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The Steepest Descent Method for Forward-Backward SDEs [1]

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Abstract. This paper aims to open a door to Monte-Carlo methods for numerically solving Forward-Backward SDEs, without computing over all Cartesian grids as usually done in the literature. We transform the FBSDE to a control problem and propose the steepest descent method to solve the latter one. We show that the original (coupled) FBSDE can be approximated by *decoupled* FBSDEs, which further boils down to computing a sequence of conditional expectations. The rate of convergence is obtained, and the key to its proof is a new well-posedness result for FBSDEs. However, the approximating decoupled FBSDEs are non-Markovian. Some Markovian type of modification is needed in order to make the algorithm efficiently implementable.

Keywords: Forward-Backward SDEs, quasilinear PDEs, stochastic control, steepest decent method, Monte-Carlo method, rate of convergence.

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1 Introduction

Since the seminal work of Pardoux-Peng [19], there have been numerous publications on Backward Stochastic Differential Equations (BSDEs) and Forward-Backward SDEs (FBSDEs). We refer the readers to the book Ma-Yong [17] and the reference therein for the details on the subject. In particular, FBSDEs of the following type are studied extensively:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s)dW_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s; \end{cases} \tag{1.1}$$

where W is a standard Brownian Motion, $T > 0$ is a deterministic terminal time, and b, σ, f, g are deterministic functions. Here for notational simplicity we assume all processes are 1-dimensional. It is well known that FBSDE (1.1) is related to the following parabolic PDE on $[0,T] \times \mathbb{R}$ (see, e.g., [13], [20], and [7])

$$\begin{cases} u_t + \frac{1}{2}\sigma^2(t,x,u)u_{xx} + b(t,x,u,\sigma(t,x,u)u_x)u_x + f(t,x,u,\sigma(t,x,u)u_x) = 0; \\ u(T,x) = g(x); \end{cases} \tag{1.2}$$

in the sense that (if a smooth solution u exists)

$$Y_t = u(t, X_t), \quad Z_t = u_x(t, X_t)\sigma(t, X_t, u(t, X_t)). \tag{1.3}$$

Due to its importance in applications, numerical methods for BSDEs have received strong attention in recent years. Bally [1] proposed an algorithm by using a random time discretization. Based on a new notion of L^2-regularity, Zhang [21] obtained rate of convergence for deterministic time discretization and transformed the problem to computing a sequence of conditional expectations. In Markovian setting, significant progress has been made in computing the conditional expectations. The following methods are of particular interest: the quantization method (see, e.g., Bally-Pagès-Printems [2]), the Malliavin calculus approach (see Bouchard-Touzi [4]), the linear regression method or the Longstaff-Schwartz algorithm (see Gobet-Lemor-Waxin [10]), and the Picard iteration approach (see Bender-Denk [3]). These methods work well in reasonably high dimensions. There are also lots of publications on numerical methods for non-Markovian BSDEs (see, e.g., [5], [6], [12], [15], [24]). But in general these methods do not work when the dimension is high.

Numerical approximations for FBSDEs, however, are much more difficult. To our knowledge, there are only very few works in the literature. The first one was Douglas-Ma-Protter [9], based on the four step scheme. Their main idea is to numerically solve the PDE (1.2). Milstein-Tretyakov [16] and Makarov [14] also proposed some numerical schemes for (1.2). Recently Delarue-Menozzi [8] proposed a probabilistic algorithm. Note that all these methods essentially need to discretize the space over regular Cartesian grids, and thus are not practical in high dimensions.

In this paper we aim to open a door to truly Monte-Carlo methods for FBSDEs, without computing over all Cartesian grids. Our main idea is to transform the FBSDE to a stochastic control problem and propose *the steepest descent method* to solve the latter one. We show that the original (coupled) FBSDE can be approximated by solving a certain number of *decoupled* FBSDEs. We then discretize the approximating decoupled FBSDEs in time and thus the problem boils down to computing a sequence of conditional expectations. The rate of convergence is obtained.

We note that the idea to approximate with a corresponding stochastic control problem is somewhat similar to the approximating solvability of FBSDEs in Ma-Yong [18] and the near-optimal control in Zhou [25]. However, in those works the original problem may have no exact solution and the authors try to find a so called approximating solution. In our case the exact solution exists and we want to approximate it with numerically computable terms. More importantly, in those works one only cares for the existence of the approximating solutions, while here for practical reasons we need explicit construction of the approximations as well as the rate of convergence.

The key to the proof is a new well-posedness result for FBSDEs. In order to obtain the rate of convergence of our approximations, we need the well-posedness of some adjoint FBSDEs, which are linear but with random coefficients. It turns out that all the existing methods in the literature do not work in our case.

At this point we should point out that, unfortunately, our approximating decoupled FBSDEs are non-Markovian (that is, the coefficients are random), and thus we cannot apply directly the existing methods for Markovian BSDEs. In order to make our algorithm efficiently implementable, some further modification of Markovian type is needed.

Although in the long term we aim to solve high dimensional FBSDEs, as a first attempt and for technical reasons (in order to apply Theorem 1.2 below), in this paper we assume all the processes are one dimensional. We also assume that $b = 0$ and f is independent of Z. That is, we will study the following FBSDE:

$$\left\{ \begin{array}{l} X_t = x + \int_0^t \sigma(s, X_s, Y_s) dW_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s) ds - \int_t^T Z_s dW_s. \end{array} \right. \tag{1.4}$$

In this case, PDE (1.2) becomes

$$\left\{ \begin{array}{l} u_t + \frac{1}{2}\sigma^2(t, x, u) u_{xx} + f(t, x, u) = 0; \\ u(T, x) = g(x); \end{array} \right. \tag{1.5}$$

Moreover, in order to simplify the presentation and to focus on the main idea, throughout the paper we assume

Assumption 1.1 *All the coefficients σ, f, g are bounded, smooth enough with bounded derivatives, and σ is uniformly nondegenerate.*

Under Assumption 1.1, it is well known that PDE (1.5) has a unique solution u which is bounded and smooth with bounded derivatives (see [11]), that FBSDE (1.4) has a unique solution (X, Y, Z), and that (1.3) holds true (see [13]). Unless otherwise specified, throughout the paper we use (X, Y, Z) and u to denote these solutions, and $C, c > 0$ to denote generic constants depending only on T, the upper bounds of the derivatives of the coefficients, and the uniform nondegeneracy of σ. We allow C, c to vary from line to line.

Finally, we cite a well-posedness result from Zhang [23] (or [22] for a weaker result) which will play an important role in our proofs.

Theorem 1.2 *Consider the following FBSDE*

$$\begin{cases} X_t = x + \int_0^t b(\omega, s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(\omega, s, X_s, Y_s)dW_s; \\ Y_t = g(\omega, X_T) + \int_t^T f(\omega, s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s; \end{cases} \tag{1.6}$$

Assume that b, σ, f, g are uniformly Lipschitz continuous with respect to (x, y, z); that there exists a constant $c > 0$ such that

$$\sigma_y b_z \leq -c|b_y + \sigma_x b_z + \sigma_y f_z|; \tag{1.7}$$

and that

$$I_0^2 \triangleq E\Big\{x^2 + |g(\omega, 0)|^2 + \int_0^T [|b|^2 + |\sigma|^2 + |f|^2](\omega, t, 0, 0, 0)dt\Big\} < \infty.$$

Then FBSDE (1.6) has a unique solution (X, Y, Z) such that

$$E\Big\{\sup_{0\leq t\leq T}[|X_t|^2 + |Y_t|^2] + \int_0^T |Z_t|^2 dt\Big\} \leq CI_0^2,$$

where C is a constant depending only on T, c and the Lipschitz constants of the coefficients.

The rest of the paper is organized as follows. In the next section we transform FBSDE (1.4) to a stochastic control problem and propose the steepest descent method; in §3 we discretize the decoupled FBSDEs introduced in §2; and in §4 we transform the discrete FBSDEs to a sequence of conditional expectations.

2 The Steepest Descent Method

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, W a standard Brownian motion, $T > 0$ a fixed terminal time, $\mathbf{F} \stackrel{\triangle}{=} \{\mathcal{F}_t\}_{0 \le t \le T}$ the filtration generated by W and augmented by the P-null sets. Let $L^2(\mathbf{F})$ denote square integrable $\mathbf{F}$-adapted processes. From now on we always assume Assumption 1.1 is in force.

2.1 The Control Problem

In order to numerically solve (1.4), we first formulate a related stochastic control problem. Given $y_0 \in \mathbb{R}$ and $z^0 \in L^2(\mathbf{F})$, consider the following 2-dimensional (forward) SDE with random coefficients (z^0 being considered as a coefficient):

$$\begin{cases} X_t^0 = x + \int_0^t \sigma(s, X_s^0, Y_s^0) dW_s; \\ Y_t^0 = y_0 - \int_0^t f(s, X_s^0, Y_s^0) ds + \int_0^t z_s^0 dW_s; \end{cases} \tag{2.1}$$

and denote

$$V(y_0, z^0) \stackrel{\triangle}{=} \frac{1}{2} E\Big\{|Y_T^0 - g(X_T^0)|^2\Big\}. \tag{2.2}$$

Our first result is

Theorem 2.1 *We have*

$$E\Big\{\sup_{0 \le t \le T} [|X_t - X_t^0|^2 + |Y_t - Y_t^0|^2] + \int_0^T |Z_t - z_t^0|^2 dt\Big\} \le CV(y_0, z^0).$$

Proof. The idea is similar to the four step scheme (see [13]).

Step 1. Denote

$$\Delta Y_t \stackrel{\triangle}{=} Y_t^0 - u(t, X_t^0); \quad \Delta Z_t \stackrel{\triangle}{=} z_t^0 - u_x(t, X_t^0)\sigma(t, X_t^0, Y_t^0).$$

Recalling (1.5) we have

$$\begin{aligned} d(\Delta Y_t) &= z_t^0 dW_t - f(t, X_t^0, Y_t^0)dt - u_x(t, X_t^0)\sigma(t, X_t^0, Y_t^0)dW_t \\ &\quad - \Big[u_t(t, X_t^0) + \frac{1}{2}u_{xx}(t, X_t^0)\sigma^2(t, X_t^0, Y_t^0)\Big]dt \\ &= \Delta Z_t dW_t - \Big[\frac{1}{2}u_{xx}(t, X_t^0)\sigma^2(t, X_t^0, Y_t^0) + f(t, X_t^0, Y_t^0)\Big]dt \\ &\quad + \Big[\frac{1}{2}u_{xx}(t, X_t^0)\sigma^2(t, X_t^0, u(t, X_t^0)) + f(t, X_t^0, u(t, X_t^0))\Big]dt \\ &= \Delta Z_t dW_t - \alpha_t \Delta Y_t dt, \end{aligned}$$

where

$$\alpha_t \triangleq \frac{1}{2\Delta Y_t} u_{xx}(t, X_t^0)[\sigma^2(t, X_t^0, Y_t^0) - \sigma^2(t, X_t^0, u(t, X_t^0))] + \frac{1}{\Delta Y_t}[f(t, X_t^0, Y_t^0) - f(t, X_t^0, u(t, X_t^0))]$$

is bounded. Note that $\Delta Y_T = Y_T^0 - g(X_T^0)$. By standard arguments one can easily get

$$E\Big\{\sup_{0\le t\le T} |\Delta Y_t|^2 + \int_0^T |\Delta Z_t|^2 dt\Big\} \le CE\{|\Delta Y_T|^2\} = CV(y_0, z^0). \tag{2.3}$$

Step 2. Denote $\Delta X_t \triangleq X_t - X_t^0$. We show that

$$E\Big\{\sup_{0\le t\le T} |\Delta X_t|^2\Big\} \le CV(y_0, z^0). \tag{2.4}$$

In fact,

$$d(\Delta X_t) = \Big[\sigma(t, X_t, u(t, X_t)) - \sigma(t, X_t^0, Y_t^0)\Big] dW_t.$$

Note that

$$u(t, X_t) - Y_t^0 = u(t, X_t) - u(t, X_t^0) - \Delta Y_t.$$

One has

$$d(\Delta X_t) = [\alpha_t^1 \Delta X_t + \alpha_t^2 \Delta Y_t] dW_t,$$

where α_t^i are defined in an obvious way and are uniformly bounded. Note that $\Delta X_0 = 0$. Then by standard arguments we get

$$E\Big\{\sup_{0\le t\le T} |\Delta X_t|^2\Big\} \le CE\Big\{\int_0^T |\Delta Y_t|^2 dt\Big\},$$

which, together with (2.3), implies (2.4).

