DOI: 10.1214/EJP.v15-754

Electronic Journal of Probability

Vol. 15 (2010), Paper no. 13, pages 346–385.

Journal URL
http://www.math.washington.edu/~ejpecp/

Functional inequalities for heavy tailed distributions and application to isoperimetry

Patrick Cattiaux* Nathael GOZLAN† Arnaud Guillin‡ Cyril Roberto§

Abstract

This paper is devoted to the study of probability measures with heavy tails. Using the Lyapunov function approach we prove that such measures satisfy different kind of functional inequalities such as weak Poincaré and weak Cheeger, weighted Poincaré and weighted Cheeger inequalities and their dual forms. Proofs are short and we cover very large situations. For product measures on $\mathbb{R}^n$ we obtain the optimal dimension dependence using the mass transportation method. Then we derive (optimal) isoperimetric inequalities. Finally we deal with spherically symmetric measures. We recover and improve many previous result.

Key words: weighted Poincaré inequalities, weighted Cheeger inequalities, Lyapunov function, weak inequalities, isoperimetric profile.

AMS 2000 Subject Classification: Primary 60E15 ; 26D10.

Submitted to EJP on January 15, 2009, final version accepted March 29, 2010.

*Institut de Mathématiques de Toulouse, CNRS UMR 5219, Université Paul Sabatier, Laboratoire de Statistique et Probabilités, 118 route de Narbonne, F-31062 Toulouse cedex 09, FRANCE patrick.cattiaux@math.univ-toulouse.fr

†Laboratoire d'Analyse et Mathématiques Appliquées UMR 8050, Universités de Paris-est, Marne la Vallée, Boulevard Descartes, Cité Descartes, Champs sur Marne, 77454 Marne la Vallée Cedex 2, FRANCE, nathael.gozlan@univ-mlv.fr

‡Université Blaise Pascal, 33, avenue des landais, 63177 Aubières Cedex , FRANCE, guillin@math.univ-bpclermont.fr

§Laboratoire d'Analyse et Mathématiques Appliquées UMR 8050, Universités de Paris-est, Marne la Vallée, Boulevard Descartes, Cité Descartes, Champs sur Marne, 77454 Marne la Vallée Cedex 2, FRANCE cyril.roberto@univ-mlv.fr

1 Introduction, definitions and first results.

The subject of functional inequalities knows an amazing growth due to the numerous fields of application: differential geometry, analysis of p.d.e., concentration of measure phenomenon, isoperimetry, trends to equilibrium in deterministic and stochastic evolutions... Let us mention Poincaré, weak Poincaré or super Poincaré inequalities, Sobolev like inequalities, F-Sobolev inequalities (in particular the logarithmic Sobolev inequality), modified log-Sobolev inequalities and so on. Each type of inequality appears to be very well adapted to the study of one (or more) of the applications listed above. We refer to [37], [2], [31], [1], [41], [57], [38], [51], [10], [30] for an introduction.

Whereas many results are known for log-concave probability measures, not so much has been proved for measures with heavy tails (let us mention [49; 9; 23; 4; 19; 25]). In this paper the focus is on such measures with heavy tails and our aim is to prove functional and isoperimetric inequalities.

Informally measures with heavy tails are measures with tails larger than exponential. Particularly interesting classes of examples are either κ-concave probability measures, or sub-exponential like laws (or tensor products of any of them) defined as follows.

We say that a probability measure μ is κ-concave with $\kappa = -1/\alpha$ if

$$d\mu(x) = V(x)^{-(n+\alpha)} dx \tag{1.1}$$

with $V : \mathbb{R}^n \to (0, \infty)$ convex and $\alpha > 0$. Such measures have been introduced by Borell [27] in more general setting. See [19] for a comprehensive introduction and the more general definition of κ-concave probability measures. Prototypes of κ-concave probability measures are the generalized Cauchy distributions

$$d\mu(x) = \frac{1}{Z} \left((1 + |x|^2)^{1/2}\right)^{-(n+\alpha)} \tag{1.2}$$

for $\alpha > 0$, which corresponds to the previous description since $x \mapsto (1 + |x|^2)^{1/2}$ is convex. In some situations we shall also consider $d\mu(x) = (1/Z)\,((1 + |x|))^{-(n+\alpha)}$. Note that these measures are Barenblatt solutions of the porous medium equations and appear naturally in weighted porous medium equations, giving the decay rate of this nonlinear semigroup towards the equilibrium measure, see [55; 33]. See also [14].

We may replace the power by an exponential yielding the notion of "extended sub-exponential law", i.e. given any convex function $V : \mathbb{R}^n \to (0, \infty)$ and $p > 0$, we shall say that

$$d\mu(x) = e^{-V(x)^p} dx$$

is an "extended sub-exponential like law". A typical example in our mind is $V(x) = |x|$, and $0 < p < 1$, which yileds to sub-exponential type law.

Heavy tails measures are now particularly important since they appear in various areas: fluid mechanics, mathematical physics, statistical mechanics, mathematical finance ... Since previous results in the literature are not optimal, our main goal is to study the isoperimetric problem for heavy tails measures. This will lead us to consider various functional inequalities (weak Cheeger, weighted Cheeger, converse weighted Cheeger). Let us explain why.

Recall the isoperimetric problem.

Denote by d the Euclidean distance on $\mathbb{R}^n$. For $h \geq 0$ the closed h-enlargement of a set $A \subset \mathbb{R}^n$ is $A_h := \{x \in M;\ d(x, A) \leq h\}$ where $d(x, A) := \inf\{d(x, a);\ a \in A\}$ is $+\infty$ by convention for $A = \emptyset$. We may define the boundary measure, in the sense of μ, of a Borel set $A \subset \mathbb{R}^n$ by

$$\mu_s(\partial A) := \liminf_{h \to 0^+} \frac{\mu(A_h \setminus A)}{h}.$$

An isoperimetric inequality is of the form

$$\mu_s(\partial A) \geq F(\mu(A)) \qquad \forall A \subset \mathbb{R}^n \tag{1.3}$$

for some function F. Their study is an important topic in geometry, see e.g. [50; 8]. The first question of interest is to find the optimal F. Then one can try to find the optimal sets for which (1.3) is an equality. In general this is very difficult and the only hope is to estimate the isoperimetric profile defined by

$$I_\mu(a) := \inf\{\mu_s(\partial A);\ \mu(A) = a\}, \quad a \in [0, 1].$$

Note that the isoperimetric inequality (1.3) is closely related to concentration of measure phenomenon, see [21; 42], or more recently the impressive work of Milman [47]: concentration is equivalent to Cheeger type, or isoperimetric one, inequality under curvature assumptions. For a large class of distributions μ on the line with exponential or faster decay, it is possible to prove [26; 52; 16; 20; 5; 10; 11; 47] that the isoperimetric profile I_{μ^n} of the n-tensor product μ^n is (up a to universal, hence dimension free constants) equal to I_μ.

Conversely, suppose that μ is a probability measure on $\mathbb{R}$ such that there exist $h > 0$ and $\varepsilon > 0$ such that for all $n \geq 1$ and all Borel sets $A \subset \mathbb{R}^n$ with $\mu^n(A) \geq \frac{1}{2}$, one has

$$\mu^n(A + [-h, h]^n) \geq \frac{1}{2} + \varepsilon, \tag{1.4}$$

then μ has exponential tails, that is there exist positive constants C_1, C_2 such that $\mu([x, +\infty)) \leq C_1 e^{-C_2 x}$, $x \in \mathbb{R}$, see [53].

Therefore, for measures with heavy tails, the isoperimetric profile as well as the concentration of measure for product measure should heavily depend on n. Some bounds on I_{μ^n}, not optimal in n, are obtained in [9] using weak Poincaré inequality. The non optimality is mainly due to the fact that $\mathbb{L}_2$ inequalities (namely weak Poincaré inequalities) and related semi-group techniques are used. We shall obtain optimal bounds, thus completing the pictures for the isoperimetric profile of tensor product of very general form of probability measures, using $\mathbb{L}_1$ inequalities called weak Cheeger inequalities that we introduce now.

As noted by Bobkov [19], for measures with heavy tails, isoperimetric inequalities are equivalent to weak Cheeger inequalities. A probability measure is said to satisfy a weak Cheeger inequality if there exists some non-increasing function $\beta : (0, \infty) \to [0, \infty)$ such that for every smooth $f : \mathbb{R}^n \to \mathbb{R}$, it holds

$$\int |f - m|\, d\mu \ \leq\ \beta(s) \int |\nabla f|\, d\mu \ +\ s\, \mathrm{Osc}_\mu(f) \qquad \forall s > 0, \tag{1.5}$$

where m is a median of f for μ and $\mathrm{Osc}_\mu(f) = \operatorname{ess\,sup}(f) - \operatorname{ess\,inf}(f)$. The relationship between β in (1.5) and F in (1.3) is explained in Lemma 4.1 below. Since $\int |f - m|\, d\mu \leq \frac{1}{2}\mathrm{Osc}_\mu(f)$, only the values $s \in (0, 1/2]$ are relevant.

Recall that similar weak Poincaré inequalities were introduced in [49], replacing the median by the mean and introducing squares.

Of course if $\beta(0) < +\infty$ we recover the usual Cheeger or Poincaré inequalities.

In order to get isoperimetric results, we thus investigate such inequalities. We use two main strategies. One is based on the Lyapunov function approach [4; 30; 3], the other is based on mass transportation method [35; 36] (see also [15; 54; 20; 22]). In the first case proofs are very short. We obtain rather poor control on the constants, in particular in terms of the dimension, but we cover very general and new situations (not at all limited to κ-concave like measures). The second strategy gives very explicit (and also new) controls on the constants for tensor products of measures on the line or spherically symmetric measures (but only for the $\mathbb{L}_2$ case).

This is not surprising in view of the analogue results known for log-concave measures for instance. Indeed recall that the important conjecture of Kannan-Lovasz-Simonovits ([40]) stating that the Poincaré constant of log-concave probability measures only depends on their variance is still a conjecture. In this situation universal equivalence between Cheeger's inequality and Poincaré inequality is known ([43; 47]), and some particular cases (for instance spherically symmetric measures) have been studied ([18]). In our situation the equivalence between weak Poincaré and weak Cheeger inequalities does not seem to be true in general, so our results are in a sense the natural extension of the state of the art to the heavy tails situation.

The Lyapunov function approach appears to be a very powerful tool not only when dealing with the $\mathbb{L}_1$ form (1.5) but also with $\mathbb{L}_2$ inequalities.

This approach is well known for dynamical systems for example. It has been introduced by Khasminski and developed by Meyn and Tweedie ([44; 45; 46]) in the context of Monte Carlo algorithm (Markov chains). This dynamical approach is in some sense natural: consider the process whose generator is symmetric with respect to the measure of interest (see next section for more precise definitions), Lyapunov conditions express that there is some drift (whose strength varies depending on the measure studied) which pushes the process to some natural, say compact, region of the state space. Once in the compact set the process behaves nicely and pushed forward to it as soon as it escapes. It is then natural to get nice qualitative (but not so quantitative) proofs of total variation convergence of the associated semigroup towards its invariant measure and find applications in the study of the decay to equilibrium of dynamical systems, see e.g. [34; 39; 56; 4; 29]. It is also widely studied in statistics, see e.g. [44] and the references therein. In [4], connections are given between Lyapunov functions and functional inequalities of weak Poincaré type, improving some existing criteria discussed in [49; 9]. In this paper we give new types of Lyapunov functions (in the spirit of [3]) leading to quantitative improvements and in some sense optimal results. Actually we obtain four types of functional inequalities: weighted Cheeger (and weighted Poincaré inequalities)

$$\int |f - m| \, d\mu \;\leq\; C \int |\nabla f| \, \omega \, d\mu \tag{1.6}$$

and their dual forms called converse weighted Cheeger (and converse weighted Poincaré inequalities)

$$\inf_c \int |f - c| \, \omega \, d\mu \;\leq\; C \int |\nabla f| \, d\mu \tag{1.7}$$

where ω are suitable "weights" (see Section 2 for precise and more general definitions definitions).

Weighted Cheeger and weighted Poincaré inequalities were very recently studied by Bobkov and Ledoux [23], using functional inequalities of Brascamp-Lieb type. Their results apply to κ-concave probability measures. We recover their results with slightly worst constants but our approach also applies to much general type laws (sub-exponential for example).

Note that converse weighted Poincaré inequalities appear in the spectral theory of Schrödinger operators, see [32]. We will not pursue this direction here.

Our approach might be summarized by the following diagram:

				Transport
				⇓
		Weighted Cheeger	⇒	Weighted Poincaré
			⇖ ⇗	
			Lyapunov	⇓
			⇙ ⇘	
		converse weighted Cheeger		converse weighted Poincaré
		⇓		⇓
Transport	⇒	Weak Cheeger	⇒	Weak Poincaré
		⇕		⇓
		Isoperimetry	⇒	Concentration

Some points have to be commented. As the diagram indicates, converse weighted inequalities are suitable for obtaining isoperimetric (or concentration like) results, while (direct) weighted inequalities, though more natural, are not. Indeed, the tensorization property of the variance immediately shows that if μ satisfies a weighted Poincaré inequality with constant C and weight ω, then the tensor product μ^n satisfies the same inequality. Since we know that the concentration property for heavy tails measures is not dimension free, this implies that contrary to the ordinary or the weak Poincaré inequality, the weighted Poincaré inequality cannot capture the concentration property of μ. The other point is that the mass transportation method can also be used to obtain some weighted Poincaré inequalities, and weighted Poincaré inequalities via a change of function lead to converse weighted Poincaré inequality (see [23]). The final point is that on most examples we obtain sharp weights, showing that (up to constants) our results are optimal. To sum up, all links between Lyapunov and inequalities are new as well as links between transports and inequalities. Are also new the links between weighted and weak inequalities.

The paper is organized as follows.

In Section 2 we prove that the existence of a Lyapunov function implies weighted Cheeger and weighted Poincaré inequalities and their converse.

Section 3 collects examples illustrating the Lyapunov function method. We will focus on multidimensional Cauchy type distributions and sub-exponential type distributions.

Section 4 shows how to derive weak Poincaré and weak Cheeger inequalities from weighted converse Poincaré and Cheeger inequalities. The weak Cheeger inequality is particularly interesting because of its link with isoperimetric inequalities. Explicit examples are given.

In Section 5, we propose another method, based on mass transport, to get weighted inequalities. We apply this method to prove weighted Poincaré inequalities for some spherically symmetric distributions, and to study the isoperimetric profile of products of one dimensional heavy tailed distributions. In both situations, the mass transport method gives explicit constants depending on the dimension in an optimal way.

Finally, the appendix is devoted to the proof of some technical results used in Section 5.

2 From ϕ-Lyapunov function to weighted inequalities and their converse

The purpose of this section is to derive weighted inequalities of Poincaré and Cheeger types, and their converse forms, from the existence of a ϕ Lyapunov function for the underlying diffusion operator. To properly define this notion let us describe the general framework we shall deal with.

Let E be some Polish state space equipped with a probability measure μ and a μ-symmetric operator L. The main assumption on L is that there exists some algebra A of bounded functions, containing constant functions, which is everywhere dense (in the $\mathbb{L}_2(\mu)$ norm) in the domain of L, and a core for L. This ensures the existence of a "carré du champ" Γ, *i.e.* for $f, g \in A$, $L(fg) = fLg + gLf + 2\Gamma(f,g)$. We also assume that Γ is a derivation (in each component), *i.e.* $\Gamma(fg,h) = f\Gamma(g,h) + g\Gamma(f,h)$. This is the standard "diffusion" case in [2] and we refer to the introduction of [28] for more details. For simplicity we set $\Gamma(f) = \Gamma(f,f)$. Note that, since Γ is a non-negative bilinear form (see [1, Proposition 2.5.2]), the Cauchy-Schwarz inequality holds: $\Gamma(f,g) \leq \sqrt{\Gamma(f)}\sqrt{\Gamma(g)}$. Furthermore, by symmetry,

$$E(f,g) := \int \Gamma(f,g)d\mu = -\int f\, Lg\, d\mu\,. \tag{2.1}$$

Also, since L is a diffusion, the following chain rule formula $\Gamma(\Psi(f),\Phi(g)) = \Psi'(f)\Phi'(g)\Gamma(f,g)$ holds. Notice that in this situation the symmetric form E extends as a Dirichlet form with domain $D(E)$, the form being nice enough (regular, local ...).

