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# SMALL-TIME ASYMPTOTIC ESTIMATES IN LOCAL DIRICHLET SPACES

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Abstract. Small-time asymptotic estimates of semigroups on a logarithmic scale are proved for all symmetric local Dirichlet forms on $\sigma$-finite measure spaces, which is an extension of the work by Hino and Ramírez [4].

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## 1. Introduction

Let $(X, \mathcal{B}, \mu)$ be a $\sigma$-finite measure space and $(\mathcal{E}, \mathbb{D})$ a symmetric Dirichlet form on the $L^2$ space of $(X, \mathcal{B}, \mu)$. Let $\{T_t\}$ denote the semi-group associated with $(\mathcal{E}, \mathbb{D})$, and set $P_t(A, B) = \int_X 1_A \cdot T_t 1_B \, d\mu$ for $A, B \in \mathcal{B}$ and $t > 0$. The small-time asymptotic behavior of $P_t(A, B)$ on a logarithmic scale is the main interest of this paper. In the paper [4], under assumptions that the total mass of $\mu$ is finite and $(\mathcal{E}, \mathbb{D})$ is conservative and local, the following small-time asymptotic estimate was proved:

$$\lim_{t \to 0} t \log P_t(A, B) = -\frac{\mathsf{d}(A, B)^2}{2}, \tag{1.1}$$

where $\mathsf{d}(A, B)$ is an intrinsic distance between $A$ and $B$ defined by

$$\mathsf{d}(A, B) = \sup_{f \in \mathbb{D}_0} \left\{ \operatorname*{essinf}_{x \in B} f(x) - \operatorname*{esssup}_{x \in A} f(x) \right\}, \tag{1.2}$$

$$\mathbb{D}_0 = \left\{ f \in \mathbb{D} \cap L^\infty(\mu) \;\middle|\; \begin{array}{l} 2\mathcal{E}(fh, h) - \mathcal{E}(f^2, h) \le \int_X |h| \, d\mu \\ \text{for every } h \in \mathbb{D} \cap L^\infty(\mu) \end{array} \right\}. \tag{1.3}$$

This result generalizes former works (see [4] and references therein) and can be regarded as an integral version of the small-time asymptotics of the transition density of Varadhan type

$$\lim_{t \to 0} t \log p_t(x, y) = -\frac{d(x, y)^2}{2},$$

which was proved in [6] for a class of symmetric and non-degenerate diffusion processes on Lipschitz manifolds.

In this paper, we further weaken the assumptions in [4] and prove the small-time estimate (1.1) holds for any $A, B \in \mathcal{B}$ with finite measure, for all local symmetric Dirichlet forms on $\sigma$-finite measure spaces. In other words, (1.1) now holds without assuming the finiteness of the total measure nor the conservativeness of $(\mathcal{E}, \mathbb{D})$, which may be considered as one of the most general results in this direction. The definition of the intrinsic distance $\mathsf{d}(A, B)$ here has to be suitably modified, by introducing the notion of nests. Note that we do not assume any topological structure of the underlying space, as in [4].

The proof is purely analytic and is done by careful modifications of the proof in [4] based on the Ramírez method [7]. In contrast to the simple statement of the result, the proof is rather technical. We will explain an idea of the proof here following the articles [7, 4] and how to generalize it.

The upper side estimate

$$\limsup_{t\to 0} t\log P_t(A,B) \le -\frac{\mathsf{d}(A,B)^2}{2}$$

is an easier part and follows from what is called Davies' method. In order to give the outline of the proof of the lower side estimate, let us consider a typical example; suppose that $X$ has a differential structure and a gradient operator $\nabla$ taking values in a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ as in the case of Riemannian manifolds, and $\mathcal{E}$ is given by $\mathcal{E}(f,g)=\frac{1}{2}\int_X\langle\nabla f,\nabla g\rangle\,d\mu$. Let us further assume that $\mu(X)$ is finite. Then, we can deduce that $\mathbb{D}_0=\{f\in\mathbb{D}\cap L^\infty(X)\mid |\nabla f|\le 1 \text{ a.e.}\}$. The function $u_t=-t\log T_t 1_A$ satisfies the equation

$$t(\partial_t u_t-\mathcal{L}u_t)=u_t-\frac{1}{2}|\nabla u_t|^2, \tag{1.4}$$

where $\mathcal{L}$ is the generator of $\{T_t\}$. Letting $t\to 0$, we expect that $|\nabla u_0|^2=2u_0$ for a limit $u_0$ of $u_t$, which implies that $|\nabla\sqrt{2u_0}|^2=1$. (What we can actually expect is $|\nabla\sqrt{2u_0}|^2\le 1$.) Since $u_0$ should vanish on $A$, this relation informally implies that

$$\lim_{t\to 0}\sqrt{-2t\log T_t 1_A(x)}\le \mathsf{d}(A,\{x\}),$$

which is close to the lower side estimate. In practice, we cannot prove the convergence of the left-hand side of (1.4) in this form and have to consider the time-average $\bar{u}_t=\frac{1}{t}\int_0^t u_s\,ds$ in place of $u_t$ and utilize the Tauberian theorem. Moreover, we have to take the integrability of $\bar{u}_t$ into consideration. In [7], this was assured by an additional assumption, the spectral gap property. To remove such assumption, a suitable cut-off function $\phi$ was introduced in [4] and the proof was done by replacing $\bar{u}_t$ by $\bar{\phi}_t=\frac{1}{t}\int_0^t\phi(u_s)\,ds$; bounded functions are always integrable as long as $\mu$ is a finite measure. When $\mu(X)=\infty$, this modification is not sufficient. In order to include this case, in this paper, we further introduce a sequence $\{\chi_k\}$ of 'cut-off functions in the space-direction' and consider $\bar{\phi}_t\chi_k$ to guarantee the integrability. By such modification, more and more extra terms appear in the argument, which have to be estimated appropriately. This makes the proof rather long.

The organization of this paper is as follows. In Section 2, we state the notion of nests and define the intrinsic distance $\mathsf{d}$, which is naturally consistent with what was given in [4]. Their basic properties are discussed in Section 3. In Section 4, we prove the main theorem. In the last section, we give a few additional claims which have also been discussed in [4].

## 2. Preliminaries

For $p \in [1,\infty]$, we denote by $L^p(\mu)$ the $L^p$-space on the $\sigma$-finite measure space $(X,\mathcal{B},\mu)$ and its norm by $\|\cdot\|_{L^p(\mu)}$. The totality of all measurable functions $f$ on $X$ will be denoted by $L^0(\mu)$. Here, as usual, two functions which are equal $\mu$-a.e. are identified. Let $L^p_+(\mu)$ denote the set of all functions $f \in L^p(\mu)$ such that $f \ge 0$ $\mu$-a.e. We set

$$C_b^1(\mathbb{R}^d) = \left\{ f \;\middle|\; \begin{array}{l} f \text{ is a } C^1\text{-function on } \mathbb{R}^d \text{ and} \\ f \text{ and } \partial f/\partial x_i \ (i=1,2,\dots,d) \text{ are all bounded} \end{array} \right\},$$
$$\hat{C}_b^1(\mathbb{R}^d) = \left\{ f \;\middle|\; \begin{array}{l} f \text{ is a } C^1\text{-function on } \mathbb{R}^d \text{ and} \\ \partial f/\partial x_i \ (i=1,2,\dots,d) \text{ are all bounded} \end{array} \right\},$$
$$C_c^1(\mathbb{R}^d) = \left\{ f \;\middle|\; f \text{ is a } C^1\text{-function on } \mathbb{R}^d \text{ with compact support} \right\}.$$

Let $(\mathcal{E},\mathbb{D})$ be a symmetric Dirichlet form on $L^2(\mu)$. The norm $\|\cdot\|_{\mathbb{D}}$ of $\mathbb{D}$ is defined by $\|f\|_{\mathbb{D}} = (\mathcal{E}(f,f) + \|f\|^2_{L^2(\mu)})^{1/2}$. We use the notation $\mathcal{E}(f)$ for $\mathcal{E}(f,f)$. We assume that $(\mathcal{E},\mathbb{D})$ is local[1], namely, for any $f \in \mathbb{D}$ and $F$, $G \in C_b^1(\mathbb{R})$ with $\operatorname{supp} F \cap \operatorname{supp} G = \emptyset$,

$$\mathcal{E}(F(f) - F(0), G(f) - G(0)) = 0.$$

This is equivalent to the condition that $\mathcal{E}(f,g) = 0$ if $f,g \in \mathbb{D}$ and $(f+a)g = 0$ $\mu$-a.e. for some $a \in \mathbb{R}$. (For the proof, see [2, Proposition I.5.1.3].) The semigroup, the resolvent, and the nonpositive self-adjoint operator on $L^2(\mu)$ associated with $(\mathcal{E},\mathbb{D})$ will be denoted by $\{T_t\}_{t>0}$, $\{G_\beta\}_{\beta>0}$, and $\mathcal{L}$, respectively. $\{T_t\}_{t>0}$ uniquely extends to a strongly continuous and contraction semigroup on $L^p(\mu)$ for $p \in [1,\infty)$. For $A \in \mathcal{B}$, set

$$\mathbb{D}_A = \{f \in \mathbb{D} \mid f = 0 \ \mu\text{-a.e. on } X \setminus A\}.$$

We also set $\mathbb{D}_b = \mathbb{D} \cap L^\infty(\mu)$, $\mathbb{D}_{A,b} = \mathbb{D}_A \cap L^\infty(\mu)$ and $\mathbb{D}_{A,b,+} = \mathbb{D}_A \cap L^\infty_+(\mu)$.

**Definition 2.1.** An increasing sequence $\{E_k\}_{k=1}^\infty$ of measurable subsets of $X$ is called *nest* if the following two conditions are satisfied.[2]

(i) For every $k \in \mathbb{N}$, there exists $\chi_k \in \mathbb{D}$ such that $\chi_k \ge 1$ $\mu$-a.e. on $E_k$.

(ii) $\bigcup_{k=1}^\infty \mathbb{D}_{E_k}$ is dense in $\mathbb{D}$.

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[1]This terminology is taken from Bouleau-Hirsch's book [2]. In Fukushima-Oshima-Takeda's book [3], the essentially same property is called *strong local*.

[2]This definition is slightly different from that in standard textbooks such as [3, 5]. Note that $X$ does not need any topology here.

*Remark* 2.2. Concerning condition (i), we can take $\chi_k$ so that $\chi_k = 1$ $\mu$-a.e. on $E_k$ and $0 \le \chi_k \le 1$ $\mu$-a.e. in addition, by considering $0 \vee \chi_k \wedge 1$ in place of $\chi_k$.

*Remark* 2.3. For every $k \in \mathbb{N}$, $\mu(E_k) < \infty$ because of condition (i). By condition (ii), we can prove $\mu(X \setminus \bigcup_{k=1}^\infty E_k) = 0$.

We will see in Section 3 that there do exist many nests.