Step 3. We now prove the theorem. Recall (1.3), we have

$$\begin{aligned}
&E\Big\{\sup_{0\le t\le T} |Y_t - Y_t^0|^2 + \int_0^T |Z_t - z_t^0|^2 dt\Big\} \\
&= E\Big\{\sup_{0\le t\le T} |u(t, X_t) - u(t, X_t^0) - \Delta Y_t|^2 \\
&\quad + \int_0^T \Big| u_x(t, X_t)\sigma(t, X_t, u(t, X_t)) - u_x(t, X_t^0)\sigma(t, X_t^0, u(t, X_t^0)) \\
&\quad + u_x(t, X_t^0)\sigma(t, X_t^0, u(t, X_t^0)) - u_x(t, X_t^0)\sigma(t, X_t^0, Y_t^0) - \Delta Z_t\Big|^2 dt\Big\} \\
&\le CE\Big\{\sup_{0\le t\le T} [|\Delta X_t|^2 + |\Delta Y_t|^2] + \int_0^T \Big[|\Delta X_t|^2 + |\Delta Y_t|^2 + |\Delta Z_t|^2\Big] dt\Big\} \\
&\le CV(y_0, z^0),
\end{aligned}$$

which, together with (2.4), ends the proof. ∎

2.2 The Steepest Descent Direction

Our idea is to modify (y_0, z^0) along the *steepest descent direction* so as to decrease V as fast as possible. First we need to find the Fréchet derivative of V along some direction $(\Delta y, \Delta z)$, where $\Delta y \in \mathbb{R}, \Delta z \in L^2(\mathbf{F})$. For $\delta \geq 0$, denote

$$y_0^\delta \stackrel{\triangle}{=} y_0 + \delta \Delta y; \quad z_t^{0,\delta} \stackrel{\triangle}{=} z_t^0 + \delta \Delta z_t;$$

and let $X^{0,\delta}, Y^{0,\delta}$ be the solution to (2.1) corresponding to $(y_0^\delta, z^{0,\delta})$. Denote:

$$\begin{cases} \nabla X_t^0 = \int_0^t [\sigma_x^0 \nabla X_s^0 + \sigma_y^0 \nabla Y_s^0] dW_s; \\ \nabla Y_t^0 = \Delta y - \int_0^t [f_x^0 \nabla X_s^0 + f_y^0 \nabla Y_s^0] ds + \int_0^t \Delta z_s dW_s; \\ \nabla V(y_0, z^0) = E\Big\{[Y_T^0 - g(X_T^0)][\nabla Y_T^0 - g'(X_T^0) \nabla X_T^0]\Big\}; \end{cases}$$

where $\varphi_s^0 \stackrel{\triangle}{=} \varphi(s, X_s^0, Y_s^0)$ for any function φ. By standard arguments, one can easily show that

$$\begin{aligned} &\lim_{\delta \to 0} \frac{1}{\delta}[X_t^{0,\delta} - X_t^0] = \nabla X_t^0; \qquad \lim_{\delta \to 0} \frac{1}{\delta}[Y_t^{0,\delta} - Y_t^0] = \nabla Y_t^0; \\ &\lim_{\delta \to 0} \frac{1}{\delta}[V(y_0^\delta, z^{0,\delta}) - V(y_0, z^0)] = \nabla V(y_0, z^0); \end{aligned}$$

where the two limits in the first line are in the $L^2(\mathbf{F})$ sense.

To investigate $\nabla V(y_0, z^0)$ further, we define some adjoint processes. Consider (X^0, Y^0) as random coefficients and let $(\bar{Y}^0, \tilde{Y}^0, \bar{Z}^0, \tilde{Z}^0)$ be the solution to the following 2-dimensional BSDE:

$$\begin{cases} \bar{Y}_t^0 = [Y_T^0 - g(X_T^0)] - \int_t^T [f_y^0 \bar{Y}_s^0 + \sigma_y^0 \tilde{Z}_s^0] ds - \int_t^T \bar{Z}_s^0 dW_s; \\ \tilde{Y}_t^0 = g'(X_T^0)[Y_T^0 - g(X_T^0)] + \int_t^T [f_x^0 \bar{Y}_s^0 + \sigma_x^0 \tilde{Z}_s^0] ds - \int_t^T \tilde{Z}_s^0 dW_s. \end{cases} \tag{2.5}$$

We note that (2.5) depends only on (y_0, z^0), but not on $(\Delta y, \Delta z)$.

Lemma 2.2 *For any $(\Delta y, \Delta z)$, we have*

$$\nabla V(y_0, z^0) = E\Big\{\bar{Y}_0^0 \Delta y + \int_0^T \bar{Z}_t^0 \Delta z_t dt\Big\}.$$

Proof. Note that

$$\nabla V(y_0, z^0) = E\Big\{\bar{Y}_T^0 \nabla Y_T^0 - \tilde{Y}_T^0 \nabla X_T^0\Big\}.$$

Applying Ito's formula one can easily check that

$$d(\bar{Y}_t^0 \nabla Y_t^0 - \tilde{Y}_t^0 \nabla X_t^0) = \bar{Z}_t^0 \Delta z_t dt + (\cdots) dW_t.$$

Then

$$\nabla V(y_0, z^0) = E\Big\{\bar{Y}_0^0 \nabla Y_0^0 - \tilde{Y}_0^0 \nabla X_0^0 + \int_0^T \bar{Z}_t^0 \Delta z_t dt\Big\} = E\Big\{\bar{Y}_0^0 \Delta y + \int_0^T \bar{Z}_t^0 \Delta z_t dt\Big\}.$$

That proves the lemma. ■

Recall that our goal is to decrease $V(y_0, z^0)$. Very naturally one would like to choose the following steepest descent direction:

$$\Delta y \stackrel{\triangle}{=} -\bar{Y}_0^0; \quad \Delta z_t \stackrel{\triangle}{=} -\bar{Z}_t^0. \tag{2.6}$$

Then

$$\nabla V(y_0, z^0) = -E\Big\{|\bar{Y}_0^0|^2 + \int_0^T |\bar{Z}_t^0|^2 dt\Big\}, \tag{2.7}$$

which depends only on (y_0, z^0) (not on $(\Delta y, \Delta z)$).

Note that if $\nabla V(y_0, z^0) = 0$, then we gain nothing on decreasing $V(y_0, z^0)$. Fortunately this is not the case.

Lemma 2.3 *Assume (2.6). Then* $\nabla V(y_0, z^0) \leq -cV(y_0, z^0)$.

Proof. Rewrite (2.5) as

$$\begin{cases} \bar{Y}_t^0 = \bar{Y}_0^0 + \int_0^t [f_y^0 \bar{Y}_s^0 + \sigma_y^0 \tilde{Z}_s^0] ds + \int_0^t \bar{Z}_s^0 dW_s; \\ \tilde{Y}_t^0 = g'(X_T^0) \bar{Y}_T^0 + \int_t^T [f_x^0 \bar{Y}_s^0 + \sigma_x^0 \tilde{Z}_s^0] ds - \int_t^T \tilde{Z}_s^0 dW_s. \end{cases} \tag{2.8}$$

One may consider (2.8) as an FBSDE with solution triple $(\bar{Y}_t, \tilde{Y}_t, \tilde{Z}_t)$, where $\bar{Y}_t$ is the forward component and $(\tilde{Y}_t, \tilde{Z}_t)$ are the backward components. Then $(\bar{Y}_0^0, \bar{Z}_t^0)$ are considered as (random) coefficients of the FBSDE. One can easily check that FBSDE (2.8) satisfies condition (1.7) (with both sides equal to 0). Applying Theorem 1.2 we get

$$E\Big\{\sup_{0\leq t\leq T}[|\bar{Y}_t^0|^2 + |\tilde{Y}_t^0|^2] + \int_0^T |\tilde{Z}_t^0| dt\Big\} \leq CI_0^2 = CE\Big\{|\bar{Y}_0^0|^2 + \int_0^T |\bar{Z}_t^0|^2 dt\Big\}.$$

In particular,

$$V(y_0, z^0) = \frac{1}{2} E\{|\bar{Y}_T^0|^2\} \leq CE\Big\{|\bar{Y}_0^0|^2 + \int_0^T |\bar{Z}_t^0|^2 dt\Big\}, \tag{2.9}$$

which, combined with (2.7), implies the lemma. ■

2.3 Iterative Modifications

We now fix a desired error level ε and pick an (y_0, z^0). If we are extremely lucky that $V(y_0, z^0) \le \varepsilon^2$, then we may use (X^0, Y^0, z^0) defined by (2.1) as an approximation of (X, Y, Z). In other cases we want to modify (y_0, z^0). From now on we assume

$$V(y_0, z^0) > \varepsilon^2; \qquad E\{|Y^0_T - g(X^0_T)|^4\} \le K_0^4; \tag{2.10}$$

where $K_0 \ge 1$ is a constant. We note that one can always assume the existence of K_0 by letting, for example, $y_0 = 0, z^0_t = 0$.

Lemma 2.4 *Assume (2.10). There exist constants $C_0, c_0, c_1 > 0$, which are independent of K_0 and ε, such that*

$$\Delta V(y_0, z^0) \stackrel{\triangle}{=} V(y_1, z^1) - V(y_0, z^0) \le -\frac{c_0\varepsilon}{K_0^2} V(y_0, z^0), \tag{2.11}$$

and

$$E\{|Y^1_T - g(X^1_T)|^4\} \le K_1^4 \stackrel{\triangle}{=} K_0^4 + 2C_0\varepsilon K_0^2, \tag{2.12}$$

where, by denoting $\lambda \stackrel{\triangle}{=} \frac{c_1\varepsilon}{K_0^2}$,

$$y_1 \stackrel{\triangle}{=} y_0 - \lambda\bar{Y}^0_0; \quad z^1_t \stackrel{\triangle}{=} z^0_t - \lambda\bar{Z}^0_t; \tag{2.13}$$

and, for $0 \le \theta \le 1$,

$$\begin{cases} X^\theta_t = x + \int_0^t \sigma(s, X^\theta_s, Y^\theta_s) dW_s; \\ Y^\theta_t = y_0 - \theta\lambda\bar{Y}^0_0 - \int_0^t f(s, X^\theta_s, Y^\theta_s) ds + \int_0^t [z^0_s - \theta\lambda\bar{Z}^0_s] dW_s; \end{cases} \tag{2.14}$$

Proof. We proceed in four steps.

