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Large Deviations for Local Times of Stable Processes and Stable Random Walks in 1 Dimension

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Abstract. In Chen and Li (2004), large deviations were obtained for the spatial L^p norms of products of independent Brownian local times and local times of random walks with finite second moment. The methods of that paper depended heavily on the continuity of the Brownian path and the fact that the generator of Brownian motion, the Laplacian, is a local operator. In this paper we generalize these results to local times of symmetric stable processes and stable random walks.

Keywords: Large deviation, intersection local time, stable processes, random walks, self-intersection local time.

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1 Introduction

In the recent paper [3] the first two authors studied large deviations for the spatial L^p norms of Brownian local time L_t^x. In particular they showed that there are explicit constants $C_1(p), C_2(p)$ such that for any $p > 1$ and $\lambda, h > 0$

$$\lim_{t\to\infty} \frac{1}{t} \log E\left(e^{\lambda \|L_t^{\cdot}\|_p}\right) = \lambda^{2p/(p+1)} C_1(p) \tag{1.1}$$

and

$$\lim_{t\to\infty} \frac{1}{t} \log P\Big\{\|L_t^{\cdot}\|_p \geq ht\Big\} = -h^{2p/(p-1)} C_2(p). \tag{1.2}$$

Similar results were obtained for products of independent local times and for local times of random walks with finite second moment. The methods of that paper depended heavily on the continuity of the Brownian path and the fact that the generator of Brownian motion, the Laplacian, is a local operator. The goal of this paper is to generalize these results to local times of symmetric stable processes and stable random walks, i.e. random walks in the domain of attraction of a symmetric stable process.

To describe our results let $\{X_t\,;\, t \geq 0\}$ denote the symmetric stable process of order $\beta > 1$ in R^1, and let L_t^x denote its local time. We normalize X_t so that $E(e^{i\lambda X_t}) = e^{-t|\lambda|^\beta}$. (Note that when $\beta = 2$ this gives a multiple of the standard Brownian motion). Let

$$\mathcal{E}_\beta(f,f) =: \int_{R^1} |\lambda|^\beta |\widehat{f}(\lambda)|^2 \, d\lambda \tag{1.3}$$

where $\widehat{f}(\lambda) = \int_{R^1} f(x) e^{-2\pi i \lambda x} dx$ denotes the Fourier transform of f, and

$$\mathcal{F}_\beta = \{f \in L^2(R^1) \,|\, \|f\|_2 = 1 \text{ and } \mathcal{E}_\beta(f,f) < \infty\}. \tag{1.4}$$

Theorem 1 *Let L_t^x be the local time for the symmetric stable process of index $\beta > 1$ in R^1. For any $p > 1$ and $\lambda > 0$*

$$\lim_{t\to\infty} \frac{1}{t} \log E\left(e^{\lambda \|L_t^{\cdot}\|_p}\right) = \lambda^{\frac{p\beta}{p\beta-(p-1)}} M_{\beta,p} \tag{1.5}$$

where

$$M_{\beta,p} = \sup_{g \in \mathcal{F}_\beta} \Big\{\|g\|_{2p}^2 - \mathcal{E}_\beta(g,g)\Big\} < \infty. \tag{1.6}$$

Equivalently, for any $h > 0$

$$\lim_{t\to\infty} \frac{1}{t} \log P\Big\{\|L_t^{\cdot}\|_p \geq ht\Big\} = -h^{p\beta/(p-1)} A_{\beta,p} \tag{1.7}$$

where

$$A_{\beta,p} = \left(\frac{p-1}{p\beta}\right) \left(\frac{p\beta - (p-1)}{p\beta M_{\beta,p}}\right)^{\frac{p\beta-(p-1)}{p-1}}. \tag{1.8}$$

We will see that the constant $M_{\beta,p}$ can be expressed in terms of the best possible constant in a Gagliardo-Nirenberg type inequality, see (2.8) and (2.9).

By the scaling property of X, for each $s \geq 0$ and $a \geq 0$, we have $L^x_{as} \stackrel{d}{=} a^{1-1/\beta} L^{a^{-1/\beta}x}_s$ so that $\|L^{\cdot}_t\|_p \stackrel{d}{=} t^{1-(p-1)/p\beta} \|L^{\cdot}_1\|_p$. Using this, our Theorem is equivalent to the fact that for any $h > 0$

$$\lim_{t\to\infty} t^{-1} \log P\Big\{\|L^{\cdot}_1\|_p^{p\beta/(p-1)} \geq ht\Big\} = -hA_{\beta,p}. \tag{1.9}$$

Thus

$$E(e^{\lambda \|L^{\cdot}_1\|_p^{p\beta/(p-1)}}) \begin{cases} < \infty & \text{if } \lambda < A^{-1}_{\beta,p} \\ = \infty & \text{if } \lambda > A^{-1}_{\beta,p}. \end{cases} \tag{1.10}$$

We also note that when p is an integer, $\|L^{\cdot}_t\|_p^p$ can be expressed as an intersection local time. To see this, let $f \in \mathcal{S}(R^d)$ be a positive, symmetric function with $\int f\,dx = 1$ and set $f_\epsilon(x) = f(x/\epsilon)/\epsilon^d$. Then it is well known that for $\beta > 1$

$$L^x_t = \lim_{\epsilon\to 0} \int_0^t f_\epsilon(X_s - x)\,ds$$

where the limit exists a.s. locally uniformly and in all L^p spaces. Thus it is clear, at least formally, that

$$\int_{R^1} (L^x_t)^p\,dx = \lim_{\epsilon\to 0} \int_{[0,t]^p} \int_{R^1} \prod_{j=1}^p f_\epsilon(X_{s_j} - x)\,dx \prod_{j=1}^p ds_j$$

which measures the 'amount' of time spent by the path in $p-$fold intersections. This can be justified.

There has also been interest in the literature in studying L^p norms for products of independent local times. Let $\{X_{j,t}\,;\, t \geq 0\}$ $j = 1, \ldots, m$ denote m independent copies of $\{X_t\,;\, t \geq 0\}$. We use $L^x_{j,t}$ to denote the local time at x of $\{X_{j,t}\,;\, t \geq 0\}$ respectively. We will develop the large deviation principle for the mixed intersection local time

$$\int_{R^1} \prod_{j=1}^m (L^x_{j,t})^p dx, \qquad t \geq 0 \tag{1.11}$$

where $m \geq 1$ is an integer and real number $p > 0$ satisfying $mp > 1$. When p is an integer, the above quantity measures the 'amount' of time that m independent trajectories intersect together, while each of them intersects itself p times.

