eta_0^{N,(\omega)}, \varphi_1\right\rangle,\ldots,\left\langle \eta_0^{N,(\omega)}, \varphi_p\right\rangle\right)$, $p \geq 1$. We write

$\Gamma_i$ for $\Gamma_i(\varphi, \varphi)$, $i = 1, 2$. One has:

$$\left\langle \eta_0^{N,(\omega)}, \varphi \right\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( \varphi(\xi^i, \omega_i) - \int_{\mathbf{S}^1} \varphi(x, \omega_i) \lambda(\mathrm{d}x) \right)$$
$$+ \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left( \int_{\mathbf{S}^1} \varphi(x, \omega_i) \lambda(\mathrm{d}x) - \langle \nu_0, \varphi \rangle \right),$$
$$=: A^{N,(\omega)} + B^{N,(\omega)}.$$

It is easy to see that $B^{N,(\omega)}$ converges in law to $Z_2 \sim \mathcal{N}(0, \Gamma_2)$. Moreover, for $\mathbb{P}$-almost every $(\omega)$, $A^{N,(\omega)}$ converges in law to $Z_1 \sim \mathcal{N}(0, \Gamma_1)$ (see Billingsley [5], Th. 27.3 p. 362). That means that for all $u \in \mathbf{R}$, $\psi_{A^N}(u) := \mathbf{E}_\lambda \left( e^{iuA^{N,(\omega)}} \right)$ converges to $\psi_Y(u) := e^{-\frac{u^2}{2\Gamma_1}}$. But, then, for all $F \in \mathscr{C}_b(\mathbf{R})$,

$$\mathbb{E} \left[ F \left( \mathbf{E}_\lambda \left[ e^{iu\left\langle \eta_0^{N,(\omega)}, \varphi \right\rangle} \right] \right) \right] = \mathbb{E} \left[ F \left( \mathbf{E}_\lambda \left[ e^{iu(A^{N,(\omega)} + B^{N,(\omega)})} \right] \right) \right] = \mathbb{E} \left[ F \left( e^{iuB^{N,(\omega)}} \psi_N(u) \right) \right].$$

Since $\psi_N(u)$ converges almost surely to a constant, the limit of the expression above exists (Slutsky's theorem) and is equal to $\mathbb{E} \left[ F \left( e^{iuZ_2 - \frac{u^2}{2\Gamma_1}} \right) \right]$. $\qquad\square$

**Proposition 4.14 (Identification of the martingale part).**
*For $\mathbb{P}$-almost every $(\omega)$, the sequence $(M^{N,(\omega)})$ converges in law in $\mathscr{C}([0,T], \mathscr{S}')$ to a Gaussian process $W$ with covariance defined in* (7).

*Proof.* For fixed $(\omega)$, $(M^{N,(\omega)})$ is a sequence of uniformly square-integrable continuous martingales (cf. Remark 4.7), which is tight in $\mathscr{C}([0,T], \mathscr{S}')$. Let $W_1$ and $W_2$ be two accumulation points (continuous square-integrable martingales which *a priori* depend on $(\omega)$) and $(M^{\phi(N),(\omega)})$ and $(M^{\psi(N),(\omega)})$ be two subsequences converging to $W_1$ and $W_2$, respectively. Note that we can suppose that $\phi(N) \leq \psi(N)$ for all $N$. For all $\varphi \in \mathscr{S}$, $\lim_{N\to\infty} \left\langle M^{\phi(N),(\omega)}(\varphi), M^{\psi(N),(\omega)}(\varphi) \right\rangle_t = \left\langle W_1(\varphi), W_2(\varphi) \right\rangle_t$, for all $t$, and

$$\left\langle M^{\phi(N),(\omega)}(\varphi), M^{\psi(N),(\omega)}(\varphi) \right\rangle_t = \int_0^t \left\langle \nu_s^{\phi(N)}, (\varphi')^2 \right\rangle \mathrm{d}s.$$

We now have to identify the limit: we already know that for $\mathbb{P}$-almost every realization of the disorder $(\omega)$, $(\nu^{N,(\omega)})$ converges in law to $P$. But, the latter expression, seen as a function of $\nu$, is continuous. So $\left\langle W_1(\varphi), W_2(\varphi) \right\rangle_t = \int_0^t \left\langle P_s, (\varphi')^2 \right\rangle$. So $W_1 - W_2$ is a continuous square integrable martingale whose Doob-Meyer process is 0. So $W_1 = W_2$ and is characterized as the Gaussian process with covariance given in (7). The convergence follows. $\qquad\square$

*Proof of the independence of $W$ and $X$.* We prove more : the triple $(Y, C, W)$ is independent. For sake of simplicity, we only consider the case of $(Y(\varphi), C(\varphi), W_t(\varphi))$ for fixed $t$ and $\varphi$.

