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Standard Spectral Dimension for the Polynomial Lower Tail Random Conductances Model

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Abstract

We study models of continuous-time, symmetric, $\mathbb{Z}^d$-valued random walks in random environment, driven by a field of i.i.d. random nearest-neighbor conductances $\omega_{xy} \in [0,1]$ with a power law with an exponent γ near 0. We are interested in estimating the quenched asymptotic behavior of the on-diagonal heat-kernel $h_t^\omega(0,0)$. We show that for $\gamma > \frac{d}{2}$, the spectral dimension is standard, i.e.,

$$-2 \lim_{t\to+\infty} \log h_t^\omega(0,0)/\log t = d.$$

As an expected consequence, the same result holds for the discrete-time case.

Key words: Markov chains, Random walk, Random environments, Random conductances, Percolation.

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*To my father Youcef Bey**

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1. Introduction

We study the model of random walk among polynomial lower tail random conductances on $\mathbb{Z}^d$, $d \geq 2$. Our aim is to derive estimates on the asymptotic behavior of the heat-kernel in the absence of uniform ellipticity assumption. This paper follows up recent results of Fontes and Mathieu [9], Berger, Biskup, Hoffman and Kozma [3], and Boukhadra [5].

1.1 Describing the model.

Let us now describe the model more precisely. We consider a family of symmetric, irreducible, nearest-neighbors Markov chains taking their values in $\mathbb{Z}^d$, $d \geq 2$, and constructed in the following way. Let Ω be the set of functions $\omega : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}_+$ such that $\omega(x,y) = \omega_{xy} > 0$ iff $x \sim y$, and $\omega_{xy} = \omega_{xy}$ ($x \sim y$ means that x and y are nearest-neighbors). We call elements of Ω environments.

Define the transition matrix

$$P_\omega(x,y) = \frac{\omega_{xy}}{\pi_\omega(x)}, \tag{1.1}$$

and the associated Markov generator

$$(\mathscr{L}^\omega f)(x) = \sum_{y \sim x} P_\omega(x,y)[f(y) - f(x)]. \tag{1.2}$$

$X = \{X(t), t \in \mathbb{R}_+\}$ will be the coordinate process on path space $(\mathbb{Z}^d)^{\mathbb{R}_+}$ and we use the notation P_x^ω to denote the unique probability measure on path space under which X is the Markov process generated by (1.2) and satisfying $X(0) = x$, with expectation henceforth denoted by E_x^ω. This process can be described as follows. The moves are those of the discrete time Markov chain with transition matrix given in (1.1) started at x, but the jumps occur after independent Poisson (1) waiting times. Thus, the probability that there have been exactly n jumps at time t is $e^{-t}t^n/n!$ and the probability to be at y after exactly n jumps at time t is $e^{-t}t^n P_\omega^n(x,y)/n!$.

Since $\omega_{xy} > 0$ for all neighboring pairs (x,y), $X(t)$ is irreducible under the "quenched law P_x^ω" for all x. The sum $\pi_\omega(x) = \sum_y \omega_{xy}$ defines an invariant, reversible measure for the corresponding (discrete) continuous-time Markov chain.

The continuous time semigroup associated with $\mathscr{L}^\omega$ is defined by

$$P_t^\omega f(x) := E_x^\omega[f(X(t))]. \tag{1.3}$$

Define the heat-kernel, that is the kernel of P_t^ω with respect to π_ω, or the transition density of $X(t)$, by

$$h_t^\omega(x,y) := \frac{P_t^\omega(x,y)}{\pi_\omega(y)} \tag{1.4}$$

Clearly, h_t^ω is symmetric, that is $h_t^\omega(x,y) = h_t^\omega(y,x)$ and satisfies the Chapman-Kolmogorov equation as a consequence of the semigroup law $P_{t+s}^\omega = P_t^\omega P_s^\omega$. We call "spectral dimension" of X the quantity

$$-2 \lim_{t\to+\infty} \frac{\log h_t^\omega(x,x)}{\log t}, \tag{1.5}$$

(if this limit exists). In the discrete-time case, we inverse the indices and denote the heat-kernel by $h_\omega^n(x,y)$.

Such walks under the additional assumptions of uniform ellipticity,

$$\exists \alpha > 0 : \quad \mathbb{Q}(\alpha < \omega_b < 1/\alpha) = 1,$$

have the standard local-CLT like decay of the heat kernel as proved by Delmotte [6]:

$$P_\omega^n(x,y) \le \frac{c_1}{n^{d/2}} \exp\left\{-c_2 \frac{|x-y|^2}{n}\right\}, \tag{1.6}$$

where c_1, c_2 are absolute constants.

Once the assumption of uniform ellipticity is relaxed, matters get more complicated. The most-intensely studied example is the simple random walk on the infinite cluster of supercritical bond percolation on $\mathbb{Z}^d$, $d \ge 2$. This corresponds to $\omega_{xy} \in \{0,1\}$ i.i.d. with $\mathbb{Q}(\omega_b = 1) > p_c(d)$ where $p_c(d)$ is the percolation threshold (cf. [10]). Here an annealed invariance principle has been obtained by De Masi, Ferrari, Goldstein and Wick [7, 8] in the late 1980s. More recently, Mathieu and Remy [15] proved the on-diagonal (i.e., $x = y$) version of the heat-kernel upper bound (1.6)—a slightly weaker version of which was also obtained by Heicklen and Hoffman [11]—and, soon afterwards, Barlow [1] proved the full upper and lower bounds on $P_\omega^n(x,y)$ of the form (1.6). (Both these results hold for n exceeding some random time defined relative to the environment in the vicinity of x and y). Heat-kernel upper bounds were then used in the proofs of quenched invariance principles by Sidoravicius and Sznitman [16] for $d \ge 4$, and for all $d \ge 2$ by Berger and Biskup [2] and Mathieu and Piatnitski [14].