By the scaling property of X, for each $t \geq 0$ and $a \geq 0$, we have $L^x_{at} = a^{1-1/\beta} L^{a^{-1/\beta}x}_t$ so that

$$\int_{R^1} \prod_{j=1}^m (L^x_{j,at})^p dx \stackrel{\mathrm{d}}{=} a^{mp(1-1/\beta)+1/\beta} \int_{R^1} \prod_{j=1}^m (L^x_{j,t})^p dx. \tag{1.12}$$

Theorem 2 *For each integer $m \geq 1$ and real number $p > 0$ with $mp > 1$, and any $\lambda > 0$*

$$\begin{aligned} &\lim_{t\to\infty} \frac{1}{t} \log E \exp\Big\{\lambda\Big(\int_{R^1} \prod_{j=1}^m (L^x_{j,t})^p dx\Big)^{1/mp}\Big\} \\ &= \lambda^{mp\beta/(mp(\beta-1)+1)} c(\beta, p, m) M_{\beta,mp}. \end{aligned} \tag{1.13}$$

with

$$c(\beta,p,m)=m^{-(mp-1)/(mp(\beta-1)+1)}. \tag{1.14}$$

Equivalently, for any $h>0$

$$\lim_{t\to\infty}\frac{1}{t}\log P\Big\{\Big(\int_{R^1}\prod_{j=1}^m (L^x_{j,t})^p dx\Big)^{1/mp}\geq ht\Big\}=-h^{mp\beta/(mp-1)}A_{\beta,m,p} \tag{1.15}$$

with

$$A_{\beta,m,p}=\left(\frac{mp-1}{mp\beta}\right)\left(\frac{mp\beta-(mp-1)}{mp\beta c(\beta,p,m)M_{\beta,mp}}\right)^{\frac{mp\beta-(mp-1)}{mp-1}} \tag{1.16}$$

We also have the following law of the iterated logarithm, which is new even in th case of $m=1$.

Theorem 3 *For each integer $m\geq 1$ and real number $p>0$ with $mp>1$,*

$$\limsup_{t\to\infty}\ t^{-(mp(1-1/\beta)+1/\beta)}(\log\log t)^{-(mp-1)/\beta}\int_{R^1}\prod_{j=1}^m (L^x_{j,t})^p dx = A_{\beta,m,p}^{-(mp-1)/\beta}\quad a.s. \tag{1.17}$$

Let now $\{S_n\,;\,n=1,2,\ldots\}$ be a symmetric random walk in Z^1 in the domain of attraction of the symmetric stable process X_t of index $\beta>1$, i.e.

$$\frac{S_n}{b(n)}\to X_1 \tag{1.18}$$

in law with $b(x)$ a function of regular variaton of index $1/\beta$. For simplicity we assume further that our random walk is strongly aperiodic.

We will use

$$l^x_n=\sum_{j=1}^n 1_{\{S_j=x\}} \tag{1.19}$$

for the local time of $\{S_n\,;\,n=1,2,\ldots\}$ at $x\in Z^1$.

Let $\{\nu_n\}$ represents a positive sequence satisfying

$$\nu_n\to\infty\quad\text{and}\quad \nu_n/n\to 0. \tag{1.20}$$

We use $\|\cdot\|_{p,Z^1}$ for the norm in $l^p(Z^1)$, i.e. $\|f\|_{p,Z^1}=(\sum_{x\in Z^1}|f(x)|^p))^{1/p}$. We have the following moderate deviation result for the local times of stable random walks.

Theorem 4 *For any positive sequence $\{\nu_n\}$ satisfying (1.20), any $p\geq 1$ and $\lambda>0$,*

$$\lim_{n\to\infty}\frac{1}{\nu_n}\log E\exp\left\{\lambda\frac{b(n/\nu_n)^{1-1/p}}{n/\nu_n}\|l^{\cdot}_n\|_{p,Z^1}\right\}=\lambda^{\frac{p\beta}{p\beta-(p-1)}}M_{\beta,p}. \tag{1.21}$$

Equivalently, for any $h>0$

$$\lim_{n\to\infty}\frac{1}{\nu_n}\log P\Big\{\|l^{\cdot}_n\|_{p,Z^1}\geq h\frac{n}{b(n/\nu_n)^{1-1/p}}\Big\} = -h^{p\beta/(p-1)}\left(\frac{p-1}{p\beta}\right)\left(\frac{p\beta-(p-1)}{p\beta M_{\beta,p}}\right)^{\frac{p\beta-(p-1)}{p-1}}. \tag{1.22}$$

An important application of the large and moderate deviations we establish is to obtain the law of the iterated logarithm. Indeed, we have

Theorem 5

$$\limsup_{n\to\infty} n^{-p} b(n/\log\log n)^{p-1} \sum_{x\in Z^1} (l_n^x)^p = A_{\beta,1,p}^{-(p-1)/\beta} \qquad a.s.$$

The analogue of Theorem 2 for independent random walks is left to the reader.

Our paper is organized as follows. In Section 2 we develop the Sobolev inequalities and Feynman-Kac formulae which are used throughout this paper. In Section 3 we study large deviations for stable local times on the circle, which is then used in section 4 to prove Theorem 1 on large deviations for stable local times in R^1. In Section 5 we prove Theorem 2 involving independent local times and Theorem 3, the law of the iterated logarithm. Section 6 contains technical material on exponential moments for local times which is needed in the paper. Section 7 explains how to get Theorems 4 and 5 for random walks.