Let us first recall some notations: let $A^{N,(\omega)}$, $B^{N,(\omega)}$ and $M_t^{N,(\omega)}(\varphi)$ be the random variables defined in the proof of Proposition 4.13 and 4.14 and let $\psi_{A^N}(u) := \mathbf{E} \left( e^{iuA^{N,(\omega)}} \right)$, $\psi_{B^N}(v) := \mathbf{E} \left( e^{ivB^{N,(\omega)}} \right)$, $\psi_{M^N}(w) := \mathbf{E} \left( e^{iwM_t^{N,(\omega)}(\varphi)} \right)$ be their characteristic functions ($u, v, w \in \mathbf{R}$). We know that, for almost

every $(\omega)$, $\psi_{A^N}(u)$ converges to $\psi_Y(u) = e^{-\frac{u^2}{2\Gamma_1}}$ and that $\psi_{M^N}(w)$ converges to the deterministic function $\psi_W(w) := \mathbf{E}\left(e^{iwW_t(\varphi)}\right)$. But, if $\psi_C(v) = \mathbb{E}\left(e^{iwC}\right)$, then, for all $u, v, w \in \mathbf{R}$, using the independence of the Brownian with the initial conditions,

$$\mathbb{E}\left(\mathbf{E}\left(e^{iuA^{N,(\omega)} + ivB^{N,(\omega)} + iwM_t^{N,(\omega)}(\varphi)}\right) - e^{ivB^{N,(\omega)}}\psi_{A^N}(u)\psi_{M^N}(w)\right) = 0.$$

Using Slutsky's theorem, we see that any limit couple $(Y, C, W)$ satisfies

$$\mathbb{E}\left(\mathbf{E}\left(e^{iuY + ivC + iwW_t(\varphi)}\right)\right) = \psi_Y(u)\psi_C(v)\psi_W(w).$$

which is the desired result. $\qquad\qquad\square$

We recall that the limit second order differential operator $\mathscr{L}_s$ is defined by

$$\mathscr{L}_s(\varphi)(y, \pi) := \frac{1}{2}\varphi''(y, \pi) + \varphi'(y, \pi)(b[y, P_s] + c(y, \pi)) + \langle P_s, \varphi'(\cdot, \cdot)b(\cdot, y, \pi)\rangle.$$

As in Lemma 4.5, we can prove the following:

**Lemma 4.15.**
*Assume $(H_{b,c})$. Then for every $N$, $s \leq T$, $(\omega)$, the operator $\mathscr{L}_s$ and $\mathscr{L}_s^{v^N}$ are linear continuous from $\mathscr{S}$ to $\mathscr{S}$ and*

$$\left\|\mathscr{L}_s(\varphi)\right\|_{6,\alpha} \leq C \left\|\varphi\right\|_{8,\alpha},$$

$$\left\|\mathscr{L}_s^{v^N}(\varphi)\right\|_{6,\alpha} \leq C \left\|\varphi\right\|_{8,\alpha}.$$

We are now in position to prove Theorem 2.10:

*Proof of Theorem 2.10.* Let $\Theta$ be an accumulation point of $\Theta_N$. Thus, for a certain subsequence (which will be also denoted as $N$ for notations purpose), the random variable $(\omega) \mapsto \mathscr{H}^{N,(\omega)}$ converges in law to a random variable $\mathscr{H}$ with values in $\mathscr{M}_1(\mathscr{C}([0, T], \mathscr{S}'))$ with law $\Theta$. Applying Skorohod's representation theorem, there exists some probability space $(\Omega^{(1)}, \mathbf{P}^{(1)}, \mathscr{F}^{(1)})$ and random variables defined on $\Omega^{(1)}$, $\omega_1 \mapsto H^N(\omega_1)$ and $\omega_1 \mapsto H(\omega_1)$ such that $H^N$ has the same law as $(\omega) \mapsto \mathscr{H}^{N,(\omega)}$, $H$ has the same law as $\mathscr{H}$, and for $\mathbf{P}^{(1)}$-almost every $\omega_1 \in \Omega^{(1)}$, $H^N(\omega_1)$ converges to $H(\omega_1)$ in $\mathscr{M}_1(\mathscr{C}([0, T], \mathscr{S}'))$.