Let $\mathbb{B}^d$ denote the set of (unordered) nearest-neighbor pairs in $\mathbb{Z}^d$. We choose in our case the family $\omega = \{\omega(x,y) \ : x \sim y, (x,y) \in \mathbb{Z}^d \times \mathbb{Z}^d\} = (\omega_b)_{b\in\mathbb{B}^d} \in (0,\infty)^{\mathbb{B}^d}$ i.i.d according to a law $\mathbb{Q}$ on $(\mathbb{R}_+^*)^{\mathbb{Z}^d}$ such that

$$\begin{aligned} &\omega_b \le 1 && \text{for all } b; \\ &\mathbb{Q}(\omega_b \le a) \sim a^\gamma && \text{when } a \downarrow 0, \end{aligned} \tag{1.7}$$

where $\gamma > 0$ is a parameter. In general, given functions f and g, we write $f \sim g$ to mean that $f(a)/g(a)$ tends to 1 when a tends to some limit a_0.

Our work is motivated by the recent study of Fontes and Mathieu [9] of continuous-time random walks on $\mathbb{Z}^d$ with conductances given by

$$\omega_{xy} = \omega(x) \wedge \omega(y)$$

for i.i.d. random variables $\omega(x) > 0$ satisfying (1.7). For these cases, it was found that the annealed heat-kernel, $\int d\mathbb{Q}(\omega)P_0^\omega(X(t) = 0)$, exhibits opposite behaviors, *standard* and *anomalous*, depending whether $\gamma \geq d/2$ or $\gamma < d/2$. Explicitly, from ([9], Theorem 4.3) we have

$$\int d\mathbb{Q}(\omega)P_0^\omega(X(t) = 0) = t^{-(\gamma \wedge \frac{d}{2})+o(1)}, \quad t \to \infty. \tag{1.8}$$

Further, in a more recent paper [5], we show that the quenched heat-kernel exhibits also opposite behaviors, anomalous and standard, for small and large values of γ. We first prove for all $d \geq 5$ that the return probability shows an anomalous decay that approaches (up to sub-polynomial terms) a random constant times n^{-2} when we push the power γ to zero. In contrast, we prove that the heat-kernel decay is as close as we want, in a logarithmic sense, to the standard decay $n^{-d/2}$ for large values of the parameter γ, i.e. : there exists a positive constant $\delta = \delta(\gamma)$ depending only on d and γ such that $\mathbb{Q} - a.s.$,

$$\limsup_{n \to +\infty} \sup_{x \in \mathbb{Z}^d} \frac{\log P_\omega^n(0, x)}{\log n} \leq -\frac{d}{2} + \delta(\gamma) \quad \text{and} \quad \delta(\gamma) \xrightarrow[\gamma \to +\infty]{} 0, \tag{1.9}$$

These results are a follow up on a paper by Berger, Biskup, Hoffman and Kozma [3], in which the authors proved a universal (non standard) upper bound for the return probability in a system of random walk among bounded (from above) random conductances. In the same paper, these authors supplied examples showing that their bounds are sharp. Nevertheless, the tails of the distribution near zero in these examples was very heavy.

1.2 Main results.

Consider random walk in reversible random environment defined by a family $(\omega_b) \in \Omega = [0, 1]^{\mathbb{B}^d}$ of i.i.d. random variables subject to the conditions given in (1.7), that we refer to as *conductances*, $\mathbb{B}^d$ being the set of unordered nearest-neighbor pairs (i.e., edges) of $\mathbb{Z}^d$. The law of the environment is denoted by $\mathbb{Q}$.

We are interested in estimating the decay of the quenched heat-kernel $h_t^\omega(0, 0)$, as t tends to $+\infty$ for the Markov process associated with the generator defined in (1.2) and we obtain, in the quenched case, a similar result to (1.8). Although our result is true for all $d \geq 2$, but it is significant when $d \geq 5$.

The main result of this paper is as follows:

Theorem 1.1 *For any $\gamma > d/2$, we have*

$$\lim_{t \to +\infty} \frac{\log h_t^{\omega}(0,0)}{\log t} = -\frac{d}{2}, \qquad \mathbb{Q} - a.s. \tag{1.10}$$

Our arguments are based on time change, percolation estimates and spectral analysis. Indeed, one operates first a time change to bring up the fact that the random walk viewed only on a strong cluster (i.e. constituted of edges of order 1) has a standard behavior. Then, we show that the transit time of the random walk in a hole is “negligible" by bounding the trace of a Markov operator that gives us the Feynman-Kac Formula and this by estimating its spectral gap.