In particular if $E = \mathbb{R}^n$, $d\mu(x) = p(x)dx$ and $L = \Delta + \nabla \log p \cdot \nabla$, we may consider the C^∞ functions with compact support (plus the constant functions) as the interesting subalgebra A, and then $\Gamma(f,g) = \nabla f \cdot \nabla g$.

Now we define the notion of Φ-Lyapunov function.

Definition 2.2. *Let $W \geq 1$ be a smooth enough function on E and ϕ be a C^1 positive increasing function defined on $\mathbb{R}^+$. We say that W is a ϕ-Lyapunov function if there exist some set $K \subset E$ and some $b \geq 0$ such that*

$$LW \;\leq\; -\phi(W) \;+\; b\,1_K\,.$$

This latter condition is sometimes called a "drift condition".

Remark 2.3. One may ask about the meaning of LW in this definition. In the $\mathbb{R}^n$ case, we shall choose C^2 functions W, so that LW is defined in the usual sense. On more general state spaces of course, the easiest way is to assume that W belongs to the ($\mathbb{L}_2$) domain of L, in particular $LW \in \mathbb{L}_2$. But in some situations one can also relax the latter, provided all calculations can be justified. ◇

2.1 Weighted Poincaré inequality and weighted Cheeger inequality.

In this section we derive weighted Poincaré and weighted Cheeger inequalities from the existence of a ϕ-Lyapunov function.

Definition 2.4. *We say that μ satisfies a weighted Cheeger (resp. Poincaré) inequality with weight ω (resp. η) if for some $C, D > 0$ and all $g \in A$ with μ-median equal to 0,*

$$\int |g|\, d\mu \;\leq\; C \int \sqrt{\Gamma(g)}\,\omega\, d\mu\,, \tag{2.5}$$

respectively, for all $g \in A$,

$$Var_\mu(g) \;\leq\; D \int \Gamma(g)\,\eta\, d\mu\,. \tag{2.6}$$

It is known that if (2.5) holds, then (2.6) also holds with $D = 4C^2$ and $\eta = \omega^2$ (see Corollary 2.14).

In order to deal with the "local" part $b1_K$ in the definition of a ϕ-Lyapunov function, we shall use the notion of local Poincaré inequality we introduce now.

Definition 2.7. *Let $U \subset E$ with $\mu(U) > 0$. We shall say that μ satisfies a local Poincaré inequality on U if there exists some constant κ_U such that for all $f \in A$*

$$\int_U f^2\, d\mu \;\leq\; \kappa_U \int_E \Gamma(f) d\mu \;+\; (1/\mu(U)) \left(\int_U f\, d\mu\right)^2 .$$

Notice that in the right hand side the energy is taken over the whole space E (unlike the usual definition). Moreover, $\int_U f^2\, d\mu - (1/\mu(U)) \left(\int_U f\, d\mu\right)^2 = \mu(U)\mathrm{Var}_{\mu_U}(f)$ with $\frac{d\mu_U}{d\mu} := \frac{1_U}{\mu(U)}$. This justifies the name "local Poincaré inequality".

Now we state our first general result.

Theorem 2.8 (Weighted Poincaré inequality)**.** *Assume that there exists some ϕ-Lyapunov function W (see Definition 2.2) and that μ satisfies a local Poincaré inequality on some subset $U \supseteq K$. Then for all $g \in A$, it holds*

$$Var_\mu(g) \;\leq\; \max\left(\frac{b\kappa_U}{\phi(1)}, 1\right) \int \left(1 + \frac{1}{\phi'(W)}\right) \Gamma(g)\, d\mu \,. \tag{2.9}$$

Proof. We shall give a rigorous complete proof in the $\mathbb{R}^n$ case where all requested definitions are well known. Similar arguments can be used in the general case, but requires to introduce the ad-hoc localization procedure.

Let $g \in A$, choose c such that $\int_U (g - c) d\mu = 0$ and set $f = g - c$. Since $\mathrm{Var}_\mu(g) = \inf_a \int (g - a)^2\, d\mu$, we have

$$\mathrm{Var}_\mu(g) \leq \int f^2 d\mu \leq \int \frac{-LW}{\phi(W)}\, f^2\, d\mu + \int f^2\, \frac{b}{\phi(W)}\, 1_K\, d\mu \,.$$

To manage the second term, we first use that $\Phi(W) \geq \Phi(1)$. Then, the definition of c and the local Poincaré inequality ensures that

$$\begin{aligned} \int_K f^2\, d\mu \;&\leq\; \int_U f^2\, d\mu \\ &\leq\; \kappa_U \int_E \Gamma(f) d\mu \;+\; (1/\mu(U)) \left(\int_U f\, d\mu\right)^2 \\ &=\; \kappa_U \int_E \Gamma(g) d\mu \,. \end{aligned}$$

For the first term, we use Lemma 2.10 below (with $\psi = \phi$ and $h = W$).

Lemma 2.10. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a C^1 increasing function. Then, for any $f \in A$ and any positive $h \in D(E)$,*

$$\int \frac{-Lh}{\psi(h)}\, f^2\, d\mu \leq \int \frac{\Gamma(f)}{\psi'(h)} d\mu$$

Proof. By (2.1), the fact that Γ is a derivation and the chain rule formula, we have

$$\int \frac{-Lh}{\psi(h)} f^2 \, d\mu \quad = \quad \int \Gamma\left(h, \frac{f^2}{\psi(h)}\right) d\mu = \int \left(\frac{2\, f\, \Gamma(f,h)}{\psi(h)} \; - \; \frac{f^2 \psi'(h) \Gamma(h)}{\psi^2(h)}\right) d\mu \,.$$

Since ψ is increasing and according to Cauchy-Schwarz inequality we get

$$\begin{aligned} \frac{f\, \Gamma(f,h)}{\psi(h)} \quad &\leq \quad \frac{f \sqrt{\Gamma(f)\Gamma(h)}}{\psi(h)} = \frac{\sqrt{\Gamma(f)}}{\sqrt{\psi'(h)}} \cdot \frac{f \sqrt{\psi'(h)\Gamma(h)}}{\psi(h)} \\ &\leq \quad \frac{1}{2} \frac{\Gamma(f)}{\psi'(h)} + \frac{1}{2} \frac{f^2 \psi'(h)\, \Gamma(h)}{\psi^2(h)}. \end{aligned}$$

The result follows. □

To be rigorous one has to check some integrability conditions. If W belongs to the domain of L, we can use the previous lemma with $h = W$. If we do not have a priori controls on the integrability of LW (and $\Gamma(f, W)$) one has to be more careful.

In the $\mathbb{R}^n$ case there is no real difficulty provided K is compact and U is for instance a ball $B(0, R)$. To overcome all difficulties in this case, we may proceed as follows : we first assume that g is compactly supported and $f = (g - c)\chi$, where χ is a non-negative compactly supported smooth function, such that $1_U \leq \chi \leq 1$. All the calculation above are thus allowed. In the end we choose some sequence χ_k satisfying $1_{kU} \leq \chi_k \leq 1$, and such that $|\nabla \chi_k| \leq 1$, and we go to the limit. □

Remark 2.11. Very recently, two of the authors and various coauthors have pushed forward the links between Lyapunov functionals (and local inequalities) and usual functional inequalities. for example if ϕ (in the Lyapunov condition) is assumed to be linear, then we recover the results in [3], namely a Poincaré inequality (and a short proof of Bobkov's result on logconcave probability measure satisfying spectral gap inequality). If ϕ is superlinear, then the authors of [30] have obtained super-Poincaré inequalities, including nice alternative proofs of Bakry-Emery or Kusuocka-Stroock criterion for logarithmic Sobolev inequality. ◊

The same ideas can be used to derive $\mathbb{L}_1$ weighted Poincaré (or weighted Cheeger) inequalities.

Consider f an arbitrary smooth function with median w.r.t. μ equal to 0. Assume that W is a ϕ-Lyapunov function. Then if $f = g - c$,

$$\begin{aligned} \int |f| d\mu \quad &\leq \quad \int |f| \frac{-LW}{\phi(W)} d\mu + b \int_K \frac{|f|}{\phi(W)} \, d\mu \\ &\leq \quad \int \Gamma\left(\frac{|f|}{\phi(W)}, W\right) + \frac{b}{\phi(1)} \int_K |f| \, d\mu \\ &\leq \quad \int \frac{\Gamma(|f|, W)}{\phi(W)} d\mu - \int \frac{|f| \Gamma(W) \phi'(W)}{\phi^2(W)} d\mu + \frac{b}{\phi(1)} \int_K |f| d\mu \,. \end{aligned}$$

Now we use Cauchy-Schwarz for the first term (i.e. $\Gamma(u, v) \leq \sqrt{\Gamma(u)} \sqrt{\Gamma(v)}$) in the right hand side, we remark that the second term is negative since ϕ' is positive, and we can control the last one as before if we assume a local Cheeger inequality, instead of a local Poincaré inequality. We have thus obtained

Theorem 2.12. *Assume that there exists a ϕ-Lyapunov function W and μ satisfies some local Cheeger inequality*

$$\int_U |f|\,d\mu \;\leq\; \kappa_U \int_E \sqrt{\Gamma(f)}d\mu\,,$$

for some $U \supseteq K$ and all f with median w.r.t. $1_U\,\mu/\mu(U)$ equal to 0. *Then for all $g \in A$ with median w.r.t. μ equal to 0, it holds*

$$\int |g|\,d\mu \;\leq\; \max\left(\frac{b\kappa_U}{\phi(1)}, 1\right) \int \left(1 + \frac{\sqrt{\Gamma(W)}}{\phi(W)}\right) \sqrt{\Gamma(g)}\,d\mu\,. \tag{2.13}$$

Again one has to be a little more careful in the previous proof, with integrability conditions, but difficulties can be overcome as before.

It is well known that Cheeger inequality implies Poincaré inequality. This is also true for weighted inequalities. Note however, that the forms of weight obtained respectively in Theorem 2.8 and next corollary are different (even if, up to constant, they are of the same order in all examples we shall treat in the following section).

Corollary 2.14. *Under the assumptions of Theorem 2.12, for all $g \in A$, it holds*

$$Var_\mu(g) \;\leq\; 8\,\max\left(\frac{b\kappa_U}{\phi(1)}, 1\right)^2 \int \left(1 + \frac{\Gamma(W)}{\phi^2(W)}\right) \Gamma(g)\,d\mu\,.$$

Proof. As suggested in the proof of Theorem 5.1 in [23], if g has a μ median equal to 0, $g_+ = \max(g, 0)$ and $g_- = \max(-g, 0)$ have zero median too. We may thus apply Theorem 2.12 to both g_+^2 and g_-^2, yielding

$$\int g_+^2\,d\mu \;\leq\; 2\max\left(\frac{b\kappa_U}{\phi(1)}, 1\right) \int g_+\,\sqrt{\Gamma(g_+)}\left(1 + \frac{\sqrt{\Gamma(W)}}{\phi(W)}\right) d\mu$$

and similarly for g_-. Applying Cauchy-Schwarz inequality, and using the elementary $(a+b)^2 \leq 2a^2+2b^2$ we get that

$$\int g_+^2\,d\mu \leq\; 8\,\max\left(\frac{b\kappa_U}{\phi(1)}, 1\right)^2 \int \left(1 + \frac{\Gamma(W)}{\phi^2(W)}\right) \Gamma(g_+)\,d\mu$$

and similarly for g_-. To conclude the proof, it remains to sum-up the positive and the negative parts and to notice that $\mathrm{Var}_\mu(g) \leq \int g^2\,d\mu$. □

2.2 Converse weighted inequalities.

This section is dedicated to the study of converse weighted inequalities from ϕ-Lyapunov function. We start with converse weighted Poincaré inequalities and then we study converse weighted Cheeger inequalities.

Definition 2.15. *We say that μ satisfies a converse weighted Cheeger (resp. Poincaré) inequality with weight ω if for some $C > 0$ and all $g \in A$*

$$\inf_c \int |g - c|\,\omega\,d\mu \;\leq\; C \int \sqrt{\Gamma(g)}\,d\mu\,, \tag{2.16}$$

respectively, for all $g \in A$,

$$\inf_c \int |g - c|^2\,\omega\,d\mu \;\leq\; C \int \Gamma(g)\,d\mu\,. \tag{2.17}$$

2.2.1 Converse weighted Poincaré inequalities

In [23, Proposition 3.3], the authors perform a change of function in the weighted Poincaré inequality to get

$$\inf_c \int (f-c)^2\,\omega\,d\mu \le \int |\nabla f|^2 d\mu.$$

This method requires that the constant D in the weighted Poincaré inequality (2.6) (with weight $\eta(x) = (1+|x|)^2$) is not too big. The same can be done in the general situation, provided the derivative of the weight is bounded and the constant is not too big.

But instead we can also use a direct approach from ϕ-Lyapunov functions.

Theorem 2.18 (Converse weighted Poincaré inequality)**.** *Under the assumptions of Theorem 2.8, for any $g \in A$, it holds*

$$\inf_c \int (g-c)^2\,\frac{\phi(W)}{W}\,d\mu \le (1+b\kappa_U)\int \Gamma(g)\,d\mu\,. \tag{2.19}$$

Proof. Rewrite the drift condition as

$$w := \frac{\phi(W)}{W} \le -\frac{LW}{W} + b\,1_K\,,$$

recalling that $W \ge 1$. Let $f = g - c$ with $\int_U (g-c)d\mu = 0$. Then,

$$\inf_c \int (g-c)^2\,\frac{\phi(W)}{W}\,d\mu \le \int f^2\,w\,d\mu \le \int -\frac{LW}{W}\,f^2\,d\mu \,+\, b\int_K f^2\,d\mu\,.$$

The second term in the right hand side of the latter can be handled using the local Poincaré inequality, as in the proof of Theorem 2.8 (we omit the details). We get $\int_K f^2\,d\mu \le \kappa_U \int \Gamma(g) d\mu$. For the first term we use Lemma 2.10 with $\psi(x) = x$. This achieves the proof. □

Remark 2.20. In the proof the previous theorem, we used the inequality

$$\int \frac{-LW}{W}\,f^2\,d\mu \,\le\, \int \Gamma(f)\,d\mu.$$

By [30, Lemma 2.12], it turns out that the latter can be obtained without assuming that Γ is a derivation. In particular the previous theorem extends to any situation where L is the generator of a μ-symmetric Markov process (including jump processes) in the form

$$\inf_c \int (g-c)^2\,\frac{\phi(W)}{W}\,d\mu \le (1+b\kappa_U)\int -g\,Lg\,d\mu\,.$$

◊

2.2.2 Converse weighted Cheeger inequalities

Here we study the harder converse weighted Cheeger inequalities. The approach by ϕ-Lyapunov functions works but some additional assumptions have to be made.

Theorem 2.21 (Converse weighted Cheeger inequality)**.** *Under the hypotheses of Theorem 2.12, assume that K is compact and that either*

(1) $|\Gamma(W,\Gamma(W))| \leq 2\delta\phi(W)\,(1+\Gamma(W))$ *outside* K*, for some* $\delta \in (0,1)$

or

(2) $\Gamma(W,\Gamma(W)) \geq 0$ *outside* K*.*

Then, there exists a constant $C > 0$ *such that for any* $g \in A$*, it holds*

$$\inf_c \int |g-c|\,\frac{\phi(W)}{\sqrt{1+\Gamma(W)}}\,d\mu \leq C\int \sqrt{\Gamma(g)}d\mu\,.$$

Remark 2.22. Note that using Cauchy-Schwarz inequality, Assumption (1) is implied by $\Gamma(\Gamma(W)) \leq 4\delta^2\phi(W)^2(1+\Gamma(W))$ outside K.
On the other hand, in dimension 1 for usual diffusions, we have $\Gamma(W,\Gamma(W)) = 2\,|W'|^2\,W''$. Hence this term is non negative as soon as W is convex outside K.