An easy application of Proposition 4.8 and Borel-Cantelli's Lemma shows that $\mathbf{P}^{(1)}$-almost surely, $\mathbf{E}\left(\sup_{t \leq T} \left\|\eta_t^{\omega_1}\right\|_{-6,\alpha}\right) < \infty$. Then we know from Lemma 4.15 that the integral term $\int_0^t \mathscr{L}_s^* \eta_s^{\omega_1} \, ds$ makes sense as a Bochner's integral in $W_0^{-8,\alpha} \subseteq \mathscr{S}'$.

Let $\eta^{N,\omega_1}$ with law $H^N(\omega_1)$; $\eta^{N,\omega_1}$ converges in law to some $\eta^{\omega_1}$ with law $H(\omega_1)$. By uniqueness in law convergence, using Propositions 4.13 and 4.14, we see that $(\eta_0^{\omega_1}, W)$ as the same law as $(X(\omega_1), W)$. For fixed $\varphi \in \mathscr{S}$, we define $F_\varphi$ from $\mathscr{C}([0, T], \mathscr{S}')$ into $\mathbf{R}$ by $F_\varphi(\gamma) := \langle \gamma_t, \varphi\rangle - \langle \gamma_0, \varphi\rangle - \int_0^t \langle \gamma_s, \mathscr{L}_s\varphi\rangle \, ds$. The function $F_\varphi$ is continuous and since $\eta^{N,\omega_1}$ converges in law to $\eta^{\omega_1}$, the sequence $(F_\varphi(\eta^{N,\omega_1}))$ converges in law to $F_\varphi(\eta^{\omega_1})$. To prove the theorem, it remains to show that the law of the term $\int_0^t \left\langle \eta_s^{N,\omega_1}, \mathscr{L}_s^{v^N}\varphi - \mathscr{L}_s\varphi\right\rangle \, ds$ converges in law to 0. We show that there

is convergence in probability: For all $\varepsilon > 0$, for all $A > 0$, using Proposition 4.8, Lemma 4.15, and Cauchy-Schwarz's inequality,

$$
\begin{aligned}
U_{N,\varepsilon} &:= \mathbf{P}^{(1)}\left(\mathbf{E}\left[\int_0^t \left|\left\langle \eta_s^{N,\omega_1}, (\mathscr{L}_s^{v^N} - \mathscr{L}_s)(\varphi)\right\rangle\right| \, ds\right] > \varepsilon\right), \\
&= \mathbb{P}\left(\mathbf{E}\left[\int_0^t \left|\left\langle \eta_s^{N,(\omega)}, (\mathscr{L}_s^{v^N} - \mathscr{L}_s)(\varphi)\right\rangle\right| \, ds\right] > \varepsilon\right), \\
&\leq \mathbb{P}\left(\int_0^t \mathbf{E}\left[\left\|\eta_s^{N,(\omega)}\right\|_{-6,\alpha}^2\right]^{1/2} \mathbf{E}\left[\left\|(\mathscr{L}_s^{v^N} - \mathscr{L}_s)(\varphi)\right\|_{6,\alpha}^2\right]^{1/2} \, ds > \varepsilon\right), \\
&\leq \mathbb{P}\left(C_N(\omega_1,\ldots,\omega_N)^{1/2} \int_0^t \mathbf{E}\left[\left\|(\mathscr{L}_s^{v^N} - \mathscr{L}_s)(\varphi)\right\|_{6,\alpha}^2\right]^{1/2} \, ds > \varepsilon\right) \quad \text{(cf. Prop 4.8),} \\
&\leq \mathbb{P}\left(\int_0^t \mathbf{E}\left[\left\|(\mathscr{L}_s^{v^N} - \mathscr{L}_s)(\varphi)\right\|_{6,\alpha}^2\right]^{1/2} \, ds > \frac{\varepsilon}{\sqrt{A}}\right) + \mathbb{P}\left(C_N > A\right).
\end{aligned}
$$

Using (4.8), it suffices to prove that, for all $\varepsilon > 0$,

$$
\limsup_{N\to\infty} \mathbb{P}\left(\int_0^t \mathbf{E}\left[\left\|(\mathscr{L}_s^{v^N} - \mathscr{L}_s)(\varphi)\right\|_{6,\alpha}^2\right]^{1/2} \, ds > \varepsilon\right) = 0. \tag{34}
$$