**Definition 2.4.** For a nest $\{E_k\}_{k=1}^\infty$, we set

$$\begin{aligned}&\mathbb{D}_{loc}(\{E_k\})\\ &= \left\{ f \in L^0(\mu) \;\middle|\; \begin{array}{l}\text{there exists a sequence of functions } \{f_k\}_{k=1}^\infty \text{ in } \mathbb{D}\\ \text{such that } f = f_k \ \mu\text{-a.e. on } E_k \text{ for each } k\end{array} \right\},\\ &\mathbb{D}_{loc,b}(\{E_k\}) = \mathbb{D}_{loc}(\{E_k\}) \cap L^\infty(\mu).\end{aligned}$$

For $f, g, h \in \mathbb{D}_b$, define

$$I_{f,g}(h) = \mathcal{E}(fh, g) + \mathcal{E}(gh, f) - \mathcal{E}(fg, h)$$

and write $I_f(h)$ for $I_{f,f}(h)$. The following are basic properties of $I$.

**Lemma 2.5.** *Let $f, g, h, h_1$, and $h_2$ be elements of $\mathbb{D}_b$.*

(i) *If $h \ge 0$ $\mu$-a.e., then $0 \le I_f(h) \le 2\|h\|_{L^\infty(\mu)}\mathcal{E}(f)$.*
(ii) *$I_{f,g}(h_1h_2)^2 \le I_f(h_1^2)I_g(h_2^2)$.*
(iii) *$\sqrt{I_{f+g}(h)} \le \sqrt{I_f(h)} + \sqrt{I_g(h)}$ if $h \ge 0$ $\mu$-a.e.*

*Proof.* For $f, g, h \in L^2(\mu) \cap L^\infty(\mu)$ and $t > 0$, define

$$I^{(t)}_{f,g}(h) = \mathcal{E}^{(t)}(fh, g) + \mathcal{E}^{(t)}(gh, f) - \mathcal{E}^{(t)}(fg, h),$$

where $\mathcal{E}^{(t)}(f, g) = t^{-1}(f - T_t f, g)$. By Lemma I.2.3.2.1 and Proposition I.2.3.3 in [2] and the limiting argument, the claims follow for $I^{(t)}$ in place of $I$. Letting $t \to 0$ reaches the conclusion. □

By the properties (i) and (iii) above, we can define $I_f(h)$ for $f \in \mathbb{D}$ and $h \in \mathbb{D}_b$ by continuity. Due to the locality of $(\mathcal{E}, \mathcal{F})$, $I_f(h) = 0$ if $(f + a)h = 0$ $\mu$-a.e. for some $a \in \mathbb{R}$. This allows us to define $I_f(h)$ for $f \in \mathbb{D}_{loc,b}(\{E_k\})$ and $h \in \mathbb{D}_{E_k,b}$ consistently by $I_f(h) = I_{f_k}(h)$, where $f_k$ is an arbitrary element in $\mathbb{D}_b$ such that $f_k = f$ $\mu$-a.e. on $E_k$. In other words, $I_f(h)$ is well-defined for $f \in \mathbb{D}_{loc,b}(\{E_k\})$ and $h \in \bigcup_{k=1}^\infty \mathbb{D}_{E_k,b}$.

**Definition 2.6.** For a nest $\{E_k\}_{k=1}^\infty$, we set

$$\begin{aligned}&\mathbb{D}_0(\{E_k\})\\ &= \left\{ f \in \mathbb{D}_{loc,b}(\{E_k\}) \;\middle|\; I_f(h) \le \|h\|_{L^1(\mu)} \text{ for every } h \in \bigcup_{k=1}^\infty \mathbb{D}_{E_k,b} \right\}.\end{aligned}$$

Clearly, we can replace $\mathbb{D}_{E_k,b}$ by $\mathbb{D}_{E_k,b,+}$ in the definition above. We will show in Proposition 3.9 that the set $\mathbb{D}_0(\{E_k\})$ is in fact independent of the choice of $\{E_k\}_{k=1}^\infty$, so we denote it simply by $\mathbb{D}_0$ below.

For $A, B \in \mathcal{B}$ with positive $\mu$-measure, we define

$$P_t(A,B) = \int_X 1_A \cdot T_t 1_B \, d\mu \, ( = \int_X T_t 1_A \cdot 1_B \, d\mu), \quad t > 0$$

and

$$\mathsf{d}(A,B) = \sup_{f \in \mathbb{D}_0} \left\{ \operatorname*{ess\,inf}_{x \in B} f(x) - \operatorname*{ess\,sup}_{x \in A} f(x) \right\}.$$

The following is our main theorem.

**Theorem 2.7.** *For any $A, B \in \mathcal{B}$ with $0 < \mu(A) < \infty$, $0 < \mu(B) < \infty$, we have*

$$\lim_{t \to 0} t \log P_t(A,B) = -\frac{\mathsf{d}(A,B)^2}{2}.$$

*Remark* 2.8. To make the meaning of $\mathbb{D}_0(\{E_k\})$ clearer, let us suppose that $X$ is a locally compact separable metric space, $\mu$ is a positive Radon measure with $\operatorname{supp} \mu = X$, $(\mathcal{E}, \mathbb{D})$ is a regular Dirichlet form on $L^2(\mu)$, and there exists a sequence of relatively compact open sets $\{O_k\}_{k=1}^\infty$ such that $\bar{O}_k \subset O_{k+1}$ for all $k$ and $\bigcup_k O_k = X$. Then, it is easy to see that $\{\bar{O}_k\}_{k=1}^\infty$ is a nest. Each $f \in \mathbb{D}_{loc}(\{\bar{O}_k\})$ provides the energy measure $\mu_{\langle f \rangle}$, a positive Radon measure on $X$ such that $I_f(h) = \int_X h \, d\mu_{\langle f \rangle}$ for every $h \in \mathcal{F} \cap C_0(X)$, where $C_0(X)$ is a space of all continuous functions on $X$ with compact support. Then, $\mathbb{D}_0(\{\bar{O}_k\})$ can be described as

$$\left\{ f \in \mathbb{D}_{loc,b}(\{\bar{O}_k\}) \;\middle|\; \begin{array}{l} \mu_{\langle f \rangle} \text{ is absolutely continuous w.r.t. } \mu \\ \text{and } \frac{d\mu_{\langle f \rangle}}{d\mu} \le 1 \ \mu\text{-a.e.} \end{array} \right\}.$$

Therefore, $\mathsf{d}$ is a natural generalization of the usual notion of intrinsic metric.

**Example 2.9.** The following are simple examples showing the necessity of introducing $\mathbb{D}_{loc,b}$ instead of $\mathbb{D}$ or $\mathbb{D}_{loc}$ in the definition of $\mathsf{d}$ and $\mathbb{D}_0$.

(i) Let $X = [-1,1]$, $\mathcal{B} =$ the Borel $\sigma$-field on $X$, $\mu =$ the Lebesgue measure on $X$, $\mathbb{D} = \{f \in H^1([-1,1]) \mid f(0) = 0\}$, and $\mathcal{E}(f,g) = \frac{1}{2}\int_X f'(x)g'(x)\,dx$, $f, g \in \mathbb{D}$. The corresponding diffusion process is the Brownian motion on $X$ killed at 0. Let $A = [-1,-1/2]$ and $B = [1/2,1]$. For each $k \in \mathbb{N}$, let $E_k = [-1,-1/k] \cup [1/k,1]$. Then, $\{E_k\}_{k=1}^\infty$ is a nest and a function $f_M(x) := M \cdot 1_{(0,1]}(x)$ belongs to $\mathbb{D}_0(\{E_k\})$ for any $M > 0$. Therefore, $\mathsf{d}(A,B) = \infty$. On the other hand, if we adopted (1.2) and (1.3) as a definition of $\mathsf{d}$, $\mathsf{d}(A,B)$ would be 1, which does not provide a correct distance. This implies the necessity of the notion of nests even if $\mu$ is a finite measure, when the Dirichlet form is not conservative.

(ii) Let $X = \mathbb{R}$ and $\mathcal{B} =$ the Borel $\sigma$-field on $X$. Let $m$ be the Lebesgue measure on $X$ and $\mu'$ a positive Radon measure on $X$ such that $\operatorname{supp}\mu'$ is $X$, $\mu'$ and $m$ are mutually singular, and $\mu'((0,\infty)) = \mu'((-\infty,0)) = \infty$. Define a Radon measure $\mu$ on $X$ by $\mu(A) = m(A\cap[-1,1]) + \mu'(A\setminus[-1,1])$, $A \in \mathcal{B}$. Let

$$\mathbb{D} = \left\{ f \in C(X)\cap L^2(\mu) \;\middle|\; \begin{array}{l} f \text{ is absolutely continuous} \\ \text{and } \int_X |f'|^2\,dm < \infty \end{array} \right\}$$

and $\mathcal{E}(f,g) = \frac{1}{2}\int_X f'g'\,dm$ for $f,g \in \mathbb{D}$. Then, $(\mathcal{E},\mathbb{D})$ is a regular Dirichlet form on $L^2(\mu)$. The corresponding diffusion process is a time changed Brownian motion. The energy measure $\mu_{\langle f\rangle}$ of $f \in \mathbb{D}$ is described as $d\mu_{\langle f\rangle} = |f'|^2\,dm$. Let $A = [-2,-1]$ and $B = [1,2]$. By Remark 2.8, $\{\bar{O}_k\}_{k=1}^\infty$ is a nest where $O_k = (-k,k)$, $k \in \mathbb{N}$, and $\mathbb{D}_0(\{\bar{O}_k\})$ is represented as

$$\left\{ f \in \mathbb{D}_{loc,b}(\{\bar{O}_k\}) \;\middle|\; \begin{array}{l} f \text{ is constant on } (-\infty,1] \text{ and on } [1,\infty), \\ \text{and } |f'| \le 1 \ m\text{-a.e. on } (-1,1) \end{array} \right\},$$

because of the singularity of $\mu$ and $m$ on $X\setminus[-1,1]$. Then, one of the functions $f$ attaining the infimum in the definition of $\mathsf{d}(A,B)$ is given by $f(x) = (-1)\vee x\wedge 1$, and $\mathsf{d}(A,B) = 2$. However, $f$ does not belong to $\mathbb{D}$. Indeed, any function $g$ in $\mathbb{D}_0(\{\bar{O}_k\})\cap\mathbb{D}$ has to satisfy $g = 0$ on $X\setminus[-1,1]$ to assure the $L^2(\mu)$-integrability. In particular, we cannot replace $\mathbb{D}_{loc,b}$ by $\mathbb{D}$ in the definition of $\mathbb{D}_0$.