Proof. Let $g \in A$ and set $f = g-c$ with c satisfying $\int_U (g-c)d\mu = 0$. Recall that $LW \leq -\phi(W) + b1_K$. Hence,

$$\frac{\phi(W)}{\sqrt{1+\Gamma(W)}} \leq -\frac{LW}{\sqrt{1+\Gamma(W)}} + \frac{b1_K}{\sqrt{1+\Gamma(W)}} \leq -\frac{LW}{\sqrt{1+\Gamma(W)}} + b1_K.$$

In turn,

$$\int |f|\,\frac{\phi(W)}{\sqrt{1+\Gamma(W)}}\,d\mu \quad\leq\quad -\int \frac{|f|}{\sqrt{1+\Gamma(W)}}\,LW\,d\mu \;+\; b\int_K |f|\,d\mu\,.$$

To control the first term we use (2.1), the fact that Γ is a derivation and Cauchy-Schwarz inequality to get that

$$\begin{aligned}
-\int \frac{|f|}{\sqrt{1+\Gamma(W)}}LW\,d\mu &= \int \Gamma\left(\frac{|f|}{\sqrt{1+\Gamma(W)}},W\right)d\mu \\
&= \int \frac{\Gamma(|f|,W)}{\sqrt{1+\Gamma(W)}}\,d\mu + \int |f|\Gamma\left(\frac{1}{\sqrt{1+\Gamma(W)}},W\right)d\mu \\
&\leq \int \sqrt{\Gamma(f)}\,d\mu - \int |f|\,\frac{\Gamma(W,\Gamma(W))}{2(1+\Gamma(W))^{\frac{3}{2}}}\,d\mu\,.
\end{aligned}$$

Now, we divide the second term of the latter in sum of the integral over K and the integral outside K. Set $M := \sup_K \frac{|\Gamma(W,\Gamma(W))|}{2(1+\Gamma(W))^{\frac{3}{2}}}$. Under Assumption (2), the integral outside K is non-positive, thus we end up with

$$\int |f|\,\frac{\phi(W)}{\sqrt{1+\Gamma(W)}}\,d\mu \leq \int \sqrt{\Gamma(f)}\,d\mu + (M+b)\int_K |f|\,d\mu$$

while under Assumption (1), we get

$$\int |f|\,\frac{\phi(W)}{\sqrt{1+\Gamma(W)}}\,d\mu \leq \int \sqrt{\Gamma(f)}\,d\mu + (M+b)\int_K |f|\,d\mu + \delta\int |f|\,\frac{\phi(W)}{\sqrt{1+\Gamma(W)}}\,d\mu$$

In any case the term $\int_K |f|\,d\mu$ can be handle using the local Cheeger inequality (we omit the details): we get $\int_K |f|d\mu \leq \kappa_U \int \sqrt{\Gamma(g)}d\mu$. This ends the proof, since $\Gamma(f) = \Gamma(g)$. □

2.3 Additional comments

Stability. As it is easily seen, the weighted Cheeger and Poincaré inequalities (and their converse) are stable under log-bounded transformations of the measure. The Lyapunov approach encompasses a similar property with compactly supported (regular) perturbations. In fact the Lyapunov approach is even more robust, as the following example illustrates: suppose that the measure $\mu = e^{-V}dx$ satisfies a ϕ-Lyapunov condition with test function W and suppose that for large x, $\nabla V \cdot \nabla W \geq \nabla V \cdot \nabla U$ for some regular (but possibly unbounded) U, then there exists $\beta > 0$ such that $d\nu = e^{-V+\beta U}dx$ satisfies a ϕ-Lyapunov condition with the same test function W and then the same weighted Poincaré or Cheeger inequality (but possibly with different constants).

Manifold case. In fact, many of the results presented here can be extended to the manifold case, as soon as we can suppose that $V(x) \to \infty$ when the geodesic distance (to some fixed points) grows to infinity and of course that a local Poincaré inequality or a local Cheeger inequality is valid. We refer to [30] for a more detailed discussion.

3 Examples

3.1 Examples in $\mathbb{R}^n$.

We consider here the $\mathbb{R}^n$ situation with $d\mu(x) = p(x)dx$ and $L = \Delta + \nabla \log p \cdot \nabla$, p being smooth enough. We can thus use the argument explained in the proof of Theorem 2.8 so that as soon as W is C^2 one may apply Theorem 2.8 and Theorem 2.12.

Recall the following elementary lemma whose proof can be found in [3].

Lemma 3.1. *If V is convex and $\int e^{-V(x)}\,dx < +\infty$, then*

(1) for all x, $x \cdot \nabla V(x) \geq V(x) - V(0)$,

(2) there exist $\delta > 0$ and $R > 0$ such that for $|x| \geq R$, $V(x) - V(0) \geq \delta\,|x|$.

We shall use this lemma in the following examples. Our first example corresponds to the convex case discussed by Bobkov and Ledoux [23].

Proposition 3.2 (Cauchy type law)**.** *Let $d\mu(x) = (V(x))^{-(n+\alpha)}\,dx$ for some positive convex function V and $\alpha > 0$. Then there exists $C > 0$ such that for all g the following weighted Poincaré and weighted Cheeger inequalities hold*

$$Var_\mu(g) \;\leq\; C \int |\nabla g(x)|^2\,(1 + |x|^2)\,d\mu(x),$$
$$\int |g - m|\,d\mu \;\leq\; C \int |\nabla g(x)|\,(1 + |x|)\,d\mu(x),$$

(where m stands for a median of g under μ), and such that for all g, the following converse weighted Poincaré and converse weighted Cheeger inequality hold

$$\inf_c \int (g(x) - c)^2\,\frac{1}{1 + |x|^2}\;d\mu(x) \;\leq\; C \int |\nabla g|^2 d\mu,$$
$$\inf_c \int |g(x) - c|\,\frac{1}{1 + |x|}\,d\mu(x) \leq C \int |\nabla g|\,d\mu\,.$$

Remark 3.3. The restriction $\alpha > 0$ is the same as in [23].

Proof. By Lemma 3.4 below, there exists a ϕ-Lyapunov function W satisfying $(1/\phi'(W))(x) = \frac{k}{c(k-2)}|x|^2$ for x large. Hence, in order to apply Theorem 2.8 it remains to recall that since $d\mu/dx$ is bounded from below and from above on any ball $B(0, R)$, μ satisfies a Poincaré inequality and a Cheeger inequality on such subset, hence a local Poincaré (and Cheeger) inequality in the sense of definition 2.7 (or Theorem 2.12). The converse weighted Poincaré inequality is a direct consequence of Theorem 2.18 and Lemma 3.4. By Lemma 3.4, we know that $W(x) = |x|^k$ (for x large) is a ϕ-Lyapunov function for $\phi(u) = c|u|^{(k-2)/k}$. Note that $\Gamma(W, \Gamma(W))(x) = (2k-2)k^2|x|^{3k-4}$ at infinity. Hence Assumption (2) of Theorem 2.21 holds and the theorem applies. □

Lemma 3.4. *Let $L = \Delta - (n+\alpha)(\nabla V/V)\nabla$ with V and α as in Proposition 3.2. Then, there exists $k > 2$, $b, R > 0$ and $W \geq 1$ such that*

$$LW \leq -\phi(W) + b1_{B(0,R)}$$

with $\phi(u) = cu^{(k-2)/k}$ for some constant $c > 0$. Furthermore, one can choose $W(x) = |x|^k$ for x large.

Proof. Let $L = \Delta - (n+\alpha)(\nabla V/V)\nabla$ and choose $W \geq 1$ smooth and satisfying $W(x) = |x|^k$ for $|x|$ large enough and $k > 2$ that will be chosen later. For $|x|$ large enough we have

$$LW(x) = k\,(W(x))^{\frac{k-2}{k}} \left(n + k - 2 - \frac{(n+\alpha)\,x \cdot \nabla V(x)}{V(x)}\right).$$

Using (1) in Lemma 3.1 (since $V^{-(n+\alpha)}$ is integrable e^{-V} is also integrable) we have

$$n + k - 2 - \frac{(n+\alpha)\,x \cdot \nabla V(x)}{V(x)} \leq k - 2 - \alpha + (n+\alpha)\frac{V(0)}{V(x)}.$$

Using (2) in Lemma 3.1 we see that we can choose $|x|$ large enough for $\frac{V(0)}{V(x)}$ to be less than ε, say $|x| > R_\varepsilon$. It remains to choose $k > 2$ and $\varepsilon > 0$ such that

$$k + n\varepsilon - 2 - \alpha(1-\varepsilon) \leq -\gamma$$

for some $\gamma > 0$. We have shown that, for $|x| > R_\varepsilon$,

$$LW \leq -k\gamma\phi(W),$$

with $\phi(u) = u^{\frac{k-2}{k}}$ (which is increasing since $k > 2$). A compactness argument completes the proof. □

Remark 3.5. The previous proof gives a non explicit constant C in terms of α and n. This is mainly due to the fact that we are not able to control properly the local Poincaré and Cheeger inequalities on balls for the general measures $d\mu = (V(x))^{-(n+\alpha)}\,dx$. More could be done on specific laws.

Our next example deals with sub-exponential distributions.

Proposition 3.6 (Sub exponential like law)**.** *Let $d\mu = (1/Z_p)\,e^{-V^p}$ for some positive convex function V and $p > 0$. Then there exists $C > 0$ such that for all g the following weighted Poincaré and weighted Cheeger inequalities hold*

$$Var_\mu(g) \;\leq\; C \int |\nabla g(x)|^2 \left(1 + (1+|x|)^{2(1-p)}\right) d\mu(x)\,,$$

$$\int |g - m|\,d\mu \;\leq\; C \int |\nabla g(x)| \left(1 + (1+|x|)^{(1-p)}\right) d\mu(x)\,,$$

(where m stands for a median of g under μ), and such that the following converse weighted Poincaré and converse weighted Cheeger inequalities hold

$$\inf_c \int (g(x)-c)^2 \, \frac{1}{1+|x|^{2(1-p)}} \, d\mu(x) \;\leq\; C \int |\nabla g|^2 \, d\mu,$$
$$\inf_c \int |g(x)-c| \, \frac{1}{1+|x|^{1-p}} \, d\mu(x) \leq C \int |\nabla g| \, d\mu,$$

Remark 3.7. For $p<1$ we get some weighted inequalities, while for $p \geq 1$ we see that (changing C into $2C$) we obtain the usual Poincaré and Cheeger inequalities. For $p=1$, one recovers the well known fact (see [40; 17]) that Log-concave distributions enjoy Poincaré and Cheeger inequalities. Moreover, if we consider the particular case $d\mu(x) = (1/Z_p)\, e^{-|x|^p}$ with $0<p<1$, and choose $g(x) = e^{|x|^p/2}\, 1_{[0,R]}(x)$ for $x \geq 0$ and $g(-x) = -g(x)$, we see that the weight is optimal in Proposition 3.6.

Proof. The proof follows the same line as the proof of Proposition 3.2, using Lemma 3.8 below. □

Lemma 3.8. *Let $L = \Delta - pV^{p-1}\nabla V \nabla$ for some positive convex function V and $p>0$. Then, there exists $b, c, R > 0$ and $W \geq 1$ such that*

$$LW \leq -\phi(W) + b 1_{B(0,R)}$$

with $\phi(u) = u \, \log^{2(p-1)/p}(c+u)$ increasing. Furthermore, one can choose $W(x) = e^{\gamma |x|^p}$ for x large.

Proof. We omit the details since we can mimic the proof of Lemma 3.4. □

Remark 3.9. Changing the values of b and R, only the values of $\Phi(u)$ in the large are relevant. In other words, one could take Φ to be an everywhere increasing function which coincides with $u \, \log^{2(p-1)/p}(u)$ for the large u's, by choosing the constants b and R large enough.

3.2 Example on the real line

In this section we give examples on the real line where other techniques can also be used.

Note that in both previous examples we used a Lyapunov function $W = p^{-\gamma}$ for some well chosen $\gamma > 0$. In the next result we give a general statement using such a Lyapunov function in dimension 1.

Proposition 3.10. *Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}$ for a smooth potential V. We assume for simplicity that V is symmetric. Furthermore, we assume that V is concave on $(R,+\infty)$ for some $R>0$ and that $\left(V''/|V'|^2\right)(x) \;\to\; r > -1/2$ as $x \to \infty$. Then for some $S > R$ and some $C>0$, the following weighted Poincaré and Cheeger inequalities hold*

$$Var_\mu(g) \;\leq\; C \int |g'(x)|^2 \left(1 + \frac{1_{|x|>S}}{|V'|^2(x)}\right) d\mu(x)\,,$$
$$\int |g-m| \, d\mu \;\leq\; C \int |g'(x)| \left(1 + \frac{1_{|x|>S}}{|V'|(x)}\right) d\mu(x)$$

where m is a median of g under μ. Furthermore, the following converse Cheeger inequality also holds

$$\inf_c \int |g-c| \, (1_{(-S,S)} + |V'|) \, d\mu \leq C \int |g'| \, d\mu\,.$$

Proof. Since V' is non-increasing on $(R, +\infty)$ it has a limit l at $+\infty$. If $l < 0$, V goes to $-\infty$ at $+\infty$ with a linear rate, contradicting $\int e^{-V} dx < +\infty$. Hence $l \geq 0$, V is increasing and goes to $+\infty$ at $+\infty$.

Now choose $W = e^{\gamma V}$ (for large $|x|$). We have

$$LW = \left(\gamma V'' - (\gamma - \gamma^2)|V'|^2\right) W$$

so that for $0 < \gamma < 1$ we have $LW \leq -(\gamma - \gamma^2)|V'|^2 W$ at infinity. We may thus choose

$$\phi(w) = (\gamma - \gamma^2)|V'|^2(W^{-1}(w))\, w, \; w \in [W(R), +\infty).$$

On the other hand,

$$\phi'(W)\, W' = (\gamma - \gamma^2)V'\, W \left(2V'' + \gamma|V'|^2\right),$$

so that, since $W' > 0$, $V' > 0$ and $V''/|V'|^2 > -1/2$ asymptotically, ϕ is non-decreasing at infinity for a well chosen γ. Then, it is possible to build ϕ on a compact interval $[0, a]$ in order to get a smooth increasing function on the whole $\mathbb{R}_+$.

Since $d\mu/dx$ is bounded from above and below on any compact interval, a local Poincaré inequality and a local Cheeger inequality hold on such interval. Hence, it remains to apply Theorem 2.8 and Theorem 2.12, since at infinity $\phi'(W)$ behaves like $|V'|^2$.

To prove the converse Cheeger inequality, observe first that the function $W = e^{\gamma V}$ is convex in the large as soon as $0 < \gamma < 1$ is chosen so that $\limsup(|V''|/|V'|^2) < \gamma$ at infinity. Hence we can use remark 2.22 and Theorem 2.21 to conclude that the converse Cheeger inequality holds with the weight $\phi(W)/\sqrt{1 + \Gamma(W)}$ which is of the order of $|V'|$ in the large. □

Remark 3.11. The example of Proposition 3.6 is within the framework of this proposition, and the general Cauchy distribution $V(x) = c \log(1 + |x|^2)$ works if $c > 1$, since $V''/|V'|^2$ behaves asymptotically as $-1/2c$. Note that the weight we obtain is of optimal order, applying the inequality with approximations of $e^{V/2}$.

It is possible to extend the previous proposition to the multi-dimensional setting, but the result is quite intricate. Assume that $V(x) \to +\infty$ as $|x| \to +\infty$, and that V is concave (at infinity). The same $W = e^{\gamma V}$ furnishes $LW/W = \gamma \Delta V - (\gamma - \gamma^2)|\nabla V|^2$. Hence we may define

$$\phi(u) = (\gamma - \gamma^2)\, u \inf_{A(u)} |\nabla V|^2 \quad \text{with} \quad A(u) = \{x; V(x) = \log(u)/\gamma\}$$

at least for large u's. The main difficulty is to check that ϕ is increasing. This could probably be done on specific examples.

It is known that Hardy-type inequalities are useful tool to deal with functional inequalities of Poincaré type in dimension 1 (see [13; 12] for recent contributions on the topic). We shall use now Hardy-type inequalities to relax the hypothesis on V and to obtain the weighted Poincaré inequality of Proposition 3.10. However no similar method (as far as we know) can be used for the weighted Cheeger inequality, making the ϕ-Lyapunov approach very efficient.