Indeed, for every $\varphi \in \mathscr{S}$,

$$
\mathscr{U}_s^N(\varphi)(y,\pi) := (\mathscr{L}_s^{v^N} - \mathscr{L}_s)(\varphi)(y,\pi) = \varphi'(y,\pi)(b[y, v_s^N] - b[y, P_s]).
$$

An analogous calculation as in Lemma 4.5 shows that, using Lipschitz assumptions on $b$, and Proposition 4.3:

$$
\mathbf{E}\left[\left\|\sup_{s\leq t} \mathscr{U}_s^N(\varphi)\right\|_{6,\alpha}^2\right] \leq \left\|\varphi\right\|_{8,\alpha}^2 (C/N + D_N(\omega_1,\ldots,\omega_N)),
$$

with the property that $\lim_{A\to\infty} \limsup_N \mathbb{P}(ND_N > A) = 0$. Equation (34) is a direct consequence.

Since there is uniqueness in law in (10), $\Theta$ is perfectly defined, and thus, unique. The convergence follows. $\qquad\square$

# 5 Proofs for the fluctuations of the order parameters

We end by the proofs of paragraph 2.3.3.

## 5.1 Proof of Proposition 2.13

1. This is straightforward since $r^{N,(\omega)} = \left|\left\langle v^{N,(\omega)}, e^{ix}\right\rangle\right|$ and since for $\mathbb{P}$-almost every disorder $(\omega)$, $v^{N,(\omega)}$ converges weakly to $P$.

2. The following sequences are well defined: $\forall k \geq 0$,

$$u_k(t) := \int_{\mathbf{S}^1 \times \mathbf{R}} e^{-|\omega|} \omega^k \cos(\theta) P_t(\mathrm{d}\theta, \mathrm{d}\omega),$$

$$v_k(t) := \int_{\mathbf{S}^1 \times \mathbf{R}} e^{-|\omega|} \omega^k \sin(\theta) P_t(\mathrm{d}\theta, \mathrm{d}\omega).$$

Let $E = (\ell_\infty(\mathbf{N}), \|\cdot\|_\infty)$ be the Banach space of real bounded sequences endowed with its usual $\|\cdot\|_\infty$ norm, ($\|u\|_\infty = \sup_{k \geq 0} |u_k|$). For all $t > 0$, let $\mathscr{A}_t : E \times E \to E \times E$, be the following linear operator (where $(u, v)$ is a typical element of $E \times E$): For all $k \geq 0$

$$\begin{cases} \mathscr{A}_t(u, 0)_k & = & -\frac{1}{2} u_k - \alpha_k(t) v_0 + \beta_k(t) u_0 - K v_{k+1}, \\ \mathscr{A}_t(0, v)_k & = & -\frac{1}{2} v_k + \gamma_k(t) v_0 - \alpha_k(t) u_0 + K u_{k+1}, \end{cases}$$

where,

$$\alpha_k(t) = \left\langle P_t, e^{-|\omega|} \omega^k \cos(\cdot) \sin(\cdot) \right\rangle,$$
$$\beta_k(t) = \left\langle P_t, e^{-|\omega|} \omega^k \sin^2(\cdot) \right\rangle,$$
$$\gamma_k(t) = \left\langle P_t, e^{-|\omega|} \omega^k \cos^2(\cdot) \right\rangle.$$

$(t, u, v) \mapsto \mathscr{A}_t \cdot (u, v)$ is globally Lipschitz-continuous map from $[0, T] \times E \times E$ into $E \times E$ and one easily verifies considering (4) (in the case of the sine-model) and developing the sine interaction that $t \mapsto (u(t), v(t))$ satisfies in $E \times E$ the following linear non-homogeneous Cauchy Problem:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(u(t), v(t)) & = & \mathscr{A}_t \cdot (u(t), v(t)), \\ u_k(0) & = & \left\langle P_0, e^{-|\omega|} \omega^k \cos(\cdot) \right\rangle, \forall k \geq 0 \\ v_k(0) & = & \left\langle P_0, e^{-|\omega|} \omega^k \sin(\cdot) \right\rangle, \forall k \geq 0. \end{cases}$$

Let us suppose that there exists some $t_0 \in [0, T]$ such that $r_{t_0} = 0$, namely $u_0(t_0) = v_0(t_0) = 0$. Then, if $(\tilde{u}, \tilde{v})$ is the constant function on $[0, T]$ such that for all $k \geq 0$, $\tilde{u}_k \equiv u_k(t_0), \tilde{v}_k \equiv v_k(t_0)$, then $(\tilde{u}, \tilde{v})$ satisfy the same Cauchy Problem as $(u, v)$ with initial condition at time $t_0$. By Cauchy-Lipschitz theorem, both functions coincide on $[0, T]$. In particular, $u_0$ and $v_0$ are always zero and thus $r \equiv 0$.