(iii) Let $X = \mathbb{R}$, $\mathcal{B} =$ the Borel $\sigma$-field on $X$, $d\mu(x) = (x^{-2}\wedge 1)\,dx$. Define $\mathcal{E}(f,g) = \frac{1}{2}\int_X f'g'\,d\mu$ for $f,g \in C_0^\infty(X)$. Then, $(\mathcal{E}, C_0^\infty(X))$ is closable on $L^2(\mu)$. The closure, which will be denoted by $(\mathcal{E},\mathbb{D})$, is a regular Dirichlet form on $L^2(\mu)$. Note that $1 \in \mathbb{D}$. Define $E_k = [-k,k]$ and $E'_k = X$ for each $k \in \mathbb{N}$. Then, both $\{E_k\}_{k=1}^\infty$ and $\{E'_k\}_{k=1}^\infty$ are nests. We define a function space $\mathbb{D}_0^\sharp(\{E_k\})$ by

$$\mathbb{D}_0^\sharp(\{E_k\}) = \{f \in \mathbb{D}_{loc}(\{E_k\}) \mid |f'| \le 1 \ \mu\text{-a.e.}\}$$

and $\mathbb{D}_0^\sharp(\{E'_k\})$ in the same way. Then, the function $f(x) = x$ belongs to $\mathbb{D}_0^\sharp(\{E_k\})$ but does not belong to $\mathbb{D}_0^\sharp(\{E'_k\})$. Indeed, $\mathbb{D}_0^\sharp(\{E'_k\}) \subset \mathbb{D}_{loc}(\{E'_k\}) = \mathbb{D} \subset L^2(\mu)$, but $f \notin L^2(\mu)$. Therefore, the set $\mathbb{D}_0^\sharp(\{E_k\})$ depends on the choice of a nest $\{E_k\}_{k=1}^\infty$. This suggests that considering $\mathbb{D}_{loc,b}$ is more natural than $\mathbb{D}_{loc}$ in the definition of $\mathbb{D}_0$.

## 3. Basic properties

Recall that a function $f \in L^2(\mu)$ is called 1-excessive if $\beta G_{\beta+1} f \le f$ $\mu$-a.e. for every $\beta \ge 0$.

**Lemma 3.1.** *Let $\xi \in \mathbb{D}$ be a 1-excessive function with $\xi > 0$ $\mu$-a.e. Define $E_k = \{\xi \ge 1/k\}$ for each $k \in \mathbb{N}$. Then, $\{E_k\}_{k=1}^\infty$ is a nest.*

*Proof.* Since $\chi_k := k\xi$ satisfies condition (i) of Definition 2.1, it is enough to prove that $\bigcup_{k=1}^\infty \mathbb{D}_{E_k}$ is dense in $\mathbb{D}$. Set

$$\mathcal{C} = \{f \mid f \in L^2_+(\mu) \text{ and there exists } c > 0 \text{ such that } f \le c\xi \ \mu\text{-a.e.}\},$$
$$\mathcal{D} = \{G_2 f \mid f \in \mathcal{C}\}.$$

Since $\mathcal{C} - \mathcal{C} = \{g - h \mid g, h \in \mathcal{C}\}$ is dense in $L^2(\mu)$, $\mathcal{D} - \mathcal{D}$ is dense in $\mathbb{D}$. For $g = G_2 f \in \mathcal{D}$, it holds that $g \le cG_2\xi \le c\xi$ $\mu$-a.e. since $\xi$ is 1-excessive. Therefore, $g_k := (g - c/k) \vee 0$ belongs to $\mathbb{D}_{E_k}$ and $g_k$ converges to $g$ in $\mathbb{D}$ as $k \to \infty$. This means that any element in $\mathcal{D} - \mathcal{D}$ can be approximated by functions in $\bigcup_{k=1}^\infty \mathbb{D}_{E_k}$, which proves the claim. □

Note that for any function $f \in L^2(\mu)$ with $f > 0$ $\mu$-a.e., $\xi = G_1 f$ satisfies the condition of the lemma above. Therefore, there exist indeed many nests.

The following claim is what is naturally expected. We give a proof for it though it is not needed in the sequel.

**Lemma 3.2.** *If both $\{E_k\}_{k=1}^\infty$ and $\{E'_k\}_{k=1}^\infty$ are nests, then so is $\{E_k \cap E'_k\}_{k=1}^\infty$.*

*Proof.* It suffices to prove that any element $f$ in $\bigcup_{l=1}^\infty \mathbb{D}_{E_l,+}$ is approximated by functions in $\bigcup_{k=1}^\infty \mathbb{D}_{E_k \cap E'_k}$. Let $f \in \mathbb{D}_{E_l,+}$ for some $l \in \mathbb{N}$. Take $g_k \in \mathbb{D}_{E'_k}$ ($k \in \mathbb{N}$) such that $g_k$ converges to $f$ in $\mathbb{D}$ and $\mu$-a.e. as $k \to \infty$. Let $f_k = 0 \vee g_k \wedge f$ for each $k$. Then, $f_k \in \mathbb{D}_{E_l \cap E'_k} \subset \mathbb{D}_{E_k \cap E'_k}$ when $k \ge l$. It also holds that $\|f_k\|_{\mathbb{D}} \le \|g_k\|_{\mathbb{D}} + \|f\|_{\mathbb{D}}$ and $f_k$ converges to $f$ $\mu$-a.e. Therefore, $\{f_k\}_{k=l}^\infty$ converges weakly to $f$ in $\mathbb{D}$. Taking the Cesàro mean of an appropriate subsequence, we obtain a desired approximating sequence. □

In order to investigate the space $\mathbb{D}_0(\{E_k\})$, we will prove some auxiliary properties.

**Lemma 3.3.** *Let $f, h, f_k$, and $h_k$ ($k \in \mathbb{N}$) be in $\mathbb{D}_b$.*

(i) *If $f_k$ converges weakly to $f$ in $\mathbb{D}$ and $h \ge 0$ $\mu$-a.e., then $\liminf_{k\to\infty} I_{f_k}(h) \ge I_f(h)$.*
(ii) *If $f_k$ converges to $f$ in $\mathbb{D}$, then $\lim_{k\to\infty} I_{f_k}(h) = I_f(h)$.*

(iii) *If* $\{\|h_k\|_{L^\infty(\mu)}\}_{k=1}^\infty$ *is bounded and* $h_k$ *converges to* $h$ *in* $\mathbb{D}$, *then* $\lim_{k\to\infty} I_f(h_k) = I_f(h)$.

*Proof.* The first and the second claims follow from the fact that $I_{\cdot,\cdot}(h)$ is a nonnegative definite continuous bilinear form on $\mathbb{D}$ when $h \geq 0$ $\mu$-a.e. For the third one, it is enough to notice that $fh_k$ converges weakly to $fh$ in $\mathbb{D}$ as $k \to \infty$. $\square$

One of the important consequences of the locality of $(\mathcal{E}, \mathbb{D})$ is the following.

**Theorem 3.4.** *For* $f = (f_1, \ldots, f_n) \in \underbrace{\mathbb{D} \times \cdots \times \mathbb{D}}_{n \text{ times}}$, *there exists a unique family* $(\sigma^f_{i,j})_{1\leq i,j\leq n}$ *of signed Radon measures on* $\mathbb{R}^n$ *of finite total variation such that*

(i) $\sigma^f_{i,j} = \sigma^f_{j,i}$ *for all* $i$ *and* $j$,

(ii) $\sum_{i,j=1}^n a_i a_j \sigma^f_{i,j}$ *is a nonnegative measure for any* $a_i \in \mathbb{R}$ $(i = 1, \ldots, n)$,

(iii) *the identity*

$$\begin{aligned} &\mathcal{E}(F(f) - F(0, \ldots, 0), G(f) - G(0, \ldots, 0)) \\ &\quad = \sum_{i,j=1}^n \int_{\mathbb{R}^n} \frac{\partial F}{\partial x_i} \frac{\partial G}{\partial x_j} \, d\sigma^f_{i,j} \end{aligned} \tag{3.1}$$

*holds for any* $F, G \in \hat{C}^1_b(\mathbb{R}^n)$.

*Moreover, these measures satisfy the following properties.*

(iv) $\sigma^f_{i,j}(\mathbb{R}^n) = \mathcal{E}(f_i, f_j)$.

(v) *If* $f_i \in [a, b]$ *and* $f_j \in [c, d]$ $\mu$-*a.e., then the topological support of* $\sigma^f_{i,j}$ *is included in* $\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \in [a, b],\ x_j \in [c, d]\}$.

*Proof.* By [2, Theorem I.5.2.1], uniquely determined are the family $(\sigma^f_{i,j})_{1\leq i,j\leq n}$ of signed Radon measures on $\mathbb{R}^n$ such that (i) and (ii) hold, and (3.1) is true for any $F$ and $G \in C^1_c(\mathbb{R}^n)$. By the way of construction of $\sigma^f_{i,j}$ (see also [8]), for any $F \in C^1_c(\mathbb{R}^n)$,

$$2 \int_{\mathbb{R}^n} F \, d\sigma^f_{i,j} = I_{\psi(f_i),\psi(f_j)}(F(f)),$$

where $\psi(x) = (-M) \vee x \wedge M$ for sufficiently large $M$ depending on $F$. Then,

$$\left| \int_{\mathbb{R}^n} F \, d\sigma^f_{i,i} \right| \leq \|F\|_\infty \mathcal{E}(\psi(f_i)) \leq \|F\|_\infty \mathcal{E}(f_i)$$

so that $\sigma^f_{i,i}(\mathbb{R}^n) \le \mathcal{E}(f_i)$. In addition, we have

$$\left|\int_{\mathbb{R}^n} F\,d\sigma^f_{i,j}\right| \le \left|\int_{\mathbb{R}^n} F\,d\sigma^f_{i,i}\right|^{1/2} \left|\int_{\mathbb{R}^n} F\,d\sigma^f_{j,j}\right|^{1/2} \le \|F\|_\infty \mathcal{E}(f_i)^{1/2}\mathcal{E}(f_j)^{1/2}.$$

Therefore, $\sigma^f_{i,j}$ is of finite variation and $|\sigma^f_{i,j}|(\mathbb{R}^n) \le \mathcal{E}(f_i)^{1/2}\mathcal{E}(f_j)^{1/2}$. Equation (3.1) now follows for $F$ and $G$ in $\hat{C}^1_b(\mathbb{R}^n)$, by taking an approximate sequence from $C^1_c(\mathbb{R}^n)$ and using the dominated convergence theorem. Letting $F(x) = x_i$ and $G(x) = x_j$, we obtain $\sigma^f_{i,j}(\mathbb{R}^n) = \mathcal{E}(f_i, f_j)$. To prove (v), it is enough to prove that the support $\sigma^f_{i,i}$ is included in $\{x \in \mathbb{R}^d \mid x_i \in [a,b]\}$ when $f_i \in [a,b]$ $\mu$-a.e. Take $F(x) = \phi(x_i)$, where $\phi$ is in $\hat{C}^1_b(\mathbb{R})$ and $\phi(x) = 0$ on $[a,b]$, $0 < \phi'(x) \le 1$ on $(-\infty,a)\cup(b,\infty)$. Then, by (3.1), $0 = \int_{\mathbb{R}^n} \phi'(x_i)^2\,d\sigma^f_{i,i}$. This implies the assertion. □

The following theorem is also necessary for subsequent arguments.