Step 1. For $0 \le \theta \le 1$, denote

$$\begin{cases} \bar{Y}^\theta_t = [Y^\theta_T - g(X^\theta_T)] - \int_t^T [f^\theta_y \bar{Y}^\theta_s + \sigma^\theta_y \tilde{Z}^\theta_s] ds - \int_t^T \bar{Z}^\theta_s dW_s; \\ \tilde{Y}^\theta_t = g'(X^\theta_T)[Y^\theta_T - g(X^\theta_T)] + \int_t^T [f^\theta_x \bar{Y}^\theta_s + \sigma^\theta_x \tilde{Z}^\theta_s] ds - \int_t^T \tilde{Z}^\theta_s dW_s; \\ \nabla X^\theta_t = \int_0^t [\sigma^\theta_x \nabla X^\theta_s + \sigma^\theta_y \nabla Y^\theta_s] dW_s; \\ \nabla Y^\theta_t = -\bar{Y}^0_0 - \int_0^t [f^\theta_x \nabla X^\theta_s + f^\theta_y \nabla Y^\theta_s] ds - \int_0^t \bar{Z}^0_s dW_s; \end{cases}$$

where $\varphi^\theta_t \stackrel{\triangle}{=} \varphi(t, X^\theta_t, Y^\theta_t)$ for any function φ. Then

$$\begin{aligned} \Delta V(y_0, z^0) &= \frac{1}{2} E\Big\{[Y^1_T - g(X^1_T)]^2 - [Y^0_T - g(X^0_T)]^2\Big\} \\ &= \lambda \int_0^1 E\Big\{[Y^\theta_T - g(X^\theta_T)][\nabla Y^\theta_T - g'(X^\theta_T)\nabla X^\theta_T]\Big\} d\theta. \end{aligned}$$

Following the proof of Lemma 2.2, we have

$$\Delta V(y_0,z^0)=-\lambda\int_0^1 E\Big\{\bar Y_0^\theta\bar Y_0^0+\int_0^T \bar Z_t^\theta\bar Z_t^0dt\Big\}d\theta. \tag{2.15}$$

Step 2. First, one can easily show that

$$E\Big\{\sup_{0\le t\le T}[|\bar Y_t^0|^4+|\tilde Y_t^0|^4]+\Big(\int_0^T[|\bar Z_t^0|^2+|\tilde Z_t^0|^2]dt\Big)^2\Big\}\le CK_0^4. \tag{2.16}$$

Denote

$$\Delta X_t^\theta\triangleq X_t^\theta-X_t^0;\quad \Delta Y_t^\theta\triangleq Y_t^\theta-Y_t^0.$$

Then

$$\begin{cases}\Delta X_t^\theta=\displaystyle\int_0^t[\alpha_s^{1,\theta}\Delta X_s^\theta+\beta_s^{1,\theta}\Delta Y_s^\theta]dW_s;\\ \Delta Y_t^\theta=-\theta\lambda\bar Y_0^0-\displaystyle\int_0^t[\alpha_s^{2,\theta}\Delta X_s^\theta+\beta_s^{2,\theta}\Delta Y_s^\theta]ds-\theta\lambda\int_0^t\bar Z_s^0dW_s;\end{cases}$$

where $\alpha^{i,\theta},\beta^{i,\theta}$ are defined in an obvious way and are bounded. Thus, by (2.16),

$$E\Big\{\sup_{0\le t\le T}[|\Delta X_t^\theta|^4+|\Delta Y_t^\theta|^4]\Big\}\le\theta^4\lambda^4E\Big\{|\bar Y_0^0|^4+\Big(\int_0^T|\bar Z_t^0|^2dt\Big)^2\Big\}\le CK_0^4\lambda^4. \tag{2.17}$$

Denote

$$\alpha_T^\theta\triangleq\frac{1}{\Delta X_T^\theta}[g(X_T^\theta)-g(X_T^0)],$$

which is bounded. For any constants $a,b>0$ and $0<\lambda<1$, applying the Young's Inequality we have

$$\begin{aligned}(a+b)^4&=a^4+4(\lambda^{\frac14}a)^3(\lambda^{-\frac34}b)+6(\lambda^{\frac14}a)^2(\lambda^{-\frac14}b)^2+4(\lambda^{\frac14}a)(\lambda^{-\frac1{12}}b)^3+b^4\\&\le[1+C\lambda]a^4+C[\lambda^{-3}+\lambda^{-1}+\lambda^{-\frac13}+1]b^4\le[1+C\lambda]a^4+C\lambda^{-3}b^4.\end{aligned}$$

Noting that the value of λ we will choose is less than 1, we have

$$\begin{aligned}&E\Big\{|Y_T^\theta-g(X_T^\theta)|^4\Big\}=E\Big\{|Y_T^0-g(X_T^0)+\Delta Y_T^\theta-\alpha_T^\theta\Delta X_T^\theta|^4\Big\}\\&\le[1+C\lambda]E\Big\{|Y_T^0-g(X_T^0)|^4\Big\}+C\lambda^{-3}E\Big\{|\Delta Y_T^\theta|^4+|\Delta X_T^\theta|^4\Big\}\\&\le[1+C\lambda]K_0^4.\end{aligned} \tag{2.18}$$

Step 3. Denote

$$\Delta\bar Y_t^\theta\triangleq\bar Y_t^\theta-\bar Y_t^0;\quad\Delta\tilde Y_t^\theta\triangleq\tilde Y_t^\theta-\tilde Y_t^0;\quad\Delta\bar Z_t^\theta\triangleq\bar Z_t^\theta-\bar Z_t^0;\quad\Delta\tilde Z_t^\theta\triangleq\tilde Z_t^\theta-\tilde Z_t^0.$$

Then

$$\begin{cases}\Delta\bar Y_t^\theta=[\Delta Y_T^\theta-\alpha_T^\theta\Delta X_T^\theta]-\displaystyle\int_t^T[f_y^\theta\Delta\bar Y_s^\theta+\sigma_y^\theta\Delta\tilde Z_s^\theta]ds-\int_t^T\Delta\bar Z_s^\theta dW_s\\ \qquad\qquad-\displaystyle\int_t^T[\bar Y_s^0\Delta f_y^\theta+\tilde Z_s^0\Delta\sigma_y^\theta]ds;\\ \Delta\tilde Y_t^\theta=g'(X_T^\theta)[\Delta Y_T^\theta-\alpha_T^\theta\Delta X_T^\theta]+\displaystyle\int_t^T[f_x^\theta\Delta\bar Y_s^\theta+\sigma_x^\theta\Delta\tilde Z_s^\theta]ds-\int_t^T\Delta\tilde Z_s^\theta dW_s\\ \qquad\qquad+[Y_T^0-g(X_T^0)]\Delta g'(\theta)+\displaystyle\int_t^T[\bar Y_s^0\Delta f_x^\theta+\tilde Z_s^0\Delta\sigma_x^\theta]ds,\end{cases}$$

where

$$\Delta f_y(\theta) \stackrel{\triangle}{=} f_y(t, X_t^\theta, Y_t^\theta) - f_y(t, X_t^0, Y_t^0);$$

and all other terms are defined in a similar way. By standard arguments one has

$$\begin{aligned}
&E\Big\{\sup_{0\le t\le T}[|\Delta\bar{Y}_t^\theta|^2 + |\Delta\tilde{Y}_t^\theta|^2] + \int_0^T [|\Delta\bar{Z}_t^\theta|^2 + |\Delta\tilde{Z}_t^\theta|^2]dt\} \\
&\le CE\Big\{|\Delta Y_T^\theta|^2 + |\Delta X_T^\theta|^2 + |Y_T^0 - g(X_T^0)|^2|\Delta g'(\theta)|^2 \\
&\quad + \int_0^T \Big[|\bar{Y}_t^0|^2[|\Delta f_x^\theta|^2 + |\Delta f_y^\theta|^2] + |\tilde{Z}_t^0|^2[|\Delta\sigma_x^\theta|^2 + |\Delta\sigma_y^\theta|^2]\Big]dt\Big\} \\
&\le CE\Big\{|\Delta Y_T^\theta|^2 + |\Delta X_T^\theta|^2 + |Y_T^0 - g(X_T^0)|^2|\Delta X_T^\theta|^2 \\
&\quad + \int_0^T [|\bar{Y}_t^0|^2 + |\tilde{Z}_t^0|^2][|\Delta X_t^\theta|^2 + |\Delta Y_t^\theta|^2]dt\Big\} \\
&\le CE^{\frac{1}{2}}\Big\{\sup_{0\le t\le T}[|\Delta X_t^\theta|^4 + |\Delta Y_t^\theta|^4]\Big\} \times \\
&\quad E^{\frac{1}{2}}\Big\{1 + |Y_T^0 - g(X_T^0)|^4 + \Big(\int_0^T [|\bar{Y}_t^0|^2 + |\tilde{Z}_t^0|^2]dt\Big)^2\Big\} \\
&\le CK_0^2\lambda^2[1 + K_0^2] \le CK_0^4\lambda^2,
\end{aligned}$$

thanks to (2.17), (2.10), and (2.16). In particular,

$$E\Big\{|\Delta\bar{Y}_0^\theta|^2 + \int_0^T |\Delta\bar{Z}_t^\theta|^2 dt\Big\} \le CK_0^4\lambda^2. \tag{2.19}$$

Step 4. Note that

$$\begin{aligned}
&\Big|E\Big\{\bar{Y}_0^\theta\bar{Y}_0^0 + \int_0^T \bar{Z}_t^\theta\bar{Z}_t^0 dt\Big\} - E\Big\{|\bar{Y}_0^0|^2 + \int_0^T |\bar{Z}_t^0|^2 dt\Big\}\Big| \\
&\le E\Big\{|\Delta\bar{Y}_0^\theta\bar{Y}_0^0| + \int_0^T |\Delta\bar{Z}_t^\theta\bar{Z}_t^0| dt\Big\} \\
&\le CE\Big\{|\Delta\bar{Y}_0^\theta|^2 + \int_0^T |\Delta\bar{Z}_t^\theta|^2 dt\Big\} + \frac{1}{2}E\Big\{|\bar{Y}_0^0|^2 + \int_0^T |\bar{Z}_t^0|^2 dt\Big\} \\
&\le CK_0^4\lambda^2 + \frac{1}{2}E\Big\{|\bar{Y}_0^0|^2 + \int_0^T |\bar{Z}_t^0|^2 dt\Big\}.
\end{aligned}$$

Then, by (2.9) we have

$$\begin{aligned}
E\Big\{\bar{Y}_0^\theta\bar{Y}_0^0 + \int_0^T \bar{Z}_t^\theta\bar{Z}_t^0 dt\Big\} &\ge \frac{1}{2}E\Big\{|\bar{Y}_0^0|^2 + \int_0^T |\bar{Z}_t^0|^2 dt\Big\} - CK_0^4\lambda^2 \\
&\ge cV(y_0, z^0) - CK_0^4\lambda^2.
\end{aligned}$$

Choose $c_1 \stackrel{\triangle}{=} \sqrt{\frac{c}{2C}}$ for the constants c, C as above, and $\lambda \stackrel{\triangle}{=} \frac{c_1\varepsilon}{K_0^2}$. Then by (2.10) we get

$$E\Big\{\bar{Y}_0^\theta\bar{Y}_0^0 + \int_0^T \bar{Z}_t^\theta\bar{Z}_t^0 dt\Big\} \ge cV(y_0, z^0) - \frac{c}{2}\varepsilon^2 \ge \frac{c}{2}V(y_0, z^0). \tag{2.20}$$

Then (2.11) follows directly from (2.15).

Finally, plug λ into (2.18) and let $\theta = 1$ we get (2.12) for some C_0. ■

Now we are ready to approximate FBSDE (1.4) iteratively. Set

$$y_0 \triangleq 0, \quad z_t^0 \triangleq 0, \quad K_0 \triangleq E^{\frac{1}{4}}\{|Y_T^0 - g(X_T^0)|^4\}. \tag{2.21}$$

For $k = 0, 1, \cdots$, let $(X^k, Y^k, \bar{Y}^k, \tilde{Y}^k, \bar{Z}^k, \tilde{Z}^k)$ be the solution to the following FBSDE:

$$\begin{cases} X_t^k = x + \int_0^t \sigma(s, X_s^k, Y_s^k) dW_s; \\ Y_t^k = y_k - \int_0^t f(s, X_s^k, Y_s^k) ds + \int_0^t z_s^k dW_s; \\ \bar{Y}_t^k = [Y_T^k - g(X_T^k)] - \int_t^T [f_y^k \bar{Y}_s^k + \sigma_y^k \tilde{Z}_s^k] ds - \int_t^T \bar{Z}_s^k dW_s; \\ \tilde{Y}_t^k = g'(X_T^k)[Y_T^k - g(X_T^k)] + \int_t^T [f_x^k \bar{Y}_s^k + \sigma_x^k \tilde{Z}_s^k] ds - \int_t^T \tilde{Z}_s^k dW_s. \end{cases} \tag{2.22}$$

We note that (2.22) is decoupled, with forward components (X^k, Y^k) and backward components $(\bar{Y}^k, \tilde{Y}^k, \bar{Z}^k, \tilde{Z}^k)$. Denote

$$\lambda_k \triangleq \frac{c_1 \varepsilon}{K_k^2}, \quad y_{k+1} \triangleq y_k - \lambda_k \bar{Y}_0^k, \quad z_t^{k+1} \triangleq z_t^k - \lambda_k \bar{Z}_t^k, \quad K_{k+1}^4 \triangleq K_k^4 + 2C_0 \varepsilon K_k^2, \tag{2.23}$$

where c_1, C_0 are the constants in Lemma 2.4.