2 Sobolev inequalities and Feynman-Kac formulae

Lemma 1 *If $p>1$ and $\beta>(p-1)/p$ then $\mathcal{F}_\beta \subseteq L^{2p}(R^1)$, and for any $\delta>0$*

$$\|f\|_{2p}^2 \le C_\delta \|f\|_2^2 + \delta \mathcal{E}_\beta(f,f) \tag{2.1}$$

for some $C_\delta<\infty$. In particular for any $\lambda>0$

$$M_{p,\beta}(\lambda) =: \sup_{f\in\mathcal{F}_\beta} \Big(\lambda \|f\|_{2p}^2 - \mathcal{E}_\beta(f,f)\Big) < \infty. \tag{2.2}$$

Proof of Lemma 1: By the Hausdorff-Young inequality

$$\|f\|_{2p} \le \|\widehat{f}\|_{2p/(2p-1)} \tag{2.3}$$

where $\widehat{f}$ denotes the Fourier transform of f. We also have that for any $r>0$

$$\begin{aligned} \|\widehat{f}\|_{2p/(2p-1)}^{2p/(2p-1)} &= \int_{R^d} \frac{(r+|\lambda|^\beta)^{p/(2p-1)}}{(r+|\lambda|^\beta)^{p/(2p-1)}} |\widehat{f}(\lambda)|^{2p/(2p-1)}\,d\lambda \\ &\le \|(r+|\lambda|^\beta)^{-p/(2p-1)}\|_{(2p-1)/(p-1)} \cdot \\ &\qquad \|(r+|\lambda|^\beta)^{p/(2p-1)} |\widehat{f}(\lambda)|^{2p/(2p-1)}\|_{(2p-1)/p}. \end{aligned} \tag{2.4}$$

Now

$$\|(r+|\lambda|^\beta)^{p/(2p-1)} |\widehat{f}(\lambda)|^{2p/(2p-1)}\|_{(2p-1)/p}^{(2p-1)/p} = r\|f\|_2^2 + \mathcal{E}_\beta(f,f) \tag{2.5}$$

and

$$c_r =: \|(r+|\lambda|^\beta)^{-p/(2p-1)}\|_{(2p-1)/(p-1)}^{(2p-1)/(p-1)} = \int_{R^1} \frac{1}{(r+|\lambda|^\beta)^{p/(p-1)}}\,d\lambda \tag{2.6}$$

which is finite if $\beta>(p-1)/p$, in which case we also have that $\lim_{r\to\infty} c_r = 0$. Summarizing,

$$\|f\|_{2p}^2 \le c_r^{(p-1)/p} \Big(r\|f\|_2^2 + \mathcal{E}_\beta(f,f)\Big). \tag{2.7}$$

This gives (2.1) on taking r sufficiently large. This completes the proof of our Lemma.

Lemma 2 *If $p > 1$ and $\beta > (p-1)/p$ then*

$$\kappa_{p,\beta} =: \inf \left\{ C \;\middle|\; \|f\|_{2p} \leq C \|f\|_2^{1-(p-1)/p\beta} [\mathcal{E}_\beta^{1/2}(f,f)]^{(p-1)/p\beta} \right\} < \infty \tag{2.8}$$

and

$$M_{p,\beta}(1) = \frac{p\beta - (p-1)}{(p-1)} \left(\frac{(p-1)\kappa_{p,\beta}^2}{p\beta} \right)^{p\beta/(p\beta-(p-1))}. \tag{2.9}$$

Proof of Lemma 2: To see that (2.8) is finite, note that if we set $f(x) = s^{1/2} g(sx)$, then $\|f\|_2 = \|g\|_2$, $\|f\|_{2p}^2 = s^{1-1/p} \|g\|_{2p}^2$ and $\mathcal{E}_\beta(f,f) = s^\beta \mathcal{E}_\beta(g,g)$ so that from (2.1) we obtain

$$\|g\|_{2p}^2 \leq C \left(\|g\|_2^2 + s^\beta \mathcal{E}_\beta(g,g) \right) s^{-(p-1)/p} \tag{2.10}$$

and the fact that (2.8) is finite follows on taking $s^\beta = \|g\|_2^2 / \mathcal{E}_\beta(g,g)$. Finally, (2.9) follows as in the proof of Lemma 8.2 of [2]. This completes the proof of our Lemma.

Let H_β be the self-adjoint operator associated to the Dirichlet form $\mathcal{E}_\beta$. Thus the form domain $Q(H_\beta) = \mathcal{F}_\beta$ and for $g \in Q(H_\beta)$ we have $(g, H_\beta g)_2 = \mathcal{E}_\beta(g,g)$. Let V_f denote the operator of multiplication by f. Note that if $f \in L^{p/(p-1)}(R^1)$, by using Hölder's inequality and then (2.1) we have that for any $g \in Q(H_\beta)$

$$(g, V_f g)_2 \leq \|g\|_{2p}^2 \|f\|_{p/(p-1)} \leq \|f\|_{p/(p-1)} \left(C_\delta \|g\|_2^2 + \delta \mathcal{E}_\beta(g,g) \right). \tag{2.11}$$

In the terminology of [15], V_f is infinitesimally form bounded with respect to H_β, written $V_f \prec\prec H_\beta$. It follows from [15], Theorem X.17 that $H_\beta - V_f$ can be defined as a self-adjoint operator with $Q(H_\beta - V_f) = Q(H_\beta) = \mathcal{F}_\beta$.

As usual, we write $E^g(\cdot) = \int g(x) E^x(\cdot)\, dx$. Aside from technical integrability issues, the lemmas below are generalizations of the Feynman-Kac formula. We include the proofs for lack of a suitable reference.

Lemma 3 *If $p > 1$ and $\beta > (p-1)/p$ then for any $f \in L^{p/(p-1)}(R^1)$ and $g, h \in L^2(R^1)$*

$$(g, e^{-t(H_\beta - V_f)} h)_2 = E^g \left(e^{\int_0^t f(X_s)\, ds} h(X_t) \right). \tag{2.12}$$

Proof of Lemma 3: If $f \in \mathcal{S}(R^1)$ then using the right-continuity of paths, (2.12) follows as in the proof of Theorem 6.1 in [16], see also Theorem 1.1 there. Let now $f_n \in \mathcal{S}(R^1)$ with $f_n \to f$ in $L^{p/(p-1)}(R^1)$. We therefore have for each n

$$(g, e^{-t(H_\beta - V_{f_n})} h)_2 = E^g \left(e^{\int_0^t f_n(X_s)\, ds} h(X_t) \right). \tag{2.13}$$

Using (2.11), it follows from [10], Theorems IX, 2.16 and VIII, 3.6, that

$$\lim_{n \to \infty} (g, e^{-t(H_\beta - V_{f_n})} h)_2 = (g, e^{-t(H_\beta - V_f)} h)_2. \tag{2.14}$$