**Theorem 3.5** ([2, Theorem I.5.2.3])**.** *When $n = 1$ in Theorem 3.4, any $\sigma^f_{1,1}$ is absolutely continuous with respect to one dimensional Lebesgue measure.*

Let $f_1, \dots, f_k, g_1, \dots, g_l, h_1 \dots, h_m$ be functions in $\mathbb{D}$ and let $f = (f_1, \dots, f_k)$, $g = (g_1, \dots, g_l)$, and $h = (h_1 \dots, h_m)$. Set $\lambda^{f,g,h}_{i,j} = 2\sigma^{(f,g,h)}_{i,k+j}$ for $i = 1, \dots, k$ and $j = 1, \dots, l$, where $\sigma^{(f,g,h)}_{\cdot,\cdot}$ is provided in Theorem 3.4 by letting $n = k + l + m$ and taking $(f_1, \dots, f_k, g_1, \dots, g_l, h_1, \dots, h_m)$ as $f$. Then,

$$\lambda^{f,g,h}_{i,j}(\mathbb{R}^{k+l+m}) = 2\mathcal{E}(f_i, g_j),$$

and simple calculation deduces the following identities.

**Proposition 3.6.** *For all $F \in \hat{C}^1_b(\mathbb{R}^k)$, $G \in \hat{C}^1_b(\mathbb{R}^l)$, and $H \in C^1_b(\mathbb{R}^{k+l+m})$ with $F(0, \dots, 0) = G(0, \dots, 0) = H(0, \dots, 0) = 0$,*

$$\begin{aligned} &I_{F(f),G(g)}(H(f,g,h)) \\ &\quad = \sum_{i=1}^k \sum_{j=1}^l \int_{\mathbb{R}^{k+l+m}} \partial_i F(x)\partial_j G(y) H(x,y,z)\,d\lambda^{f,g,h}_{i,j}(x,y,z). \end{aligned} \tag{3.2}$$

When $k = 1$ and $l = 1$, we will write $\lambda^{f,g,h}$ for $\lambda^{f,g,h}_{1,1}$. By Proposition 3.6, we have the integral expression

$$I_{f,g}(F(h)) = \int_{\mathbb{R}^3} F(z)\,d\lambda^{f,g,h}(x,y,z) \tag{3.3}$$

for $f, g, h \in \mathbb{D}$ and $F \in C^1_b(\mathbb{R})$ with $F(0) = 0$. We can define $I_{f,g}(F(h))$ for $F \in C^1_b(\mathbb{R})$ (possibly with $F(0) \ne 0$) by the right-hand side of (3.3).

In other words, when we set $\hat{\mathbb{D}}_b = \{h \mid h = h_0 + \alpha,\ h_0 \in \mathbb{D}_b,\ \alpha \in \mathbb{R}\}$,

$$I_{f,g}(h) = \int_{\mathbb{R}^3} (z+\alpha)\, d\lambda^{f,g,h_0}(x,y,z) = I_{f,g}(h_0) + 2\alpha\mathcal{E}(f,g)$$

for $f, g \in \mathbb{D}$ and $h = h_0 + \alpha \in \hat{\mathbb{D}}_b$ is well-defined. Then, Lemma 2.5 and Lemma 3.3 (i) (ii) are true for $h, h_1, h_2 \in \hat{\mathbb{D}}_b$.

**Lemma 3.7.** (i) *Let $f = (f_1, \dots, f_k)$, $f_i \in \mathbb{D}$, $g = (g_1, \dots, g_l)$, $g_j \in \mathbb{D}$, and $h \in \hat{\mathbb{D}}_b$. Then,*

$$I_{F(f),G(g)}(h) = \sum_{i=1}^{k}\sum_{j=1}^{l} I_{f_i,g_j}(\partial_i F(f)\partial_j G(g)h)$$

*for $F \in \hat{C}_b^1(\mathbb{R}^k)$, $G \in \hat{C}_b^1(\mathbb{R}^l)$ with $F(0,\dots,0) = G(0,\dots,0) = 0$.*

(ii) *For $f, g \in \mathbb{D}_b$, $\{2\mathcal{E}(fg)\}^{1/2} = I_{fg}(1)^{1/2} \le I_f(g^2)^{1/2} + I_g(f^2)^{1/2}$.*

*Proof.* This is immediately proved by the integral representation of $I$. □

**Lemma 3.8.** *Let $f \in \mathbb{D}_0(\{E_k\})$ for some nest $\{E_k\}_{k=1}^{\infty}$ and $g \in \mathbb{D}_b$. Then, $fg \in \mathbb{D}$ and $\|fg\|_{\mathbb{D}} \le \sqrt{2}\|f\|_{L^\infty(\mu)}\|g\|_{\mathbb{D}} + \|g\|_{L^2(\mu)}$.*

*Proof.* We can take sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ from $\mathbb{D}$ such that $f_k = f$ $\mu$-a.e. on $E_k$, $\|f_k\|_{L^\infty(\mu)} \le \|f\|_{L^\infty(\mu)}$, $g_k \in \mathbb{D}_{E_k}$, $\|g_k\|_{L^\infty(\mu)} \le \|g\|_{L^\infty(\mu)}$ for every $k$, and $g_k$ converges to $g$ in $\mathbb{D}$ and $\mu$-a.e. as $k \to \infty$. By Lemma 3.7 (ii) and Lemma 2.5 (i), we have

$$\begin{aligned}
\{2\mathcal{E}(f_k g_k)\}^{1/2} &\le I_{f_k}(g_k^2)^{1/2} + I_{g_k}(f_k^2)^{1/2} \\
&\le \|g_k^2\|_{L^1(\mu)}^{1/2} + \|f_k\|_{L^\infty(\mu)}\{2\mathcal{E}(g_k)\}^{1/2} \\
&\le \|g_k\|_{L^2(\mu)} + \|f\|_{L^\infty(\mu)}\{2\mathcal{E}(g_k)\}^{1/2}, \\
\|f_k g_k\|_{L^2(\mu)} &\le \|f\|_{L^\infty(\mu)}\|g_k\|_{L^2(\mu)}.
\end{aligned}$$

Therefore, $\{f_k g_k\}_{k=1}^{\infty}$ is bounded in $\mathbb{D}$. Since $f_k g_k$ converges to $fg$ $\mu$-a.e., we obtain that $fg \in \mathbb{D}$ and

$$\begin{aligned}
\|fg\|_{\mathbb{D}} &\le \liminf_{k\to\infty} \|f_k g_k\|_{\mathbb{D}} \\
&\le \liminf_{k\to\infty} \left(\mathcal{E}(f_k g_k)^{1/2} + \|f_k g_k\|_{L^2(\mu)}\right) \\
&\le 2^{-1/2}\|g\|_{L^2(\mu)} + \|f\|_{L^\infty(\mu)}\mathcal{E}(g)^{1/2} + \|f\|_{L^\infty(\mu)}\|g\|_{L^2(\mu)} \\
&\le \|g\|_{L^2(\mu)} + \sqrt{2}\|f\|_{L^\infty(\mu)}\|g\|_{\mathbb{D}}.
\end{aligned}$$

□

**Proposition 3.9.** *The set $\mathbb{D}_0(\{E_k\})$ does not depend on the choice of the nest $\{E_k\}_{k=1}^{\infty}$.*

*Proof.* Let $\{E_k\}_{k=1}^\infty$ and $\{E'_k\}_{k=1}^\infty$ be two nests and $f \in \mathbb{D}_0(\{E'_k\})$. It is enough to prove $f \in \mathbb{D}_0(\{E_k\})$. Take $\{\chi_k\}_{k=1}^\infty \subset \mathbb{D}$ as in Definition 2.1 and Remark 2.2 for the nest $\{E_k\}_{k=1}^\infty$. From Lemma 3.8, $f\chi_k \in \mathbb{D}$ for each $k$. Therefore, $f \in \mathbb{D}_{loc,b}(\{E_k\})$. Let $h \in \mathbb{D}_{E_k,b,+}$ for some $k \in \mathbb{N}$. As in the proof of Lemma 3.2, we can take $\{h_l\}_{l=1}^\infty$ such that $h_l \in \mathbb{D}_{E_k \cap E'_l}$, $0 \le h_l \le h$ $\mu$-a.e. for all $l$ and $h_l$ converges to $h$ in $\mathbb{D}$ as $l \to \infty$. Then, for all $l$,

$$\|h_l\|_{L^1(\mu)} \ge I_f(h_l) = I_{f\chi_k}(h_l).$$

Letting $l \to \infty$, we obtain from Lemma 3.3 (iii) that

$$\|h\|_{L^1(\mu)} \ge I_{f\chi_k}(h) = I_f(h).$$

Hence, we conclude $f \in \mathbb{D}_0(\{E_k\})$. □

By this proposition, we will use the notation $\mathbb{D}_0$ as well as $\mathbb{D}_0(\{E_k\})$ from now on. Note that $1 \in \mathbb{D}_0$. When $\mu(X) < \infty$ and $1 \in \mathbb{D}$, the space $\mathbb{D}_0$ is the same as given in [4], namely (1.3), because we can take $E_k = X$ ($k \in \mathbb{N}$) as a nest.

**Proposition 3.10.** *Let $f$, $g \in \mathbb{D}_0$. Then, $-f$, $f \vee g$, and $f \wedge g$ also belong to $\mathbb{D}_0$.*

*Proof.* It is trivial that $f \in \mathbb{D}_0$ implies $-f \in \mathbb{D}_0$. Let $f$ and $g$ be in $\mathbb{D}_0$. Take an arbitrary nest $\{E_k\}_{k=1}^\infty$. It is easy to see that $f \vee g \in \mathbb{D}_{loc,b}(\{E_k\})$. Take $\{f_k\}_{k=1}^\infty$ and $\{g_k\}_{k=1}^\infty$ from $\mathbb{D}$ so that $f_k = f$, $g_k = g$ $\mu$-a.e. on $E_k$, $\|f_k\|_{L^\infty(\mu)} \le \|f\|_{L^\infty(\mu)}$ and $\|g_k\|_{L^\infty(\mu)} \le \|g\|_{L^\infty(\mu)}$ for each $k$. Take any $h \in \mathbb{D}_{E_k,b,+}$, $k \in \mathbb{N}$. Define for $l \in \mathbb{N}$

$$h_{1,l} = (0 \vee l(f_k - g_k) \wedge 1)h, \quad h_{2,l} = (0 \vee l(g_k - f_k + 1/l) \wedge 1)h.$$