Theorem 2.5 *There exist constants C_1, C_2 and $N \le C_1 \varepsilon^{-C_2}$ such that*

$$V(y_N, z^N) \le \varepsilon^2.$$

Proof. Assume $V(y_k, z^k) > \varepsilon^2$ for $k = 0, \cdots, N-1$. Note that $K_{k+1}^4 \le (K_k^2 + C_0 \varepsilon)^2$. Then $K_{k+1}^2 \le K_k^2 + C_0 \varepsilon$, which implies that

$$K_k^2 \le K_0^2 + C_0 k \varepsilon.$$

Thus by Lemma 2.4 we have

$$V(y_{k+1}, z^{k+1}) \le \Big[1 - \frac{c_0 \varepsilon}{K_0^2 + C_0 k \varepsilon}\Big] V(y_k, z^k).$$

Note that $\log(1-x) \le -x$ for $x \in [0, 1)$. For ε small enough, we get

$$\begin{aligned} &\log(V(y_N, z^N)) \le \log(V(0,0)) + \sum_{k=0}^{N-1} \log\Big(1 - \frac{c_0 \varepsilon}{K_0^2 + C_0 k \varepsilon}\Big) \\ &\le C - c \sum_{k=0}^{N-1} \frac{1}{k + \varepsilon^{-1}} \le C - c \int_0^{N-1} \frac{dx}{x + \varepsilon^{-1}} \\ &= C - c\Big[\log(N - 1 + \varepsilon^{-1}) - \log(\varepsilon^{-1})\Big] = C - c \log(1 + \varepsilon(N-1)). \end{aligned}$$

For c, C as above, choose N to be the smallest integer such that

$$N \geq 1 + \varepsilon^{-1}[e^{\frac{C}{c}}\varepsilon^{-\frac{2}{c}} - 1].$$

We get

$$\log(V(y_N, z^N)) \leq C - c[\frac{C}{c} - \frac{2}{c}\log(\varepsilon)] = \log(\varepsilon^2),$$

which obviously proves the theorem. ■

3 Time Discretization

We now investigate the time discretization of FBSDEs (2.22). Fix n and denote

$$t_i \triangleq \frac{i}{n}T; \quad \Delta t \triangleq \frac{T}{n}; \qquad i = 0, \cdots, n.$$

3.1 Discretization of the FSDEs

Given $y_0 \in \mathbb{R}$ and $z^0 \in L^2(\mathbf{F})$, denote

$$\begin{cases} X^{n,0}_{t_0} \triangleq x; \quad Y^{n,0}_{t_0} \triangleq y_0; \\ X^{n,0}_t \triangleq X^{n,0}_{t_i} + \sigma(t_i, X^{n,0}_{t_i}, Y^{n,0}_{t_i})[W_t - W_{t_i}], \quad t \in (t_i, t_{i+1}]; \\ Y^{n,0}_t \triangleq Y^{n,0}_{t_i} - f(t_i, X^{n,0}_{t_i}, Y^{n,0}_{t_i})[t - t_i] + \int_{t_i}^t z^0_s dW_s, \quad t \in (t_i, t_{i+1}]. \end{cases} \tag{3.1}$$

Note that we do not discretize z^0 here. For notational simplicity, we denote

$$X^{n,0}_i \triangleq X^{n,0}_{t_i}; \quad Y^{n,0}_i \triangleq Y^{n,0}_{t_i}; \qquad i = 0, \cdots, n.$$

Define

$$V_n(y_0, z^0) \triangleq \frac{1}{2}E\{|Y^{n,0}_n - g(X^{n,0}_n)|^2\}. \tag{3.2}$$

First we have

Theorem 3.1 *Denote*

$$I^{n,0} \triangleq E\Big\{\max_{0\leq i\leq n}[|X_{t_i} - X^{n,0}_i|^2 + |Y_{t_i} - Y^{n,0}_i|^2] + \int_0^T |Z_t - z^0_t|^2 dt\Big\}. \tag{3.3}$$

Then

$$I^{n,0} \leq CV_n(y_0, z^0) + \frac{C}{n}. \tag{3.4}$$

We note that (see, e.g. Zhang [21]),

$$\max_{0\le i\le n-1} E\Big\{\sup_{t_i\le t\le t_{i+1}}[|X_t-X_{t_i}|^2+|Y_t-Y_{t_i}|^2]\Big\}\le \frac{C}{n};$$
$$E\Big\{\max_{0\le i\le n-1}\sup_{t_i\le t\le t_{i+1}}[|X_t-X_{t_i}|^2+|Y_t-Y_{t_i}|^2]\Big\}\le \frac{C\log n}{n};$$
$$E\Big\{\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|Z_t-\frac{1}{\Delta t}E_i\{\int_{t_i}^{t_{i+1}}Z_s ds\}|^2dt\Big\}\le\frac{C}{n};$$

where $E_i\{\cdot\}\stackrel{\triangle}{=}E\{\cdot|\mathcal{F}_{t_i}\}$. Then one can easily show the following estimates:

Corollary 3.2 *We have*

$$\max_{0\le i\le n-1} E\Big\{\sup_{t_i\le t\le t_{i+1}}[|X_t-X_i^{n,0}|^2+|Y_t-Y_i^{n,0}|^2]\Big\}\le CV_n(y_0,z^0)+\frac{C}{n};$$
$$E\Big\{\max_{0\le i\le n-1}\sup_{t_i\le t\le t_{i+1}}[|X_t-X_i^{n,0}|^2+|Y_t-Y_i^{n,0}|^2]\Big\}\le CV_n(y_0,z^0)+\frac{C\log n}{n};$$
$$E\Big\{\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|Z_t-\frac{1}{\Delta t}E_i\{\int_{t_i}^{t_{i+1}}z^0_s ds\}|^2dt\Big\}\le CV_n(y_0,z^0)+\frac{C}{n}.$$

Proof of Theorem 3.1. Recall (2.1). For $i=0,\cdots,n$, denote

$$\Delta X_i\stackrel{\triangle}{=}X^0_{t_i}-X_i^{n,0};\quad \Delta Y_i\stackrel{\triangle}{=}Y^0_{t_i}-Y_i^{n,0}.$$

Then

$$\begin{cases}\Delta X_0=0;\quad \Delta Y_0=0;\\ \Delta X_{i+1}=\Delta X_i+\int_{t_i}^{t_{i+1}}\Big[[\alpha_i^1\Delta X_i+\beta_i^1\Delta Y_i]+[\sigma(t,X_t^0,Y_t^0)-\sigma(t_i,X^0_{t_i},Y^0_{t_i})]\Big]dW_t;\\ \Delta Y_{i+1}=\Delta Y_i-\int_{t_i}^{t_{i+1}}\Big[[\alpha_i^2\Delta X_i+\beta_i^2\Delta Y_i]+[f(t,X_t^0,Y_t^0)-f(t_i,X^0_{t_i},Y^0_{t_i})]\Big]dt;\end{cases}$$

where $\alpha_i^j,\beta_i^j\in\mathcal{F}_{t_i}$ are defined in an obvious way and are uniformly bounded. Then

$$\begin{aligned}&E\{|\Delta X_{i+1}|^2\}\\ &=E\Big\{|\Delta X_i|^2+\int_{t_i}^{t_{i+1}}\Big[[\alpha_i^1\Delta X_i+\beta_i^1\Delta Y_i]+[\sigma(t,X_t^0,Y_t^0)-\sigma(t_i,X^0_{t_i},Y^0_{t_i})]\Big]^2dt\Big\}\\ &\le E\Big\{|\Delta X_i|^2+\frac{C}{n}[|\Delta X_i|^2+|\Delta Y_i|^2]+C\int_{t_i}^{t_{i+1}}[|X_t^0-X^0_{t_i}|^2+|Y^0_t-Y^0_{t_i}|^2]dt\Big\};\end{aligned}$$

and, similarly,

$$\begin{aligned}&E\{|\Delta Y_{i+1}|^2\}\\ &\le E\Big\{|\Delta Y_i|^2+\frac{C}{n}[|\Delta X_i|^2+|\Delta Y_i|^2]+C\int_{t_i}^{t_{i+1}}[|X_t^0-X^0_{t_i}|^2+|Y^0_t-Y^0_{t_i}|^2]dt\Big\}.\end{aligned}$$

Denote

$$A_i \triangleq E\{|\Delta X_i|^2 + |\Delta Y_i|^2\}.$$

Then $A_0 = 0$, and

$$A_{i+1} \le [1 + \frac{C}{n}]A_i + CE\Big\{\int_{t_i}^{t_{i+1}} [|X_t^0 - X_{t_i}^0|^2 + |Y_t^0 - Y_{t_i}^0|^2]dt\Big\}.$$

By the discrete Gronwall inequality we get

$$\begin{aligned}
&\max_{0\le i\le n} A_i \le C\sum_{i=0}^{n-1} E\Big\{\int_{t_i}^{t_{i+1}} [|X_t^0 - X_{t_i}^0|^2 + |Y_t^0 - Y_{t_i}^0|^2]dt\Big\} \\
&\le C\sum_{i=0}^{n-1} E\Big\{\int_{t_i}^{t_{i+1}} \Big[\int_{t_i}^{t} |\sigma(s, X_s^0, Y_s^0)|^2 ds + |\int_{t_i}^{t} f(s, X_s^0, Y_s^0)ds|^2 + \int_{t_i}^{t} |z_s^0|^2 ds\Big]dt\Big\} \\
&\le C\sum_{i=0}^{n-1} E\Big\{|\Delta t|^2 + |\Delta t|^3 + \Delta t\int_{t_i}^{t_{i+1}} |z_t^0|^2 dt\Big\} \\
&\le \frac{C}{n} + \frac{C}{n}E\Big\{\int_0^T |z_t^0|^2 dt\Big\}.
\end{aligned} \tag{3.5}$$

Next, note that

$$\begin{aligned}
\Delta X_i &= \sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} \Big[[\alpha_j^1 \Delta X_j + \beta_j^1 \Delta Y_j] + [\sigma(t, X_t^0, Y_t^0) - \sigma(t_j, X_{t_j}^0, Y_{t_j}^0)]\Big]dW_t; \\
\Delta Y_i &= \sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}} \Big[[\alpha_j^2 \Delta X_j + \beta_j^2 \Delta Y_j] - [f(t, X_t^0, Y_t^0) - f(t_j, X_{t_j}^0, Y_{t_j}^0)]\Big]dt.
\end{aligned}$$

Applying the Burkholder-Davis-Gundy Inequality and by (3.5) we get

$$E\Big\{\max_{0\le i\le n}[|\Delta X_i|^2 + |\Delta Y_i|^2]\Big\} \le \frac{C}{n} + \frac{C}{n}E\Big\{\int_0^T |z_t^0|^2 dt\Big\},$$

which, together with Theorem 2.1, implies that

$$I^{n,0} \le CV(y_0, z^0) + \frac{C}{n} + \frac{C}{n}E\Big\{\int_0^T |z_t^0|^2 dt\Big\}.$$

Finally, note that

$$V(y_0, z^0) \le CV_n(y_0, z^0) + CE\Big\{|\Delta X_n|^2 + |\Delta Y_n|^2\Big\} = CV_n(y_0, z^0) + CA_n.$$

We get

$$I^{n,0} \le CV_n(y_0, z^0) + \frac{C}{n} + \frac{C}{n}E\Big\{\int_0^T |z_t^0|^2 dt\Big\}.$$

Moreover, noting that $Z_t = u_x(t, X_t)\sigma(t, X_t, Y_t)$ is bounded, we have

$$
\begin{aligned}
E\Big\{\int_0^T |z_t^0|^2 dt\Big\} &\leq CE\Big\{\int_0^T |Z_t - z_t^0|^2 dt\Big\} + CE\Big\{\int_0^T |Z_t|^2 dt\Big\} \\
&\leq CE\Big\{\int_0^T |Z_t - z_t^0|^2 dt\Big\} + C.
\end{aligned}
$$

Thus

$$
I^{n,0} \leq CV_n(y_0, z^0) + \frac{C}{n} + \frac{C}{n}E\Big\{\int_0^T |Z_t - z_t^0|^2 dt\Big\}.
$$

Choose $n \geq 2C$ for C as above, by (3.3) we prove (3.4) immediately. ■

3.2 Discretization of the BSDEs

Define the adjoint processes (or say, discretize BSDE (2.5)) as follows.