3. We suppose $(H_r)$. A simple calculation shows that the fluctuations process $\mathscr{R}^N$ verifies for all $t \in [0, T]$,

$$\mathscr{R}_t^{N,(\omega)} = \frac{\left\langle \eta_t^{N,(\omega)}, \cos(\cdot) \right\rangle \left\langle v_t^{N,(\omega)} + P_t, \cos(\cdot) \right\rangle + \left\langle \eta_t^{N,(\omega)}, \sin(\cdot) \right\rangle \left\langle v_t^{N,(\omega)} + P_t, \sin(\cdot) \right\rangle}{r_t^{N,(\omega)} + r_t},$$

$$= \frac{\Re \left( \left\langle \eta_t^{N,(\omega)}, e^{ix} \right\rangle \overline{\left\langle v_t^{N,(\omega)} + P_t, e^{ix} \right\rangle} \right)}{\left| \left\langle v_t^{N,(\omega)}, e^{ix} \right\rangle \right| + r_t}.$$

Let $u^{N,(\omega)} := \left\langle v^{N,(\omega)}, e^{ix} \right\rangle$, $v^{N,(\omega)} := \left\langle \eta^{N,(\omega)}, e^{ix} \right\rangle$ and $u := \left\langle P, e^{ix} \right\rangle$, $v^\omega := \left\langle \eta^\omega, e^{ix} \right\rangle$ be their corresponding limits. The result follows if we prove the following property:

the random variables $(\omega) \mapsto \mathscr{L}\left(u^{N,(\omega)}, v^{N,(\omega)}\right)$ converges in law to the random variable $\omega \mapsto \mathscr{L}(u, v^{\omega})$. The tightness of this random variable follows from the convergence of both empirical measure and fluctuations process. As already said in Remark 2.14, it suffices to prove the convergence of the finite-dimensional marginals $(\underline{u}_{\underline{t}}^{N,(\omega)}, \underline{v}_{\underline{t}}^{N,(\omega)}) = \left((u_{t_1}^{N,(\omega)}, \ldots, u_{t_p}^{N,(\omega)}), (v_{t_1}^{N,(\omega)}, \ldots, v_{t_p}^{N,(\omega)})\right)$, for all element of $[0, T]$, $t_1, \ldots, t_p$, $p \geq 1$.

Since the limit of $(u^{N,(\omega)})$ is a constant, this is mainly a consequence of Slutsky's theorem. But since this is a convergence *in law with respect to the disorder*, one has to adapt the proof. We prove the following: $\forall G \in \mathscr{C}_b^1(\mathbf{R})$, $\forall \underline{r} = \left(r_1, \ldots, r_p\right) \in \mathbf{R}^p$, $\forall \underline{s} = \left(s_1, \ldots, s_p\right) \in \mathbf{R}^p$,

$$\mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}^{N,(\omega)}, \underline{v}_{\underline{t}}^{N,(\omega)})}(\underline{r}, \underline{s})\right)\right] \underset{N\to\infty}{\to} \mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{\omega})}(\underline{r}, \underline{s})\right)\right],$$

where $\varphi_{(\underline{X},\underline{Y})}(\underline{r}, \underline{s}) = \mathbb{E}\left[e^{i\underline{r}\cdot\underline{X} + i\underline{s}\cdot\underline{Y}}\right]$ is the characteristic function of the couple $(\underline{X}, \underline{Y})$. Indeed, we have successively:

$$a_N := \left|\mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}^{N,(\omega)}, \underline{v}_{\underline{t}}^{N,(\omega)})}(\underline{r}, \underline{s})\right)\right] - \mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{\omega})}(\underline{r}, \underline{s})\right)\right]\right|,$$

$$\leq \left|\mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}^{N,(\omega)}, \underline{v}_{\underline{t}}^{N,(\omega)})}(\underline{r}, \underline{s})\right)\right] - \mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{N,(\omega)})}(\underline{r}, \underline{s})\right)\right]\right|$$

$$+ \left|\mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{N,(\omega)})}(\underline{r}, \underline{s})\right)\right] - \mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{\omega})}(\underline{r}, \underline{s})\right)\right]\right|,$$

$$\leq C\mathbb{E}\left|\varphi_{(\underline{u}_{\underline{t}}^{N,(\omega)}, \underline{v}_{\underline{t}}^{N,(\omega)})}(\underline{r}, \underline{s}) - \varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{N,(\omega)})}(\underline{r}, \underline{s})\right|$$