On the other hand the integrand on the right-hand side of (2.13) converges a.s. to the corresponding integrand in (2.12). Thus to finish the proof we need only show uniform integrabilty. We have

$$
\begin{aligned}
(2.15)\qquad & E^g\left(e^{(4/3)\int_0^t f_n(X_s)\,ds}h^{4/3}(X_t)\right) \\
&\le \int\left(E^x\left(e^{4\int_0^t f_n(X_s)\,ds}\right)\right)^{1/3}\left(E^x\left(h^2(X_t)\right)\right)^{2/3} g(x)\,dx \\
&\le \sup_{x_0}\left(E^{x_0}\left(e^{4\int_0^t f_n(X_s)\,ds}\right)\right)^{1/3}\int\left(E^x\left(h^2(X_t)\right)\right)^{2/3} g(x)\,dx
\end{aligned}
$$

We then use $E^x\left(h^2(X_t)\right) = p_t * h^2(x)$ so that

$$
(2.16)\qquad \int\left(E^x\left(h^2(X_t)\right)\right)^{2/3} g(x)\,dx \le \|g\|_2\|p_t * h^2\|_{4/3}^{2/3}
$$

and

$$
(2.17)\qquad \|p_t * h^2\|_{4/3} \le \|p_t\|_{4/3}\|h^2\|_1
$$

which is finite. In fact, for any $r > 1$

$$
(2.18)\qquad \|p_t\|_r \le C/t^{(r-1)/r\beta}.
$$

This follows from the fact that p_t is a probability density function so that $\|p_t\|_r \le \|p_t\|_\infty^{(r-1)/r}$ and $\|p_t\|_\infty = \sup_x |\int e^{i\lambda x}e^{-t\lambda^\beta}\,d\lambda| \le \int e^{-t\lambda^\beta}\,d\lambda \le C/t^{1/\beta}$. (Alternatively, one can use scaling: $p_t(x) = t^{-1/\beta}p_1(xt^{-1/\beta})$).

We now bound

$$
\begin{aligned}
(2.19)\qquad & \sup_{x_0} E^{x_0}\left(e^{4\int_0^t f_n(X_s)\,ds}\right) \\
&= \sum_{k=0}^{\infty} 4^k \int_{\{0\le t_1\le\ldots\le t_k\le t\}}\int_{R^k}\prod_{j=1}^{k} f_n(x_j)p_{t_j-t_{j-1}}(x_j - x_{j-1})\,dx_j\,dt_j.
\end{aligned}
$$

By Hölder's inequality

$$
\begin{aligned}
(2.20)\qquad & \int_{R^k}\prod_{j=1}^{k} f_n(x_j)p_{t_j}(x_j - x_{j-1})\,dx_j \\
&\le \|f_n\|_{p/(p-1)}^k\|\prod_{j=1}^{k} p_{t_j-t_{j-1}}(x_j - x_{j-1})\|_p \\
&\le \|f_n\|_{p/(p-1)}^k\prod_{j=1}^{k}\|p_{t_j-t_{j-1}}\|_p.
\end{aligned}
$$

Using (2.18) we have

$$
(2.21)\qquad \prod_{j=1}^{k}\|p_{t_j-t_{j-1}}\|_p = c^k\prod_{j=1}^{k}(t_j - t_{j-1})^{-(p-1)/p\beta}
$$

Since by assumption $(p-1)/p\beta < 1$ we have that

$$\int_{\{0\le t_1\le \ldots \le t_k\le t\}} \prod_{j=1}^{k} (t_j - t_{j-1})^{-(p-1)/p\beta}\, dt_j = \frac{c^k t^{k(1-(p-1)/p\beta)}}{\Gamma(k(1-(p-1)/p\beta))}. \tag{2.22}$$

Thus we have shown that for fixed t

$$\sup_{x_0} E^{x_0}\left(e^{4\int_0^t f_n(X_s)\,ds}\right) \le \sum_{k=0}^{\infty} \frac{c^k \|f_n\|_{p/(p-1)}^k}{\Gamma(k(1-(p-1)/p\beta))} \tag{2.23}$$

which is bounded uniformly in n. This completes the proof of Lemma 3.

Fix $M > 0$ and let $T_M = R^1/MZ^1$ denote the circle of circumference M. We use the notation $\|f\|_{p,T_M}$ to denote the $L^p(T_M)$ norm with the usual Lebesgue measure on T_M. Set

$$\mathcal{E}_{\beta,T_M}(h,h) =: \sum_{\lambda\in(\frac{2\pi}{M})Z^1} |\lambda|^\beta\, |\widehat{h}(\lambda)|^2 \frac{1}{M} \tag{2.24}$$

where $\widehat{h}$ denotes the usual Fourier transform for functions on T_M. Let

$$\mathcal{F}_{\beta,T_M} = \{f \in L^2(T_M)\,|\,\|f\|_{2,T_M} = 1 \text{ and } \mathcal{E}_{\beta,T_M}(f,f) < \infty\}. \tag{2.25}$$

We introduce T_M to deal with two technical problems in the proof of the upper bound in Theorem 1. First, the stable infinitesimal generator is not a local operator when $\beta < 2$. As a consequence, we will not have the upper bound for the Feymann-Kac large deviation estimate which corresponds to the lower bound given in (4.2) below. Second, as pointed out in [3] (p. 225-226), the family $\{t^{-1}L_t^{\cdot}\}$ is not exponentially tight as a stochastic process taking values in the Banach space $\mathcal{L}^p(R^1)$. To fix these two problems we map the process X_t into the compact space T_M. It is crucial that the image process is Markovian.

An almost identical proof gives the following analogue of Lemma 1.

Lemma 4 *If $p > 1$ and $\beta > (p-1)/p$ then $\mathcal{F}_{\beta,T_M} \subseteq L^{2p}(T_M)$, and for any $\delta > 0$*

$$\|f\|_{2p,T_M}^2 \le C_\delta \|f\|_{2,T_M}^2 + \delta\mathcal{E}_{\beta,T_M}(f,f) \tag{2.26}$$

for some $C_\delta < \infty$. In particular for any $\lambda > 0$

$$M_{p,\beta,T_M}(\lambda) =: \sup_{f\in\mathcal{F}_{\beta,T_M}} \left(\lambda\|f\|_{2p,T_M}^2 - \mathcal{E}_{\beta,T_M}(f,f)\right) < \infty. \tag{2.27}$$

Let Y_t be the image of X_t under the quotient map $x \in R^1 \mapsto \bar{x} \in T_M$. It is easily seen that Y_t is a Markov process with independent increments. Y_t is called the symmetric stable process of order β on T_M.