An expected consequence of this Theorem is the following corollary, whose proof is given in part 3.3 and that gives the same result for the discrete-time case. For the random walk associated with the transition probabilities given in (1.1) for an environment ω with conductances satisfying the assumption (1.7), we have

Corollary 1.2 *For any $\gamma > d/2$, we have*

$$\lim_{n \to +\infty} \frac{\log h_{\omega}^{2n}(0,0)}{\log n} = -\frac{d}{2}, \qquad \mathbb{Q} - a.s. \tag{1.11}$$

Remark 1.3 As it has been pointed out in [4], Remark 2.2, the invariance principle (CLT) (cf. Theorem 1.3 in [13]) and the Spatial Ergodic Theorem automatically imply the standard lower bound on the heat-kernel under weaker conditions on the conductances. Indeed, let $\mathscr{C}$ represents the set of sites that have a path to infinity along bonds with positive conductances. For $\omega_{xy} \in [0,1]$ and the conductance law is i.i.d. subject to the condition that the probability of $\omega_{xy} > 0$ exceeds the threshold for bond percolation on $\mathbb{Z}^d$, we have then by the Markov property, reversibility of X and Cauchy-Schwarz

$$\begin{aligned} h_t^{\omega}(0,0) &\geq \frac{1}{2d} \sum_{\substack{x \in \mathscr{C} \\ |x| \leq \sqrt{t}}} P_0^{\omega}(X(t/2) = x)^2 \\ &\geq \frac{1}{2d} \frac{P_0^{\omega}\left(|X(t/2)| \leq \sqrt{t}\right)^2}{|\mathscr{C} \cap [-\sqrt{t}, +\sqrt{t}]^d|} \\ &\geq \frac{c(\omega)}{t^{d/2}}, \end{aligned}$$

with $c(\omega) > 0$ a.s. on the set $\{0 \in \mathscr{C}\}$ and t large enough. Note that, in $d = 2,3$, this complements nicely the "universal" upper bounds derived in [3], Theorem 2.1. Thus, for $d \geq 2$, we have

$$\liminf_{t\to+\infty} \frac{\log h_t^\omega(0,0)}{\log t} \geq -\frac{d}{2} \qquad \mathbb{Q} - a.s. \tag{1.12}$$

and for the cases $d = 2,3,4$, we have already the limit (1.11) under weaker conditions on the conductances (see [3], Theorem 2.2). So, under assumption (1.7), it remains to study the cases where $d \geq 5$ and prove that for $\gamma > d/2$,

$$\limsup_{t\to+\infty} \frac{\log h_t^\omega(0,0)}{\log t} \leq -\frac{d}{2} \qquad \mathbb{Q} - a.s.$$

or equivalently, since the conductances are $\mathbb{Q} - a.s.$ positive by (1.7),

$$\limsup_{t\to+\infty} \frac{\log P_t^\omega(X(t)=0)}{\log t} \leq -\frac{d}{2} \qquad \mathbb{Q} - a.s. \tag{1.13}$$

2. A time changed process

In this section we introduce a concept that is becoming a standard fact in the fields of random walk in random environment, namely random walk on the infinite strong cluster.

Choose a threshold parameter $\xi > 0$ such that $\mathbb{Q}(\omega_b \geq \xi) > p_c(d)$. The i.i.d. nature of the measure $\mathbb{Q}$ ensures that for $\mathbb{Q}$ almost any environment ω, the percolation graph $(\mathbb{Z}^d, \{e \in \mathbb{B}^d; \omega_b \geq \xi\})$ has a unique infinite cluster that we denote with $\mathscr{C}^\xi = \mathscr{C}^\xi(\omega)$.

We will refer to the connected components of the complement of $\mathscr{C}^\xi(\omega)$ in $\mathbb{Z}^d$ as *holes*. By definition, holes are connected sub-graphs of the grid. The sites which belong to $\mathscr{C}^\xi(\omega)$ are all the endvertices of its edges. The other sites belong to the holes. Note that holes may contain edges such that $\omega_b \geq \xi$.

First, we will give some important characterization of the volume of the holes, see Lemma 5.2 in [13]. $\mathscr{C}$ denotes the *infinite cluster*.

Lemma 2.1 *There exists $\bar{p} < 1$ such that for $p > \bar{p}$, for almost any realization of bond percolation of parameter p and for large enough n, any connected component of the complement of the infinite cluster $\mathscr{C}$ that intersects the box $[-n,n]^d$ has volume smaller than* $(\log n)^{5/2}$.

For the rest of the section, choose $\xi > 0$ such that $\mathbb{Q}(\omega_b \geq \xi) > \bar{p}$.

Define the conditioned measure

$$\mathbb{Q}_0^\xi(\cdot) = \mathbb{Q}(\cdot \mid 0 \in \mathscr{C}^\xi).$$

Consider the following additive functional of the random walk :

$$A^{\xi}(t) = \int_0^t \mathbf{1}_{\{X(s)\in\mathscr{C}^{\xi}\}} \mathrm{d}s,$$

its inverse $(A^{\xi})^{-1}(t) = \inf\{s; A^{\xi}(s) > t\}$ and define the corresponding time changed process

$$X^{\xi}(t) = X((A^{\xi})^{-1}(t)).$$

Thus the process X^{ξ} is obtained by suppressing in the trajectory of X all the visits to the holes. Note that, unlike X, the process X^{ξ} may perform long jumps when straddling holes.

As X performs the random walk in the environment ω, the behavior of the random process X^{ξ} is described in the next

Proposition 2.2 *Assume that the origin belongs to $\mathscr{C}^{\xi}$. Then, under P_0^{ω}, the random process X^{ξ} is a symmetric Markov process on $\mathscr{C}^{\xi}$.*

The Markov property, which is not difficult to prove, follows from a very general argument about time changed Markov processes. The reversibility of X^{ξ} is a consequence of the reversibility of X itself as will be discussed after equation (2.2).