Then, $h_{1,l}, h_{2,l} \in \mathbb{D}_{E_k,b,+}$, $h = h_{1,l} + h_{2,l}$, $h_{1,l} = 0$ $\mu$-a.e. on $\{f_k \vee g_k \ne f_k\}$, and $h_{2,l} \le F_l(f_k \vee g_k - g_k)\|h\|_{L^\infty(\mu)}$ for every $l$. Here, $F_l$ is an arbitrary $C^1$-function on $\mathbb{R}$ such that $0 \le F_l \le 1$, $F_l(x) = 0$ on $(-\infty, -1/l] \cup [2/l, \infty)$, and $F_l(x) = 1$ on $[0, 1/l]$. Then, we have

$$\begin{aligned}
&I_{f\vee g}(h) - \|h\|_{L^1(\mu)} \\
&= I_{f_k \vee g_k}(h_{1,l}) - \|h_{1,l}\|_{L^1(\mu)} + I_{f_k \vee g_k}(h_{2,l}) - \|h_{2,l}\|_{L^1(\mu)} \\
&\le I_{f_k}(h_{1,l}) - \|h_{1,l}\|_{L^1(\mu)} \\
&\quad + \left(I_{f_k \vee g_k - g_k}(h_{2,l})^{1/2} + I_{g_k}(h_{2,l})^{1/2}\right)^2 - \|h_{2,l}\|_{L^1(\mu)} \\
&\le (1+\varepsilon^{-1})\|h\|_{L^\infty(\mu)} I_{f_k \vee g_k - g_k}(F_l(f_k \vee g_k - g_k)) + (1+\varepsilon) I_{g_k}(h_{2,l}) \\
&\quad - \|h_{2,l}\|_{L^1(\mu)} \\
&\le (1+\varepsilon^{-1})\|h\|_{L^\infty(\mu)} \int_{\mathbb{R}} F_l(x)\, \sigma_{1,1}^{f_k \vee g_k - g_k}(dx) + \varepsilon I_{g_k}(h),
\end{aligned}$$

for any $\varepsilon > 0$, where $\sigma_{1,1}^{f_k \vee g_k - g_k}$ is a measure on $\mathbb{R}$ given in Theorem 3.4 with $n = 1$. Since Theorem 3.5 implies $\sigma_{1,1}^{f_k \vee g_k - g_k}(\{0\}) = 0$, we obtain $I_{f\vee g}(h) - \|h\|_{L^1(\mu)} \le 0$ by letting $l \to \infty$, then $\varepsilon \to 0$. This means $f \vee g \in \mathbb{D}_0$. We also have $f \wedge g = -((-f) \vee (-g)) \in \mathbb{D}_0$. $\square$

For $A \in \mathcal{B}$ and $M \ge 0$, define

$$\mathbb{D}_{A,M} := \{f \in \mathbb{D}_0 \mid f = 0 \text{ on } A \text{ and } f \le M \ \mu\text{-a.e.}\}$$

**Proposition 3.11.** *For each $A \in \mathcal{B}$, there exists a unique $[0,\infty]$-valued measurable function $\mathsf{d}_A$ such that, for every $M > 0$, $\mathsf{d}_A \wedge M$ is the maximal element of $\mathbb{D}_{A,M}$. Namely, $\mathsf{d}_A = 0$ $\mu$-a.e. on $A$ and $f \le \mathsf{d}_A \wedge M$ for every $f \in \mathbb{D}_{A,M}$. Moreover, $\mathsf{d}(A,B) = \operatorname{essinf}_{x\in B} \mathsf{d}_A(x)$ for every $B \in \mathcal{B}$.*

*Proof.* Take a finite measure $\nu$ on $X$ such that $\nu$ and $\mu$ are mutually absolutely continuous. Fix $M > 0$ and let $a = \sup\{\int_X f\,d\nu \mid f \in \mathbb{D}_{A,M}\}(<\infty)$. There is a sequence $\{g_k\}_{k=1}^\infty$ in $\mathbb{D}_{A,M}$ such that $\lim_{k\to\infty} \int_X g_k\,d\nu = a$. Let $f_k = g_1 \vee g_2 \vee \cdots \vee g_k$. By Proposition 3.10, $f_k \in \mathbb{D}_{A,M}$. It also holds that $f_k$ converges to some $f$ $\mu$-a.e. and $\int_X f\,d\nu = a$. We will prove $f \in \mathbb{D}_{A,M}$. Take a nest $\{E_k\}_{k=1}^\infty$ and functions $\{\chi_k\}_{k=1}^\infty$ as in Definition 2.1 and Remark 2.2. By Lemma 3.8, $f_k\chi_l \in \mathbb{D}$ for every $k$ and $l$, and $\{f_k\chi_l\}_{k=1}^\infty$ is bounded in $\mathbb{D}$ for each $l$. Therefore, $f\chi_l \in \mathbb{D}$ and $f_k\chi_l$ converges to $f\chi_l$ weakly in $\mathbb{D}$ as $k \to \infty$. For any $h \in \mathbb{D}_{E_l,b,+}$, we have

$$I_{f_k\chi_l}(h) = I_{f_k}(h) \le \|h\|_{L^1(\mu)}.$$

Lemma 3.3 (i) assures that $I_f(h) = I_{f\chi_l}(h) \le \|h\|_{L^1(\mu)}$ by letting $k \to \infty$. Therefore, $f \in \mathbb{D}_0$. It is easy to see that $f$ is the maximal element in $\mathbb{D}_{A,M}$. Denote $f$ by $f^M$ to indicate the dependency of $M$. Since $f^M$ has consistency in $M$, namely, $f^{M'} = f^M \wedge M'$ when $M > M'$, the existence of $\mathsf{d}_A$ follows. The uniqueness of $\mathsf{d}_A$ is clear.

To prove the latter part of the proposition, let $B \in \mathcal{B}$. By definition, $\mathsf{d}(A,B) \ge \operatorname{essinf}_{x\in B} \mathsf{d}_A(x) \wedge M$ for every $M$. Letting $M \to \infty$, we get $\mathsf{d}(A,B) \ge \operatorname{essinf}_{x\in B} \mathsf{d}_A(x)$. For the converse inequality, let $f \in \mathbb{D}_0$ and define $\hat{f} = (f - \operatorname{esssup}_{x\in A} f(x)) \vee 0$. Then, $\hat{f} \in \mathbb{D}_{A,M}$ for some $M > 0$ and

$$\operatorname*{essinf}_{x\in B} f(x) - \operatorname*{esssup}_{x\in A} f(x) \le \operatorname*{essinf}_{x\in B} \hat{f}(x) \le \operatorname*{essinf}_{x\in B} \mathsf{d}_A(x).$$

Taking a supremum over $f$, we obtain $\mathsf{d}(A,B) \le \operatorname{essinf}_{x\in B} \mathsf{d}_A(x)$. $\square$

## 4. Proof of Theorem 2.7

We first prove the upper side estimate.

**Theorem 4.1.** *For any* $A, B \in \mathcal{B}$ *with* $0 < \mu(A) < \infty$ *and* $0 < \mu(B) < \infty$*,*

$$P_t(A, B) \le \sqrt{\mu(A)\mu(B)} \exp\left(-\frac{\mathsf{d}(A, B)^2}{2t}\right), \qquad t > 0.$$

*In particular,* $\limsup_{t\to 0} t \log P_t(A, B) \le -\mathsf{d}(A, B)^2/2$*.*

*Proof.* Let $w \in \mathbb{D}_{A,M}$ and let $\{E_k\}_{k=1}^\infty$ be an arbitrary nest. There exists $\{w_k\}_{k=1}^\infty \subset \mathbb{D}$ such that $\|w_k\|_{L^\infty(\mu)} \le M$ and $w_k = w$ $\mu$-a.e. on $E_k$ for all $k$. Note that $w_k$ converges to $w$ $\mu$-a.e. Set $v_t = T_t 1_A$ for $t > 0$. For $\alpha \in \mathbb{R}$, define $q(t) = \int_X (e^{\alpha w} v_t)^2 \, d\mu$, $t > 0$. Fix $t > 0$. We can take $\{u_k\}_{k=1}^\infty$ from $\mathbb{D}$ so that $u_k \in \mathbb{D}_{E_k}$, $0 \le u_k \le 1$ $\mu$-a.e. for each $k$, and $u_k$ converges to $v_t$ in $\mathbb{D}$ and $\mu$-a.e. as $k \to \infty$. Then,

$$\begin{aligned}
q'(t) &= \int_X 2\mathcal{L}v_t \cdot v_t e^{2\alpha w} \, d\mu \\
&= \lim_{k\to\infty} \int_X 2\mathcal{L}v_t \cdot u_k e^{2\alpha w_k} \, d\mu \\
&= \lim_{k\to\infty} -2\mathcal{E}(v_t, u_k e^{2\alpha w_k}).
\end{aligned}$$

Since $u_k \to v_t$ in $\mathbb{D}$ as $k \to \infty$, we have

$$\lim_{k\to\infty} (\mathcal{E}(v_t, u_k) - \mathcal{E}(u_k, u_k)) = 0.$$

In the inequality

$$\begin{aligned}
&\Big|\mathcal{E}(v_t, u_k(e^{2\alpha w_k} - 1)) - \mathcal{E}(u_k, u_k(e^{2\alpha w_k} - 1))\Big| \\
&\le \mathcal{E}(v_t - u_k)^{1/2} \mathcal{E}(u_k(e^{2\alpha w_k} - 1))^{1/2},
\end{aligned}$$

the right-hand side converges to 0 as $k \to \infty$, since

$$\begin{aligned}
&\Big\{2\mathcal{E}(u_k(e^{2\alpha w_k} - 1))\Big\}^{1/2} \\
&\le I_{u_k}((e^{2\alpha w_k} - 1)^2)^{1/2} + I_{e^{2\alpha w_k}-1}(u_k^2)^{1/2} \quad \text{(by Lemma 3.7 (ii))} \\
&\le (e^{2|\alpha|M} + 1)\{2\mathcal{E}(u_k)\}^{1/2} + I_{w_k}((2\alpha e^{2\alpha w_k})^2 u_k^2)^{1/2} \\
&\qquad\qquad \text{(by Lemma 2.5 (i) and Lemma 3.7 (i))} \\
&\le (e^{2|\alpha|M} + 1)\{2\mathcal{E}(u_k)\}^{1/2} + 2|\alpha| e^{2|\alpha|M} \|u_k\|_{L^2(\mu)},
\end{aligned}$$

which is bounded in $k$. Thus we have

$$q'(t) = \lim_{k\to\infty} -2\mathcal{E}(u_k, u_k e^{2\alpha w_k}).$$

Letting $f = (u_k, w_k)$ in Theorem 3.4, we have

$$\begin{aligned}-2\mathcal{E}(u_k, u_k e^{2\alpha w_k}) &= -2\int_{\mathbb{R}^2} e^{2\alpha y}\, d\sigma^f_{1,1}(x,y) - 2\int_{\mathbb{R}^2} 2\alpha x e^{2\alpha y}\, d\sigma^f_{1,2}(x,y)\\ &\le 2\int_{\mathbb{R}^2} \alpha^2 x^2 e^{2\alpha y}\, d\sigma^f_{2,2}(x,y)\\ &= \alpha^2 I_{w_k}(u_k^2 e^{2\alpha w_k}) \le \alpha^2 \|u_k^2 e^{2\alpha w_k}\|_{L^1(\mu)}.\end{aligned}$$