$$+ \left|\mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{N,(\omega)})}(\underline{r}, \underline{s})\right)\right] - \mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{\omega})}(\underline{r}, \underline{s})\right)\right]\right|,$$

$$\leq C\mathbb{E}\mathbf{E}\left|e^{i\underline{r}\cdot\underline{u}_{\underline{t}}^{N,(\omega)}} - e^{i\underline{r}\cdot\underline{u}_{\underline{t}}}\right|$$

$$+ \left|\mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{N,(\omega)})}(\underline{r}, \underline{s})\right)\right] - \mathbb{E}\left[G\left(\varphi_{(\underline{u}_{\underline{t}}, \underline{v}_{\underline{t}}^{\omega})}(\underline{r}, \underline{s})\right)\right]\right|.$$

But, we have $\mathbf{E}\left|e^{i\underline{r}_{\underline{t}}\cdot\underline{u}_{\underline{t}}^{N,(\omega)}} - e^{i\underline{r}\cdot\underline{u}_{\underline{t}}}\right| \leq \min\left(2, |\underline{r}||\underline{u}_{\underline{t}}^{N,(\omega)} - \underline{u}_{\underline{t}}|\right)$. So, for all $\varepsilon > 0$,

$$\mathbf{E}\left|e^{i\underline{r}\cdot\underline{u}_{\underline{t}}^{N,(\omega)}} - e^{i\underline{r}\cdot\underline{u}_{\underline{t}}}\right| \leq \varepsilon|\underline{r}| + 2\mathbf{P}\left(\left|\underline{u}_{\underline{t}}^{N,(\omega)} - \underline{u}_{\underline{t}}\right| > \varepsilon\right).$$

Taking $\limsup_{N\to\infty}$, and letting $\varepsilon \to 0$, we get $\lim a_N = 0$. The result follows.

## 5.2 Proof of Proposition 2.15

The proof is similar to the previous one and relies on the two following equalities :

$$\zeta^{N,(\omega)} = \frac{\left\langle P, e^{ix}\right\rangle}{r^{N,(\omega)}},$$

$$\sqrt{N}\left(\zeta^{N,(\omega)} - \zeta\right) = \frac{1}{r \cdot r^{N,(\omega)}}\left(r\left\langle \eta^{N,(\omega)}, e^{ix}\right\rangle + \left\langle P, e^{ix}\right\rangle \mathscr{R}^{N,(\omega)}\right).$$

## A Proof of Proposition 4.3

Thanks to the Lipschitz continuity of $b$ and $c$, introducing $v$ as the empirical measure corresponding to $(x^{\omega_i}, \omega_i)$, we have, (inserting $b[x_s^{\omega_i}, v_s^N] - b[x_s^{\omega_i}, v_s]$ in the $b$ term),

$$
\mathbf{E}\left[\sup_{s\leq t}\left|x_s^{i,N} - x_s^{\omega_i}\right|^2\right] \leq C\left(\int_0^t \mathbf{E}\left[\left(b[x_s^{i,N}, v_s^N] - b[x_s^{\omega_i}, P_s]\right)^2\right]ds\right.
$$

$$
+ \int_0^t \mathbf{E}\left[\left(c(x_s^{i,N}, \omega_i) - c(x_s^{\omega_i}, \omega_i)\right)^2\right]ds\bigg),
$$

$$
\leq C\left(2\int_0^t \mathbf{E}\left[\sup_{u\leq s}\left|x_u^{i,N} - x_u^{\omega_i}\right|^2\right]ds + \int_0^t \sup_{1\leq j\leq N}\mathbf{E}\left[\sup_{u\leq s}\left|x_u^{\omega_j} - x_u^{j,N}\right|^2\right]ds\right.
$$

$$
+ \int_0^t \mathbf{E}\left[\left(b[x_s^{\omega_i}, v_s] - b[x_s^{\omega_i}, P_s]\right)^2\right]ds\bigg).
$$

Applying Gronwall's Lemma to $\sup_{1\leq j\leq N}\mathbf{E}\left[\sup_{u\leq t}\left|x_u^{\omega_j} - x_u^{j,N}\right|^2\right]$, it suffices to prove that for some $Z_N$:

$$
\int_0^t \mathbf{E}\left[\left(b[x_s^{\omega_i}, v_s] - b[x_s^{\omega_i}, P_s]\right)^2\right]ds \leq C/N + Z_N(\omega_1, \ldots, \omega_N).
$$