The generator of the process X^{ξ} has the form

$$\mathscr{L}_{\xi}^{\omega} f(x) = \frac{1}{\eta^{\omega}(x)} \sum_y \omega^{\xi}(x,y)(f(y) - f(x)), \tag{2.1}$$

where

$$\begin{aligned} \frac{\omega^{\xi}(x,y)}{\eta^{\omega}(x)} &= \lim_{t\to 0} \frac{1}{t} P_x^{\omega}(X^{\xi}(t) = y) \\ &= P_x^{\omega}(y \text{ is the next point in } \mathscr{C}^{\xi} \text{ visited by the random walk}), \end{aligned} \tag{2.2}$$

if both x and y belong to $\mathscr{C}^{\xi}$ and $\omega^{\xi}(x,y) = 0$ otherwise.

The function ω^{ξ} is symmetric : $\omega^{\xi}(x,y) = \omega^{\xi}(y,x)$ as follows from the reversibility of X and formula (2.2), but it is no longer of nearest-neighbor type i.e. it might happen that $\omega^{\xi}(x,y) \neq 0$ although x and y are not neighbors. More precisely, one has the following picture : $\omega^{\xi}(x,y) = 0$ unless either x and y are neighbors and $\omega(x,y) \geq \xi$, or there exists a hole, h, such that both x and y have neighbors in h. (Both conditions may be fulfilled by the same pair (x,y).)

Consider a pair of neighboring points x and y, both of them belonging to the infinite cluster $\mathscr{C}^{\xi}$ and such that $\omega(x,y) \geq \xi$, then

$$\omega^{\xi}(x,y) \geq \xi. \tag{2.3}$$

This simple remark will play an important role. It implies, in a sense that the parts of the trajectory of X^ξ that consist in nearest-neighbors jumps are similar to what the simple symmetric random walk on $\mathscr{C}^\xi$ does. Precisely, we will need the following important fact that X^ξ obeys the standard heat-kernel bound :

Lemma 2.3 *There exists a constant c such that* $\mathbb{Q}_0^\xi - a.s.$ *for large enough* t*, we have*

$$P_0^\omega(X^\xi(t) = y) \le \frac{c}{t^{d/2}}, \quad \forall y \in \mathbb{Z}^d \tag{2.4}$$

For a proof, we refer to [13], Lemma 4.1. In the discrete-time case, see [3], Lemma 3.2.

3. Proof of Theorem 1.1 and Corollary 1.2

The upper bound (1.13) will be discussed in part 3.2 and the proof of Corollary 1.2 is given in part 3.3. We first start with some preliminary lemmata.

3.1 Preliminaries.

First, let us recall the following standard fact from Markov chain theory :

Lemma 3.1 *The function* $t \longmapsto P_0^\omega(X(t) = 0)$ *is non increasing.*

Proof. This is an immediate consequence of the semigroup property of the family of bounded self-adjoint operators $(P_t^\omega)_{t\ge 0}$ in the space $L^2(\mathbb{Z}^d, \pi_\omega)$ endowed with the inner product defined by

$$\langle f, g\rangle_\omega = \sum_{x\in\mathbb{Z}^d} f(x)g(x)\pi_\omega(x).$$

□

Next, let $B_r = [-r, r]^d$ be the box centered at the origin and of radius r that we choose as a function of time such that $t \sim r^2(\log r)^{-b}$ when $t \to \infty$, with $b > 1$, and let $\mathscr{B}_r$ denote the set of nearest-neighbor bonds of B_r, i.e., $\mathscr{B}_r = \{b = (x, y) : x, y \in B_r, x \sim y\}$. We have

Lemma 3.2 *Under assumption* (1.7)*,*

$$\lim_{r\to+\infty} \frac{\log \inf_{b\in\mathscr{B}_r} \omega_b}{\log r} = -\frac{d}{\gamma}, \qquad \mathbb{Q} - a.s. \tag{3.1}$$

Thus, for arbitrary $\mu > 0$, we can write $\mathbb{Q} - a.s.$ for r large enough,

$$\inf_{b \in \mathscr{B}_r} \omega_b \geq r^{-(\frac{d}{\gamma}+\mu)}. \tag{3.2}$$

Proof. This is a restatement of Lemma 3.6 of [9]. □

Consider now the following formula

$$R_t^\omega f(x) = E_x^\omega \left[f(X(t)) e^{-\lambda A^\xi(t)} \right], \qquad t \geq 0, \lambda \geq 0, x \in \mathbb{Z}^d \tag{3.3}$$

This object will play a key role in our proof of Theorem 1.1. Let $L_b^2(\mathbb{Z}^d, \pi_\omega)$ denote the set of bounded functions of $L^2(\mathbb{Z}^d, \pi_\omega)$. We have

Proposition 3.3 *$R = \{R_t^\omega, t \geq 0\}$ defines a semigroup (of symmetric operators) on $L_b^2(\mathbb{Z}^d, \pi_\omega)$, with generator*

$$\mathscr{G}^\omega f = \mathscr{L}^\omega f - \lambda \varphi f \tag{3.4}$$

where $\varphi = \mathbf{1}_{\{\cdot \in \mathscr{C}^\xi\}}$. One also has the perturbation identities

$$\begin{aligned} R_t^\omega f(x) &= P_t^\omega f(x) - \lambda \int_0^t P_t^\omega (\varphi R_{t-s}^\omega f)(x) ds \\ &= P_t^\omega f(x) - \lambda \int_0^t R_s^\omega (\varphi P_{t-s}^\omega f)(x) ds, \\ t \geq 0, x &\in \mathbb{Z}^d, f \in L_b^2(\mathbb{Z}^d, \pi_\omega). \end{aligned} \tag{3.5}$$