Therefore, we have

$$q'(t) \le \alpha^2 \|v_t^2 e^{2\alpha w}\|_{L^1(\mu)} = \alpha^2 q(t).$$

Solving this differential inequality, we have

$$q(t) \le q(0) e^{\alpha^2 t}, \quad t > 0. \tag{4.1}$$

By setting $w = \mathsf{d}_A \wedge M$, (4.1) implies that

$$\|e^{\alpha w} T_t 1_A\|_{L^2(\mu)} \le \sqrt{\mu(A)}\, e^{\alpha^2 t/2}.$$

A similar calculation for $v_t = T_t 1_B$ gives, for $\alpha \ge 0$,

$$\|e^{-\alpha w} T_t 1_B\|_{L^2(\mu)} \le \sqrt{\mu(B)}\, e^{-\alpha(\mathsf{d}(A,B)\wedge M) + \alpha^2 t/2}.$$

Therefore,

$$\begin{aligned}P_t(A,B) &\le \|e^{\alpha w} T_{t/2} 1_A\|_{L^2(\mu)} \|e^{-\alpha w} T_{t/2} 1_B\|_{L^2(\mu)}\\ &\le \sqrt{\mu(A)\mu(B)}\, e^{-\alpha(\mathsf{d}(A,B)\wedge M) + \alpha^2 t/2}.\end{aligned}$$

The conclusion follows by optimizing the right-hand side in $\alpha$ and letting $M \to \infty$. □

We turn to the lower side estimate. Fix a nest $\{E_k\}_{k=1}^\infty$ and associated functions $\{\chi_k\}_{k=1}^\infty$ as in Definition 2.1 and Remark 2.2. Take functions $\phi^K$, $\Phi^K$, $\Psi^K$ for $K > 0$ as in Section 2.1 of [4]. That is, using an arbitrary concave function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that

- $g$ is bounded and three times continuously differentiable;
- $g(x) = x$ for $x \le 1$ and $0 < g'(x) \le 1$ for any $x \in \mathbb{R}_+$;
- there is a positive constant $C$ such that $0 \le -g''(x) \le Cg'(x)$ for all $x \ge 0$,

define

$$\phi^K(x) = Kg(x/K), \quad \Phi^K(x) = \int_0^x (\phi^K)'(s)^2\, ds, \quad \Psi^K(x) = x(\phi^K)'(x)^2.$$

In what follows, we suppress the symbol $K$ from the notation since $K$ is fixed in most of the argument. The following are some basic properties for these functions, proved in [4].

- $0 < \phi'(x) \le 1$, $0 \le -\phi''(x) \le (C/K)\phi'(x)$,

- $0 \le \Psi(x) \le \Phi(x) \le \phi(x) \le LK$, where $L := \lim_{x\to\infty} g(x) < \infty$.
- $\Phi(x) = \Psi(x) = x$ for $x \in [0, K]$.

We also adopt the same abbreviations there; for functions $u_t^\delta$ on $X$ with parameters $t$ and $\delta$, we write $\phi_t^\delta$ for $\phi(u_t^\delta)$, $\Phi_t^\delta$ for $\Phi(u_t^\delta)$, $\bar\phi_t^\delta$ for $t^{-1}\int_0^t \phi_s^\delta\,ds$, and so on. We denote $\int_X fg\,d\mu$ by $(f,g)$ for functions $f$ and $g$ on $X$.

For $\delta \in (0,1]$ and $f \in L^2(\mu)$ with $0 \le f \le 1-\delta$ $\mu$-a.e., define for $t > 0$

$$u_t^\delta(x) = -t\log(T_t f(x) + \delta), \quad e_t^\delta = -t\log\delta.$$

**Lemma 4.2.** $u_t^\delta - e_t^\delta \in \mathbb{D}$ *and*

$$\begin{aligned}(4.2)\qquad &\partial_t(\Phi_t^\delta, \rho)\\ &= -\mathcal{E}(((\phi')_t^\delta)^2\rho, u_t^\delta - e_t^\delta) - \frac{1}{2t} I_{u_t^\delta - e_t^\delta}(((\phi')_t^\delta)^2\rho) + \frac{1}{t}(\Psi_t^\delta, \rho)\end{aligned}$$

*for* $\rho \in L^1(\mu) \cap \mathbb{D}_b$.

*Proof.* Since $u_t^\delta - e_t^\delta = F(T_t f)$, where $F(s) = -t\log((s+\delta)/\delta)$ is a Lipschitz function on $[0,\infty)$ with $F(0) = 0$, we conclude that $u_t^\delta - e_t^\delta \in \mathbb{D}$. The identity (4.2) is proved in the same way as [4, Lemma 2.10], but we give the proof for completeness.

Using Lemma 3.7, it holds that

$$\mathcal{E}(u_t^\delta - e_t^\delta, \rho) = -t\mathcal{E}\left(T_t f, \frac{\rho}{T_t f + \delta}\right) - \frac{1}{2t} I_{u_t^\delta - e_t^\delta}(\rho).$$

Then,

$$\begin{aligned}\partial_t(u_t^\delta, \rho) &= \frac{1}{t}(u_t^\delta, \rho) - \left(\frac{t\,\mathcal{L}T_t f}{T_t f + \delta}, \rho\right)\\ &= \frac{1}{t}(u_t^\delta, \rho) - t\mathcal{E}\left(T_t f, \frac{\rho}{T_t f + \delta}\right)\\ &= \frac{1}{t}(u_t^\delta, \rho) - \mathcal{E}(u_t^\delta - e_t^\delta, \rho) - \frac{1}{2t} I_{u_t^\delta - e_t^\delta}(\rho).\end{aligned}$$

By using the identity $\partial_t(\Phi_t^\delta, \rho) = (\partial_t u_t^\delta, ((\phi')_t^\delta)^2\rho)$ and replacing $\rho$ by $((\phi')_t^\delta)^2\rho$ in the relation above, we obtain (4.2). $\square$

Define $u_t^\delta = -t\log((1-\delta)T_t 1_A + \delta)$ for $A \in \mathcal{B}$ with $\mu(A) < \infty$.

**Lemma 4.3.** *There exists* $T_0 > 0$ *such that both* $\{\mathcal{E}(\bar\phi_t^\delta \chi_k)\}_{0<t\le T_0,\,0<\delta\le 1}$ *and* $\{\mathcal{E}(\bar\Phi_t^\delta \chi_k)\}_{0<t\le T_0,\,0<\delta\le 1}$ *are bounded for each* $k$.

*Proof.* Let $U_t^\delta = 2\mathcal{E}(\phi_t^\delta \chi_k)$ and $a_k = 2\mathcal{E}(\chi_k)$. Since $\phi_t^\delta = \phi((u_t^\delta - e_t^\delta) + e_t^\delta)$, Lemma 3.7 and Lemma 2.5 imply

$$
\begin{aligned}
U_t^\delta &= I_{u_t^\delta - e_t^\delta}(((\phi')_t^\delta)^2\chi_k^2) + 2I_{u_t^\delta - e_t^\delta,\chi_k}(\phi_t^\delta(\phi')_t^\delta\chi_k) + I_{\chi_k}((\phi_t^\delta)^2)\\
&\le 2I_{u_t^\delta - e_t^\delta}(((\phi')_t^\delta)^2\chi_k^2) + 2I_{\chi_k}((\phi_t^\delta)^2)\\
&\le 2I_{u_t^\delta - e_t^\delta}(((\phi')_t^\delta)^2\chi_k^2) + 2K^2L^2a_k.
\end{aligned}
\tag{4.3}
$$

Set $V_t^\delta = I_{u_t^\delta - e_t^\delta}(((\phi')_t^\delta)^2\chi_k^2)$. By letting $\rho = \chi_k^2$ in Lemma 4.2,

$$V_t^\delta = -2t\partial_t(\Phi_t^\delta, \chi_k^2) - 2t\mathcal{E}(((\phi')_t^\delta)^2\chi_k^2, u_t^\delta - e_t^\delta) + 2(\Psi_t^\delta, \chi_k^2).$$

The last term of the right-hand side is dominated by $2KL$. Regarding the second term, we have

$$
\begin{aligned}
&\mathcal{E}(((\phi')_t^\delta)^2\chi_k^2, u_t^\delta - e_t^\delta)\\
&= I_{\chi_k,\phi_t^\delta\chi_k}((\phi')_t^\delta) - I_{\chi_k}((\phi')_t^\delta\phi_t^\delta) + I_{u_t^\delta - e_t^\delta}((\phi'')_t^\delta(\phi')_t^\delta\chi_k^2)\\
&\ge -\{2\mathcal{E}(\chi_k)\}^{1/2}\{2\mathcal{E}(\phi_t^\delta\chi_k)\}^{1/2} - KL\cdot 2\mathcal{E}(\chi_k)\\
&\quad - (C/K)I_{u_t^\delta - e_t^\delta}(((\phi')_t^\delta)^2\chi_k^2)\\
&= -a_k^{1/2}(U_t^\delta)^{1/2} - KLa_k - (C/K)V_t^\delta.
\end{aligned}
$$

Here, the first equality follows from Lemma 3.7. By combining these estimates,

$$
\begin{aligned}
\left(1 - \frac{2Ct}{K}\right)V_t^\delta &\le -2t\partial_t(\Phi_t^\delta, \chi_k^2) + 2a_k^{1/2}t(U_t^\delta)^{1/2} + 2KLa_kt + 2KL\\
&\le -2t\partial_t(\Phi_t^\delta, \chi_k^2) + U_t^\delta/8 + 8a_kt^2 + 2KLa_kt + 2KL.
\end{aligned}
$$

Take $T_0 = K/(4C)$. For $t \in (0, T_0]$, $V_t^\delta/2 \le (1 - (2Ct)/K)V_t^\delta$. Putting this and the above inequalities into (4.3),

$$U_t^\delta \le 4\{-2t\partial_t(\Phi_t^\delta, \chi_k^2) + U_t^\delta/8 + 8a_kt^2 + 2KLa_kt + 2KL\} + 2K^2L^2a_k,$$

so that

$$U_t^\delta \le -16t\partial_t(\Phi_t^\delta, \chi_k^2) + c,$$

where $c$ is a constant independent of $t$ and $\delta$. Therefore,

$$
\begin{aligned}
\int_\varepsilon^t \mathcal{E}(\phi_s^\delta\chi_k)\,ds &= \frac{1}{2}\int_\varepsilon^t U_s^\delta\,ds\\
&\le -8\int_\varepsilon^t s(\partial_s\Phi_s^\delta, \chi_k^2)\,ds + \frac{c}{2}(t-\varepsilon)\\
&= -8s(\Phi_s^\delta, \chi_k^2)|_{s=\varepsilon}^{s=t} + 8\int_\varepsilon^t(\Phi_s^\delta, \chi_k^2)\,ds + \frac{c}{2}(t-\varepsilon).
\end{aligned}
$$

Letting $\varepsilon \to 0$ and dividing by $t$, we obtain

$$\mathcal{E}(\bar\phi_t^\delta\chi_k) \le \frac{1}{t}\int_0^t \mathcal{E}(\phi_s^\delta\chi_k)\,ds \le 8KL + \frac{cT_0}{2}.$$