Indeed, for all $1 \leq i \leq N$, (we write $x^i$ instead of $x^{\omega_i}$ to simplify notations):

$$
u_{i,N} := \left(b[x_s^i, v_s] - b[x_s^i, P_s]\right)^2 = \frac{1}{N^2}\left(\sum_{j=1}^N T(x^i, x^j)^2 + \sum_{k\neq l} T(x^i, x^k)T(x^i, x^l)\right),
$$

where $T(x^i, x^j) := b(x_s^i, x_s^j, \omega_j) - \int b(x_s^i, y, \pi)P_s(dy, d\pi)$. Since $b$ is bounded, we see that the first term is of order $(1/N)$. We only have to study the remaining term:

$$
\mathbf{E}\left[\sum_{k\neq l} T(x^i, x^k)T(x^i, x^l)\right] \leq CN + \mathbf{E}\left[\sum_{\substack{k\neq i, l\neq i \\ k\neq l}} T(x^i, x^k)T(x^i, x^l)\right].
$$

Since the $(x^i)$ are independent, if we take conditional expectation w.r.t. $(x^r, r\neq l)$ in the last term, we get:

$$
\mathbf{E}\left[\sum_{\substack{k\neq i, l\neq i \\ k\neq l}} T(x^i, x^k)T(x^i, x^l)\right] = \mathbf{E}\left[\mathbf{E}\left[\sum T(x^i, x^k)T(x^i, x^l)\Big| x^r, r\neq l\right]\right],
$$

$$
= \mathbf{E}\left[\sum_{\substack{k\neq i, l\neq i \\ k\neq l}} T(x^i, x^k)G_l(x^i)\right] = \mathbf{E}\left[\sum_{\substack{k\neq i, l\neq i \\ k\neq l}} G_k(x^i)G_l(x^i)\right],
$$

826

where $G_l(x) = G(x, \omega_l) = \int b(x, y, \omega_l) P_s^{\omega_l}(\mathrm{d}y) - \int b(x, y, \pi) P_s(\mathrm{d}y, \mathrm{d}\pi)$. Defining

$$Z_N(\omega_1, \ldots, \omega_N) := \frac{C}{N} \int_0^T \mathbf{E}\left[\left(\frac{1}{\sqrt{N}} \sum_{l=1}^N G(x_s^i, \omega_l)\right)^2\right] \mathrm{d}s,$$

in order to prove (24) it suffices to show that for some constant $C$,

$$\mathbb{E}\left[\int_0^T \mathbf{E}\left[\left(\frac{1}{\sqrt{N}} \sum_{l=1}^N G(x_s^i, \omega_l)\right)^2\right] \mathrm{d}s\right] \leq C.$$

The rest of the proof is devoted to prove this last assertion: we have successively (setting $U_N(x_s^{\omega_i}, \underline{\omega}) := \frac{1}{\sqrt{N}} \sum_{l=1}^N G(x_s^{\omega_i}, \omega_l)$)

$$\mathbb{E}\left[\int_0^T \mathbf{E}\left[U_N(x_s^{\omega_i}, \underline{\omega})^2\right] \mathrm{d}s\right] \leq \int_0^T \mathbf{E}\left[\mathbb{E}\left[U_N(x_s^{\omega_i}, \underline{\omega})^2\right]\right] \mathrm{d}s,$$

$$\leq \frac{1}{N} \int_0^T \mathbf{E}\left[\mathbb{E}\left[\sum_{k=1}^N \sum_{l=1}^N G(x_s^{\omega_i}, \omega_k) G(x_s^{\omega_i}, \omega_l)\right]\right] \mathrm{d}s,$$

$$\leq \frac{1}{N} \int_0^T \mathbf{E}\left[\mathbb{E}\left[\sum_{l=1}^N G(x_s^{\omega_i}, \omega_l)^2\right]\right] \mathrm{d}s + C$$

$$+ \frac{1}{N} \int_0^T \mathbf{E}\left[\mathbb{E}\left[\sum_{l \neq k,\, l \neq i,\, k \neq i} G(x_s^{\omega_i}, \omega_k) G(x_s^{\omega_i}, \omega_l)\right]\right] \mathrm{d}s.$$

The first term of the RHS of the last inequality is bounded, since $b$ is bounded. But, if we condition w.r.t. $\omega_r$ for $r \neq i, r \neq k$, we see that the second term is zero. The result follows.