$$
\begin{cases}
\bar{Y}_n^{n,0} \triangleq Y_n^{n,0} - g(X_n^{n,0}); \quad \tilde{Y}_n^{n,0} \triangleq g'(X_n^{n,0})[Y_n^{n,0} - g(X_n^{n,0})]; \\
\bar{Y}_{i-1}^{n,0} = \bar{Y}_i^{n,0} - f_{y,i-1}^{n,0}\bar{Y}_{i-1}^{n,0}\Delta t - \sigma_{y,i-1}^{n,0}\int_{t_{i-1}}^{t_i} \tilde{Z}_t^{n,0} dt - \int_{t_{i-1}}^{t_i} \bar{Z}_t^{n,0} dW_t; \\
\tilde{Y}_{i-1}^{n,0} = \tilde{Y}_i^{n,0} + f_{x,i-1}^{n,0}\bar{Y}_{i-1}^{n,0}\Delta t + \sigma_{x,i-1}^{n,0}\int_{t_{i-1}}^{t_i} \tilde{Z}_t^{n,0} dt - \int_{t_{i-1}}^{t_i} \tilde{Z}_t^{n,0} dW_t;
\end{cases}
\tag{3.6}
$$

where $\varphi_i^{n,0} \triangleq \varphi(t_i, X_i^{n,0}, Y_i^{n,0})$ for any function φ. We note again that $\bar{Z}^{n,0}, \tilde{Z}^{n,0}$ are not discretized. Denote $\Delta W_{i+1} \triangleq W_{t_{i+1}} - W_{t_i}, i = 0, \cdots, n-1$. Following the direction $(\Delta y, \Delta z)$, by (3.1) we have the following gradients:

$$
\begin{cases}
\nabla X_0^{n,0} = 0, \quad \nabla Y_0^{n,0} = \Delta y; \\
\nabla X_{i+1}^{n,0} = \nabla X_i^{n,0} + \Big[\sigma_{x,i}^{n,0}\nabla X_i^{n,0} + \sigma_{y,i}^{n,0}\nabla Y_i^{n,0}\Big]\Delta W_{i+1}; \\
\nabla Y_{i+1}^{n,0} = \nabla Y_i^{n,0} - \Big[f_{x,i}^{n,0}\nabla X_i^{n,0} + f_{y,i}^{n,0}\nabla Y_i^{n,0}\Big]\Delta t + \int_{t_i}^{t_{i+1}} \Delta z_t dW_t; \\
\nabla V_n(y_0, z^0) = E\Big\{[Y_n^{n,0} - g(X_n^{n,0})][\nabla Y_n^{n,0} - g'(X_n^{n,0})\nabla X_n^{n,0}]\Big\}.
\end{cases}
$$

Then

$$
\begin{aligned}
\nabla V_n(y_0, z^0) &= E\Big\{\bar{Y}_n^{n,0}\nabla Y_n^{n,0} - \tilde{Y}_n^{n,0}\nabla X_n^{n,0}\Big\} \\
&= E\Big\{\Big[\bar{Y}_{n-1}^{n,0} + f_{y,n-1}^{n,0}\bar{Y}_{n-1}^{n,0}\Delta t + \sigma_{y,n-1}^{n,0}\int_{t_{n-1}}^{t_n} \tilde{Z}_t^{n,0} dt + \int_{t_{n-1}}^{t_n} \bar{Z}_t^{n,0} dW_t\Big] \times \\
&\quad \Big[\nabla Y_{n-1}^{n,0} - [f_{x,n-1}^{n,0}\nabla X_{n-1}^{n,0} + f_{y,n-1}^{n,0}\nabla Y_{n-1}^{n,0}]\Delta t + \int_{t_{n-1}}^{t_n} \Delta z_t dW_t\Big] \\
&\quad -\Big[\tilde{Y}_{n-1}^{n,0} - f_{x,n-1}^{n,0}\bar{Y}_{n-1}^{n,0}\Delta t - \sigma_{x,n-1}^{n,0}\int_{t_{n-1}}^{t_n} \tilde{Z}_t^{n,0} dt + \int_{t_{n-1}}^{t_n} \tilde{Z}_t^{n,0} dW_t\Big] \times \\
&\quad \Big[\nabla X_{n-1}^{n,0} + [\sigma_{x,n-1}^{n,0}\nabla X_{n-1}^{n,0} + \sigma_{y,n-1}^{n,0}\nabla Y_{n-1}^{n,0}]\Delta W_n\Big]\} \\
&= E\Big\{\bar{Y}_{n-1}^{n,0}\nabla Y_{n-1}^{n,0} - \tilde{Y}_{n-1}^{n,0}\nabla X_{n-1}^{n,0} + \int_{t_{n-1}}^{t_n} \bar{Z}_t^{n,0}\Delta z_t dt + I_n^{n,0}\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
I_i^{n,0} \quad \stackrel{\triangle}{=} \quad & \sigma_{y,i-1}^{n,0} \int_{t_{i-1}}^{t_i} \tilde{Z}_t^{n,0} dt \int_{t_{i-1}}^{t_i} \Delta z_t dW_t \\
& + \sigma_{x,i-1}^{n,0} [\sigma_{x,i-1}^{n,0} \nabla X_{i-1}^{n,0} + \sigma_{y,i-1}^{n,0} \nabla Y_{i-1}^{n,0}] \Delta W_i \int_{t_{i-1}}^{t_i} \tilde{Z}_t^{n,0} dt \\
& - \sigma_{y,i-1}^{n,0} [f_{x,i-1}^{n,0} \nabla X_{i-1}^{n,0} + f_{y,i-1}^{n,0} \nabla Y_{i-1}^{n,0}] \Delta t \int_{t_{i-1}}^{t_i} \tilde{Z}_t^{n,0} dt \\
& - f_{y,i-1}^{n,0} \bar{Y}_{i-1}^{n,0} [f_{x,i-1}^{n,0} \nabla X_{i-1}^{n,0} + f_{y,i-1}^{n,0} \nabla Y_{i-1}^{n,0}] |\Delta t|^2.
\end{aligned} \tag{3.7}
$$

Repeating the same arguments and by induction we get

$$
\nabla V_n(y_0, z^0) = E\Big\{ \bar{Y}_0^{n,0} \Delta y + \int_0^T \bar{Z}_t^{n,0} \Delta z_t dt + \sum_{i=1}^n I_i^{n,0} \Big\}. \tag{3.8}
$$

¿From now on, we choose the following "almost" steepest descent direction:

$$
\Delta y \stackrel{\triangle}{=} -\bar{Y}_0^{n,0}; \quad \int_{t_{i-1}}^{t_i} \Delta z_t dW_t \stackrel{\triangle}{=} E_{i-1}\{\bar{Y}_i^{n,0}\} - \bar{Y}_i^{n,0}. \tag{3.9}
$$

We note that Δz is well defined here. Then we have

Lemma 3.3 *Assume (3.9). Then for n large, we have*

$$
\nabla V_n(y_0, z^0) \leq -c V_n(y_0, z^0).
$$

Proof. We proceed in several steps.

Step 1. We show that

$$
E\Big\{ \max_{0 \leq i \leq n} [|\bar{Y}_i^{n,0}|^2 + |\tilde{Y}_i^{n,0}|^2] + \int_0^T [|\bar{Z}_t^{n,0}|^2 + |\tilde{Z}_t^{n,0}|^2] dt \Big\} \leq C V_n(y_0, z^0). \tag{3.10}
$$

In fact, for any i,

$$
\begin{aligned}
& E\Big\{ |\bar{Y}_{i-1}^{n,0}|^2 + |\tilde{Y}_{i-1}^{n,0}|^2 + \int_{t_{i-1}}^{t_i} [|\bar{Z}_t^{n,0}|^2 + |\tilde{Z}_t^{n,0}|^2] dt \Big\} \\
& = E\Big\{ \Big| \bar{Y}_i^{n,0} - f_{y,i-1}^{n,0} \bar{Y}_{i-1}^{n,0} \Delta t - \sigma_{y,i-1}^{n,0} \int_{t_{i-1}}^{t_i} \tilde{Z}_t^{n,0} dt \Big|^2 \\
& \quad + \Big| \tilde{Y}_i^{n,0} + f_{x,i-1}^{n,0} \bar{Y}_{i-1}^{n,0} \Delta t + \sigma_{x,i-1}^{n,0} \int_{t_{i-1}}^{t_i} \tilde{Z}_t^{n,0} dt \Big|^2 \Big\} \\
& \leq [1 + \frac{C}{n}] E\Big\{ |\bar{Y}_i^{n,0}|^2 + |\tilde{Y}_i^{n,0}|^2 \Big\} + \frac{C}{n} E\{|\bar{Y}_{i-1}^{n,0}|^2\} + \frac{1}{2} E\Big\{ \int_{t_{i-1}}^{t_i} [|\bar{Z}_t^{n,0}|^2 + |\tilde{Z}_t^{n,0}|^2] dt \Big\}.
\end{aligned}
$$

Then

$$
E\Big\{ |\bar{Y}_{i-1}^{n,0}|^2 + |\tilde{Y}_{i-1}^{n,0}|^2 + \frac{1}{2} \int_{t_{i-1}}^{t_i} [|\bar{Z}_t^{n,0}|^2 + |\tilde{Z}_t^{n,0}|^2] dt \Big\} \leq [1 + \frac{C}{n}] E\Big\{ |\bar{Y}_i^{n,0}|^2 + |\tilde{Y}_i^{n,0}|^2 \Big\}.
$$

By standard arguments we get

$$\begin{aligned}
&\max_{0\le i\le n} E\Big\{|\bar Y_i^{n,0}|^2+|\tilde Y_i^{n,0}|^2\Big\}+E\Big\{\int_0^T[|\bar Z_t^{n,0}|^2+|\tilde Z_t^{n,0}|^2]dt\Big\}\\
&\le CE\Big\{|\bar Y_n^{n,0}|^2+|\tilde Y_n^{n,0}|^2\Big\}\le CV_n(y_0,z^0).
\end{aligned}$$

Then (3.10) follows from the Burkholder-Davis-Gundy Inequality.

Step 2. We show that

$$V_n(y_0,z^0)\le CE\Big\{|\bar Y_0^{n,0}|^2+\int_0^T|\bar Z_t^{n,0}|^2dt\Big\}. \tag{3.11}$$

In fact, for $t\in(t_i,t_{i+1}]$, let

$$\begin{aligned}
\bar Y_t^{n,0}&\overset{\triangle}{=}\bar Y_i^{n,0}+f_{y,i}^{n,0}\bar Y_i^{n,0}[t-t_i]+\sigma_{y,i}^{n,0}\int_{t_i}^t\tilde Z_s^{n,0}ds+\int_{t_i}^t\bar Z_s^{n,0}dW_s;\\
\tilde Y_t^{n,0}&=\tilde Y_i^{n,0}-f_{x,i}^{n,0}\bar Y_i^{n,0}[t-t_i]-\sigma_{x,i}^{n,0}\int_{t_i}^t\tilde Z_s^{n,0}ds+\int_{t_i}^t\tilde Z_s^{n,0}dW_s;
\end{aligned}$$

Denote $\pi(t)\overset{\triangle}{=}t_i$ for $t_i\in[t_i,t_{i+1})$. Then one can write them as

$$\begin{aligned}
\bar Y_t^{n,0}&=\bar Y_0^{n,0}+\int_0^t f_y^{n,0}(\pi(s))[\bar Y_{\pi(s)}^{n,0}-\bar Y_s^{n,0}]ds\\
&\quad+\int_0^t[f_y^{n,0}(\pi(s))\bar Y_s^{n,0}+\sigma_y^{n,0}(\pi(s))\tilde Z_s^{n,0}]ds+\int_0^t\bar Z_s^{n,0}dW_s;\\
\tilde Y_t^{n,0}&=g'(X_n^{n,0})\bar Y_T^{n,0}+\int_t^T f_x^{n,0}(\pi(s))[\bar Y_{\pi(s)}^{n,0}-\bar Y_s^{n,0}]ds\\
&\quad+\int_t^T[f_x^{n,0}(\pi(s))\bar Y_s^{n,0}+\sigma_x^{n,0}(\pi(s))\tilde Z_s^{n,0}]ds-\int_t^T\tilde Z_s^{n,0}dW_s.
\end{aligned}$$

Applying Theorem 1.2, we get

$$\begin{aligned}
&V_n(y_0,z^0)=\frac12E\{|\bar Y_n^{n,0}|^2\}\\
&\le CE\Big\{|\bar Y_0^{n,0}|^2+\int_0^T|\bar Z_t^{n,0}|^2dt+\int_0^T|\bar Y_t^{n,0}-\bar Y_{\pi(t)}^{n,0}|^2dt\Big\}\\
&=CE\Big\{|\bar Y_0^{n,0}|^2+\int_0^T|\bar Z_t^{n,0}|^2dt+\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|\bar Y_t^{n,0}-\bar Y_{t_i}^{n,0}|^2dt\Big\}\\
&\le CE\Big\{|\bar Y_0^{n,0}|^2+\int_0^T|\bar Z_t^{n,0}|^2dt\\
&\quad+C\Delta t\sum_{i=0}^{n-1}\Big[|\bar Y_i^{n,0}|^2|\Delta t|^2+\Delta t\int_{t_i}^{t_{i+1}}|\tilde Z_t^{n,0}|^2dt+\int_{t_i}^{t_{i+1}}|\bar Z_t^{n,0}|^2dt\Big]\Big\}\\
&\le CE\Big\{|\bar Y_0^{n,0}|^2+\int_0^T|\bar Z_t^{n,0}|^2dt\Big\}+\frac{C}{n}V_n(y_0,z^0),
\end{aligned}$$

thanks to (3.10). Choosing $n \geq 2C$, we get (3.11) immediately.