As before, let H_{β,T_M} be the self-adjoint operator associated to the Dirichlet form $\mathcal{E}_{\beta,T_M}$. Thus $Q(H_{\beta,T_M}) = \mathcal{F}_{\beta,T_M}$ and for $g \in Q(H_{\beta,T_M})$ we have $(g, H_{\beta,T_M}g)_2 = \mathcal{E}_{\beta,T_M}(g,g)$. Let V_f denote the operator of multiplication by f. Using Lemma 4, we see that $V_f \prec\prec H_{\beta,T_M}$ so that, as before we can define $H_{\beta,T_M} - V_f$ as a self-adjoint operator with $Q(H_{\beta,T_M} - V_f) = Q(H_{\beta,T_M}) = \mathcal{F}_{\beta,T_M}$. An almost identical proof gives the following analogue of Lemma 3.

Lemma 5 *If $p > 1$ and $\beta > (p-1)/p$ then for any $f \in L^{p/(p-1)}(T_M)$ and $g, h \in L^2(T_M)$*

$$(g, e^{-t(H_\beta - V_f)} h)_2 = E^g \left(e^{\int_0^t f(Y_s)\, ds} h(Y_t) \right). \tag{2.28}$$

We next present an important large deviation result. It is possible to derive this result from the methods of Donsker and Varadhan, see in particular [7], but we prefer to give a simple self-contained proof.

Lemma 6 *If $p > 1$ and $\beta > (p-1)/p$ then for any non-negative function $f \in L^{p/(p-1)}(T_M)$*

$$\lim_{t \to \infty} \frac{1}{t} \log E \left(e^{\int_0^t f(Y_s)\, ds} \right) = \sup_{g \in \mathcal{F}_{\beta, T_M}} \Big((g, fg)_{2, T_M} - \mathcal{E}_{\beta, T_M}(g, g) \Big). \tag{2.29}$$

Note that using Hölder's inequality and Lemma 4, the sup on the right-hand side is finite.

Proof of Lemma 6: Let $\bar{p}_t$ denote the density of Y_t. Fix $t_0 > 0$, and recall from (2.18), (more precisely the analogue for T_M), that $\bar{p}_{t_0} \in L^2(T_M)$. Then using the non-negativity of f, the Markov property and (2.28) we have

$$\begin{aligned} E \left(e^{\int_0^t f(Y_s)\, ds} \right) &\geq E^{\bar{p}_{t_0}} \left(e^{\int_0^{t-t_0} f(Y_s)\, ds} \right) \\ &= (\bar{p}_{t_0}, e^{-(t-t_0)(H_{\beta, T_M} - V_f)} 1)_{2, T_M}. \end{aligned} \tag{2.30}$$

By (2.24) we can see that $\sigma(H_{\beta, T_M})$, the spectrum of H_{β, T_M}, is purely discrete. In fact

$$\sigma(H_{\beta, T_M}) = \{ \left(\frac{2\pi j}{M} \right)^\beta \mid j = 0, 1, \ldots \}$$

with a complete set of corresponding eigenvectors

$$\{ \frac{1}{\sqrt{M}} \} \cup \{ \frac{e^{\pm (2\pi i) j x / M}}{\sqrt{M}} \mid j = 1, 2, \ldots \}.$$

Hence, using the fact that $V_f \prec\prec H_{\beta, T_M}$ and [15], Theorem XIII.64, (iv), (v), see also Theorem XIII.68, we find that $H_{\beta, T_M} - a' V_f$ also has purely discrete spectrum for any a'. (We note for later that these Theorem's show that $H_{\beta, T_M} - a' V_f$ has compact resolvent). From Lemma 5 it follows that $e^{-t(H_{\beta, T_M} - a' V_f)}$ is positivity preserving and ergodic. It follows from [15], Theorem XIII.43 that $\inf \sigma(H_{\beta, T_M} - a' V_f)$ is a simple eigenvalue and the associated eigenvector is strictly positive. Since $\bar{p}_{t_0}$ is also strictly positive, we find from (2.30) that

$$\liminf_{t \to \infty} \frac{1}{t} \log E \left(e^{\int_0^t f(Y_s)\, ds} \right) \geq - \inf \sigma(H_{\beta, T_M} - V_f). \tag{2.31}$$

The lower bound for (2.29) follows by the Rayleigh-Ritz principle, see [15].

For the upper bound use the Markov property to see that for any a, a' with $1/a+1/a' = 1$,

$$
\begin{aligned}
(2.32) \qquad & E\left(e^{\int_0^t f(Y_s)\,ds}\right) \\
&= E\left(e^{\int_0^{t_0} f(Y_s)\,ds} E^{Y_{t_0}}\left(e^{\int_0^{t-t_0} f(Y_s)\,ds}\right)\right) \\
&\leq \left\{E\left(e^{a\int_0^{t_0} f(Y_s)\,ds}\right)\right\}^{1/a} \left\{E\left(\left(E^{Y_{t_0}}\left(e^{\int_0^{t-t_0} f(Y_s)\,ds}\right)\right)^{a'}\right)\right\}^{1/a'} \\
&\leq \left\{E\left(e^{a\int_0^{t_0} f(Y_s)\,ds}\right)\right\}^{1/a} \left\{E\left(E^{Y_{t_0}}\left(e^{a'\int_0^{t-t_0} f(Y_s)\,ds}\right)\right)\right\}^{1/a'} \\
&= \left\{E\left(e^{a\int_0^{t_0} f(Y_s)\,ds}\right)\right\}^{1/a} \left\{E^{\bar{p}_{t_0}}\left(e^{a'\int_0^{t-t_0} f(Y_s)\,ds}\right)\right\}^{1/a'}.
\end{aligned}
$$

By (2.23), (more precisely the analogue for T_M), we have that the first factor on the right hand side is bounded for any fixed t_0 and a. On the other hand, by (2.28)

$$
(2.33) \qquad E^{\bar{p}_{t_0}}\left(e^{a'\int_0^{t-t_0} f(Y_s)\,ds}\right) = (\bar{p}_{t_0}, e^{-(t-t_0)(H_{\beta,T_M}-a'V_f)}1)_2.
$$