Therefore, $\{\mathcal{E}(\bar\phi_t^\delta\chi_k)\}_{0<t\le T_0,\,0<\delta\le 1}$ is bounded. Moreover, since

$$\begin{aligned}\mathcal{E}(\Phi_t^\delta\chi_k) &= I_{u_t^\delta-e_t^\delta}(((\phi')_t^\delta)^4\chi_k^2) + 2I_{u_t^\delta-e_t^\delta,\chi_k}(\Phi_t^\delta((\phi')_t^\delta)^2\chi_k) + I_{\chi_k}((\Phi_t^\delta)^2)\\ &\le 2I_{u_t^\delta-e_t^\delta}(((\phi')_t^\delta)^4\chi_k^2) + 2I_{\chi_k}((\Phi_t^\delta)^2)\\ &\le 2V_t^\delta + 2K^2L^2a_k,\end{aligned}$$

we can prove the boundedness of $\{\mathcal{E}(\bar\Phi_t^\delta\chi_k)\}_{0<t\le T_0,\,0<\delta\le 1}$ in the same way. □

Since $\bar\phi_t^\delta\chi_k$ converges to $\bar\phi_t\chi_k$ $\mu$-a.e. as $\delta\to 0$, we conclude that $\bar\phi_t\chi_k\in\mathbb{D}$ and $\{\bar\phi_t\chi_k\}_{t\in(0,T_0]}$ is bounded in $\mathbb{D}$ for each $k$. By the diagonal argument, for any decreasing sequence $\{t_n\}\downarrow 0$, we can take a subsequence $\{t_{n'}\}$ such that for every $k$, $\bar\phi_{t_{n'}}\chi_k$ converges weakly to some $\psi_k$ in $\mathbb{D}$. Since $\chi_k=1$ on $E_l$ if $k\ge l$, $\psi_k$ and $\psi_l$ should be identical on $E_l$ for $k\ge l$. Therefore, there exists $\bar\phi_0\in\mathbb{D}_{loc,b,+}(\{E_k\})$ such that $\psi_k=\bar\phi_0$ on $E_k$ for every $k$.

We may also assume, by taking a further subsequence if necessary, that there exist $\Phi_0$, $\bar\Phi_0$, and $\bar\Psi_0$ in $L_+^\infty(\mu)$ such that $\Phi_{t_{n'}}\to\Psi_0$, $\bar\Phi_{t_{n'}}\to\bar\Phi_0$, $\bar\Psi_{t_{n'}}\to\bar\Psi_0$ both in weak-$L^2(\nu)$ sense and in weak*-$L^\infty(\mu)$ sense. Here, $\nu$ is an arbitrarily fixed *finite* measure on $X$ such that $\nu$ and $\mu$ are mutually absolutely continuous, and $L^\infty(\mu)$ is regarded as the dual space of $L^1(\mu)$.

Now, fix $k$ and let $h\in\mathbb{D}_{E_k,b,+}$. By Lemma 3.7 and Lemma 4.2,

$$\begin{aligned}I_{\phi_t^\delta\chi_k}(h) &= I_{u_t^\delta-e_t^\delta}(((\phi')_t^\delta)^2\chi_k^2h) + 2I_{u_t^\delta-e_t^\delta,\chi_k}((\phi')_t^\delta\phi_t^\delta\chi_kh) + I_{\chi_k}((\phi_t^\delta)^2h)\\ &= I_{u_t^\delta-e_t^\delta}(((\phi')_t^\delta)^2h)\\ &= -2t\partial_t(\Phi_t^\delta,h) - 2t\mathcal{E}(((\phi')_t^\delta)^2h, u_t^\delta-e_t^\delta) + 2(\Psi_t^\delta,h).\end{aligned}$$

Regarding the second term, we have

$$\begin{aligned}\mathcal{E}(((\phi')_t^\delta)^2h, u_t^\delta-e_t^\delta) &= \mathcal{E}(\Phi_t^\delta\chi_k,h) + I_{u_t^\delta-e_t^\delta}((\phi')_t^\delta(\phi'')_t^\delta h)\\ &\ge \mathcal{E}(\Phi_t^\delta\chi_k,h) - (C/K)I_{\phi_t^\delta-\phi(e_t^\delta)}(h).\end{aligned}$$

It also holds that

$$I_{\phi_t^\delta-\phi(e_t^\delta)}(h) = I_{\phi_t^\delta\chi_k-\phi(e_t^\delta)\chi_k}(h) = I_{\phi_t^\delta\chi_k}(h).$$

Then,

$$
\begin{aligned}
\frac{1}{2}\int_0^T I_{\phi_t^\delta\chi_k}(h)\,dt &\le -t(\Phi_t^\delta,h)|_{t=0}^{t=T}+\int_0^T(\Phi_t^\delta,h)\,dt\\
&\quad-\int_0^T t\{\mathcal{E}(\Phi_t^\delta\chi_k,h)-(C/K)I_{\phi_t^\delta\chi_k}(h)\}\,dt+T(\bar\Psi_T^\delta,h)\\
&\le -T(\Phi_T^\delta,h)+T(\bar\Phi_T^\delta,h)-\int_0^T t\mathcal{E}(\Phi_t^\delta\chi_k,h)\,dt\\
&\quad+\frac{CT}{K}\int_0^T I_{\phi_t^\delta\chi_k}(h)\,dt+T(\bar\Psi_T^\delta,h),
\end{aligned}
$$

which implies

$$
\begin{aligned}
(4.4)\qquad &\left(\frac{1}{2}-\frac{CT}{K}\right)\frac{1}{T}\int_0^T I_{\phi_t^\delta\chi_k}(h)\,dt\\
&\le -(\Phi_T^\delta,h)+(\bar\Phi_T^\delta,h)-\frac{1}{T}\int_0^T t\mathcal{E}(\Phi_t^\delta\chi_k,h)\,dt+(\bar\Psi_T^\delta,h).
\end{aligned}
$$

For the third term of the right-hand side, the integration by parts formula gives

$$
\begin{aligned}
&\frac{1}{T}\int_0^T t\mathcal{E}(\Phi_t^\delta\chi_k,h)\,dt\\
&=\frac{t}{T}\int_0^t\mathcal{E}(\Phi_s^\delta\chi_k,h)\,ds\Big|_{t=0}^{t=T}-\frac{1}{T}\int_0^T\!\!\int_0^t\mathcal{E}(\Phi_s^\delta\chi_k,h)\,ds\,dt\\
&=T\mathcal{E}(\bar\Phi_T^\delta\chi_k,h)-\frac{1}{T}\int_0^T t\mathcal{E}(\bar\Phi_t^\delta\chi_k,h)\,dt,
\end{aligned}
$$

which converges to 0 as $\delta\to0$ and $T\to0$ because $\mathcal{E}(\bar\Phi_t^\delta\chi_k)$ is bounded in $\delta$ and $t$ by Lemma 4.3.

Letting $\delta\to0$, dividing by $T$ and letting $T\to0$ along the sequence $\{t_{n'}\}$ in (4.4), we obtain

$$
(4.5)\qquad \frac{1}{2}I_{\bar\phi_0}(h)\le(-\Phi_0+\bar\Phi_0+\bar\Psi_0,h)\le(2\bar\phi_0,h).
$$

Then, for $\varepsilon>0$,

$$
I_{\sqrt{\bar\phi_0+\varepsilon}-\sqrt{\varepsilon}}(h)=\frac{1}{4}I_{\bar\phi_0}\left(\frac{h}{\bar\phi_0+\varepsilon}\right)\le\frac{1}{2}\left(2\bar\phi_0,\frac{h}{\bar\phi_0+\varepsilon}\right)\le\|h\|_{L^1(\mu)}.
$$

Hence $\sqrt{\bar\phi_0+\varepsilon}-\sqrt{\varepsilon}\in\mathbb{D}_0$, which implies $\sqrt{\bar\phi_0}\in\mathbb{D}_0$ by Lemma 3.3 (i).

**Lemma 4.4.** *$\bar\phi_0=0$ $\mu$-a.e. on $A$.*

*Proof.* Since $T_s1_A$ converges to $1_A$ in $L^2(\mu)$ as $s\to0$, we can take a subsequence $\{s_{k'}\}$ from an arbitrary sequence $\{s_k\}\downarrow0$ such that $T_{s_{k'}}1_A\to1_A$ $\mu$-a.e. as $k'\to\infty$. By the dominated convergence theorem,

$\lim_{k'\to\infty}\int_A \phi_{s_{k'}}\,d\mu = 0$. This means that $\lim_{t\to 0}\int_A \phi_t\,d\mu = 0$. Then, by letting $t\to 0$ along the sequence $\{t_{n'}\}$ in the identity

$$\int_A \bar{\phi}_t\,d\mu = \frac{1}{t}\int_0^t\!\!\int_A \phi_s\,d\mu\,ds,$$

we obtain $\int_A \bar{\phi}_0\,d\mu = 0$, which implies the claim. □

From these arguments, we conclude that $\sqrt{\bar{\phi}_0}\in\mathbb{D}_{A,\sqrt{KL}}$ and therefore, $\bar{\phi}_0 \le \mathsf{d}_A^2$ $\mu$-a.e. But this inequality is not optimal; a sharper estimate is obtained by the following lemma.

**Lemma 4.5.** *If the inequality*

$$\bar{\phi}_0^K(x) \le c\frac{\mathsf{d}_A(x)^2}{2}$$

*holds true $\mu$-a.e. for some $c>1$ for every $K$ and every limit $\bar{\phi}_0^K$, then*

$$\bar{\phi}_0^K(x) \le (2-c^{-1})\frac{\mathsf{d}_A(x)^2}{2} \quad \mu\text{-a.e.}$$

*Proof.* The proof is a modification of Lemma 2.12 in [4]. Given $K$, we can choose $M<\infty$ such that $\Phi^K(M)\le \sup_x \Psi^K(x)$. Then $\Phi^K(\phi_t^M)\ge \Psi_t^K$ holds $\mu$-a.e. Let $D$ be a measurable set with $0<\mu(D)<\infty$. Using the convexity of $\Phi(-t\log(\cdot))$ for small $t$ (see Lemma 2.1 in [4]), we have

$$\begin{aligned}(\Phi_t^K, 1_D) &= \int_D \Phi^K(-t\log T_t 1_A))\,d\mu \\ &\ge \mu(D)\Phi^K\left(-t\log\left(\frac{1}{\mu(D)}(T_t 1_A, 1_D)\right)\right).\end{aligned}$$

Also, by Theorem 4.1,

$$\liminf_{t\to 0} -t\log(T_t 1_A, 1_D) = \liminf_{t\to 0} -t\log P_t(A,D) \ge \frac{\mathsf{d}(A,D)^2}{2}.$$

Therefore, in the limit,

$$(\Phi_0^K, 1_D) \ge \mu(D)\Phi^K\left(\frac{\mathsf{d}(A,D)^2}{2}\right) \ge \frac{\mu(D)}{c}\operatorname*{essinf}_{x\in D}\Phi^K(\bar{\phi}_0^M(x)).$$