Proof. The proof, that we give here, very closely mimics the arguments of [17], Theorem 1.1. Indeed, we begin with the proof of (3.5). Observe that $P_x^\omega - a.s.$, for every $x \in \mathbb{Z}^d$, the time function $t \mapsto e^{-\lambda A^\xi(t)}$ is continuous, and

$$\lim_{h \to 0} \frac{1}{h} \left(e^{-\lambda A^\xi(s+h)} - e^{-\lambda A^\xi(s)} \right) = -\lambda \varphi(X(s)) e^{-\lambda A^\xi(s)}, \quad \forall s \in [0,t],\ t > 0,$$

except possibly for a countable set $\{\alpha_i\}_{i \in I} \subset (0,t], |I| \subset \mathbb{N}^*$. Then, for $t \geq 0$, we have

$$\begin{aligned} e^{-\lambda A^\xi(t)} &= 1 - \lambda \int_0^t \varphi(X(s)) e^{-\lambda A^\xi(s)} \mathrm{d}s \\ &= 1 - \lambda \int_0^t \varphi(X(s)) \exp\left\{ - \int_s^t \lambda \varphi(X(u)) du \right\} \mathrm{d}s. \end{aligned} \tag{3.6}$$

Multiplying both sides of the first equality of (3.6) by $f(X(t))$ and integrating we find :

$$\begin{aligned} R_t^\omega f(x) &= P_t^\omega f(x) - \lambda \int_0^t E_x^\omega \left[\varphi(X(s)) \exp\left\{ -\lambda \int_s^t \varphi(X(u)) du \right\} f(X(t)) \right] \mathrm{d}s \\ &= P_t^\omega f(x) - \lambda \int_0^t P_s^\omega (\varphi R_{t-s}^\omega f)(x) \mathrm{d}s, \quad \text{(Markov property)}, \end{aligned}$$

which is the first identity of (3.5). Analogously we find the second identity of (3.5) with the help of the second line of (3.6). This completes the proof of (3.5).

Clearly $||R_t^\omega f||_2^2 \le c(t,\varphi)||P_t^\omega f||_2^2$, then $R_t^\omega f \in L_b^2(\mathbb{Z}^d, \pi_\omega)$, for $f \in L_b^2(\mathbb{Z}^d, \pi_\omega)$. Moreover for $s, t \ge 0$,

$$\begin{aligned} R_{s+t}^\omega f(x) &= E_x^\omega \left[e^{-\lambda A^\xi(s)} \exp\left\{ -\lambda \int_s^{s+t} \varphi(X(u)) du \right\} f(X(t)) \right] \\ &= R_s^\omega (R_t^\omega f)(x), \qquad \text{(Markov property)}. \end{aligned}$$

We thus proved that R_t^ω defines a semigroup on $L_b^2(\mathbb{Z}^d, \pi_\omega)$. The strong continuity of this semigroup follows readily by letting t tend to 0 in (3.5).

Let us finally prove (3.4). To this end notice that for $f \in L_b^2(\mathbb{Z}^d, \pi_\omega)$:

$$\frac{1}{t} \int_0^t P_t^\omega (\varphi R_{t-s}^\omega f)(x) \mathrm{d}s \underset{t \to 0}{\longrightarrow} \varphi(x) f(x), \quad \text{uniformly in } x,$$

since $\varphi(X(t)) \mapsto \varphi(x)$, $P_x^\omega - a.s.$ Coming back to the first line of (3.5), this proves that the convergence of $\frac{1}{t}(R_t^\omega - f)$ or $\frac{1}{t}(P_t^\omega - f)$ as t tends to 0, are equivalent and (3.4) holds. $\square$

Let $\mathscr{L}_r^\omega$ and $\mathscr{G}_r^\omega$ be respectively the restrictions of the operators $\mathscr{L}^\omega$ and $\mathscr{G}^\omega$ (cf. (1.2)–(3.4)) to the set of functions on B_r with Dirichlet boundary conditions outside B_r, that we denote by $L^2(B_r, \pi_\omega)$ (that is, $\mathscr{L}_r^\omega$ and $\mathscr{G}_r^\omega$ are respectively the generators of the process X and of the semigroup R, which coincide with the ones given by $\mathscr{L}^\omega$ and $\mathscr{G}^\omega$ until the process X leaves B_r for the first time, and then it is killed). Then $-\mathscr{L}_r^\omega$ and $-\mathscr{G}_r^\omega$ are positive symmetric operators and we have

$$\mathscr{G}_r^\omega f = \mathscr{L}_r^\omega f - \lambda \varphi f,$$

with associated semigroup defined by

$$(R_t^{\omega,r} f)(0) := E_0^\omega \left[f(X(t)) e^{-\lambda A^\xi(t)}; t < \tau_r \right].$$

where τ_r is the exit time for the process X from the box B_r.

Let $\{\lambda_i^\omega(r), i \in [1, \#B_r]\}$ be the set of eigenvalues of $-\mathscr{G}_r^\omega$ labelled in increasing order, and $\{\psi_i^{\omega,r}, i \in [1, \#B_r]\}$ the corresponding eigenfunctions with due normalization in $L^2(B_r, \pi_\omega)$.

3.2 Proof of the upper bound.

In this last part, we will complete the proof of Theorem 1.1 by giving the proof of the upper bound (1.13).

Proof of Theorem 1.1.