Hence

$$
(2.34) \qquad \limsup_{t\to\infty} \frac{1}{t} \log E\left(e^{\int_0^t f(Y_s)\,ds}\right) \leq -\frac{\inf \sigma(H_{\beta,T_M} - a'V_f)}{a'}.
$$

Using once more the fact that $V_f \prec\prec H_{\beta,T_M}$ we find that $H_{\beta,T_M} - zV_f$ is an analytic family of type (B). We have noted in the last paragraph that $H_{\beta,T_M} - a'V_f$ has compact resolvent. It follows from [10], VII, Remark 4.22, that

$$
(2.35) \qquad \lim_{a'\to 1} \inf \sigma(H_{\beta,T_M} - a'V_f) = \inf \sigma(H_{\beta,T_M} - V_f).
$$

The upper bound for (2.29) follows by taking $a' \to 1$ and then applying the Rayleigh-Ritz principle.

3 Large deviations for stable local times on the circle

We use the notation of the previous section. $M > 0$ is fixed throughout.

Theorem 6 *Let $\bar{L}_t^x$ be the local time for the symmetric stable process of index $\beta > 1$ in T_M. For any $p > 1$ and $\lambda > 0$*

$$
(3.1) \qquad \lim_{t\to\infty} \frac{1}{t} \log E\left(e^{\lambda\|\bar{L}_t\|_{p,T_M}}\right) = \sup_{g\in\mathcal{F}_{\beta,T_M}} \left\{\lambda\|g\|_{2p,T_M}^2 - \mathcal{E}_{\beta,T_M}(g,g)\right\}.
$$

Proof of Theorem 6: We first establish the lower bound for (3.1). We claim that for any $\lambda > 0$

$$
(3.2) \qquad \liminf_{t\to\infty} \frac{1}{t} \log E\left(e^{\lambda\|\bar{L}_t\|_{p,T_M}}\right) \geq \sup_{g\in\mathcal{F}_{\beta,T_M}} \left\{\lambda\|g\|_{2p,T_M}^2 - \mathcal{E}_{\beta,T_M}(g,g)\right\}.
$$

Indeed, if we take any $f \in L^{(p-1)/p}(T_M)$ with $\|f\|_{(p-1)/p,T_M} = 1$ then

$$\|\bar{L}_t^{\cdot}\|_{p,T_M} \geq \int_{T_M} \bar{L}_t^x f(x)dx = \int_0^t f(Y_s)ds. \tag{3.3}$$

Consequently, taking f non-negative, by Lemma 6

$$\begin{aligned} &\liminf_{t\to\infty} \frac{1}{t} \log E\left(e^{\lambda\|\bar{L}_t^{\cdot}\|_{p,T_M}}\right) \\ &\geq \sup_{g\in\mathcal{F}_{\beta,T_M}} \Big\{\lambda(g,fg)_{2,T_M} - \mathcal{E}_{\beta,T_M}(g,g)\Big\}. \end{aligned} \tag{3.4}$$

Taking supremum on the right hand side over such f we obtain (3.2).

To establish the upper bound and complete the proof of (3.1) we shall prove that for any $\lambda > 0$

$$\limsup_{t\to\infty} \frac{1}{t} \log E\left(e^{\lambda\|\bar{L}_t^{\cdot}\|_{p,T_M}}\right) \leq \sup_{g\in\mathcal{F}_{\beta,T_M}} \Big\{\lambda\|g\|^2_{2p,T_M} - \mathcal{E}_{\beta,T_M}(g,g)\Big\}. \tag{3.5}$$

By (3.3) and Lemma 6, for any non-negative $f \in L^{(p-1)/p}(T_M)$

$$\begin{aligned} &\lim_{t\to\infty} t^{-1} \log E \exp\Big\{\lambda \int_{T_M} \bar{L}_t^x f(x)dx\Big\} \\ &= \sup_{g\in\mathcal{F}_{\beta,T_M}} \Big\{\lambda(g,fg)_{2,T_M} - \mathcal{E}_{\beta,T_M}(g,g)\Big\}. \end{aligned} \tag{3.6}$$

Let $\epsilon > 0$ and $\gamma > 0$ be fixed and let $K \subset L^p(T_M)$ be the compact set given in Lemma 11. By the fact that the set of bounded measurable functions on T_M is dense in the unit ball of $L^q(T_M)$, and by the Hahn-Banach Theorem, for each $h \in \gamma K$, there is a bounded function f such that $\|f\|_{(p-1)/p,T_M} = 1$ and

$$\int_{T_M} f(\bar{x})h(\bar{x})\lambda(d\bar{x}) > \Big(\int_{T_M} |h(\bar{x})|^p \lambda(d\bar{x})\Big)^{1/p} - \epsilon.$$

Consequently, there are finitely many bounded functions $f_1, \cdots, f_N$ in the unit sphere of $L^q(T_M)$ such that

$$\Big(\int_{T_M} |h(\bar{x})|^p \lambda(d\bar{x})\Big)^{1/p} < \max_{1\leq i\leq N} \int_{T_M} f_i(\bar{x})h(\bar{x})\lambda(d\bar{x}) + \epsilon \quad \forall h \in \gamma K.$$

Therefore,

$$\begin{aligned} &E\Big(\exp\Big\{\lambda\|\bar{L}_t^{\cdot}\|_{p,T_M}\Big\};\ t^{-1}\bar{L}_t^{\cdot} \in \gamma K\Big) \\ &\leq e^{\epsilon t} \sum_{i=1}^N E \exp\Big\{\int_{T_M} f_i(x)\bar{L}_t^x\, dx\Big\}. \end{aligned} \tag{3.7}$$