Proposition 3.12. *Let $d\mu(x) = e^{-V(x)}dx$ be a probability measure on $\mathbb{R}$ for a smooth potential V that we suppose for simplicity to be even. Let $\varepsilon \in (0, 1)$. Assume that there exists $x_0 \geq 0$ such that V is twice differentiable on $[x_0, \infty)$ and*

$$V'(x) \neq 0, \qquad \frac{|V''(x)|}{V'(x)^2} \leq 1 - \varepsilon, \qquad \forall x \geq x_0.$$

Then, for some $C > 0$, it holds

$$Var_\mu(g) \le C \int |g'(x)|^2 \left(1 + \frac{1_{|x|>x_0}}{|V'|^2(x)}\right) d\mu(x)\,.$$

Proof. Given η and using a result of Muckenhoupt [48], one has for any G smooth enough

$$\int_0^{+\infty} (G(x) - G(0))^2\, d\mu(x) \le 4B \int_0^{+\infty} G'(x)^2\,(1 + \eta^2(x))\, d\mu(x)\,,$$

with $B = \sup_{y>0} \left(\int_y^{+\infty} e^{-V(x)} dx\right)\left(\int_0^y \frac{e^{V(x)}}{1+\eta^2(x)}\, dx\right)$. Hence, since V is even and $\mathrm{Var}_\mu(g) \le \int_{-\infty}^0 (G(x) - G(0))^2 d\mu + \int_0^{+\infty} (G(x) - G(0))^2 d\mu$, the previous bound applied twice leads to

$$\mathrm{Var}_\mu(g) \le 4B \int |g'(x)|^2 \left(1 + \eta(x)^2\right) d\mu.$$

In particular, one has to prove that

$$B = \sup_{y>0} \left(\int_y^{+\infty} e^{-V(x)} dx\right)\left(\int_0^y \frac{e^{V(x)}}{1 + \frac{1_{|x|>x_0}}{|V'|^2(x)}}\, dx\right) < \infty$$

Consider $y \ge x_0$. Then, (note that $V' > 0$ since it cannot change sign and e^{-V} is integrable),

$$\begin{aligned}
\int_{x_0}^y \frac{e^{V(x)}}{1 + \frac{1_{|x|>x_0}}{|V'|^2(x)}}\, dx &= \int_{x_0}^y \frac{V'(x)e^{V(x)}}{V'(x) + \frac{1}{V'(x)}}\, dx = \left[\frac{e^V}{V' + \frac{1}{V'}}\right]_{x_0}^y + \int_{x_0}^y e^V \frac{V''((V')^2 - 1)}{((V')^2 + 1)^2}\\
&\le \frac{e^{V(y)}}{V'(y) + \frac{1}{V'(y)}} + (1 - \varepsilon) \int_{x_0}^y e^V \frac{(V')^2|(V')^2 - 1|}{((V')^2 + 1)^2}\\
&\le \frac{e^{V(y)}}{V'(y) + \frac{1}{V'(y)}} + (1 - \varepsilon) \int_{x_0}^y \frac{e^V}{1 + \frac{1}{(V')^2}}
\end{aligned}$$

where in the last line we used that $x^2|x^2 - 1|/(x^2 + 1)^2 \le 1/(1 + \frac{1}{x^2})$ for $x = V' > 0$. This leads to

$$\int_{x_0}^y \frac{e^{V(x)}}{1 + \frac{1_{|x|>x_0}}{|V'|^2(x)}}\, dx \le \frac{1}{\varepsilon} \frac{e^{V(y)}}{V'(y) + \frac{1}{V'(y)}}.$$

Similar calculations give (we omit the proof)

$$\int_y^{+\infty} e^{-V(x)} dx \le \frac{1}{\varepsilon} \frac{e^{-V(y)}}{V'(y)} \qquad \forall y \ge x_0.$$

Combining these bounds and using a compactness argument on $[0, x_0]$, one can easily show that B is finite. □

We end this section with distributions in dimension 1 that do not fit into the framework of the two previous propositions. Moreover, the laws we have considered so far are κ concave for $\kappa > -\infty$. The last examples shall satisfy $\kappa = -\infty$.

Example 3.13. Let $q > 1$ and define

$$d\mu(x) = (1/Z_q)\ \left((2 + |x|)\ \log^q(2 + |x|)\right)^{-1}\ dx\ =\ V_q^{-1}(x)dx \qquad x \in \mathbb{R}.$$

The function V_q is convex but V_q^γ is not convex for $\gamma < 1$ (hence $\kappa = -\infty$). We may choose $W(x) = (2 + |x|)^2\ \log^a(2 + |x|)$ (at least far from 0), which is a ϕ-Lyapunov function for $\phi(u) = \log^{a-1}(2 + |u|)$ provided $q > a > 1$ (details are left to the reader). We thus get a weighted inequality

$$\mathrm{Var}_\mu(g)\ \leq\ C \int |\nabla g(x)|^2 \left(1 + x^2\ \log^2(2 + |x|)\right) d\mu(x)\,. \tag{3.14}$$

Unfortunately we do not know whether the weight is of the smallest rate of growth as $x \to \infty$ in this situation. The usual choice g behaving like $\sqrt{(2 + |x|)\ \log^q(2 + |x|)}$ on $(-R, R)$ furnishes a variance behaving like R but the right hand side behaves like $R\ \log^2 R$.

We may even find a Lyapunov functional in the case $V(x) = x \log x \log^q(\log x)$ for large x and $q > 1$, *i.e* choose $W(x) = 1 + |x|^2 \log(2 + |x|) \log^c \log(2e + |x|)$ with $1 < c < q$ for which $\phi(x)$ is merely $\log^{c-1} \log(2e + |x|)$ so that the weight in the Poincaré inequality is $1 + |x|^2 \log^2(2 + |x|) \log^2 \log(2e + |x|)$. ◊

4 Applications to weak inequalities and to isoperimetry.

In this section we recall first a result of Bobkov that shows the equivalence between the isoperimetric inequality and what we have called a weak Cheeger inequality (see 1.5).

Lemma 4.1 (Bobkov [19])**.** *Let μ be a probability measure on $\mathbb{R}^n$. There is an equivalence between the following two statements (where I is symmetric around $1/2$)*

(1) for all $s > 0$ and all smooth f with μ median equal to 0,

$$\int |f|\, d\mu\ \leq\ \beta(s) \int |\nabla f|\, d\mu\ +\ s\, \mathrm{Osc}_\mu(f)\,,$$

(2) for all Borel set A with $0 < \mu(A) < 1$,

$$\mu_s(\partial A)\ \geq\ I(\mu(A)),$$

where β and I are related by the duality relation

$$\beta(s)\ =\ \sup_{s \leq t \leq \frac{1}{2}} \frac{t - s}{I(t)}\,, \quad I(t)\ =\ \sup_{0<s\leq t} \frac{t - s}{\beta(s)}\ \textit{for}\ t \leq \frac{1}{2}\,.$$

Here as usual $\mathrm{Osc}_\mu(f) = \mathrm{ess\,sup} f - \mathrm{ess\,inf} f$ *and* $\mu_s(\partial A) = \liminf_{h \to 0} \frac{\mu(0<d(x,A)<h)}{h}$.

Recall that in the weak Cheeger inequality, only the values $s \in (0, 1/2)$ are relevant since $\int |f| d\mu \leq \frac{1}{2} \mathrm{Osc}_\mu(f)$.

Thanks to the previous lemma, we see that isoperimetric results can be derived from weak Cheeger inequalities. We now explain how to obtain such weak Cheeger inequalities from the weighted Cheeger inequality obtained via the ϕ-Lyapunov approach in the previous sections.

4.1 From converse weighted Cheeger to weak Cheeger inequalities

Here we shall first relate converse weighted inequalities to weak inequalities, and then deduce some isoperimetric results on concrete examples.

Theorem 4.2. *Let μ be a probability measure and ω be a non-negative function satisfying $\bar{\omega} = \int \omega\, d\mu < +\infty$. Assume that there exists $C > 0$ such that*

$$\inf_c \int |g-c|\,\omega\, d\mu \le C \int \sqrt{\Gamma(g)}\, d\mu \qquad \forall g \in A.$$

Define $F(u) = \mu(\omega < u)$ and $G(s) = F^{-1}(s) := \inf\{u; \mu(\omega \le u) > s\}$. Then, for all $s > 0$ and all $g \in A$, it holds

$$\inf_c \int |g-c|\, d\mu \ \le\ \frac{C}{G(s)} \int \sqrt{\Gamma(g)}\, d\mu \ +\ s\, \mathrm{Osc}_\mu(f)\,.$$

Proof. Let $g \in A$. Define $m_\omega \in \mathbb{R}$ to be a median of g under $\omega d\mu/\bar{\omega}$. We have

$$\begin{aligned}
\inf_c \int |g-c|\, d\mu \ &\le\ \int |g-m_\omega| d\mu \\
&\le\ \int_{\omega \ge u} |g-m_\omega|\, \frac{\omega}{u}\, d\mu \ +\ \int_{\omega<u} |g-m_\omega|\, d\mu \\
&\le\ \frac{1}{u} \int |g-m_\omega|\, \omega\, d\mu \ +\ \mathrm{Osc}_\mu(g)\, F(u) \\
&=\ \frac{1}{u} \inf_c \int |g-c|\, \omega\, d\mu \ +\ \mathrm{Osc}_\mu(g)\, F(u)\,.
\end{aligned}$$

It remains to apply the converse weighted Cheeger inequality and the definition of G. Note that if $F(u) = 0$ for $u \le u_0$ then $G(s) \ge u_0$. □

We illustrate this result on two examples.

Proposition 4.3 (Cauchy type laws)**.** *Let $d\mu(x) = V^{-(n+\alpha)}(x)dx$ with V convex and $\alpha > 0$. Recall that $\kappa = -1/\alpha$. Then, there exists a constant $C > 0$ such that for any f with μ-median 0,*

$$\int |f|\, d\mu \ \le\ C s^\kappa \int |\nabla f|\, d\mu \ +\ s \mathrm{Osc}_\mu(f) \qquad \forall s > 0.$$

Equivalently there exists $C' > 0$ such that for any $A \subset \mathbb{R}^n$,

$$\mu_s(\partial A) \ge C' \min\left(\mu(A), 1-\mu(A)\right)^{1-\kappa}.$$

Proof. By Proposition 3.2, μ satisfies a converse weighted Cheeger inequality with weight $\omega(x) = \frac{1}{1+|x|}$. So $F(u) = \mu(\omega < u) = \mu(u^{-1} - 1 < |x|)$. Since V is convex, $V(x) \ge \rho|x|$ for large $|x|$ (recall Lemma 3.1), hence using polar coordinates we have

$$\mu(|x| > R) = \int_{|x|>R} V^{-\beta}(x)\, dx \ \le\ \int_{|x|>R} \rho^{-\beta} |x|^{-\beta} dx \le\ c R^{n-\beta}\,,$$

for some $c = c(n, \alpha, \rho)$. The result follows by Theorem 4.2. The isoperimetric inequality follows at once by Lemma 4.1. □

Remark 4.4. The previous result recover Corollary 8.4 in [19] (up to the constants). Of course we do not attain the beautiful Theorem 1.2 in [19], where S. Bobkov shows that the constant C' only depends on κ and the median of $|x|$.

Proposition 4.5 (Sub exponential type laws)**.** *Let $d\mu = (1/Z_p)\, e^{-V^p}$ for some positive convex function V and $p \in (0,1)$. Then there exists $C > 1$ such that for all f with μ-median* 0*,*

$$\int |f|\, d\mu \;\leq\; C \log^{\frac{1}{p}-1}(C/s) \int |\nabla f|\, d\mu \;+\; s\mathrm{Osc}_\mu(f) \qquad \forall s \in (0,1).$$

Equivalently there exists $C' > 0$ such that for any $A \subset \mathbb{R}^n$,

$$\mu_s(\partial A) \geq C' \min\left(\mu(A), 1-\mu(A)\right) \log\left(\frac{1}{\min\left(\mu(A), 1-\mu(A)\right)}\right)^{1-\frac{1}{p}}.$$

Proof. According to Proposition 3.6, μ verifies the converse weighted Cheeger inequality with the weight function ω defined by $\omega(x) = 1/(1+|x|^{1-p})$ for all $x \in \mathbb{R}^n$. Moreover, since V is convex, it follows from Lemma 3.1 that there is some $\rho > 0$ such that $\int e^{\rho|x|^p}\, d\mu(x) < \infty$. Hence, applying Markov's inequality gives $\mu(|x| > R) \leq Ke^{-\rho R^p}$, for some $K \geq 1$. Elementary calculations give the result. □

4.2 Links with weak Poincaré inequalities.

In this section we deal with weak Poincaré inequalities and work under the general setting of Section 2. One says that a probability measure μ verifies the weak Poincaré inequality if for all $f \in A$,

$$\mathrm{Var}_\mu(f) \;\leq\; \beta(s) \int \Gamma(f)\, d\mu \;+\; s\, \mathrm{Osc}_\mu(f)^2, \qquad \forall s \in (0, 1/4),$$

where $\beta : (0,1/4) \to \mathbb{R}^+$ is a non-increasing function. Note that the limitation $s \in (0,1/4)$ comes from the bound $\mathrm{Var}_\mu(f) \leq \mathrm{Osc}_\mu(f)^2/4$.

Weak Poincaré inequalities were introduced by Röckner and Wang in [49]. In the symmetric case, they describe the decay of the semi-group P_t associated to L (see [49; 4]). Namely for all bounded centered function f, there exists $\psi(t)$ tending to zero at infinity such that $\|P_t f\|_{L_2(\mu)} \leq \psi(t)\|f\|_\infty$.

They found another application in concentration of measure phenomenon for sub-exponential laws in [9, Thm 5.1]. The approach proposed in [9] to derive weak Poincaré inequalities was based on capacity-measure arguments (following [13]). In this section, we give alternative arguments. One is based on converse weighted Poincaré inequalities, and the second approach is based on a direct implication of weak Poincaré inequalities from weak Cheeger inequalities.

Converse weighted Poincaré inequalities imply weak Poincaré inequalities as shown in the following theorem.

Theorem 4.6. *Assume that μ satisfies a converse weighted Poincaré inequality*

$$\inf_c \int (g-c)^2\, \omega\, d\mu \leq C \int \Gamma(g)\, d\mu$$

for some non-negative weight ω, such that $\bar{\omega} = \int \omega d\mu < +\infty$. Define $F(u) = \mu(\omega < u)$ and $G(s) = F^{-1}(s) := \inf\{u; \mu(\omega \leq u) > s\}$ for $s < 1$.

Then, for all $f \in A$,

$$Var_\mu(f) \le \frac{C}{G(s)} \int \Gamma(f)\, d\mu + s\mathrm{Osc}_\mu(f)^2, \qquad \forall s \in (0, 1/4).$$

Proof. The proof follows the same line of reasoning as the one of Theorem 4.2. □

Weak Poincaré inequalities are also implied by weak Cheeger inequalities as stated in the following lemma. The proof of the lemma is a little bit more tricky than the usual one from Cheeger to Poincaré. We give it for completeness.

Lemma 4.7. *Let μ be a probability measure and $\beta : \mathbb{R}^+ \to \mathbb{R}^+$. Assume that for any $f \in A$ it holds*

$$\int |f - m|\, d\mu \le \beta(s) \int \sqrt{\Gamma(f)}\, d\mu + s\mathrm{Osc}(f) \qquad \forall s \in (0, 1)$$

where m is a median of f under μ. Then, any $f \in A$ satisfies

$$\mathrm{Var}_\mu(f) \le 4\beta\left(\frac{s}{2}\right)^2 \int \Gamma(f) d\mu + s\mathrm{Osc}(f)^2 \qquad \forall s \in (0, 1/4). \tag{4.8}$$

Proof. let $f \in A$. Assume that 0 is a median of f and by homogeneity of (4.8) that $\mathrm{Osc}(f) = 1$ (which implies in turn that $\|f\|_\infty \le 1$). Let m be a median of f^2. Applying the weak Cheeger inequality to f^2, using the definition of the median and the chain rule formula, we obtain

$$\int f^2\, d\mu \le \int |f^2 - m|\, d\mu \le 2\beta(s) \int |f| \sqrt{\Gamma(f)}\, d\mu + s\mathrm{Osc}(f^2) \qquad \forall s \in (0, 1).$$

Since $\|f\|_\infty \le 1$ and $\mathrm{Osc}(f) = 1$, one has $\mathrm{Osc}(f^2) \le 2$. Hence, by the Cauchy-Schwarz inequality, we have

$$\int f^2\, d\mu \le 2\beta(s) \left(\int \Gamma(f)\, d\mu\right)^{\frac{1}{2}} \left(\int |f|^2\, d\mu\right)^{\frac{1}{2}} + 2s \qquad \forall s \in (0, 1).$$

Hence,

$$\left(\int f^2\, d\mu\right)^{\frac{1}{2}} \le \beta(s) \left(\int \Gamma(f)\, d\mu\right)^{\frac{1}{2}} + \left(\beta(s)^2 \int \Gamma(f)\, d\mu + s\right)^{\frac{1}{2}}.$$

Since $\mathrm{Var}_\mu(f) \le \int f^2\, d\mu$, we finally get

$$\mathrm{Var}_\mu(f) \le 4\beta(s)^2 \int \Gamma(f)\, d\mu + 2s \qquad \forall s \in (0, 1)$$

which is the expected result. □

Two examples follow.