Assume that the origin belongs to $\mathscr{C}^\xi$. By lemma 3.1, we have

$$P_0^\omega(X(t)=0) \le \frac{2}{t}\int_{t/2}^t P_0^\omega(X(s)=0)\mathrm{d}s = \frac{2}{t}E_0^\omega\left[\int_{t/2}^t \mathbf{1}_{\{X(s)=0\}}\mathrm{d}s\right].$$

The additive functional A^ξ being a continuous increasing function of the time, so by operating a variable change by setting $s=(A^\xi)^{-1}(u)$ (i.e. $u=A^\xi(s)$), we get

$$\begin{aligned} E_0^\omega\left[\int_{t/2}^t \mathbf{1}_{\{X(s)=0\}}\mathrm{d}s\right] &= E_0^\omega\left[\int_{t/2}^t \mathbf{1}_{\{X(s)=0\}}\varphi(X(s))\mathrm{d}s\right] \\ &= E_0^\omega\left[\int_{A^\xi(t/2)}^{A^\xi(t)} \mathbf{1}_{\{X^\xi(u)=0\}}\mathrm{d}u\right], \end{aligned}$$

which is bounded by

$$E_0^\omega\left[\int_{A^\xi(t/2)}^{t} \mathbf{1}_{\{X^\xi(u)=0\}}\mathrm{d}u\right],$$

since $A^\xi(t)\le t$.

Therefore, for $\epsilon\in(0,1)$

$$\begin{aligned} P_0^\omega(X(t)=0) \le{}& \frac{2}{t}E_0^\omega\left[\int_{A^\xi(t/2)}^{t} \mathbf{1}_{\{A^\xi(t/2)\ge t^\epsilon\}}\mathbf{1}_{\{X^\xi(u)=0\}}\mathrm{d}u\right] \\ &+\frac{2}{t}E_0^\omega\left[\int_{A^\xi(t/2)}^{t} \mathbf{1}_{\{A^\xi(t/2)\le t^\epsilon\}}\mathbf{1}_{\{X^\xi(u)=0\}}\mathrm{d}u\right] \\ \le{}& \frac{2}{t}\int_{t^\epsilon}^t P_0^\omega(X^\xi(u)=0)\mathrm{d}u + \frac{2}{t}\int_0^t P_0^\omega(A^\xi(t/2)\le t^\epsilon)\mathrm{d}u \end{aligned}$$

and using lemma 2.3,

$$
\begin{aligned}
P_0^\omega(X(t)=0) &\leq \frac{c}{t}\int_{t^\epsilon}^{t} u^{-d/2}\mathrm{d}u + \frac{2}{t}P_0^\omega(A^\xi(t/2)\leq t^\epsilon)t \\
&\leq \frac{c}{t^{\epsilon\frac{d}{2}-\epsilon+1}} + 2P_0^\omega(A^\xi(t/2)\leq t^\epsilon) \qquad (3.7)
\end{aligned}
$$

It remains to estimate the second term in the right-hand side of the last inequality, i.e. $P_0^\omega(A^\xi(t/2) \leq t^\epsilon)$ or more simply $P_0^\omega(A^\xi(t) \leq 2^\varepsilon t^\epsilon)$, but we can neglect the constant 2^ϵ in the calculus as one will see in (3.15).

For each $\lambda \geq 0$, Chebychev inequality gives

$$
\begin{aligned}
P_0^\omega(A^\xi(t)\leq t^\epsilon) &= P_0^\omega(A^\xi(t)\leq t^\epsilon; t<\tau_r) + P_0^\omega(A^\xi(t)\leq t^\epsilon; \tau_r\leq t) \\
&\leq P_0^\omega\left(e^{-\lambda A^\xi(t)} \geq e^{-\lambda t^\epsilon}; t<\tau_r\right) + P_0^\omega(\tau_r\leq t) \\
&\leq e^{\lambda t^\epsilon}E_0^\omega\left[e^{-\lambda A^\xi(t)}; t<\tau_r\right] + P_0^\omega(\tau_r\leq t). \qquad (3.8)
\end{aligned}
$$

From the Carne-Varopoulos inequality, it follows that

$$
P_0^\omega(\tau_r\leq t)\leq Ctr^{d-1}e^{-\frac{r^2}{4t}} + e^{-ct}, \qquad (3.9)
$$

where C and c are numerical constants, see Appendix C in [15]. With our choice of r such that $t\sim r^2(\log r)^{-b}$ $(b>1)$, we get that $P_0^\omega(\tau_r\leq t)$ decays faster than any polynomial as t tends to $+\infty$.

Thus Theorem 1.1 will be proved if we can check, for a particular choice of $\lambda>0$ that may depend on t, that

$$
\limsup_{t\to+\infty}\frac{\log\left(e^{\lambda t^\epsilon}E_0^\omega\left[e^{-\lambda A^\xi(t)}; t<\tau_r\right]\right)}{\log t} \leq -\frac{d}{2}. \qquad (3.10)
$$

That will be true if $e^{\lambda t^\epsilon}E_0^\omega\left[e^{-\lambda A^\xi(t)}; t<\tau_r\right]$ decays faster than any polynomial in t as t tends to $+\infty$.