Step 3. By (3.9) we have

$$
\begin{aligned}
&E\Big\{\int_0^T |\Delta z_t + \bar Z_t^{n,0}|^2 dt\Big\} = \sum_{i=0}^{n-1} E\Big\{\Big|\int_{t_i}^{t_{i+1}} \Delta z_t dW_t + \int_{t_i}^{t_{i+1}} \bar Z_t^{n,0} dW_t\Big|^2\Big\}\\
&= \sum_{i=0}^{n-1} E\Big\{\Big|\bar Y_i^{n,0} + f_{y,i}^{n,0}\bar Y_i^{n,0}\Delta t + \sigma_{y,i}^{n,0} E_i\{\int_{t_i}^{t_{i+1}} \tilde Z_t^{n,0} dt\} - \bar Y_{i+1}^{n,0} + \int_{t_i}^{t_{i+1}} \bar Z_t^{n,0} dW_t\Big|^2\Big\}\\
&= \sum_{i=0}^{n-1} E\Big\{|\sigma_{y,i}^{n,0}|^2\Big|\int_{t_i}^{t_{i+1}} \tilde Z_t^{n,0} dt - E_i\{\int_{t_i}^{t_{i+1}} \tilde Z_t^{n,0} dt\}\Big|^2\Big\}\\
&\leq C\Delta t \sum_{i=0}^{n-1} E\Big\{\int_{t_i}^{t_{i+1}} |\tilde Z_t^{n,0}|^2 dt\Big\} \leq \frac{C}{n} V_n(y_0, z^0), \qquad (3.12)
\end{aligned}
$$

where we used (3.10) for the last inequality. Then,

$$
\begin{aligned}
&\Big|E\Big\{\bar Y_0^{n,0}\Delta y + \int_0^T \bar Z_t^{n,0} \Delta z_t dt\Big\} + E\Big\{|\bar Y_0^{n,0}|^2 + \int_0^T |\bar Z_t^{n,0}|^2 dt\Big\}\Big|\\
&= \Big|E\Big\{\int_0^T \bar Z_t^{n,0}[\Delta z_t + \bar Z_t^{n,0}] dt\Big\}\Big|\\
&\leq C E^{\frac12}\Big\{\int_0^T |\bar Z_t^{n,0}|^2 dt\Big\} E^{\frac12}\Big\{\int_0^T |\Delta z_t + \bar Z_t^{n,0}|^2 dt\Big\}\\
&\leq C\sqrt{V_n(y_0, z^0)}\sqrt{\frac{C}{n} V_n(y_0, z^0)} = \frac{C}{\sqrt n} V_n(y_0, z^0).
\end{aligned}
$$

Assume n is large. By (3.11) we get

$$
E\Big\{\bar Y_0^{n,0}\Delta y + \int_0^T \bar Z_t^{n,0}\Delta z_t dt\Big\} \leq -\frac12 E\Big\{|\bar Y_0^{n,0}|^2 + \int_0^T |\bar Z_t^{n,0}|^2 dt\Big\} \leq -cV_n(y_0, z^0). \qquad (3.13)
$$

Step 4. It remains to estimate $I_i^{n,0}$. First, by standard arguments and recalling (3.9), (3.12), and (3.10), we have

$$
\begin{aligned}
&E\Big\{\max_{0\leq i\leq n}[|\nabla X_i^{n,0}|^2 + |\nabla Y_i^{n,0}|^2]\Big\} \leq CE\Big\{|\Delta y|^2 + \int_0^T |\Delta z_t|^2 dt\Big\}\\
&\leq CE\Big\{|\bar Y_0^{n,0}|^2 + \int_0^T [|\Delta z_t + \bar Z_t^{n,0}|^2 + |\bar Z_t^{n,0}|^2] dt\Big\} \leq CV_n(y_0, z^0). \qquad (3.14)
\end{aligned}
$$

Then

$$
\begin{aligned}
\Big|\sum_{i=1}^n E\{I_i^{n,0}\}\Big| \;\leq\; & \frac{C}{\sqrt n}\sum_{i=0}^{n-1} E\Big\{\int_{t_i}^{t_{i+1}} |\tilde Z_t^{n,0}|^2 dt + \int_{t_i}^{t_{i+1}} |\Delta z_t|^2 dt\\
& + [|\nabla X_i^{n,0}|^2 + |\nabla Y_i^{n,0}|^2][E_{t_i}\{|\Delta W_{i+1}|^2\} + |\Delta t|^2] + |\bar Y_i^{n,0}|^2 \Delta t\Big\}\\
\leq\; & \frac{C}{\sqrt n} E\Big\{\int_0^T [|\tilde Z_t^{n,0}|^2 + |\bar Z_t^{n,0}|^2 + |\Delta z_t + \bar Z_t^{n,0}|^2] dt\Big\}
\end{aligned}
$$

$$+\frac{C}{\sqrt{n}}\max_{0\le i\le n} E\Big\{|\nabla X_i^{n,0}|^2+|\nabla Y_i^{n,0}|^2+|\bar{Y}_i^{n,0}|^2\Big\}$$
$$\le \frac{C}{\sqrt{n}}V_n(y_0,z^0). \tag{3.15}$$

Recall (3.8). Combining the above inequality with (3.13) we prove the lemma for large n. ■

3.3 Iterative Modifications

We now fix a desired error level ε. In light of Theorem 3.1, we set $n=\varepsilon^{-2}$. So it suffices to find (y,z) such that $V_n(y,z)\le\varepsilon^2$. As in §2.3, we assume

$$V_n(y_0,z^0)>\varepsilon^2;\qquad E\Big\{|Y_n^{n,0}-g(X_n^{n,0})|^4\Big\}\le K_0^4. \tag{3.16}$$

Lemma 3.4 *Assume (3.16). There exist constants $C_0,c_0,c_1>0$, which are independent of K_0 and ε, such that*

$$\Delta V_n(y_0,z^0)\stackrel{\triangle}{=}V_n(y_1,z^1)-V_n(y_0,z^0)\le-\frac{c_0\varepsilon}{K_0^2}V_n(y_0,z^0), \tag{3.17}$$

and

$$E\{|Y_n^{n,1}-g(X_T^{n,1})|^4\}\le K_1^4\stackrel{\triangle}{=}K_0^4+2C_0\varepsilon K_0^2, \tag{3.18}$$

where, recalling (3.9) and denoting $\lambda\stackrel{\triangle}{=}\frac{c_1\varepsilon}{K_0^2}$,

$$y_1\stackrel{\triangle}{=}y_0+\lambda\Delta y;\quad z_t^1\stackrel{\triangle}{=}z_t^0+\lambda\Delta z_t; \tag{3.19}$$

and, for $0\le\theta\le1$,

$$\left\{\begin{array}{l} X_0^{n,\theta}\stackrel{\triangle}{=}x;\quad Y_0^{n,\theta}\stackrel{\triangle}{=}y_0+\theta\lambda\Delta y;\\ X_{i+1}^{n,\theta}\stackrel{\triangle}{=}X_i^{n,\theta}+\sigma(t_i,X_i^{n,\theta},Y_i^{n,\theta})\Delta W_{i+1};\\ Y_{i+1}^{n,\theta}\stackrel{\triangle}{=}Y_i^{n,\theta}-f(t_i,X_i^{n,\theta},Y_i^{n,\theta})\Delta t+\displaystyle\int_{t_i}^{t_{i+1}}[z_t+\theta\lambda\Delta z_t]dW_t.\end{array}\right. \tag{3.20}$$

Proof. We shall follow the proof for Lemma 2.4.
Step 1. For $0\le\theta\le1$, denote

$$\left\{\begin{array}{l} \bar{Y}_n^{n,\theta}\stackrel{\triangle}{=}Y_n^{n,\theta}-g(X_n^{n,\theta});\quad \tilde{Y}_n^{n,\theta}\stackrel{\triangle}{=}g'(X_n^{n,\theta})[Y_n^{n,\theta}-g(X_n^{n,\theta})];\\ \bar{Y}_{i-1}^{n,\theta}=\bar{Y}_i^{n,\theta}-f_{y,i-1}^{n,\theta}\bar{Y}_{i-1}^{n,\theta}\Delta t-\sigma_{y,i-1}^{n,\theta}\displaystyle\int_{t_{i-1}}^{t_i}\tilde{Z}_t^{n,\theta}dt-\int_{t_{i-1}}^{t_i}\bar{Z}_t^{n,\theta}dW_t;\\ \tilde{Y}_{i-1}^{n,\theta}=\tilde{Y}_i^{n,\theta}+f_{x,i-1}^{n,\theta}\bar{Y}_{i-1}^{n,\theta}\Delta t+\sigma_{x,i-1}^{n,\theta}\displaystyle\int_{t_{i-1}}^{t_i}\tilde{Z}_t^{n,\theta}dt-\int_{t_{i-1}}^{t_i}\tilde{Z}_t^{n,\theta}dW_t,\end{array}\right.$$

and

$$\begin{cases} \nabla X_0^{n,\theta} = 0, \quad \nabla Y_0^{n,\theta} = \Delta y; \\ \nabla X_{i+1}^{n,\theta} = \nabla X_i^{n,\theta} + \Big[\sigma_{x,i}^{n,\theta}\nabla X_i^{n,\theta} + \sigma_{y,i}^{n,\theta}\nabla Y_i^{n,\theta}\Big]\Delta W_{i+1}; \\ \nabla Y_{i+1}^{n,\theta} = \nabla Y_i^{n,\theta} - \Big[f_{x,i}^{n,\theta}\nabla X_i^{n,\theta} + f_{y,i}^{n,\theta}\nabla Y_i^{n,\theta}\Big]\Delta t + \int_{t_i}^{t_{i+1}} \Delta z_t dW_t; \end{cases}$$

where $\varphi_i^{n,\theta} \triangleq \varphi(t_i, X_i^{n,\theta}, Y_i^{n,\theta})$ for any function φ. Then

$$\begin{aligned} \Delta V_n(y_0, z^0) &= E\Big\{[Y_n^{n,1} - g(X_n^{n,1})]^2 - [Y_n^{n,0} - g(X_n^{n,0}]^2\Big\} \\ &= \lambda\int_0^1 E\Big\{[Y_n^{n,\theta} - g(X_n^{n,\theta})][\nabla Y_n^{n,\theta} - g'(X_n^{n,\theta})\nabla X_n^{n,\theta}]\Big\}d\theta. \end{aligned}$$