Proposition 4.9 (Cauchy type laws)**.** *Let $d\mu(x) = V^{-(n+\alpha)}(x)dx$ with V convex on $\mathbb{R}^n$ and $\alpha > 0$. Recall that $\kappa = -1/\alpha$. Then there exists a constant $C > 0$ such that for all smooth enough $f : \mathbb{R}^n \to \mathbb{R}$,*

$$Var_\mu(f) \le Cs^{2\kappa} \int |\nabla f|^2\, d\mu + s\mathrm{Osc}_\mu(f)^2, \qquad \forall s \in (0, 1/4).$$

Proof. The proof is a direct consequence of Proposition 4.3 together with Lemma 4.7 above. □

Remark 4.10. For the generalized Cauchy distribution $d\mu(x) = c_\beta\,(1+|x|)^{-(n+\alpha)}$, this result is optimal for $n = 1$ and was shown in [49] (see also [9, Example 2.5]). For $n \geq 2$ the result obtained in [49] is no more optimal. In [4], a weak Poincaré inequality is proved in any dimension with rate function $\beta(s) \leq c(p)\, s^{2p}$ for any $p < \kappa$. Here we finally get the optimal rate. Note however that the constant C may depend on n. ◊

Proposition 4.11 (Sub exponential type laws)**.** *Let $d\mu = (1/Z_p)\, e^{-V^p}\, dx$ for some positive convex function V on $\mathbb{R}^n$ and $p \in (0,1)$. Then there exists $C > 0$ such that for all f*

$$Var_\mu(f) \;\leq\; C\left(\log\left(\frac{1}{s}\right)\right)^{2(\frac{1}{p}-1)} \int |\nabla f|^2\, d\mu \;+\; s\mathrm{Osc}_\mu(f)^2, \qquad \forall s \in (0, 1/4).$$

Proof. The proof is a direct consequence of Proposition 4.5 together with Lemma 4.7 above. □

Remark 4.12. According to an argument of Talagrand (recalled in the introduction), if for all k, μ^k satisfies the same concentration property as μ, then the tail distribution of μ is at most exponential. So no heavy tails measure can satisfy a dimension-free concentration property. The concentration properties of heavy tailed measure are thus particularly interesting to study, and in particular the dimension dependence of the result. The first results in this direction using weak Poincaré inequalities were done in [9]. As converse weighted Poincaré inequalities plus control of the tail of the weight lead to weak Poincaré inequality, and thus concentration, it is interesting to remark that in Theorem 4.1 and Corollary 4.2 in [23], Bobkov and Ledoux proved that if a weighted Poincaré inequality with weight $1 + \eta^2$ holds, then any 1-Lipschitz function with zero mean satisfies

$$\| f \|_p \leq \frac{Dp}{\sqrt{2}} \,\| \sqrt{1+\eta^2} \|_p$$

for all $p \geq 2$. It follows that for all t large enough ($t > Dpe \| \sqrt{1+\eta^2} \|_p$),

$$\mu(|f| > t) \leq 2\left(\frac{D\,p\,\| \sqrt{1+\eta^2} \|_p}{t}\right)^p .$$

Hence the concentration function is controlled by some moment of the weight. Dimension dependence is hidden in this moment control. However if one is only interested in concentration properties, one could use directly weighted Poincaré inequalities. ◊

5 Weighted inequalities and isoperimetry via mass transport

In this section, we present another method to obtain weighted functional inequalities. The idea is to use a change of variable to derive new inequalities from a known functional inequality satisfied by a given reference probability measure. To be more precise, suppose that a probability ν verifies an inequality of the form

$$\Phi_\nu(f) \leq \int \alpha(|\nabla f|)\, d\nu,$$

for all f smooth enough, where Φ_ν is some functional, and α a non-decreasing function. Now suppose that μ is the image of ν under a map T, that is to say that

$$\int f\,d\mu = \int f \circ T\,d\nu,$$

for all f. If the functional J_ν verifies the following invariance property:

$$\Phi_\nu(f \circ T) = \Phi_\mu(f)$$

(which holds in all the situations we shall study below), then μ verifies the inequality

$$\Phi_\mu(f) \leq \int \alpha(|\nabla(f \circ T)|) \circ T^{-1}\,d\mu.$$

We will see that the right hand side can sometimes be bounded from above by a quantity of the form $\int \alpha(|\nabla f|)\omega_T\,d\mu$, where ω_T is some weight function, thus yielding a weighted functional inequality. If the inequality satisfied by ν is sharp then we can hope that the inequality obtained by this method will be rather good. This will be the case in the examples studied below. In particular, we will be able to derive weighted inequalities and isoperimetric results with constants depending explicitly on the dimension.

5.1 Weighted Poincaré inequalities for some spherically symmetric probability measures with heavy tails

In this section we deal with spherically symmetric probability measures $d\mu(x) = h(|x|)dx$ on $\mathbb{R}^n$ with $|\cdot|$ the Euclidean distance. In polar coordinates, the measure μ with density h can be viewed as the distribution of $\xi\theta$, where θ is a random vector uniformly distributed on the unit sphere S^{n-1}, and ξ (the radial part) is a random variable independent of θ with distribution function

$$\mu\left(\{|x| \leq r\}\right) = n\omega_n \int_0^r s^{n-1}h(s)ds\,, \tag{5.1}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. We shall denote by $\rho_\mu(r) = n\omega_n r^{n-1}h(r)$ the density of the distribution of ξ, defined on $\mathbb{R}_+$.

Our aim is to obtain weighted Poincaré inequalities with explicit constants for μ on $\mathbb{R}^n$ of the forms $d\mu(x) = \frac{1}{Z}\frac{1}{(1+|x|)^{(n+\alpha)}}\,dx$ with $\alpha > 0$ or $d\mu(x) = \frac{1}{Z}e^{-|x|^p}\,dx$, with $p \in (0,1)$. To do so we will apply a general radial transportation technique which is explained in the following result.

Recall that the image of ν under a map T is by definition the unique probability measure μ such that

$$\int f\,d\mu = \int f \circ T\,d\nu, \qquad \forall f.$$

In the sequel, $T_\#\mu$ denotes this probability measure.

Theorem 5.2 (Transportation method)**.** *Let μ and ν be two spherically symmetric probability measures on $\mathbb{R}^n$ and suppose that $\mu = T_\#\nu$ with T a radial transformation of the form: $T(x) = \varphi(|x|)\frac{x}{|x|}$, with $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ an increasing function with $\varphi(0) = 0$.*

If ν satisfies Poincaré inequality with constant C, then μ verifies the following weighted Poincaré inequality

$$Var_\mu(f) \leq C \int \omega(|x|)^2|\nabla f|^2\,d\mu(x), \qquad \forall f,$$

with the weight ω defined by

$$\omega(r) = \max\left(\varphi' \circ \varphi^{-1}(r), \frac{r}{\varphi(r)}\right).$$

If ν verifies Cheeger's inequality with constant C, then μ verifies the following weighted Cheeger inequality

$$\int |f - m|\, d\mu \leq C \int \omega(|x|)|\nabla f|(x)\, d\mu(x), \qquad \forall f,$$

with the same weight ω as above and m being a median of f.

Finally, if the function φ is convex, then $\omega(r) = \varphi' \circ \varphi^{-1}(r)$.

Remark 5.3. In [58], Wang has used a similar technique to get weighted logarithmic Sobolev inequalities.

Proof. Consider a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$; it follows from the minimizing property of the variance and the Poincaré inequality verified by ν that

$$\mathrm{Var}_\mu(f) \leq \int \left(f - \int f\, d\nu\right)^2 d\mu = \int \left(f \circ T - \int f d\nu\right)^2 d\nu \leq C \int |\nabla(f \circ T)|^2\, d\nu.$$

In polar coordinates we have

$$\begin{aligned} |\nabla(f \circ T)|^2 &= \left[\frac{\partial}{\partial r}(f \circ T)\right]^2 + \frac{1}{r^2}\left|\nabla_\theta(f \circ T)\right|^2 = \left(\frac{\partial f}{\partial r}\right)^2 \circ T \times \varphi'^2 + \frac{1}{r^2}\left|\nabla_\theta f\right|^2 \\ &= \left(\frac{\partial f}{\partial r}\right)^2 \circ T \times \left(\varphi' \circ \varphi^{-1} \circ \varphi\right)^2 + \frac{1}{(\varphi^{-1} \circ \varphi)^2}\left|\nabla_\theta f\right|^2 \circ T. \end{aligned}$$

Moreover, denoting by $d\theta$ the normalized Lebesgue measure on S^{n-1}, and using the notations introduced in the beginning of the section, the previous inequality reads

$$\begin{aligned} \mathrm{Var}_\mu(f) &\leq C \iint \left(\left(\frac{\partial f}{\partial r}\right)^2 \circ T \times \left(\varphi' \circ \varphi^{-1} \circ \varphi\right)^2 + \frac{1}{(\varphi^{-1} \circ \varphi)^2}\left|\nabla_\theta f\right|^2 \circ T\right)\rho_\nu(r) dr d\theta \\ &= C \iint \left(\left(\frac{\partial f}{\partial r}\right)^2 \times \left(\varphi' \circ \varphi^{-1}\right)^2 + \frac{1}{(\varphi^{-1})^2}\left|\nabla_\theta f\right|^2\right)\rho_\mu(r) dr d\theta \\ &\leq C \iint \omega^2(r)\left(\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left|\nabla_\theta f\right|^2\right)\rho_\mu(r) dr d\theta \\ &= C \int \omega^2(|x|)|\nabla f|^2 d\mu \end{aligned}$$

where we used the fact that the map φ transports $\rho_\nu\, dr$ onto $\rho_\mu\, dr$. The proof of the Cheeger case follows exactly in the same way.

Now, let us suppose that φ is convex. Since φ is convex and $\varphi(0) = 0$, one has $\frac{\varphi(r)}{r} \leq \varphi'(r)$. This implies at once that $\omega(r) = \varphi' \circ \varphi^{-1}$ and achieves the proof. □

To apply Theorem 5.2, one needs a criterion for Poincaré inequality. The following theorem is a slight adaptation of a result by Bobkov [18, Theorem 1].

Theorem 5.4. *Let $d\nu(x) = h(|x|)\,dx$ be a spherically symmetric probability measure on $\mathbb{R}^n$. Define as before ρ_ν as the density of the law of $|X|$ where X is distributed according to ν and suppose that ρ_ν is a log-concave function. Then ν verifies the following Poincaré inequality*

$$Var_\nu(f) \le C_\nu \int |\nabla f|^2\, d\nu, \qquad \forall f$$

with $C_\nu = 12\left(\int r^2\rho_\nu(r)\,dr - \left(\int r\rho_\nu(r)\,dr\right)^2\right) + \frac{1}{n}\int r^2\rho_\nu(r)\,dr.$

Proof. We refer to [18]. □

Proposition 5.5 (Generalized Cauchy distributions)**.** *The probability measure $d\mu(x) = \frac{1}{Z}\frac{dx}{(1+|x|)^{(n+\alpha)}}$ on $\mathbb{R}^n$ with $\alpha > 0$ verifies the weighted Poincaré inequality*

$$Var_\mu(f) \le C_{opt} \int (1+|x|)^2\, |\nabla f|^2\, d\mu(x), \qquad \forall f.$$

where the optimal constant C_{opt} is such that

$$\sum_{k=0}^{n-1} \frac{1}{(\alpha+k)^2} \le C_{opt} \le 14 \sum_{k=0}^{n-1} \frac{1}{(\alpha+k)^2}.$$

Remark 5.6. Note that, comparing to integrals, we have

$$\frac{1}{\alpha^2} + \frac{n-1}{(\alpha+1)(\alpha+n)} \le \sum_{k=0}^{n-1} \frac{1}{(\alpha+k)^2} \le \frac{1}{\alpha^2} + \frac{n-1}{\alpha(\alpha+n-1)}.$$

Since $\alpha^2 \sum_{k=0}^{n-1} \frac{1}{(\alpha+k)^2} \to n$ when $\alpha \to \infty$, applying the previous weighted Poincaré inequality to $g(\alpha x)$, making a change of variables, and letting α tend to infinity lead to

$$\mathrm{Var}_\nu(f) \le 14n \int |\nabla f|^2\, d\nu$$

with $d\nu(x) = (1/Z)e^{-|x|}dx$. Moreover, the optimal constant in the latter is certainly greater than n. This recover (with 14 instead of 13) one particular result of Bobkov [18].

Proof. Define $\psi(r) = \ln(1+r)$, $r > 0$ and let ν be the image of μ under the radial map $S(x) = \psi(|x|)\frac{x}{|x|}$. Conversely, one has evidently that μ is the image of ν under the radial map $T(x) = \varphi(|x|)\frac{x}{|x|}$, with $\varphi(r) = \psi^{-1}(r) = e^r - 1$ (which is convex). To apply Theorem 5.2, one has to check that ν verifies Poincaré inequality.

Elementary computations yield

$$\frac{d\nu}{dx}(x) = \frac{1}{Z}\left(\frac{e^{|x|}-1}{|x|}\right)^{n-1} e^{(1-n-\alpha)|x|} \qquad \text{and} \qquad \rho_\nu(r) = \frac{n\omega_n}{Z}\left(1-e^{-r}\right)^{n-1} e^{-\alpha r}$$

It is clear that $\log\rho_\nu$ is concave. So we may apply Theorem 5.4 and conclude that ν verifies Poincaré inequality with the constant C_ν defined above.

Define

$$H(\alpha) = \int_0^{+\infty} e^{-\alpha r}(1-e^{-r})^{n-1}\,dr = \int_0^1 u^{\alpha-1}(1-u)^{n-1}\,du.$$

Then $\int r\rho_\nu(r)\,dr = -\frac{H'(\alpha)}{H(\alpha)}$ and $\int r^2\rho_\nu(r)\,dr = \frac{H''(\alpha)}{H(\alpha)}$. Integrations by parts yield

$$H(\alpha) = \frac{(n-1)!}{(\alpha+n-1)(\alpha+n-2)\cdots(\alpha)}.$$

So,

$$H'(\alpha) = -H(\alpha)\sum_{k=0}^{n-1}\frac{1}{\alpha+k} \quad \text{and} \quad H''(\alpha) = H(\alpha)\left[\left(\sum_{k=0}^{n-1}\frac{1}{\alpha+k}\right)^2 + \sum_{k=0}^{n-1}\frac{1}{(\alpha+k)^2}\right].$$

This gives, using Cauchy-Schwarz inequality

$$C_\nu = 13\sum_{k=0}^{n-1}\frac{1}{(\alpha+k)^2} + \frac{1}{n}\left(\sum_{k=0}^{n-1}\frac{1}{\alpha+k}\right)^2 \le 14\sum_{k=0}^{n-1}\frac{1}{(\alpha+k)^2}.$$

Now, suppose that there is some constant C such that the inequality $\mathrm{Var}_\mu(f) \le C\int(1+|x|)^2|\nabla f|^2\,d\mu$ holds for all f. We want to prove that $C \ge \sum_{k=0}^{n-1}\frac{1}{(\alpha+k)^2}$. To do so let us test this inequality on the functions $f_a(x) = \frac{1}{(1+|x|)^a}$, $a > 0$. Defining $F(r) = \int \frac{1}{(1+|x|)^{n+r}}\,dx$, for all $r > 0$, one obtains immediately

$$C \ge \frac{1}{a^2}\frac{F(2a+\alpha)F(\alpha) - F(a+\alpha)^2}{F(\alpha)F(2a+\alpha)}.$$

But a Taylor expansion easily shows that the right hand side goes to $K = \frac{F''(\alpha)}{F(\alpha)} - \left(\frac{F'(\alpha)}{F(\alpha)}\right)^2$, so $C \ge K$. Easy computations give that $F(\alpha) = n\omega_n H(\alpha)$ and so $K = \sum_{k=0}^{n-1}\frac{1}{(\alpha+k)^2}$. □

Proposition 5.7 (Sub-exponential laws)**.** *The probability measure $d\mu(x) = \frac{1}{Z}e^{-|x|^p}\,dx$ on $\mathbb{R}^n$ with $p \in (0,1)$ verifies the weighted Poincaré inequality*

$$Var_\mu(f) \le C_{opt}\int |\nabla f|^2|x|^{2(1-p)}\,d\mu(x),$$

where the optimal constant C_{opt} is such that

$$\frac{n}{p^3} \le C_{opt} \le 12\frac{n}{p^3} + \frac{n+p}{p^4}.$$

Remark 5.8. As for the Cauchy law, letting p go to 1 leads to

$$\mathrm{Var}_\nu(f) \le (13n+1)\int |\nabla f|^2\,d\nu$$

with $d\nu(x) = (1/Z)e^{-|x|}dx$. Again this recover (with $13n+1$ instead of $13n$) one particular result of Bobkov [18].