The Dirichlet form of $-\mathscr{L}_r^\omega$ on $L^2(B_r,\pi_\omega)$ endowed with the usual scalar product (see Lemma 3.1), can be written as

$$
\mathscr{E}^{\omega,r}(f,f) = \left\langle -\mathscr{L}_r^\omega f, f\right\rangle_\omega = \frac{1}{2}\sum_{b\in\mathscr{B}_{r+1}}(df(b))^2\omega_b,
$$

where $df(b) = f(y) - f(x)$ and the sum ranges over $b = (x, y) \in \mathscr{B}^{\omega}_{r+1}$. By the min-max Theorem (see [12]) and (3.4), we have

$$\lambda_1^{\omega}(r) = \inf_{f \not\equiv 0} \frac{\mathscr{E}^{\omega,r}(f,f) + \lambda \sum_{x \in \mathscr{C}_r^{\xi}} f^2(x)\pi_{\omega}(x)}{\pi_{\omega}(f^2)}. \tag{3.11}$$

where $\mathscr{C}_r^{\xi}$ is the largest connected component of $\mathscr{C}^{\xi} \cap B_r$, and the infimum is taken over functions with Dirichlet boundary conditions. (Recall that $\lambda_1^{\omega}(r)$ is the first eigenvalue of $-\mathscr{G}_r^{\omega}$.)

To estimate the decay of the first term in the right-hand side of (3.8), we will also need to estimate the first eigenvalue $\lambda_1^{\omega}(r)$. Recall that μ denotes an arbitrary positive constant.

Lemma 3.4 *Under assumption* (1.7)*, for any $d \geq 2$ and $\gamma > 0$, we have $\mathbb{Q} - a.s.$ for r large enough,*

$$\lambda_1^{\omega}(r) \geq (8d)^{-1} r^{-(\frac{d}{\gamma}+\mu)} (\log n)^{-5}, \tag{3.12}$$

for λ proportional to $r^{-(\frac{d}{\gamma}+\mu)}$.

Proof. For some arbitrary $\mu > 0$, let r be large enough so that (3.2) holds. Let h be a hole that intersects the box B_r, and for notational ease we will use the same notation for $h \cap B_r$. Define ∂h to be the outer boundary of h, i.e. the set of sites in $\mathscr{C}_r^{\xi}$ which are adjacent to some vertex in h. Let us associate to each hole h a fixed site $h^* \in \mathscr{C}_r^{\xi}$ situated at the outer boundary of h and for $x \in h$ call $\kappa(x, h^*)$ a self-avoiding path included in h with end points x and h^*, and let $|\kappa(x, h^*)|$ denote the length of such a path.

Now let $f \in L^2(B_r, \pi_{\omega})$ and let $\mathscr{B}_{\omega}(h)$ denote the set of the bonds of h. For each $x \in h$, write

$$f(x) = \sum_{b \in \kappa(x,h^*)} df(b) + f(h^*)$$

and, using Cauchy-Schwarz

$$f^2(x) \leq 2|\kappa(x, h^*)| \sum_{b \in \kappa(x,h^*)} |df(b)|^2 + 2f^2(h^*).$$

In every path $\kappa(x, h^*)$, we see each bond only one time. Multiply the last inequality by $\pi_{\omega}(x)$

and sum over $x \in h$ to obtain

$$\begin{aligned}
&\sum_{x\in h} f^2(x)\pi_\omega(x) \\
&\quad \le 2\sum_{x\in h}|\kappa(x,h^*)| \sum_{b\in\kappa(x,h^*)} |df(b)|^2\pi_\omega(x) + 2\sum_{x\in h} f^2(h^*)\pi_\omega(x) \\
&\quad \le 4d \max_{x\in h}|\kappa(x,h^*)| \max_{b\in\mathscr{B}_\omega(h)} \frac{1}{\omega_b} \#h \sum_{b\in\mathscr{B}_\omega(h)} |df(b)|^2\omega_b \\
&\qquad\qquad\qquad + 2\sum_{x\in h} f^2(h^*)\pi_\omega(x),
\end{aligned} \tag{3.13}$$

which, by virtue of lemma 2.1, (1.7), (3.2) and since $\pi_\omega(h^*) \ge \xi$, is bounded by

$$4dr^{\frac{d}{\gamma}+\mu}(\log r)^5 \sum_{b\in\mathscr{B}_\omega(h)} |df(b)|^2\omega_b + \frac{4d}{\xi}\#hf^2(h^*)\pi_\omega(h^*),$$

Thus,

$$\sum_{x\in h} f^2(x)\pi_\omega(x) \le 4dr^{\frac{d}{\gamma}+\mu}(\log r)^5 \sum_{b\in\mathscr{B}_\omega(h)} |df(b)|^2\omega_b + \frac{4d}{\xi}\#hf^2(h^*)\pi_\omega(h^*).$$

Let $\mathscr{C}_r^c(\xi)$ denote the complement of $\mathscr{C}_r^\xi$ in the box B_r and sum over h to obtain

$$\sum_{x\in\mathscr{C}_r^c(\xi)} f^2(x)\pi_\omega(x) \le 8dr^{\frac{d}{\gamma}+\mu}(\log r)^5\mathscr{E}^{\omega,r}(f,f) + \frac{8d^2}{\xi}\#h\sum_{x\in\mathscr{C}_r^\xi} f^2(x)\pi_\omega(x),$$

where in the last term, we multiply by $2d$ since we may associate the same h^* to $2d$ different holes. Then

$$\begin{aligned}
&\sum_{x\in B_r} f^2(x)\pi_\omega(x) \\
&\quad \le (1+8d^2(\log r)^{5/2}\xi^{-1}) \sum_{x\in\mathscr{C}_r^\xi} f^2(x)\pi_\omega(x) + 8dr^{\frac{d}{\gamma}+\mu}(\log r)^5\mathscr{E}^{\omega,r}(f,f). \\
&\quad \le 8d^2(\log r)^5\xi^{-1} \sum_{x\in\mathscr{C}_r^\xi} f^2(x)\pi_\omega(x) + 8dr^{\frac{d}{\gamma}+\mu}(\log r)^5\mathscr{E}^{\omega,r}(f,f).
\end{aligned}$$