In view of (3.6),

$$\begin{aligned}
(3.8)\qquad &\limsup_{t\to\infty} t^{-1}\log E\Big(\exp\Big\{\lambda\|\bar L_t^{\cdot}\|_{p,T_M}\Big\};\ t^{-1}\bar L_t^{\cdot}\in\gamma K\Big)\\
\le\ & \epsilon+\max_{1\le i\le N}\sup_{g\in\mathcal F_{\beta,T_M}}\Big\{\lambda(g,f_ig)_{2,T_M}-\mathcal E_{\beta,T_M}(g,g)\Big\}\\
\le\ & \epsilon+\sup_{g\in\mathcal F_{\beta,T_M}}\Big\{\lambda\|g\|^2_{2p,T_M}-\mathcal E_{\beta,T_M}(g,g)\Big\}
\end{aligned}$$

where the second step follows from the Hölder's inequality and the fact $\|f_i\|_{(p-1)/p,T_M}=1$ for $1\le i\le N$. Letting $\epsilon\longrightarrow 0$ gives

$$\begin{aligned}
(3.9)\qquad &\limsup_{t\to\infty} t^{-1}\log E\Big(\exp\Big\{\lambda\|\bar L_t^{\cdot}\|_{p,T_M}\Big\};\ t^{-1}\bar L_t^{\cdot}\in\gamma K\Big)\\
\le\ & \sup_{g\in\mathcal F_{\beta,T_M}}\Big\{\lambda\|g\|^2_{2p,T_M}-\mathcal E_{\beta,T_M}(g,g)\Big\}.
\end{aligned}$$

By the Cauchy-Schwarz inequality, on the other hand,

$$\begin{aligned}
(3.10)\qquad & E\Big(\exp\Big\{\lambda\|\bar L_t^{\cdot}\|_{p,T_M}\Big\};\ t^{-1}\bar L_t^{\cdot}\notin\gamma K\Big)\\
\le\ & \Big(E\exp\Big\{2\lambda\|\bar L_t^{\cdot}\|_{p,T_M}\Big\}\Big)^{1/2}\Big(P\Big\{t^{-1}\bar L_t^{\cdot}\notin\gamma K\Big\}\Big)^{1/2}.
\end{aligned}$$

Note that Lemma 8 and scaling (1.12) imply that

$$(3.11)\qquad \sup_{t\le 1} E\exp\Big\{2\lambda\|\bar L_t^{\cdot}\|_{p,T_M}\Big\}<\infty$$

so that using the additivity of local time and the Markov property

$$\limsup_{t\to\infty} t^{-1}\log E\exp\Big\{2\lambda\|\bar L_t^{\cdot}\|_{p,T_M}\Big\}\equiv C_1<\infty.$$

By (6.37)

$$\limsup_{t\to\infty} t^{-1}\log P\Big\{t^{-1}\bar L_t^{\cdot}\notin\gamma K\Big\}\le -N(\gamma)$$

with $\lim_{\gamma\to\infty}N(\gamma)=\infty$. Combining above observations we have

$$\begin{aligned}
(3.12)\qquad &\limsup_{t\to\infty}\frac{1}{t}\log E\Big(\exp\Big\{\lambda\|\bar L_t^{\cdot}\|_{p,T_M}\Big\};\ t^{-1}\bar L_t^{\cdot}\notin\gamma K\Big)\\
&\le (C_1-N(\gamma))/2.
\end{aligned}$$

Note that $\gamma>0$ and hence $N(\gamma)$ can be arbitrarily large. Combining (3.9) and (3.12) we obtain (3.5) completing the proof of our theorem.

4 Large deviations for stable local times in R^1

Proof of Theorem 1: We first establish the lower bound for (1.5). We claim that for any $\lambda > 0$

$$\liminf_{t\to\infty} \frac{1}{t} \log E\left(e^{\lambda \|L_t^\cdot\|_p}\right) \geq \sup_{g\in\mathcal{F}_\beta} \Big\{\lambda\|g\|_{2p}^2 - \mathcal{E}_\beta(g,g)\Big\}. \tag{4.1}$$

This will follow exactly as in the proof of Theorem 6 once we establish that for any non-negative $f \in L^{(p-1)/p}(R^1)$

$$\liminf_{t\to\infty} \frac{1}{t} \log E\left(e^{\int_0^t f(X_s)ds}\right) \geq \sup_{g\in\mathcal{F}_\beta} \Big\{(g, fg)_2 - \mathcal{E}_\beta(g,g)\Big\}. \tag{4.2}$$

But as in the proof of (2.30) we have, for any bounded g_1, g_2

$$\begin{aligned} E\left(e^{\int_0^t f(X_s)\,ds}\right) &\geq E^{p_{t_0}}\left(e^{\int_0^{t-t_0} f(X_s)\,ds}\right) \\ &\geq E^{g_1 p_{t_0}}\left(e^{\int_0^{t-t_0} f(X_s)\,ds} g_2(X_{t-t_0})\right)/\|g_1\|_\infty\|g_2\|_\infty \\ &= (g_1 p_{t_0}, e^{-(t-t_0)(H_\beta - V_f)} g_2)_2/\|g_1\|_\infty\|g_2\|_\infty. \end{aligned} \tag{4.3}$$

Since $p_{t_0}(x) > 0$ for all x, by varying g_1, g_2 we obtain

$$\lim_{t\to\infty} \frac{1}{t} \log E\left(e^{\int_0^t f(X_s)\,ds}\right) \geq -\inf \sigma(H_\beta - V_f) \tag{4.4}$$

which gives (4.2) by the Rayleigh-Ritz principle.

We next establish the upper bound for (1.5). We claim that for any $\lambda > 0$

$$\limsup_{t\to\infty} \frac{1}{t} \log E\left(e^{\lambda \|L_t^\cdot\|_p}\right) \leq \sup_{g\in\mathcal{F}_\beta} \Big\{\lambda\|g\|_{2p}^2 - \mathcal{E}_\beta(g,g)\Big\}. \tag{4.5}$$

Fix $M > 0$ and recall from the last section the symmetric stable process Y_t of index β in T_M and its local time $\bar{L}_t^x$. It can be easily verified that

$$\bar{L}_t^x = \sum_{k\in Z^1} L_t^{x+kM}, \quad t \geq 0, \;\; x \in R^1$$

Consequently, for any $p > 1$

$$\begin{aligned} \int_{R^1} (L_t^x))^p\,dx &= \sum_{k\in Z^1} \int_0^M (L_t^{x+kM}))^p\,dx \\ &\leq \int_0^M \Big(\sum_{k\in Z^1} L_t^{x+kM}\Big)^p dx = \int_{T_M} (\bar{L}_t^x)^p\,dx. \end{aligned} \tag{4.6}$$

Hence (4.5), and thus our theorem, will follow from Theorem 6 once we verify the following lemma.