By (3.8) we have

$$\Delta V_n(y_0, z^0) = \lambda\int_0^1 E\Big\{\bar{Y}_0^{n,\theta}\Delta y + \int_0^T \bar{Z}_t^{n,\theta}\Delta z_t dt + \sum_{i=1}^n I_i^{n,\theta}\Big\}d\theta; \tag{3.21}$$

where

$$\begin{aligned} I_i^{n,\theta} \triangleq\ & \sigma_{y,i-1}^{n,\theta}\int_{t_{i-1}}^{t_i}\tilde{Z}_t^{n,\theta}dt\int_{t_{i-1}}^{t_i}\Delta z_t dW_t \\ & +\sigma_{x,i-1}^{n,\theta}[\sigma_{x,i-1}^{n,\theta}\nabla X_{i-1}^{n,\theta} + \sigma_{y,i-1}^{n,\theta}\nabla Y_{i-1}^{n,\theta}]\Delta W_i\int_{t_{i-1}}^{t_i}\tilde{Z}_t^{n,\theta}dt \\ & -\sigma_{y,i-1}^{n,\theta}[f_{x,i-1}^{n,\theta}\nabla X_{i-1}^{n,\theta} + f_{y,i-1}^{n,\theta}\nabla Y_{i-1}^{n,\theta}]\Delta t\int_{t_{i-1}}^{t_i}\tilde{Z}_t^{n,\theta}dt \\ & -f_{y,i-1}^{n,\theta}\bar{Y}_{i-1}^{n,\theta}[f_{x,i-1}^{n,\theta}\nabla X_{i-1}^{n,\theta} + f_{y,i-1}^{n,\theta}\nabla Y_{i-1}^{n,\theta}]|\Delta t|^2. \end{aligned} \tag{3.22}$$

Step 2. First, similarly to (3.10) and (3.12) one can show that

$$E\Big\{\max_{0\le i\le n}[|\bar{Y}_i^{n,0}|^4 + |\tilde{Y}_i^{n,0}|^4] + \Big(\int_0^T[|\bar{Z}_t^{n,0}|^2 + |\tilde{Z}_t^{n,0}|^2 + |\Delta z_t|^2]dt\Big)^2\Big\} \le CK_0^4. \tag{3.23}$$

Denote

$$\Delta X_i^{n,\theta} \triangleq X_i^{n,\theta} - X_i^{n,0}; \quad \Delta Y_i^{n,\theta} \triangleq Y_i^{n,\theta} - Y_i^{n,0}.$$

Then

$$\begin{cases} \Delta X_0^{n,\theta} = 0; \quad \Delta Y_0^{n,\theta} = \theta\lambda\Delta y; \\ \Delta X_{i+1}^{n,\theta} = \Delta X_i^{n,\theta} + [\alpha_i^{1,\theta}\Delta X_i^{n,\theta} + \beta_i^{1,\theta}\Delta Y_i^{n,\theta}]\Delta W_{i+1}; \\ \Delta Y_{i+1}^{n,\theta} = \Delta Y_i^{n,\theta} - [\alpha_i^{2,\theta}\Delta X_i^{n,\theta} + \beta_i^{2,\theta}\Delta Y_i^{n,\theta}]\Delta t - \theta\lambda\int_{t_i}^{t_{i+1}}\Delta z_t dW_t; \end{cases}$$

where $\alpha_i^{j,\theta}, \beta_i^{j,\theta}$ are defined in an obvious way and are bounded. Thus, by (3.23),

$$E\Big\{\max_{0\le i\le n}[|\Delta X_i^{n,\theta}|^4 + |\Delta Y_i^{n,\theta}|^4]\Big\} \le C\theta^4\lambda^4 E\Big\{|\Delta y|^4 + \Big(\int_0^T|\Delta z_t|^2 dt\Big)^2\Big\} \le CK_0^4\lambda^4. \tag{3.24}$$

Therefore, similarly to (2.18) one can show that

$$E\Big\{|Y_n^{n,\theta}-g(X_n^{n,\theta})|^4\Big\}\le[1+C\lambda]K_0^4. \tag{3.25}$$

Step 3. Denote

$$\begin{array}{ll}\Delta\bar{Y}_i^{n,\theta}\triangleq\bar{Y}_i^{n,\theta}-\bar{Y}_i^{n,0}; & \Delta\tilde{Y}_i^{n,\theta}\triangleq\tilde{Y}_i^{n,\theta}-\tilde{Y}_i^{n,0};\\ \Delta\bar{Z}_t^{n,\theta}\triangleq\bar{Z}_t^{n,\theta}-\bar{Z}_t^{n,0}; & \Delta\tilde{Z}_t^{n,\theta}\triangleq\tilde{Z}_t^{n,\theta}-\tilde{Z}_t^{n,0}.\end{array}$$

Then

$$\left\{\begin{array}{l}\Delta\bar{Y}_n^{n,\theta}=\Delta Y_n^{n,\theta}-\alpha_n^{n,\theta}\Delta X_n^{n,\theta};\\ \Delta\tilde{Y}_n^{n,\theta}=g'(X_n^{n,\theta})[\Delta Y_n^{n,\theta}-\alpha_n^{n,\theta}\Delta X_n^{n,\theta}]+[Y_n^{n,0}-g(X_n^{n,0})]\Delta g'(n,\theta);\\ \Delta\bar{Y}_{i-1}^{n,\theta}=\Delta\bar{Y}_i^{n,\theta}-f_{y,i-1}^{n,\theta}\Delta\bar{Y}_{i-1}^{n,\theta}\Delta t-\sigma_{y,i-1}^{n,\theta}\displaystyle\int_{t_{i-1}}^{t_i}\Delta\tilde{Z}_t^{n,\theta}dt-\int_{t_{i-1}}^{t_i}\Delta\bar{Z}_t^{n,\theta}dW_t\\ \qquad\qquad -\bar{Y}_{i-1}^{n,0}\Delta f_{y,i-1}^{n,\theta}\Delta t-\displaystyle\int_{t_{i-1}}^{t_i}\tilde{Z}_t^{n,0}dt\Delta\sigma_{y,i-1}^{n,\theta};\\ \Delta\tilde{Y}_{i-1}^{n,\theta}=\Delta\tilde{Y}_i^{n,\theta}+f_{x,i-1}^{n,\theta}\Delta\bar{Y}_{i-1}^{n,\theta}\Delta t+\sigma_{x,i-1}^{n,\theta}\displaystyle\int_{t_{i-1}}^{t_i}\Delta\tilde{Z}_t^{n,\theta}dt-\int_{t_{i-1}}^{t_i}\Delta\tilde{Z}_t^{n,\theta}dW_t\\ \qquad\qquad +\bar{Y}_{i-1}^{n,0}\Delta f_{x,i-1}^{n,\theta}\Delta t+\displaystyle\int_{t_{i-1}}^{t_i}\tilde{Z}_t^{n,0}dt\Delta\sigma_{x,i-1}^{n,\theta},\end{array}\right.$$

where

$$\alpha_n^{n,\theta}\triangleq\frac{g(X_n^{n,\theta})-g(X_n^{n,0})}{\Delta X_n^{n,\theta}};\quad \Delta\varphi_i^{n,\theta}\triangleq\varphi(t_i,X_i^{n,\theta},Y_i^{n,\theta})-\varphi(t_i,X_i^{n,0},Y_i^{n,0});$$

and all other terms are defined in a similar way. By standard arguments one has

$$\begin{array}{l}E\Big\{\displaystyle\max_{0\le i\le n}[|\Delta\bar{Y}_i^{n,\theta}|^2+|\Delta\tilde{Y}_i^{n,\theta}|^2]+\int_0^T[|\Delta\bar{Z}_t^{n,\theta}|^2+|\Delta\tilde{Z}_t^{n,\theta}|^2]dt\Big\}\\ \le CE\Big\{|\Delta Y_n^{n,\theta}|^2+|\Delta X_n^{n,\theta}|^2+|Y_n^{n,0}-g(X_n^{n,0})|^2|\Delta g'(n,\theta)|^2\\ \quad+\displaystyle\sum_{i=0}^{n-1}\Big[|\bar{Y}_i^{n,0}|^2[|\Delta f_{y,i}^{n,\theta}|^2+|\Delta f_{x,i}^{n,\theta}|^2]\Delta t+\int_{t_i}^{t_{i+1}}|\tilde{Z}_t^{n,0}|^2dt[|\Delta\sigma_{y,i}^{n,\theta}|^2+|\Delta\sigma_{x,i}^{n,\theta}|^2]\Big]\Big\}\\ \le CE\Big\{|\Delta Y_n^{n,\theta}|^2+|\Delta X_n^{n,\theta}|^2+|Y_n^{n,0}-g(X_n^{n,0})|^2|\Delta X_n^{n,\theta}|^2\\ \quad+\displaystyle\sum_{i=0}^{n-1}[|\bar{Y}_i^{n,0}|^2\Delta t+\int_{t_i}^{t_{i+1}}|\tilde{Z}_t^{n,0}|^2dt][|\Delta X_i^{n,\theta}|^2+|\Delta Y_i^{n,\theta}|^2]\Big\}\\ \le CE^{\frac12}\Big\{\displaystyle\max_{0\le i\le n}[|\Delta X_i^{n,\theta}|^4+|\Delta Y_i^{n,\theta}|^4]\Big\}\times\\ \qquad E^{\frac12}\Big\{1+|Y_n^{n,0}-g(X_n^{n,0})|^4+\displaystyle\max_{0\le i\le n}|\bar{Y}_i^{n,0}|^4+\Big(\int_0^T|\tilde{Z}_t^{n,0}|^2dt\Big)^2\Big\}\\ \le CK_0^2\lambda^2[1+K_0^2]\le CK_0^4\lambda^2,\end{array}$$

thanks to (3.24), (3.16), and (3.23). In particular,

$$E\Big\{|\Delta\bar{Y}_0^{n,\theta}|^2+\int_0^T|\Delta\bar{Z}_t^{n,\theta}|^2dt\Big\}\leq CK_0^4\lambda^2. \tag{3.26}$$

Step 4. Recall (3.12). Note that

$$\begin{aligned}
&\Big|E\Big\{\bar{Y}_0^{n,\theta}\Delta y+\int_0^T\bar{Z}_t^{n,\theta}\Delta z_t dt\Big\}+E\Big\{|\bar{Y}_0^{n,0}|^2+\int_0^T|\bar{Z}_t^{n,0}|^2dt\Big\}\Big|\\
&\leq E\Big\{|\Delta\bar{Y}_0^{n,\theta}\bar{Y}_0^{n,0}|+\int_0^T\Big[|\Delta\bar{Z}_t^{n,\theta}||\bar{Z}_t^{n,0}|+(|\Delta\bar{Z}_t^{n,\theta}|+|\bar{Z}_t^{n,0}|)|\Delta z_t+\bar{Z}_t^{n,0}|\Big]dt\Big\}\\
&\leq CE\Big\{|\Delta\bar{Y}_0^{n,\theta}|^2+\int_0^T[|\Delta\bar{Z}_t^{n,\theta}|^2+|\Delta z_t+\bar{Z}_t^{n,0}|^2]dt\Big\}+\frac{1}{2}E\Big\{|\bar{Y}_0^{n,0}|^2+\int_0^T|\bar{Z}_t^{n,0}|^2dt\Big\}\\
&\leq CK_0^4\lambda^2+\frac{C}{n}V_n(y_0,z^0)+\frac{1}{2}E\Big\{|\bar{Y}_0^{n,0}|^2+\int_0^T|\bar{Z}_t^{n,0}|^2dt\Big\}.
\end{aligned}$$

Then

$$E\Big\{\bar{Y}_0^{n,\theta}\Delta y+\int_0^T\bar{Z}_t^{n,\theta}\Delta z_t dt\Big\}\leq-\frac{1}{2}E\Big\{|\bar{Y}_0^{n,0}|^2+\int_0^T|\bar{Z}_t^{n,0}|^2dt\Big\}+CK_0^4\lambda^2+\frac{C}{n}V_n(y_0,z^0).$$

Choose n large and by (3.11) we get

$$E\Big\{\bar{Y}_0^{n,\theta}\Delta y+\int_0^T\bar{Z}_t^{n,\theta}\Delta z_t dt\Big\}\leq-cV_n(y_0,z^0)+CK_0^4\lambda^2. \tag{3.27}$$

Moreover, similarly to (3.14) and (3.15) we have

$$E\Big\{\max_{0\leq i\leq n}[|\nabla X_i^{n,\theta}|^2+|\nabla Y_i^{n,\theta}|^2]\Big\}\leq CV_n(y_0,z^0);\quad |\sum_{i=1}^n E\{I_i^{n,\theta}\}|\leq\frac{C}{\sqrt{n}}V_n(y_0,z^0).$$