Since $\Phi^K$ is concave,

$$\Phi^K(\bar{\phi}_t^M) = \Phi^K\left(\frac{1}{t}\int_0^t \phi_s^M\,ds\right) \ge \frac{1}{t}\int_0^t \Psi_s^K\,ds = \bar{\Psi}_t^K \quad \mu\text{-a.e.}$$

Lemma 2.2 in [4] is applied to obtain that $\Phi^K(\bar{\phi}_0^M)\ge\bar{\Psi}_0^K$ $\mu$-a.e. Therefore,

$$(\Phi_0^K, 1_D) \ge \frac{\mu(D)}{c}\operatorname*{essinf}_{x\in D}\bar{\Psi}_0^K(x). \tag{4.6}$$

We will prove

$$\Phi_0^K \geq c^{-1}\bar{\Psi}_0^K \quad \mu\text{-a.e.} \tag{4.7}$$

from the estimate (4.6). If (4.7) is false, there exists some $D' \in \mathcal{B}$ with $0 < \mu(D') < \infty$ and $\varepsilon > 0$ such that $\Phi_0^K \leq c^{-1}\bar{\Psi}_0^K - \varepsilon$ $\mu$-a.e. on $D'$. Let

$$D = \{x \in D' \mid \bar{\Psi}_0^K(x) \leq \operatorname*{essinf}_{x \in D'} \bar{\Psi}_0^K + c\varepsilon/2\}.$$

Then $\mu(D) > 0$ and $\Phi_0^K \leq c^{-1} \operatorname{essinf}_{x\in D} \bar{\Psi}_0^K - \varepsilon/2$ $\mu$-a.e. on $D$. This is contradictory to (4.6).

Combining (4.7) with (4.5), we obtain

$$\frac{1}{2} I_{\bar{\phi}_0^K}(h) \leq (-c^{-1}\bar{\Psi}_0^K + \bar{\Phi}_0^K + \bar{\Psi}_0^K, h) \leq ((2 - c^{-1})\bar{\phi}_0^K, h)$$

for every $h \in \bigcup_{k=1}^\infty \mathbb{D}_{E_k,b,+}$. The claim follows by the same argument after Eq. (4.5). $\square$

By the iterated use of Lemma 4.5, we obtain that $\bar{\phi}_0 \leq \mathsf{d}_A^2/2$ $\mu$-a.e. and, therefore, $\bar{\Phi}_0 \leq \mathsf{d}_A^2/2$ $\mu$-a.e.

Now, Lemmas 2.13 and 2.14 in [4] are valid in the present setting (by replacing $\mu$ with $\nu$ suitably in the proof), and we know that $\bar{\Phi}_t$ converges both in weak $L^2(\nu)$ sense and in weak* $L^\infty(\mu)$ sense as $t \to 0$ and the limit $\bar{\Phi}_0$ is equal to $\Phi(\mathsf{d}_A^2/2)$.

**Lemma 4.6.** *Let $\tau > 0$ and $B \in \mathcal{B}$ with $\mu(B) < \infty$. Then,*

$$\lim_{t\to 0}(T_{\tau-t}1_B, \Phi_t) = (T_\tau 1_B, \bar{\Phi}_0) = (T_\tau 1_B, \Phi(\mathsf{d}_A^2/2)).$$

*Proof.* Let $f(t) = (T_{\tau-t}1_B, \Phi_t - \Phi(\infty))$, $t > 0$. If we check the following two conditions:

(i) $T^{-1}\int_0^T f(t)\,dt \to (T_\tau 1_B, \bar{\Phi}_0 - \Phi(\infty))$ as $T \to 0$,

(ii) there exist $M > 0$ and $t_0 > 0$ such that $f(t) - f(s) \leq M(t-s)/s$ for any $0 < s < t \leq t_0$,

then we can apply the Tauberian theorem (Lemma 3.11 in [7]) to obtain that $\lim_{t\to 0} f(t) = (T_\tau 1_B, \bar{\Phi}_0 - \Phi(\infty))$, which implies the assertion.

Condition (i) is proved as follows:

$$\begin{aligned}
&\left|\frac{1}{T}\int_0^T f(t)\,dt - (T_\tau 1_B, \bar{\Phi}_0 - \Phi(\infty))\right| \\
&\leq \left|\frac{1}{T}\int_0^T f(t)\,dt - \frac{1}{T}\int_0^T (T_\tau 1_B, \Phi_t - \Phi(\infty))\,dt\right| + \left|(T_\tau 1_B, \bar{\Phi}_T - \bar{\Phi}_0)\right| \\
&\leq \frac{2LK}{T}\int_0^T \|1_B - T_t 1_B\|_{L^1(\mu)}\,dt + \left|(T_\tau 1_B, \bar{\Phi}_T - \bar{\Phi}_0)\right| \\
&\to 0 \qquad \text{as } T \to 0.
\end{aligned}$$

Regarding (ii), for $\delta \in (0,1)$, Lemma 4.2 implies

$$
\begin{aligned}
&(T_{\tau-r}1_B, \Phi_r^\delta - \Phi(e_r^\delta))|_{r=s}^{r=t}\\
&= \int_s^t (T_{\tau-r}1_B, \partial_r \Phi_r^\delta)\,dr - \int_s^t (T_{\tau-r}1_B, \partial_r \Phi(e_r^\delta))\,dr\\
&\quad + \int_s^t (\partial_r T_{\tau-r}1_B, \Phi_r^\delta - \Phi(e_r^\delta))\,dr\\
&= -\int_s^t \mathcal{E}(((\phi')_r^\delta)^2 T_{\tau-r}1_B, u_r^\delta - e_r^\delta)\,dr - \int_s^t \frac{1}{2r} I_{u_r^\delta - e_r^\delta}(((\phi')_r^\delta)^2 T_{\tau-r}1_B)\,dr\\
&\quad + \int_s^t \frac{1}{r}(\Psi_r^\delta, T_{\tau-r}1_B)\,dr - \int_s^t (T_{\tau-r}1_B, \frac{1}{r}\Psi(e_r^\delta))\,dr\\
&\quad + \int_s^t \mathcal{E}(T_{\tau-r}1_B, \Phi_r^\delta - \Phi(e_r^\delta))\,dr\\
&=: -J_1 - J_2 + J_3 - J_4 + J_5.
\end{aligned}
$$

We have

$$
\begin{aligned}
J_3 &\le \int_s^t \frac{KL\mu(B)}{s}\,dr \le KL\mu(B)(t-s)/s, \quad J_4 \ge 0,\\
J_1 &= \frac{1}{2}\int_s^t I_{u_r^\delta - e_r^\delta}(2(\phi')_r^\delta(\phi'')_r^\delta T_{\tau-r}1_B)\,dr + \frac{1}{2}\int_s^t I_{T_{\tau-r}1_B, u_r^\delta - e_r^\delta}(((\phi')_r^\delta)^2)\,dr\\
&\ge -\frac{C}{K}\int_s^t I_{u_r^\delta - e_r^\delta}(((\phi')_r^\delta)^2 T_{\tau-r}1_B)\,dr + J_5.
\end{aligned}
$$

If $t \le t_0 := K/(2C)$, then $J_1 \ge -J_2 + J_5$. Therefore,

$$(T_{\tau-r}1_B, \Phi_r^\delta - \Phi(e_r^\delta))|_{r=s}^{r=t} \le KL\mu(B)(t-s)/s$$

for $s < t \le t_0$. By letting $\delta \to 0$, condition (ii) follows. $\square$

In the identity

$$\int_B \Phi_t\,d\mu = (\Phi_t, T_{\tau-t}1_B) + (\Phi_t, 1_B - T_{\tau-t}1_B),$$

the first term of the right-hand side converges to $\int_B \bar{\Phi}_0\,d\mu$ as $t \to 0$, then $\tau \to 0$, by the lemma above. The modulus of the second term is dominated by $KL\|1_B - T_{\tau-t}1_B\|_{L^1(\mu)}$, which also converges to 0 as $t \to 0$ and $\tau \to 0$. Therefore,

$$\lim_{t\to 0}\int_B \Phi_t\,d\mu = \int_B \bar{\Phi}_0\,d\mu = \int_B \Phi(\mathsf{d}_A^2/2)\,d\mu$$

for any $B \in \mathcal{B}$ with $\mu(B) < \infty$. Now, by the exactly same argument as the end of Section 2.6 of [4], it follows that

$$\limsup_{t\to 0} -t\log P_t(A,B) \le \frac{\mathsf{d}(A,B)^2}{2}.$$

Combining Theorem 4.1, we finish the proof of Theorem 2.7.

## 5. Additional results

**Proposition 5.1.** *For every $t > 0$ and $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$, it holds that $\{T_t 1_A = 0\} = \{\mathsf{d}_A = \infty\}$ $\mu$-a.e. Moreover, the following are equivalent for $A$, $B \in \mathcal{B}$ with $0 < \mu(A) < \infty$, $0 < \mu(B) < \infty$.*

- $\mathsf{d}(A, B) = \infty$.
- $P_t(A, B) = 0$ *for every* $t > 0$.
- $P_t(A, B) = 0$ *for some* $t > 0$.

*Proof.* This is almost the same as Lemma 2.16 in [4]. By Theorem 4.1, it holds that $\{T_t 1_A = 0\} \supset \{\mathsf{d}_A = \infty\}$ $\mu$-a.e. Let $0 < s < t$ and suppose $P_t(A, B) = 0$. Then we have

$$\begin{aligned} 0 = P_t(A, B) &= (1_A, T_{t-s} T_s 1_B) \geq (1_A \cdot T_s 1_B, T_{t-s}(1_A \cdot T_s 1_B)) \\ &= \|T_{(t-s)/2}(1_A \cdot T_s 1_B)\|_{L^2(\mu)}. \end{aligned}$$

Therefore, $1_A \cdot T_s 1_B = 0$, in particular, $P_s(A, B) = 0$. By Theorem 2.7, we obtain $\{T_t 1_A = 0\} \subset \{\mathsf{d}_A = \infty\}$ $\mu$-a.e. The second assertion follows from the first one. $\square$

The proof of Theorem 1.3 of [4] is also valid in our setting here with slight modification, and we have the following counterpart.

**Theorem 5.2.** *Let $A \in \mathcal{B}$ with $0 < \mu(A) < \infty$ take any probability measure $\nu$ which is mutually absolutely continuous with respect to $\mu$. Then, the functions $u_t = -t \log T_t 1_A$ converges to $\mathsf{d}_A^2/2$ as $t \to 0$ in the following senses.*

(i) *$u_t \cdot 1_{\{u_t < \infty\}}$ converges to $\mathsf{d}_A^2/2 \cdot 1_{\{\mathsf{d}_A < \infty\}}$ in $\nu$-probability.*
(ii) *If $F$ is a bounded function on $[0, \infty]$ that is continuous on $[0, \infty)$, then $F(u_t)$ converges to $F(\mathsf{d}_A^2/2)$ in $L^2(\nu)$.*

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