Proof. We mimic the proof of the preceding example. Let $\psi(r) = \frac{1}{p} r^p$, $r \geq 0$ and define ν as the image of μ under the radial map $S(x) = \psi(|x|)\frac{x}{|x|}$. Easy calculations give that the radial part of ν has density ρ_ν defined by

$$\rho_\nu(r) = \frac{n\omega_n}{Z}\,(\beta u)^{\frac{n-p}{p}}\, e^{-pu}.$$

It is clearly a log-concave function on $[0, +\infty)$. Let us compute the constant C_ν appearing in Theorem 5.4. One has

$$\int r\rho_\nu(r)\,dr = \frac{1}{p}\frac{\Gamma(\frac{n}{p}+1)}{\Gamma(\frac{n}{p})} = \frac{n}{p^2},$$

and

$$\int r^2\rho_\nu(r)\,dr = \frac{1}{p^2}\frac{\Gamma(\frac{n}{p}+2)}{\Gamma(\frac{n}{p})} = \frac{n(n+p)}{p^4}.$$

Consequently,

$$C_\nu = 12\frac{n}{p^3} + \frac{n+p}{p^4}.$$

Now suppose that there is some C such that $\mathrm{Var}_\mu(f) \leq C\int |\nabla f|^2|x|^{2(1-p)}\,d\mu(x)$ holds for all f. To prove that $C \geq \frac{n}{p^3}$, we will test this inequality on the functions $f_a(x) = e^{-a|x|^p}$, $a > 0$. Letting $G(t) = \int e^{-t|x|^p}\,d\mu(x)$, we arrive at the relation

$$C \geq \frac{1}{\beta^2 a^2}\frac{G(1)G(2a+1) - G(a+1)^2}{G(1)G(2a+1)}, \qquad \forall a > 0.$$

Letting $a \to 0$, one obtains $C \geq \frac{1}{p^2}\left[\frac{G''(1)}{G(1)} - \left(\frac{G'(1)}{G(1)}\right)^2\right]$. The change of variable formula immediately yields $G(t) = t^{-\frac{n}{p}}G(1)$, and so $C \geq \frac{1}{p^2}\left[\frac{n(n+p)}{p^2} - \left(\frac{n}{p}\right)^2\right]$, which achieves the proof. □

5.2 Weak Cheeger inequalities via mass transport

The aim of this section is to study how the isoperimetric inequality, or equivalently the weak Cheeger inequality, behave under tensor products. More precisely, we shall start with a probability measure μ on the real line $\mathbb{R}$ and derive weak Cheeger inequalities for μ^n with explicit constants.

We need some notations. For any probability measure μ (on $\mathbb{R}$) we denote by F_μ the cumulative distribution function of μ which is defined by

$$F_\mu(x) = \mu(-\infty, x], \quad \forall x \in \mathbb{R}.$$

It will be also convenient to consider the tail distribution function $\overline{F}_\mu$ defined by

$$\overline{F}_\mu(x) = 1 - F_\mu(x) = \mu(x, +\infty), \quad \forall x \in \mathbb{R}.$$

The isoperimetric function of μ is defined by

$$J_\mu = F'_\mu \circ F_\mu^{-1}. \tag{5.9}$$

In all the sequel, the two sided exponential measure $d\nu(x) = \frac{1}{2}e^{-|x|}\,dx$, $x \in \mathbb{R}$ will play the role of a reference probability measure. We will set $F_\nu = F$ and $J_\nu = J$ for simplicity. Note that the isoperimetric function J can be explicitly computed: $J(t) = \min(t, 1-t)$, $t \in [0, 1]$.

5.2.1 A general result

We are going to derive a weak Cheeger inequality starting from a well known Cheeger inequality for ν^n obtained in [20] and using a transportation idea developed in [35]. Our result will be available for a special class of probability measures on $\mathbb{R}$ which is described in the following lemma.

Lemma 5.10. *Let μ be a symmetric probability measure on $\mathbb{R}$; the following propositions are equivalent*

1. *The function $\log \overline{F}_\mu$ is convex on $\mathbb{R}^+$,*
2. *The function J/J_μ is non increasing on $(0, 1/2]$ and non decreasing on $[1/2, 1)$.*

Furthermore, if $d\mu(x) = e^{-\Phi(|x|)}\,dx$ with $\Phi : \mathbb{R}^+ \to \mathbb{R}$ concave, then $\log \overline{F}_\mu$ is convex on $\mathbb{R}^+$.

Proof. The equivalence between (1) and (2) is easy to check. Now suppose that μ is of the form $d\mu(x) = e^{-\Phi(|x|)}\,dx$ with a concave Φ. Then for $r \in \mathbb{R}^+$,

$$(\log \overline{F}_\mu)''(r) = \frac{e^{-\Phi(r)}}{\left(\int_r^\infty e^{-\Phi(s)}\,ds\right)^2}\left(\Phi'(r)\int_r^\infty e^{-\Phi(s)}\,ds - e^{-\Phi(r)}\right)$$

where Φ' is the right derivative. Since Φ is concave, Φ' is non-increasing. It follows that

$$\Phi'(r)\int_r^\infty e^{-\Phi(s)}\,ds \geq \int_r^\infty \Phi'(s)e^{-\Phi(s)}\,ds = e^{-\Phi(r)}.$$

The result follows. □

Recall that distributions satisfying (1) in the previous lemma are known as "Decreasing Hazard Rate" distributions. We refer to [6] for some very interesting properties of these distributions.

Using a mass transportation technique, we are now able to derive a weak Cheeger inequality for product measures on $\mathbb{R}^n$. Dimension dependence is explicit, as well as the constants.

Theorem 5.11. *Let μ be a symmetric probability measure on $\mathbb{R}$ absolutely continuous with respect to the Lebesgue measure. Assume that $\log \overline{F}_\mu$ is convex on $\mathbb{R}^+$.*
Then, for any n, any bounded smooth function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies

$$\int |f - m|\,d\mu^n \leq \kappa_1 \frac{s}{J_\mu(s)} \int |\nabla f|\,d\mu^n + \kappa_2 ns\mathrm{Osc}(f), \quad \forall s \in (0, 1/2), \tag{5.12}$$

where m is a median of f under μ^n, $\kappa_1 = 2\sqrt{6}$ and $\kappa_2 = 2(1 + 2\sqrt{6})$.

Remark 5.13. Note that $\int |f - m| d\mu^n \leq \frac{1}{2}\mathrm{Osc}(f)$. Hence only the values $s \leq (2\kappa_2 n)^{-1}$ are of interest in (5.12). ◇

Proof. Recall that ν is the two sided exponential distribution. Fix the dimension n and $r > 0$. By [20, Inequality (6.9)], any locally Lipschitz function $h : \mathbb{R}^n \to \mathbb{R}$ with $\int |h|\,d\nu^n < \infty$ satisfies

$$\int |h - m_{\nu^n}(h)|\,d\nu^n \leq \kappa_1 \int |\nabla h|\,d\nu^n \tag{5.14}$$

where $m_{\nu^n}(h)$ is a median of h for ν^n and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$.

Consider the map $T^n : \mathbb{R}^n \to \mathbb{R}^n$, that pushes forward ν^n onto μ^n, defined by $(x_1, \ldots, x_n) \mapsto (T(x_1), \ldots, T(x_n))$ with $T = F_\mu^{-1} \circ F$. By construction, any $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $\int f(T^n)\, d\nu^n = \int f\, d\mu^n$.

Next, for $t \geq 0$ let $B(t) = \{x = (x_1, \ldots, x_n) : \max_i |x_i| \leq t\}$. Fix $a > 0$ that will be chosen later and consider $g : \mathbb{R} \to [0,1]$ defined by $g(x) = \left(1 - \frac{1}{a}(x-r)_+\right)_+$ with $X_+ = \max(X, 0)$. Set $\varphi(x) = g(\max_i(|x_i|))$, $x \in \mathbb{R}^n$. The function φ is locally Lipschitz.

Finally let $f : \mathbb{R}^n \to \mathbb{R}$ be smooth and bounded. We assume first that 0 is a μ^n-median of f. Furthermore, by homogeneity of (5.12) we may assume that $\mathrm{Osc}(f) = 1$ in such a way that $\|f\|_\infty \leq 1$. It follows from the definition of the median that

$$\begin{aligned}\int |f|\, d\mu^n &\leq \int |f - m_{\nu^n}((f\varphi)(T^n))|\, d\mu^n \\ &\leq \int |f\varphi - m_{\nu^n}((f\varphi)(T^n))|\, d\mu^n + \int |f(1-\varphi)|\, d\mu^n \\ &\leq \int |f\varphi - m_{\nu^n}((f\varphi)(T^n))|\, d\mu^n + \mu^n\left(B(r)^c\right).\end{aligned}$$

Note that the assumption on $\log \overline{F}_\mu$ guarantees that $T' \circ T^{-1}$ is non-decreasing on $\mathbb{R}^+$. Hence, using (5.14), the triangle inequality in $\ell^2(\mathbb{R}^n)$, the fact that $0 \leq \varphi \leq 1$ on $\mathbb{R}^n$ and $\varphi = \partial_i \varphi = 0$ on $B(r+a)^c$ imply that

$$\begin{aligned}&\int |f\varphi - m_{\nu^n}((f\varphi)(T^n))|\, d\mu^n = \int |(f\varphi)(T^n) - m_{\nu^n}((f\varphi)(T^n))|\, d\nu^n \\ &\leq \kappa_1 \int \sqrt{\sum_{i=1}^n T'(x_i)^2 \left((\varphi\partial_i f)(T^n) + (f\partial_i\varphi)(T^n)\right)^2}\, d\nu^n \\ &= \kappa_1 \int \sqrt{\sum_{i=1}^n T' \circ T^{-1}(x_i)^2 \left(\varphi\partial_i f + f\partial_i\varphi\right)^2}\, d\mu^n \\ &\leq \kappa_1 \int \sqrt{\sum_{i=1}^n T' \circ T^{-1}(x_i)^2 \left(\varphi\partial_i f\right)^2}\, d\mu^n + \kappa_1 \int \sqrt{\sum_{i=1}^n T' \circ T^{-1}(x_i)^2 \left(f\partial_i\varphi\right)^2}\, d\mu^n \\ &\leq \kappa_1 T' \circ T^{-1}(r+a)\left(\int |\nabla f|\, d\mu^n + \int |\nabla\varphi|\, d\mu^n\right).\end{aligned}$$

Note that $|\nabla\varphi| \leq 1/a$ on $B(r+a) \setminus B(r)$ and $|\nabla\varphi| = 0$ elsewhere μ^n-almost surely. Hence,

$$\int |f|\, d\mu^n \leq \kappa_1 T' \circ T^{-1}(r+a)\left(\int |\nabla f| d\mu^n + \frac{1}{a}\mu^n\left(B(r+a) \setminus B(r)\right)\right) + \mu^n\left(B(r)^c\right). \quad (5.15)$$

Since μ is symmetric, we have

$$G(t) := \mu^n\left(B(t)\right) = \left(1 - 2\overline{F}_\mu(t)\right)^n.$$

Hence,

$$\begin{aligned}\lim_{a\to 0} \frac{1}{a}\mu^n\left(B(r+a) \setminus B(r)\right) = G'(r) &= 2nF'_\mu(r)\left(1 - 2\overline{F}_\mu(t)\right)^{n-1} \\ &\leq 2nF'_\mu(r).\end{aligned}$$

On the other hand, since the function $x \mapsto 1-(1-2x)^n$ is concave on $[0,1/2]$, one has: $1-(1-2x)^n \le 2nx$ for all $x \in [0,1/2]$. As a consequence,

$$\mu^n(B(r)^c) = 1 - G(r) = 1 - (1 - 2\overline{F}_\mu(r))^n \le 2n\overline{F}_\mu(r),$$

for all $r \ge 0$.

Letting a go to 0 in (5.15) leads to

$$\int |f|\, d\mu^n \le \kappa_1 T' \circ T^{-1}(r) \int |\nabla f|\, d\mu^n + 2n\kappa_1 T' \circ T^{-1}(r) F'_\mu(r) + 2n\overline{F}_\mu(r).$$

Note that $T' \circ T^{-1} = J \circ F_\mu / F'_\mu = \min(F_\mu, 1 - F_\mu)/F'_\mu$. Hence, for $r \ge 0$,

$$T' \circ T^{-1}(r) F'_\mu(r) = \frac{1 - F_\mu(r)}{F'_\mu(r)} F'_\mu(r) = \overline{F}_\mu(r)\,.$$

It follows that

$$\int |f|\, d\mu^n \le \kappa_1 \frac{\overline{F}_\mu(r)}{F'_\mu(r)} \int |\nabla f|\, d\mu^n + n\kappa_2 \overline{F}_\mu(r),$$

for all $r \ge 0$. Using the symmetry of μ it is easy to see that $F'_\mu \circ \overline{F}_\mu^{-1}(t) = J_\mu(t)$ for all $t \in (0,1/2)$. Consequently, one has

$$\int |f|\, d\mu^n \le \kappa_1 \frac{s}{J_\mu(s)} \int |\nabla f|\, d\mu^n + \kappa_2 n s,$$

for all $s \in (0,1/2)$. For general $f : \mathbb{R}^n \to \mathbb{R}$ with μ^n-median m, we apply the result to $f - m$. This ends the proof. □

Combining this theorem with Bobkov's Lemma 4.1 we immediately deduce

Corollary 5.16. *Let μ be a symmetric probability measure on $\mathbb{R}$ absolutely continuous with respect to the Lebesgue measure. Assume that $\log \overline{F}_\mu$ is convex on $\mathbb{R}^+$. Then, for any n, any Borel set $A \subset \mathbb{R}^n$ satisfies*

$$(\mu^n)_s(\partial A) \ge \frac{n\kappa_2}{\kappa_1} J_\mu\left(\frac{\min(\mu^n(A), 1 - \mu^n(A))}{2n\kappa_2}\right). \tag{5.17}$$

Proof. According to Lemma 4.1, if $\mu(A) \le 1/2$ (the other case is symmetric), $(\mu^n)_s(\partial A) \ge I(\mu^n(A))$ with $I(t) = \sup_{0<s\le t} \frac{t-s}{\beta(s)}$, for $t \le 1/2$, where according to the previous theorem

$$\beta(s) = \frac{\kappa_1}{n\,\kappa_2} \frac{s}{J_\mu(s/n\kappa_2)}\,,$$

for $s \le n\kappa_2/2$ hence for $s \le 1/2$. This yields

$$I(t) = \sup_{0<s\le t} \frac{t-s}{\kappa_1} \frac{J_\mu(s/n\kappa_2)}{(s/n\kappa_2)}\,.$$

In order to estimate I we use the following: first a lower bound is obtained for $s = t/2$ yielding the statement of the corollary. But next according to Lemma 5.10, the slope function $J_\mu(v)/v$ is non-decreasing, so that

$$I(t) \le \sup_{0<s\le t} \frac{t-s}{\kappa_1} \frac{J_\mu(t/n\kappa_2)}{(t/n\kappa_2)} \le \frac{n\kappa_2}{\kappa_1} J_\mu(t/n\kappa_2)\,.$$

Remark that we have shown that for $t \le 1/2$

$$\frac{n\kappa_2}{\kappa_1} J_\mu(t/2n\kappa_2) \le I(t) \le \frac{n\kappa_2}{\kappa_1} J_\mu(t/n\kappa_2), \tag{5.18}$$

so that up to a factor 2 our estimate is of optimal order. □

5.2.2 Application: Isoperimetric profile for product measures with heavy tails

Here we apply the previous results to product of the measures

$$d\mu(x) = \mu_\Phi(dx) = Z_\Phi^{-1} \exp\{-\Phi(|x|)\}dx, \tag{5.19}$$

$x \in \mathbb{R}$, with Φ concave.