Then by (3.27) and choosing n large, we get

$$E\Big\{\bar{Y}_0^{n,\theta}\Delta y+\int_0^T\bar{Z}_t^{n,\theta}\Delta z_t dt+\sum_{i=1}^n I_i^{n,\theta}\Big\}\leq-cV_n(y_0,z^0)+CK_0^4\lambda^2.$$

Choose $c_1\stackrel{\triangle}{=}\sqrt{\frac{c}{2C}}$ for the constants c,C as above, and $\lambda\stackrel{\triangle}{=}\frac{c_1\varepsilon}{K_0^2}$. Then by (3.8) and (3.16), we have

$$\Delta V_n(y_0,z^0)\leq\lambda[-\frac{c}{2}V_n(y_0,z^0)]=-\frac{c_0\varepsilon}{K_0^2}V_n(y_0,z^0).$$

Finally, plug λ into (3.25) and let $\theta=1$ to get (3.18) for some C_0. ∎

We now iteratively modify the approximations. Set

$$y_0\stackrel{\triangle}{=}0,\quad z^0\stackrel{\triangle}{=}0,\quad K_0\stackrel{\triangle}{=}E^{\frac{1}{4}}\{|Y_n^{n,0}-g(X_n^{n,0})|^4\}. \tag{3.28}$$

For $k = 0, 1, \cdots$, define $(X^{n,k}, Y^{n,k}, \bar{Y}^{n,k}, \tilde{Y}^{n,k}, \bar{Z}^{n,k}, \tilde{Z}^{n,k})$ as follows:

$$\begin{cases} X_0^{n,k} \triangleq x; \quad Y_0^{n,k} \triangleq y_k; \\ X_{i+1}^{n,k} \triangleq X_i^{n,k} + \sigma(t_i, X_i^{n,k}, Y_i^{n,k})\Delta W_{i+1}; \\ Y_{i+1}^{n,k} \triangleq Y_i^{n,k} - f(t_i, X_i^{n,k}, Y_i^{n,k})\Delta t + \int_{t_i}^{t_{i+1}} z_t^k dW_t; \end{cases} \tag{3.29}$$

and

$$\begin{cases} \bar{Y}_n^{n,k} \triangleq Y_n^{n,k} - g(X_n^{n,k}); \quad \tilde{Y}_n^{n,k} \triangleq g'(X_n^{n,k})[Y_n^{n,k} - g(X_n^{n,k})]; \\ \bar{Y}_{i-1}^{n,k} = \bar{Y}_i^{n,k} - f_{y,i-1}^{n,k}\bar{Y}_{i-1}^{n,k}\Delta t - \sigma_{y,i-1}^{n,k}\int_{t_{i-1}}^{t_i} \tilde{Z}_t^{n,k}dt - \int_{t_{i-1}}^{t_i} \bar{Z}_t^{n,k}dW_t; \\ \tilde{Y}_{i-1}^{n,k} = \tilde{Y}_i^{n,k} + f_{x,i-1}^{n,k}\bar{Y}_{i-1}^{n,k}\Delta t + \sigma_{x,i-1}^{n,k}\int_{t_{i-1}}^{t_i} \tilde{Z}_t^{n,k}dt - \int_{t_{i-1}}^{t_i} \tilde{Z}_t^{n,k}dW_t, \end{cases} \tag{3.30}$$

Denote

$$\Delta y_k \triangleq -\bar{Y}_0^{n,k}; \quad \int_{t_{i-1}}^{t_i} \Delta z_t^k dW_t \triangleq E_{i-1}\{\bar{Y}_i^{n,k}\} - \bar{Y}_i^{n,k}; \tag{3.31}$$

and

$$\lambda_k \triangleq \frac{c_1\varepsilon}{K_k^2}; \quad y_{k+1} \triangleq y_k + \lambda_k\Delta y_k; \quad z_t^{k+1} \triangleq z_t^k + \lambda_k\Delta z_t^k; \quad K_{k+1}^4 \triangleq K_k^4 + 2C_0\varepsilon K_k^2, \tag{3.32}$$

where c_1, C_0 are the constants in Lemma 3.4. Then following exactly the same arguments as in Theorem 2.5, we can prove

Theorem 3.5 *Set $n = \varepsilon^{-2}$. There exist constants C_1, C_2 and $N \le C_1\varepsilon^{-C_2}$ such that*

$$V_n(y_N, z^N) \le \varepsilon^2.$$

4 Further Simplification

We now transform (3.30) into conditional expectations. First,

$$z_i^{n,k} \triangleq \frac{1}{\Delta t}E_i\Big\{\int_{t_i}^{t_{i+1}} z_t^k dt\Big\} = \frac{1}{\Delta t}E_i\{Y_{i+1}^{n,k}\Delta W_{i+1}\}.$$

Second, denote

$$M_i^{n,k} \triangleq \exp\Big(\sigma_{x,i-1}^{n,k}\Delta W_i - \frac{1}{2}|\sigma_{x,i-1}^{n,k}|^2\Delta t\Big). \tag{4.1}$$

Then

$$\begin{aligned} \tilde{Y}_{i-1}^{n,k} &= E_{i-1}\Big\{M_i^{n,k}\tilde{Y}_i^{n,k}\Big\} + f_{x,i-1}^{n,k}\bar{Y}_{i-1}^{n,k}\Delta t; \\ \sigma_{x,i-1}^{n,k}\bar{Y}_{i-1}^{n,k} + \sigma_{y,i-1}^{n,k}\tilde{Y}_{i-1}^{n,k} &= \sigma_{x,i-1}^{n,k}E_{i-1}\{\bar{Y}_i^{n,k}\} + \sigma_{y,i-1}^{n,k}E_{i-1}\{\tilde{Y}_i^{n,k}\} \\ &\quad + [\sigma_{y,i-1}^{n,k}f_{x,i-1}^{n,k} - \sigma_{x,i-1}^{n,k}f_{y,i-1}^{n,k}]\bar{Y}_{i-1}^{n,k}\Delta t. \end{aligned}$$

Thus

$$
\begin{aligned}
\bar{Y}^{n,k}_{i-1} &= \frac{1}{1+f^{n,k}_{y,i-1}\Delta t}\Big[E_{i-1}\{\bar{Y}^{n,k}_i\} - \frac{\sigma^{n,k}_{y,i-1}}{\sigma^{n,k}_{x,i-1}}E_{i-1}\{\tilde{Y}^{n,k}_i[M^{n,k}_i - 1]\}\Big];\\
\tilde{Y}^{n,k}_{i-1} &= E_{i-1}\{M^{n,k}_i\tilde{Y}^{n,k}_i\} + f^{n,k}_{x,i-1}\bar{Y}^{n,k}_{i-1}\Delta t.
\end{aligned} \tag{4.2}
$$

When $\sigma^{n,k}_{x,i-1} = 0$, by solving (3.30) directly, we see that (4.2) becomes

$$
\begin{aligned}
\bar{Y}^{n,k}_{i-1} &= \frac{1}{1+f^{n,k}_{y,i-1}\Delta t}\Big[E_{i-1}\{\bar{Y}^{n,k}_i\} - \sigma^{n,k}_{y,i-1}E_{i-1}\{\tilde{Y}^{n,k}_i\Delta W_i\}\Big];\\
\tilde{Y}^{n,0}_{i-1} &= E_{i-1}\{\tilde{Y}^{n,k}_i\} + f_x\bar{Y}^{n,k}_{i-1}\Delta t.
\end{aligned} \tag{4.3}
$$

Now fix ε and, in light of (3.4), set $n \triangleq \varepsilon^{-2}$. Let c_1, C_0 be the constants in Lemma 3.4. We have the following algorithm.

First, set

$$
\begin{cases}
X^{n,0}_0 \triangleq x; \quad Y^{n,0}_0 \triangleq 0;\\
X^{n,0}_{i+1} \triangleq X^{n,0}_i + \sigma(t_i, X^{n,0}_i, Y^{n,0}_i)\Delta W_{i+1};\\
Y^{n,0}_{i+1} \triangleq Y^{n,0}_i - f(t_i, X^{n,0}_i, Y^{n,0}_i)\Delta t;
\end{cases}
$$

and

$$
z^{n,0}_i \triangleq 0; \quad K_0 \triangleq E^{\frac{1}{4}}\Big\{|Y^{n,0}_n - g(X^{n,0}_n)|^4\Big\}.
$$

For $k = 0, 1, \cdots$, if $E\{|Y^k_n - g(X^k_n)|^2\} \le \varepsilon^2$, we quit the loop and by Theorems 3.1, 3.4, and Corollary 3.2, we have

$$
E\Big\{\max_{0\le i\le n}[|X_{t_i} - X^{n,k}_i|^2 + |Y_{t_i} - Y^{n,k}_i|^2] + \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|Z_t - z^{n,k}_i|^2 dt\Big\} \le C\varepsilon^2.
$$

Otherwise, we proceed the loop as follows:

Step 1. Define $(\bar{Y}^{n,k}_n, \tilde{Y}^{n,k}_n)$ by the first line of (3.30); and for $i = n, \cdots, 1$, define $(\bar{Y}^{n,k}_{i-1}, \tilde{Y}^{n,k}_{i-1})$ by (4.2) or (4.3).

Step 2. Let $\lambda_k \triangleq \dfrac{c_1\varepsilon}{K_k^2}$, $K^4_{k+1} \triangleq K^4_k + 2C_0\varepsilon K^2_k$. Define $(X^{n,k+1}, Y^{n,k+1}, z^{n,k+1})$ by

$$
\begin{cases}
X^{n,k+1}_0 \triangleq x; \quad Y^{n,k+1}_0 \triangleq Y^{n,k}_0 - \lambda_k\bar{Y}^{n,k}_0;\\
X^{n,k+1}_{i+1} \triangleq X^{n,k+1}_i + \sigma(t_i, X^{n,k+1}_i, Y^{n,k+1}_i)\Delta W_{i+1};\\
Y^{n,k+1}_{i+1} \triangleq Y^{n,k+1}_i - f(t_i, X^{n,k+1}_i, Y^{n,k+1}_i)\Delta t\\
\qquad + \Big[Y^{n,k}_{i+1} - Y^{n,k}_i + f(t_i, X^{n,k}_i, Y^{n,k}_i)\Delta t\Big] + \lambda_k\Big[E_i\{\bar{Y}^{n,k}_{i+1}\} - \bar{Y}^{n,k}_{i+1}\Big];
\end{cases} \tag{4.4}
$$

and

$$z_i^{n,k+1} \triangleq \frac{1}{\Delta t} E_i\Big\{Y_{i+1}^{n,k+1}\Delta W_{i+1}\Big\}. \tag{4.5}$$

We note that in the last line of (4.4), the two terms stand for $\int_{t_i}^{t_{i+1}} z_t^k dW_t$ and $\int_{t_i}^{t_{i+1}} \Delta z_t^k dW_t$, respectively.

By Theorem 3.4, the above loop should stop after at most $C_1\varepsilon^{-C_2}$ steps.

We note that in the above algorithm the only costly terms are the conditional expectations:

$$E_i\{\bar{Y}_{i+1}^{n,k}\}, \quad E_i\{\tilde{Y}_{i+1}^{n,k}\}, \quad E_i\{\Delta W_{i+1}Y_{i+1}^{n,k}\}, \quad E_i\{M_{i+1}^{n,k}\tilde{Y}_{i+1}^{n,k}\} \text{ or } E_i\{\Delta W_{i+1}\tilde{Y}_{i+1}^{n,k}\}. \tag{4.6}$$

By induction, one can easily show that

$$Y_i^{n,k} = u_i^{n,k}(X_0^{n,k}, \cdots, X_i^{n,k}),$$

for some deterministic function $u_i^{n,k}$. Similar properties hold true for $(\bar{Y}_i^{n,k}, \tilde{Y}_i^{n,k})$. However, they are not Markovian in the sense that one cannot write $Y_i^{n,k}, \bar{Y}_i^{n,k}, \tilde{Y}_i^{n,k}$ as functions of $X_i^{n,k}$ only. In order to use Monte-Carlo methods to compute the conditional expectations in (4.6) efficiently, some Markovian type modification of our algorithm is needed.

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