## Acknowledgements

## References

[1] J. A. Acebrón, L. L. Bonilla, C. J. Pérez Vicente, F. Ritort, and R. Spigler. The Kuramoto model: A simple paradigm for synchronization phenomena. *Rev. Mod. Phys.*, 77(1):137–185, Apr 2005.

[2] N. J. Balmforth and R. Sassi. A shocking display of synchrony. *Physica D: Nonlinear Phenomena*, 143(1-4):21 – 55, 2000. MR1783383

[3] L. Bertini, G. Giacomin, and K. Pakdaman. Dynamical aspects of mean field plane rotators and the Kuramoto model. *J. Statist. Phys.*, 138:270–290, 2010. MR2594897

[4] P. Billingsley. *Probability and measure*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York, third edition, 1995. MR1324786

[5] P. Billingsley. *Convergence of probability measures*. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. MR1700749

[6] E. Bolthausen. Laplace approximations for sums of independent random vectors. *Probab. Theory Relat. Fields*, 72(2):305–318, 1986. MR0836280

[7] A. Budhiraja, P. Dupuis, and F. Markus. Large deviation properties of weakly interacting processes via weak convergence methods. http://arxiv.org/pdf/1009.603.

[8] F. Collet, P. Dai Pra, and E. Sartori. A simple mean field model for social interactions: dynamics, fluctuations, criticality. *J. Stat. Phys.*, 139(5):820–858, 2010. MR2639892

[9] P. Dai Pra and F. den Hollander. McKean-Vlasov limit for interacting random processes in random media. *J. Statist. Phys.*, 84(3-4):735–772, 1996. MR1400186

[10] D. A. Dawson. Critical dynamics and fluctuations for a mean-field model of cooperative behavior. *J. Statist. Phys.*, 31(1):29–85, 1983. MR0711469

[11] F. den Hollander. *Large deviations*, volume 14 of *Fields Institute Monographs*. American Mathematical Society, Providence, RI, 2000. MR1739680

[12] B. Fernandez and S. Méléard. A Hilbertian approach for fluctuations on the McKean-Vlasov model. *Stochastic Process. Appl.*, 71(1):33–53, 1997. MR1480638

[13] J. Gärtner. On the McKean-Vlasov limit for interacting diffusions. *Math. Nachr.*, 137:197–248, 1988. MR0968996

[14] I. M. Gel'fand and N. Y. Vilenkin. *Generalized functions. Vol. 4*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977]. MR0173945

[15] M. Hitsuda and I. Mitoma. Tightness problem and stochastic evolution equation arising from fluctuation phenomena for interacting diffusions. *J. Multivariate Anal.*, 19(2):311–328, 1986. MR0853061

[16] B. Jourdain and S. Méléard. Propagation of chaos and fluctuations for a moderate model with smooth initial data. *Ann. Inst. H. Poincaré Probab. Statist.*, 34(6):727–766, 1998. MR1653393

[17] Y. Kuramoto. *Chemical oscillations, waves, and turbulence*, volume 19 of *Springer Series in Synergetics*. Springer-Verlag, Berlin, 1984. MR0762432

[18] F. Malrieu. Convergence to equilibrium for granular media equations and their Euler schemes. *Ann. Appl. Probab.*, 13(2):540–560, 2003. MR1970276

[19] I. Mitoma. Tightness of probabilities on $C([0,1]; \mathscr{S}')$ and $D([0,1]; \mathscr{S}')$. *Ann. Probab.*, 11(4):989–999, 1983. MR0714961

[20] K. Oelschläger. A martingale approach to the law of large numbers for weakly interacting stochastic processes. *Ann. Probab.*, 12(2):458–479, 1984. MR0735849

[21] S. Roelly-Coppoletta. A criterion of convergence of measure-valued processes: application to measure branching processes. *Stochastics*, 17(1-2):43–65, 1986. MR0878553

[22] H. Sakaguchi. Cooperative phenomena in coupled oscillator systems under external fields. *Progr. Theoret. Phys.*, 79(1):39–46, 1988. MR0937229

[23] T. Shiga and H. Tanaka. Central limit theorem for a system of Markovian particles with mean field interactions. *Z. Wahrsch. Verw. Gebiete*, 69(3):439–459, 1985. MR0787607

[24] S. H. Strogatz and R. E. Mirollo. Stability of incoherence in a population of coupled oscillators. *J. Statist. Phys.*, 63(3-4):613–635, 1991. MR1115806

[25] A.-S. Sznitman. Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated. *J. Funct. Anal.*, 56(3):311–336, 1984. MR0743844

[26] A.-S. Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991. MR1108185