So, according to (3.11) and for $\lambda = d\xi^{-1}r^{-(\frac{d}{\gamma}+\mu)}$, we get

$$\lambda_1^\omega(r) \ge (8d)^{-1}r^{-(\frac{d}{\gamma}+\mu)}(\log r)^{-5}. \tag{3.14}$$

□

Let us get back to the proof of the upper bound. Let

$$\lambda = d\xi^{-1} r^{-(\frac{d}{\gamma}+\mu)}; \quad m(r) := (8d)^{-1} r^{-(\frac{d}{\gamma}+\mu)} (\log r)^{-5}.$$

For $f \equiv \mathbf{1}$, observe that

$$\begin{aligned}(R_t^{\omega,r} f)(0) &= E_0^{\omega}\left[e^{-\lambda A^{\xi}(t)}; t < \tau_r\right] \\ &= \sum_i e^{-\lambda_i^{\omega}(r)t} \left\langle \mathbf{1}, \psi_i^{\omega,r}\right\rangle \psi_i^{\omega,r}(0),\end{aligned}$$

and

$$\begin{aligned}(R_t^{\omega,r} f)^2(0)\pi_{\omega}(0) &\leq \sum_{x\in B_r} (R_t^{\omega,r} f)^2(x)\pi_{\omega}(x) \\ &= \sum_i e^{-2\lambda_i^{\omega}(r)t} \left\langle \mathbf{1}, \psi_i^{\omega,r}\right\rangle^2 \\ &\leq e^{-2\lambda_1^{\omega}(r)t} \sum_x \mathbf{1}^2(x)\pi_{\omega}(x) \\ &\leq 2^{d+2} d r^d e^{-2\lambda_1^{\omega}(r)t}.\end{aligned}$$

Then, for large enough t and by (3.14), we have

$$\begin{aligned}e^{\lambda t^{\epsilon}} E_0^{\omega}\left[e^{-\lambda A^{\xi}(t)}; t < \tau_r\right] &\leq (2^{d+2}d)^{1/2} e^{\lambda t^{\epsilon}} e^{-t\lambda_1^{\omega}(r)} r^{d/2} \\ &\leq (2^{d+2}d)^{1/2} r^{d/2} \exp\{\lambda t^{\epsilon} - t m(r)\} \\ &\leq (2^{d+2}d)^{1/2} r^{d/2} e^{-\frac{t}{2}m(r)}, \qquad (3.15)\end{aligned}$$

since $\epsilon < 1$.

By our choice of $t \sim r^2(\log r)^{-b}$ $(b > 1)$, we deduce

$$\begin{aligned}e^{\lambda t^{\epsilon}} E_0^{\omega}\left[e^{-\lambda A^{\xi}(t)}; t < \tau_r\right] &\leq (2^{d+2}d)^{1/2} r^{d/2} \exp\left\{-[16d(\log r)^{b+5}]^{-1} r^2 r^{-(\frac{d}{\gamma}+\mu)}\right\} \\ &\ll t^{-\frac{d}{2}} \qquad \text{if} \quad \gamma > \frac{d}{2-\mu}, \qquad (3.16)\end{aligned}$$

which yields (3.10).

In conclusion, as μ is arbitrary and according to (3.9)–(3.10), we obtain

$$\limsup_{t\to+\infty} \frac{\log P_0^{\omega}(A^{\xi}(t) \leq t^{\epsilon})}{\log t} \leq -\frac{d}{2} \qquad \text{for} \quad \gamma > \frac{d}{2},$$

and finally, by (3.7)

$$\lim_{\epsilon\to 1}\limsup_{t\to+\infty}\frac{\log P_0^{\omega}(X(t)=0)}{\log t}\leq -\frac{d}{2}\qquad\text{for}\quad \gamma>\frac{d}{2},$$

which gives (1.13).

We conclude that for any sufficiently small ξ, then $\mathbb{Q}_0^{\xi}-$a.s. (1.11) is true and since $\mathbb{Q}\left(\cup_{\xi>0}\{0\in\mathscr{C}^{\xi}(\omega)\}\right)=1$, it remains true $\mathbb{Q}$-a.s. □

3.3 Proof of the discrete-time case.

Proof. In the same way, the lower bound holds by the Invariance Principle (cf. [4]) and the Spatial Ergodic Theorem (see Remark 1.3).

For the upper bound, let $(N_t)_{t\geq 0}$ be a Poisson process of rate 1. Set $n=\lfloor t\rfloor$. $P_{\omega}^{2n}(0,0)$ being a non increasing function of n (cf. Lemma 3.1), then

$$\begin{aligned}
P_0^{\omega}(X(t)=0) &= e^{-t}\sum_{k\geq 0}\frac{t^k}{k!}P_{\omega}^k(0,0)\\
&\geq e^{-t}\sum_{k=0}^{2n}\frac{t^k}{k!}P_{\omega}^k(0,0)\\
&\geq P_{\omega}^{2n}(0,0)\left[e^{-t}\sum_{k=0}^{2n}\frac{t^k}{k!}\right]\\
&= P_{\omega}^{2n}(0,0)\mathrm{Prob}(N_t\leq 2n).
\end{aligned}$$

By virtue of the LLN, we have

$$\mathrm{Prob}(N_t\leq 2n)\xrightarrow[t\to+\infty]{}1.$$

From here the claim follows. □

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