For even measures on $\mathbb{R}$ with positive density on a segment, Bobkov and Houdré [21, Corollary 13.10] proved that solutions to the isoperimetric problem can be found among half-lines, symmetric segments and their complements. More precisely, one has for $t \in (0,1)$

$$I_\mu(t) = \min\left(J_\mu(t), 2J_\mu\Big(\frac{\min(t,1-t)}{2}\Big)\right). \tag{5.20}$$

Under few assumptions on Φ, I_μ has the same order as $t \to 0$ of the function

$$L_\Phi(t) = \min(t, 1-t)\Phi' \circ \Phi^{-1}\left(\log \frac{1}{\min(t,1-t)}\right),$$

where Φ' denotes the right derivative. More precisely,

Proposition 5.21. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a non-decreasing concave function satisfying $\Phi(x)/x \to 0$ as $x \to \infty$. Assume that in a neighborhood of $+\infty$ the function Φ is C^2 and there exists $\theta > 1$ such that Φ^θ is convex. Let μ_Φ be defined in (5.19). Define F_μ and J_μ as in (5.9).*

Then,

$$\lim_{t\to 0} \frac{J_\mu(t)}{t\Phi' \circ \Phi^{-1}(\log \frac{1}{t})} = 1.$$

Consequently, if $\Phi(0) < \log 2$, L_Φ is defined on $[0,1]$ and there exist constants $k_1, k_2 > 0$ such that for all $t \in [0,1]$,

$$k_1 L_\Phi(t) \le J_\mu(t) \le k_2 L_\Phi(t).$$

Remark 5.22. This result appears in [7; 24] in the particular case $\Phi(x) = |x|^p$ and in [11] for Φ convex and $\sqrt{\Phi}$ concave. ◊

The previous results together with Corollary 5.16 lead to the following (dimensional) isoperimetric inequality.

Corollary 5.23. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a non-decreasing concave function satisfying $\Phi(x)/x \to 0$ as $x \to \infty$ and $\Phi(0) < \log 2$. Assume that in a neighborhood of $+\infty$ the function Φ is C^2 and there exists $\theta > 1$ such that Φ^θ is convex. Let $d\mu(x) = Z_\Phi^{-1} e^{-\Phi(|x|)}dx$ be a probability measure on $\mathbb{R}$. Then,*

$$I_{\mu^n}(t) \ge c \min(t, 1-t)\Phi' \circ \Phi^{-1}\left(\log \frac{n}{\min(t,1-t)}\right) \qquad \forall t \in [0,1],\ \forall n$$

for some constant $c > 0$ independent on n.

Remark 5.24. Note that there is a gain of a square root with respect to the results in [9]. ◊

For the clarity of the exposition, the rather technical proofs of Proposition 5.21 and Corollary 5.23 are postponed to the Appendix.

We end this section with two examples.

Proposition 5.25 (Sub-exponential law)**.** *Consider the probability measure μ on $\mathbb{R}$, with density $Z_p^{-1}e^{-|x|^p}$, $p \in (0,1]$. There is a constant c depending only on p such that for all $n \geq 1$ and all $A \subset \mathbb{R}^n$,*

$$\mu_s^n(\partial A) \geq c \min(\mu^n(A), 1-\mu^n(A)) \log\left(\frac{n}{\min(\mu^n(A), 1-\mu^n(A))}\right)^{1-\frac{1}{\beta}}.$$

Proof. The proof follows immediately from Corollary 5.23. □

Remark 5.26. Let $I_{\mu^n}(t)$ be the isoperimetric profile of μ^n. The preceding bound combined with the upper bound of [9, Inequality (4.10)] gives

$$c(p)\, t\left(\log\left(\frac{n}{t}\right)\right)^{1-1/p} \leq I_{\mu^n}(t) \leq c'(p) t \log(1/t)\left(\log\left(\frac{n}{\log(1/t)}\right)\right)^{1-1/p}$$

for any $n \geq \log(1/t)/\log 2$ and $t \in (0,1/2)$. Hence, we obtain the right logarithmic behavior of the isoperimetric profile in term of the dimension n. This result completes Proposition 4.5 obtained in Section 4 for this class of examples. ◊

More generally consider the probability measure $\mu = Z^{-1}e^{-|x|^p \log(\gamma+|x|)^\alpha}$, $p \in (0,1]$, $\alpha \in \mathbb{R}$ and $\gamma = \exp\{2|\alpha|/(p(1-p))\}$ chosen in such a way that $\Phi(x) = |x|^p \log(\gamma+|x|)^\alpha$ is concave on $\mathbb{R}^+$. The assumptions of Corollary 5.23 are satisfied. Hence, we get that

$$I_{\mu^n}(t) \geq c(p,\alpha) t\left(\log\left(\frac{n}{t}\right)\right)^{1-1/p}\left(\log\log\left(e+\frac{n}{t}\right)\right)^{\frac{\alpha}{p}}, \qquad t \in (0,1/2).$$

Cauchy laws do not enter the framework of Corollary 5.23. Nevertheless, explicit computations can be done.

Proposition 5.27 (Cauchy distributions)**.** *Consider $d\mu(x) = \frac{\alpha}{2(1+|x|)^{1+\alpha}}\, dx$ on $\mathbb{R}$, with $\alpha > 0$. There is $c > 0$ depending only on α such that for all $n \geq 1$ and all $A \subset \mathbb{R}^n$,*

$$\mu_s^n(\partial A) \geq c \frac{\min(\mu^n(A), 1-\mu^n(A))^{1+\frac{1}{\alpha}}}{n^{\frac{1}{\alpha}}}.$$

Proof. Since $1 - F_\mu(r) = \frac{1}{2(1+r)^\alpha}$ for $r \in \mathbb{R}^+$, $\log(1-F_\mu)$ is convex on $\mathbb{R}^+$. Moreover $J_\mu(t) = \alpha 2^{1/\alpha}\min(t,1-t)^{1+1/\alpha}$, and so the result follows by Corollary 5.16. □

Remark 5.28. Note that, since $J_\mu(t) = \alpha 2^{1/\alpha}\min(t,1-t)^{1+1/\alpha}$, one has

$$I_\mu(t) = \alpha t^{1+1/\alpha}, \qquad \forall t \in (0,1/2).$$

Hence, our results reads as

$$I_{\mu^n}(t) \geq c\, \frac{t}{n^{1/\alpha}} t^{1/\alpha}$$

for some constant c depending only on α. Together with [9, Inequality (4.9)] (for the upper bound) our results gives for any $n \geq \log(1/t)/\log 2$ and $t \in (0, 1/2)$

$$c\,\frac{t}{n^{1/\alpha}}t^{1/\alpha} \leq I_{\mu^n}(t) \leq c'\,\frac{t}{n^{1/\alpha}}\log(1/t)^{1+1/\alpha}.$$

Again, the polynomial behavior in the dimension n is of optimal order. ◊

5.2.3 Applications of transport to weak Poincaré inequalities

By Lemma 4.7 above, we see that weak Poincaré inequalities can be derived from mass-transport arguments using Theorem 5.11. This is stated in the next corollary.

Corollary 5.29. *Let μ be a symmetric probability measure on $\mathbb{R}$ absolutely continuous with respect to the Lebesgue measure. Assume that* $\log \overline{F}_\mu$ *is convex on $\mathbb{R}^+$. Then, for any n, every function $f : \mathbb{R}^n \to \mathbb{R}$ smooth enough satisfies*

$$Var_{\mu^n}(f) \leq \kappa_1^2 \frac{s^2}{J_\mu(s/2)^2} \int |\nabla f|^2\, d\mu^n + 2\kappa_2 n s \mathrm{Osc}(f)^2, \qquad \forall s > 0. \tag{5.30}$$

with $\kappa_1 = 2\sqrt{6}$ and $\kappa_2 = 2(1 + 2\sqrt{6})$.

Proof. Applying Lemma 4.7 to μ^n together with Theorem 5.11 immediately yields the result. □

We illustrate this corollary on two examples.

Proposition 5.31 (Cauchy distributions)**.** *Consider $d\mu(x) = \frac{\alpha}{2(1+|x|)^{1+\alpha}}\,dx$ on $\mathbb{R}$, with $\alpha > 0$. Then, there is a constant C depending only on α such that for all $n \geq 1$*

$$Var_{\mu^n}(f) \leq C\left(\frac{n}{s}\right)^{\frac{2}{\alpha}} \int |\nabla f|^2\, d\mu^n + s\mathrm{Osc}_{\mu^n}(f)^2, \qquad \forall s \in (0, 1/4).$$

Proof. Since $J_{m_\alpha}(t) = \alpha 2^{1/\alpha} t^{1+1/\alpha}$ for $t \in (0, 1/2)$, by Corollary 5.29, on $\mathbb{R}^n$, μ^n satisfies a weak Poincaré inequality with rate function $\beta(s) = C\left(\frac{n}{s}\right)^{\frac{2}{\alpha}}$, $s \in (0, \frac{1}{4})$. □

Proposition 5.32 (Sub-exponential law)**.** *Consider the probability measure μ on $\mathbb{R}$, with density $Z^{-1}e^{-|x|^p}$,* $p \in (0, 1]$. *Then, there is a constant C depending only on p such that for all $n \geq 1$*

$$Var_{\mu^n}(f) \leq C\left(\log\left(\frac{n}{s}\right)\right)^{2(\frac{1}{p}-1)} \int |\nabla f|^2\, d\mu^n + s\mathrm{Osc}_{\mu^n}(f)^2, \qquad \forall s \in (0, 1/4).$$

Proof. By Corollary 5.23, $J_\mu(t)$ is, up to a constant, greater than or equal to $t\,(\log(1/t))^{1-\frac{1}{p}}$ for $t \in [0, 1/2]$. Hence, by Corollary 5.29, μ^n satisfies a weak Poincaré inequality on $\mathbb{R}^n$, with the rate function $\beta(s) = C\left(\log\left(\frac{n}{s}\right)\right)^{2(\frac{1}{p}-1)}$, $s \in (0, \frac{1}{4})$. □

Remark 5.33. The two previous results recover the results of [9]. Note the difference between the results of Proposition 4.11 (applied to $V(x) = |x|$) and Proposition 5.32. This is mainly due to the fact that Proposition 4.11 holds in great generality, while Proposition 5.32 deals with a very specific distribution. The same remark applies to Propositions 4.9 and 5.31 since in the setting of Proposition 5.31, $2/\alpha = -2\kappa$.

However, it is possible to recover the results of Proposition 5.32 (resp. Propositions 5.31) applying Proposition 4.11 (resp. Propositions 4.9) to the sub-exponential (resp. Cauchy) measure on $\mathbb{R}$ and then to use the tensorization property [9, Theorem 3.1]. $\diamond$

6 Appendix

This appendix is devoted to the proofs of Proposition 5.21 and Corollary 5.23. Let us recall the first of these statements.

Proposition. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be a non-decreasing concave function satisfying $\Phi(x)/x \to 0$ as $x \to \infty$. Assume that in a neighborhood of $+\infty$ the function Φ is C^2 and there exists $\theta > 1$ such that Φ^θ is convex. Let μ_Φ be defined in (5.19). Define F_μ and J_μ as in (5.9).*

Then,

$$\lim_{t\to 0} \frac{J_\mu(t)}{t\Phi' \circ \Phi^{-1}(\log \frac{1}{t})} = 1.$$

Proof of Proposition 5.21. The proof follows the line of [11, Proposition 13]. By Point (*iii*) of Lemma 6.2 below, Φ' never vanishes. Under our assumptions on Φ we have $F_\mu(y) = \int_{-\infty}^{y} Z_\Phi^{-1} e^{-\Phi(|x|)} dx \sim Z_\Phi^{-1} e^{-\Phi(|y|)}/\Phi'(|y|)$ when y tends to $-\infty$. Thus using the change of variable $y = F_\mu^{-1}(t)$, we get

$$\begin{aligned}\lim_{t\to 0} \frac{J_\mu(t)}{t\Phi' \circ \Phi^{-1}(\log \frac{1}{t})} &= \lim_{y\to-\infty} \frac{e^{-\Phi(|y|)}}{Z_\Phi F_\mu(y)\Phi' \circ \Phi^{-1}(\log \frac{1}{F_\mu(y)})} \\ &= \lim_{y\to-\infty} \frac{\Phi'(|y|)}{\Phi' \circ \Phi^{-1}(\log \frac{1}{F_\mu(y)})}.\end{aligned}$$

By concavity of Φ we have $F_\mu(y) \geq Z_\Phi^{-1} e^{-\Phi(|y|)}/\Phi'(|y|)$ for all $y \leq 0$. Hence, since $\lim_{x\to+\infty} \Phi'(x) = 0$, we have $\log \frac{1}{F_\mu(y)} \leq \Phi(|y|)$ when $y \ll -1$.

Then, the Mean Value theorem applied to $\Phi' \circ \Phi^{-1}$ between $\log \frac{1}{F_\mu(y)}$ and $\Phi(|y|)$ gives

$$\frac{\Phi' \circ \Phi^{-1}(\log \frac{1}{F_\mu(y)})}{\Phi'(|y|)} = 1 + \frac{1}{\Phi'(|y|)} \left(\log \frac{1}{F_\mu(y)} - \Phi(|y|)\right) \frac{\Phi'' \circ \Phi^{-1}(c_y)}{\Phi' \circ \Phi^{-1}(c_y)}$$

for some $c_y \in [\log \frac{1}{F_\mu(y)}, \infty)$.

For $y \ll -1$, we have

$$\frac{e^{-\Phi(|y|)}}{Z_\Phi \Phi'(|y|)} \leq F_\mu(y) \leq 2 \frac{e^{-\Phi(|y|)}}{Z_\Phi \Phi'(|y|)}. \tag{6.1}$$

Hence, using Point (*iii*) of Lemma 6.2 below,

$$\begin{aligned}\left|\log \frac{1}{F_\mu(y)} - \Phi(|y|)\right| &= \Phi(|y|) - \log \frac{1}{F_\mu(y)} \\ &\leq \log \frac{2}{Z_\Phi} + \log\left(\frac{1}{\Phi'(|y|)}\right) \\ &\leq \log \frac{2}{Z_\Phi} + c\log(|y|)\end{aligned}$$

for some constant c and all $y \ll -1$.

On the other hand, when Φ^θ is convex and C^2, $(\Phi^\theta)''$ is non negative. This, together with Point (*i*) of Lemma 6.2, lead to

$$\left|\frac{\Phi''(x)}{\Phi'(x)}\right| = -\frac{\Phi''(x)}{\Phi'(x)} \leq (\theta - 1)\frac{\Phi'(x)}{\Phi(x)} \leq \frac{c'}{x}$$

for some constant c' and $x \gg 1$. It follows that

$$\frac{\Phi'' \circ \Phi^{-1}(c_y)}{\Phi' \circ \Phi^{-1}(c_y)} \leq \frac{c'}{\Phi^{-1}\left(c_y\right)} \leq \frac{c'}{\Phi^{-1}\left(\log \frac{1}{F_\mu(y)}\right)}.$$

Now, by (6.1) and Point (*iii*) and (*ii*) of Lemma 6.2, we note that

$$\begin{aligned}\log \frac{1}{F_\mu(y)} &\geq \Phi(|y|) + \log\left(\frac{Z_\Phi}{2}\right) + \log(\Phi'(|y|)) \\ &\geq \Phi(|y|) + \log\left(\frac{Z_\Phi}{2}\right) - c_3\log(|y|)) \\ &\geq \Phi(|y|) + \log\left(\frac{Z_\Phi}{2}\right) - \frac{c_3}{c_2}\log(\Phi(|y|))) \\ &\geq \frac{1}{2}\Phi(|y|)\end{aligned}$$

provided $y \ll -1$. In turn, by Point (*iv*) of Lemma 6.2,

$$\frac{\Phi'' \circ \Phi^{-1}(c_y)}{\Phi' \circ \Phi^{-1}(c_y)} \leq \frac{c''}{|y|}$$

for some constant c''.

All these computations together give

$$\left|\frac{1}{\Phi'(|y|)}\left(\log \frac{1}{H(y)} - \Phi(|y|)\right)\frac{\Phi'' \circ \Phi^{-1}(c_y)}{\Phi' \circ \Phi^{-1}(c_y)}\right| \leq c''\frac{\log \frac{2}{Z_\Phi} + c\log(|y|)}{|y|\Phi'(|y|)}$$

which goes to 0 as y goes to $-\infty$ by Point (*i*) and (*ii*) of Lemma 6.2. This ends the proof. □

Lemma 6.2. *Let $\Phi : \mathbb{R}^+ \to \mathbb{R}$ be an increasing concave function satisfying $\Phi(x)/x \to 0$ as $x \to \infty$. Assume that in a neighborhood of $+\infty$ the function Φ is C^2 and there exists $\theta > 1$ such that Φ^θ is convex. Assume that $\int e^{-\Phi(|x|)}dx < \infty$. Then, there exist constants $c_1, c_3 > 1$, $c_2, c_4 \in (0, 1)$ such that for x large*

enough,
(i) $c_1^{-1}x\Phi'(x) \leq \Phi(x) \leq c_1 x\Phi'(x)$*;*
(ii) $\Phi(x) \geq x^{c_2}$*;*
(iii) $\Phi'(x) \geq x^{-c_3}$*;*
(iv) $\frac{1}{2}\Phi(x) \geq \Phi(c_4 x)$.

Proof. Let $\widetilde{\Phi} = \Phi - \Phi(0)$. Then, in the large, $\widetilde{\Phi}$ is concave and $(\widetilde{\Phi})^\theta$ is convex. Hence, the slope functions $\widetilde{\Phi}(x)/x$ and $(\widetilde{\Phi})^\theta/x$ are non-increasing and non-decreasing respectively. In turn, for x large enough,

$$x\Phi'(x) = x\widetilde{\Phi}'(x) \leq \widetilde{\Phi}(x) \leq \theta x\widetilde{\Phi}'(x) = \theta x\Phi'(x).$$

This bound implies in particular that $x\Phi'(x) \to \infty$ as x tends to infinity. Point (i) follows.

The second inequality in (i) implies that for x large enough,

$$\frac{\Phi'(x)}{\Phi(x)} \geq \frac{1}{c_1 x}. \tag{6.3}$$

Hence, for some x_0 large enough, integrating, we get

$$\log\Phi(x) \geq \log\Phi(x_0) + \frac{1}{c_1}(\log(x) - \log(x_0)) \geq \frac{1}{2c_1}\log(x) \quad \forall x \gg x_0.$$

Point (ii) follows.

Point (iii) follows from the latter and Inequality (6.3).

Take $c = \exp\{1/c_1\}$. By Point (i), we have for x large enough

$$\begin{aligned} \Phi(cx) &= \Phi(x) + \int_x^{cx} \Phi'(t)dt \\ &\geq \Phi(x) + \int_x^{cx} \frac{\Phi(t)}{c_1 t}dt \\ &\geq \Phi(x)\left(1 + \int_x^{cx} \frac{1}{c_1 t}\right)dt \\ &= \Phi(x)\left(1 + \frac{\log c}{c_1}\right) = 2\Phi(x). \end{aligned}$$

Point (iv) follows. □

Now let us recall the statement of Corollary 5.23.

Corollary. *Let* $\Phi : \mathbb{R}^+ \to \mathbb{R}$ *be a non-decreasing concave function satisfying* $\Phi(x)/x \to 0$ *as* $x \to \infty$ *and* $\Phi(0) < \log 2$*. Assume that in a neighborhood of* $+\infty$ *the function* Φ *is* C^2 *and there exists* $\theta > 1$ *such that* Φ^θ *is convex. Let* $d\mu(x) = Z_\Phi^{-1}e^{-\Phi(|x|)}dx$ *be a probability measure on* $\mathbb{R}$*. Then,*

$$I_{\mu^n}(t) \geq c\min(t, 1-t)\Phi' \circ \Phi^{-1}\left(\log\frac{n}{\min(t, 1-t)}\right) \qquad \forall t \in [0,1],\ \forall n$$

for some constant $c > 0$ *independent on n.*

Proof of Corollary 5.23. Since Φ is concave, $\log(1 - F_\mu)$ is convex on $\mathbb{R}^+$. Applying Corollary 5.16 together with Proposition 5.21 lead to

$$I_{\mu^n}(t) \geq c \min(t, 1-t)\Phi' \circ \Phi^{-1}\left(\log \frac{n}{c' \min(t, 1-t)}\right) \qquad \forall t \in [0,1],\ \forall n$$

for some constant $c > 0$ and $c' > 1$ independent on n. It remains to prove that for all $t \in [0, 1/2]$,

$$t\Phi' \circ \Phi^{-1}\left(\log \frac{n}{c't}\right) \geq c'' t\Phi' \circ \Phi^{-1}\left(\log \frac{n}{t}\right)$$

for some constant $c'' > 0$. For $t \leq 1/2$ we have $1/(c't) \leq (1/t)^C$ for some $C > 1$. Hence, since $\Phi' \circ \Phi^{-1}$ is non-increasing,

$$\Phi' \circ \Phi^{-1}(\log \frac{n}{c't}) \geq \Phi' \circ \Phi^{-1}(C \log \frac{n}{t}).$$

Now note that Point (iv) of Lemma 6.2 is equivalent to say $\Phi^{-1}(2x) \leq \frac{1}{c_4}\Phi^{-1}(x)$ for x large enough. Hence $\Phi^{-1}(Cx) \leq \left(\frac{1}{c_4}\right)^{\lfloor \log_2 C \rfloor + 1} \Phi^{-1}(x)$. It follows that

$$\Phi' \circ \Phi^{-1}(\log \frac{n}{c't}) \geq \Phi'\left(\left(\frac{1}{c_4}\right)^{\lfloor \log_2 C \rfloor + 1} \Phi^{-1}(\log \frac{n}{t})\right)$$

for t small enough. Finally, Point (i) and (iv) of Lemma 6.2 ensure that

$$\Phi'\left(\frac{1}{c_4}x\right) \geq \frac{c_4}{c_1}\frac{\Phi\left(\frac{x}{c_4}\right)}{x} \geq \frac{2c_4}{c_1}\frac{\Phi(x)}{x} \geq \frac{2c_4}{c_1^2}\Phi'(x).$$

Hence

$$t\Phi' \circ \Phi^{-1}\left(\log \frac{n}{c't}\right) \geq c'' t\Phi' \circ \Phi^{-1}\left(\log \frac{n}{t}\right)$$

for some constant $c'' > 0$ and t small enough, say for $t \leq t_0$. The expected result follows by continuity of $t \mapsto t\Phi' \circ \Phi^{-1}(\log \frac{n}{t})/t\Phi' \circ \Phi^{-1}(\log \frac{n}{c't})$ (on $[t_0, 1/2]$). □

References

[1] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer. *Sur les inégalités de Sobolev logarithmiques*, volume 10 of *Panoramas et Synthèses*. Société Mathématique de France, Paris, 2000.

[2] D. Bakry. L'hypercontractivité et son utilisation en théorie des semigroupes. In *Lectures on Probability theory. École d'été de Probabilités de St-Flour 1992*, volume 1581 of *Lecture Notes in Math.*, pages 1–114. Springer, Berlin, 1994. MR1307413

[3] D. Bakry, F. Barthe, P. Cattiaux, and A. Guillin. A simple proof of the Poincaré inequality for a large class of probability measures. *Electronic Communications in Probability.*, 13:60–66, 2008. MR2386063

[4] D. Bakry, P. Cattiaux, and A. Guillin. Rate of convergence for ergodic continuous Markov processes : Lyapunov versus Poincaré. *J. Func. Anal.*, 254:727–759, 2008. MR2381160

[5] D. Bakry and M. Ledoux. Levy-Gromov isoperimetric inequality for an infinite dimensional diffusion generator. *Invent. Math.*, 123:259–281, 1996. MR1374200

[6] R. E. Barlow, A. W. Marshall, and F. Proschan. Properties of probability distributions with monotone hazard rate. *The Annals of Math. Statistics*, 34(2):375–389, 1963. MR0171328

[7] F. Barthe. Levels of concentration between exponential and Gaussian. *Ann. Fac. Sci. Toulouse Math. (6)*, 10(3):393–404, 2001. MR1923685

[8] F. Barthe. Isoperimetric inequalities, probability measures and convex geometry. In *European Congress of Mathematics*, pages 811–826. Eur. Math. Soc., Zürich, 2005. MR2185783

[9] F. Barthe, P. Cattiaux, and C. Roberto. Concentration for independent random variables with heavy tails. *AMRX*, 2005(2):39–60, 2005. MR2173316

[10] F. Barthe, P. Cattiaux, and C. Roberto. Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry. *Rev. Mat. Iber.*, 22(3):993–1066, 2006. MR2320410

[11] F. Barthe, P. Cattiaux, and C. Roberto. Isoperimetry between exponential and Gaussian. *Electronic J. Prob.*, 12:1212–1237, 2007. MR2346509

[12] F. Barthe and C. Roberto. Sobolev inequalities for probability measures on the real line. *Studia Math.*, 159(3), 2003. MR2052235

[13] F. Barthe and C. Roberto. Modified logarithmic Sobolev inequalities on $\mathbb{R}$. To appear in Potential Analysis, 2008. MR2430612

[14] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, and J.L. Vázquez. Asymptotics of the fast diffusion equation via entropy estimates. To appear in Archive for Rational Mechanics and Analysis, 2008.

[15] S. G. Bobkov. Isoperimetric inequalities for distributions of exponential type. *Ann. Probab.*, 22(2):978–994, 1994. MR1288139

[16] S. G. Bobkov. A functional form of the isoperimetric inequality for the Gaussian measure. *J. Funct. Anal*, 135:39–49, 1996. MR1367623

[17] S. G. Bobkov. Isoperimetric and analytic inequalities for log-concave probability measures. *Ann. Prob.*, 27(4):1903–1921, 1999. MR1742893

[18] S. G. Bobkov. Spectral gap and concentration for some spherically symmetric probability measures. In *Geometric Aspects of Functional Analysis, Israel Seminar 2000-2001,*, volume 1807 of *Lecture Notes in Math.*, pages 37–43. Springer, Berlin, 2003. MR2083386

[19] S. G. Bobkov. Large deviations and isoperimetry over convex probability measures. *Electr. J. Prob.*, 12:1072–1100, 2007. MR2336600

[20] S. G. Bobkov and C. Houdré. Isoperimetric constants for product probability measures. *Ann. Prob.*, 25:184–205, 1997. MR1428505

[21] S. G. Bobkov and C. Houdré. Some connections between isoperimetric and Sobolev-type inequalities. *Mem. Amer. Math. Soc.*, 129(616), 1997. MR1396954

[22] S. G. Bobkov and C. Houdré. Weak dimension-free concentration of measure. *Bernoulli*, 6(4):621–632, 2000. MR1777687

[23] S. G. Bobkov and M. Ledoux. Weighted Poincaré-type inequalities for Cauchy and other convex measures. To appear in Annals of Probability., 2007.

[24] S. G. Bobkov and B. Zegarlinski. Entropy bounds and isoperimetry. *Memoirs of the American Mathematical Society*, 176(829), 2005. MR2146071

[25] S. G. Bobkov and B. Zegarlinski. Distribution with slow tails and ergodicity of markov semigroups in infinite dimensions. Preprint, 2007.

[26] C. Borell. The Brunn-Minkowski inequality in Gauss space. *Invent. Math.*, 30(2):207–216, 1975. MR0399402

[27] C. Borell. Convex set functions in d-space. *Period. Math. Hungar.*, 6(2):111–136, 1975. MR0404559

[28] P. Cattiaux. A pathwise approach of some classical inequalities. *Potential Analysis*, 20:361–394, 2004. MR2032116

[29] P. Cattiaux and A. Guillin. Trends to equilibrium in total variation distance. Ann. Inst. Henri Poincar Probab. Stat. 45 (2009), no. 1, 117–145. Available on Mathematics ArXiv.math.PR/0703451, 2007. MR2500231

[30] P. Cattiaux, A. Guillin, F. Y. Wang, and L. Wu. Lyapunov conditions for logarithmic Sobolev and super Poincaré inequality. J. Funct. Anal. 256 (2009), no. 6, 1821–1841. MR2498560

[31] E. B. Davies. *Heat kernels and spectral theory*. Cambridge University Press, 1989. MR0990239

[32] J. Denzler and R. J. McCann. Fast diffusion to self-similarity: complete spectrum, long-time asymptotics and numerology. *Arch. Ration. Mech. Anal.*, 175(3):301–342, 2005. MR2126633

[33] J. Dolbeault, I. Gentil, A. Guillin, and F.Y.. Wang. l^q functional inequalities and weighted porous media equations. *Pot. Anal.*, 28(1):35–59, 2008. MR2366398

[34] R. Douc, G. Fort, and A. Guillin. Subgeometric rates of convergence of f-ergodic strong Markov processes. Stochastic Process. Appl. 119 (2009), no. 3, 897–923. MR2499863

[35] N. Gozlan. Characterization of Talagrand's like transportation-cost inequalities on the real line. *J. Func. Anal.*, 250(2):400–425, 2007. MR2352486

[36] N. Gozlan. Poincaré inequalities and dimension free concentration of measure. Preprint. To appear in Ann. Inst. Henri Poincaré. Prob. Stat., 2007.

[37] L. Gross. Logarithmic Sobolev inequalities and contractivity properties of semi-groups. in Dirichlet forms. Dell'Antonio and Mosco eds. *Lect. Notes Math.*, 1563:54–88, 1993. MR1292277

[38] A. Guionnet and B. Zegarlinski. Lectures on logarithmic Sobolev inequalities. Séminaire de Probabilités XXXVI. *Lect. Notes Math.*, 1801, 2002.

[39] M. Hairer and J. C. Mattingly. Slow energy dissipation in anharmonic oscillator chains. Comm. Pure Appl. Math. 62 (2009), no. 8, 999–1032. MR2531551

[40] R. Kannan, L. Lovász, and M. Simonovits. Isoperimetric problems for convex bodies and a localization lemma. *Discrete Comput. Geom.*, 13(3-4):541–559, 1995. MR1318794

[41] M. Ledoux. Concentration of measure and logarithmic Sobolev inequalities. In *Séminaire de Probabilités XXXIII*, volume 1709 of *Lecture Notes in Math.*, pages 120–216. Springer, Berlin, 1999. MR1767995

[42] M. Ledoux. *The concentration of measure phenomenon*, volume 89 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001. MR1849347

[43] M. Ledoux. Spectral gap, logarithmic Sobolev constant, and geometric bounds. In *Surveys in differential geometry.*, volume IX, pages 219–240. Int. Press, Somerville MA, 2004. MR2195409

[44] S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability*. Communications and Control Engineering Series. Springer-Verlag London Ltd., London, 1993. MR1287609

[45] S. P. Meyn and R. L. Tweedie. Stability of markovian processes II: continuous-time processes and sampled chains. *Adv. Appl. Proba.*, 25:487–517, 1993. MR1234294

[46] S. P. Meyn and R. L. Tweedie. Stability of markovian processes III: Foster-Lyapunov criteria for continuous-time processes. *Adv. Appl. Proba.*, 25:518–548, 1993. MR1234295

[47] E. Milman. On the role of convexity in isoperimetry, spectral-gap and concentration. Invent. Math. 177 (2009), no. 1, 1–43. MR2507637

[48] B. Muckenhoupt. Hardy's inequality with weights. *Studia Math.*, 44:31–38, 1972. collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, **I**. MR0311856

[49] M. Röckner and F. Y. Wang. Weak Poincaré inequalities and L^2-convergence rates of Markov semigroups. *J. Funct. Anal.*, 185(2):564–603, 2001. MR1856277

[50] A. Ros. The isoperimetric problem. In *Global theory of minimal surfaces*, volume 2 of *Clay Math. Proc.*, pages 175–209. Amer. Math. Soc., 2005. MR2167260

[51] G. Royer. *Une initiation aux inégalités de Sobolev logarithmiques*. S.M.F., Paris, 1999.

[52] V. N. Sudakov and B. S. Cirel′son. Extremal properties of half-spaces for spherically invariant measures. *Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 41:14–24, 165, 1974. Problems in the theory of probability distributions, II. MR0365680

[53] M. Talagrand. A new isoperimetric inequality and the concentration of measure phenomenon. In *Geometric aspects of functional analysis (1989–90)*, volume 1469 of *Lecture Notes in Math.*, pages 94–124. Springer, Berlin, 1991. MR1122615

[54] M. Talagrand. The supremum of some canonical processes. *Amer. J. Math.*, 116(2):283–325, 1994. MR1269606

[55] J. L. Vázquez. An introduction to the mathematical theory of the porous medium equation. In *Shape optimization and free boundaries (Montreal, PQ, 1990)*, volume 380 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 347–389. Kluwer Acad. Publ., Dordrecht, 1992. MR1260981

[56] A. Yu. Veretennikov. On polynomial mixing bounds for stochastic differential equations. *Stochastic Process. Appl.*, 70(1):115–127, 1997. MR1472961

[57] F. Y. Wang. *Functional inequalities, Markov processes and Spectral theory*. Science Press, Beijing, 2005.

[58] F. Y. Wang. From Super Poincaré to Weighted Log-Sobolev and Entropy-Cost Inequalities. To appear in J. Math. Pures Appl., 2008. MR2446080