Lemma 7 *For any $p > 1, r > 0$*

$$\limsup_{M\to\infty} \sup_{\bar g\in\mathcal{F}_{\beta,T_M}} \Big\{ r\|\bar g\|^2_{2p,T_M} - \mathcal{E}_{\beta,T_M}(\bar g,\bar g)\Big\} \le \sup_{g\in\mathcal{F}_\beta} \Big\{ r\|g\|^2_{2p} - \mathcal{E}_\beta(g,g)\Big\}. \tag{4.7}$$

Proof of Lemma 7: Without loss of generality we will prove this lemma with $r = 1$. Recall the definition

$$\mathcal{E}_\beta(f,f) =: \int_{R^1} |\lambda|^\beta |\widehat{f}(\lambda)|^2\, d\lambda. \tag{4.8}$$

Using

$$|\lambda|^\beta = c_\beta \int_{R^1} \frac{1-\cos(\lambda y)}{|y|^{1+\beta}}\, dy$$

where $c_\beta^{-1} = \int_{R^1} \frac{1-\cos(y)}{|y|^{1+\beta}}\, dy$ and Parseval's formula we find that

$$\mathcal{E}_\beta(f,f) = c_\beta \int_{R^1}\int_{R^1} \frac{|f(y)-f(x)|^2}{|y-x|^{1+\beta}}\, dy\, dx. \tag{4.9}$$

Similarly, for any M-periodic function h

$$\bar{\mathcal{E}}_\beta(h,h) = \sum_{\lambda\in(\frac{2\pi}{M})Z^1} |\lambda|^\beta \widehat{h}(\lambda)|^2 \frac{1}{M} = c_\beta \int_0^M \int_{R^1} \frac{|h(x+y)-h(x)|^2}{|y|^{1+\beta}}\, dy\, dx \tag{4.10}$$

where the last equality follows as in the proof of (4.9).

Let $\bar g$ be an M-periodic function in $\mathcal{F}_{\beta,T_M}$. We need to construct a function $f \in \mathcal{F}_\beta$ which is equal to $\bar g$ on $[M^{1/2}, M - M^{1/2}]$ and is negligible in some suitable sense on the rest of the real line as M gets large. Let $E = [0, M^{1/2}] \cup [M - M^{1/2}, M]$. By Lemma 3.4 in Donsker-Varadhan (1975), there is a real number a such that $\int_E \bar g^2(x-a)dx \le 2M^{-1/2}$. We may assume $a = 0$, i.e,

$$\int_E \bar g^2(x)dx \le 2M^{-1/2} \tag{4.11}$$

for otherwise we can replace $\bar g(\cdot)$ by $\bar g(\cdot + a)$. Define

$$\varphi(x) = \begin{cases} xM^{-1/2} & 0 \le x \le M^{1/2} \\ 1 & M^{-1/2} \le x \le M - M^{1/2} \\ M^{1/2} - xM^{-1/2} & M - M^{1/2} \le x \le M \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to verify that

$$0 \le \varphi(x) \le 1, \quad |\varphi'(x)| \le M^{-1/2}, \quad |(\varphi^2(x))'| \le 2M^{-1/2}, \qquad -\infty < x < \infty.$$

Define

$$f(x) = \bar g(x)\varphi(x) \cdot \Big(\int_{-\infty}^\infty \bar g^2(x)\varphi^2(x)dx \Big)^{-1/2}.$$

Set $\alpha = \int_{-\infty}^{\infty} \bar{g}^2(x)\varphi^2(x)dx$. Note that

$$
\begin{aligned}
(4.12) \qquad & |\bar{g}(y)\varphi(y) - \bar{g}(x)\varphi(x)|^2 \\
&= |(\bar{g}(y) - \bar{g}(x))\varphi(y) + \bar{g}(x)(\varphi(y) - \varphi(x))|^2 \\
&= |\bar{g}(y) - \bar{g}(x)|^2\varphi^2(y) + \bar{g}^2(x)|\varphi(y) - \varphi(x)|^2 \\
&\quad +2|\bar{g}(y) - \bar{g}(x)|\varphi(y)\bar{g}(x)|\varphi(y) - \varphi(x)|
\end{aligned}
$$

Now

$$
\begin{aligned}
(4.13) \qquad & c_\beta \int_{R^1}\int_{R^1} \frac{|\bar{g}(y) - \bar{g}(x)|^2\varphi^2(y)}{|y-x|^{1+\beta}}\,dy\,dx \\
&\leq c_\beta \int_{R^1}\int_0^M \frac{|\bar{g}(y) - \bar{g}(x)|^2}{|y-x|^{1+\beta}}\,dy\,dx \\
&= \bar{\mathcal{E}}_\beta(\bar{g}, \bar{g}).
\end{aligned}
$$

Using

$$
\begin{aligned}
(4.14) \qquad & \bar{g}^2(x)|\varphi(y) - \varphi(x)|^2 \\
&\leq \bar{g}^2(x)|\varphi(y) - \varphi(x)|^2(1_{[0,M]}(x) + 1_{[0,M]}(y)) \\
&\leq 2M^{-1/2}\bar{g}^2(x)(1_{[0,M]}(x) + 1_{[0,M]}(y))(|y-x| \wedge |y-x|^2)
\end{aligned}
$$

we have

$$
\begin{aligned}
(4.15) \qquad & c_\beta \int_{R^1}\int_{R^1} \frac{\bar{g}^2(x)|\varphi(y) - \varphi(x)|^2}{|y-x|^{1+\beta}}\,dy\,dx \\
&\leq 2M^{-1/2}c_\beta \int_0^M\int_{R^1} \frac{|\bar{g}(x)|^2(|y-x| \wedge |y-x|^2)}{|y-x|^{1+\beta}}\,dy\,dx \\
&\quad +2M^{-1/2}c_\beta \int_{R^1}\int_0^M \frac{|\bar{g}(x)|^2(|y-x| \wedge |y-x|^2)}{|y-x|^{1+\beta}}\,dy\,dx \\
&\leq cM^{-1/2}
\end{aligned}
$$

where for the first integral we used the change of variables $y \to y + x$ and for the second we used the change of variables $x \to y + x$ and the periodicity of $\bar{g}(x)$. Finally we use

$$
\begin{aligned}
(4.16) \qquad & 2|\bar{g}(y) - \bar{g}(x)|\varphi(y)\bar{g}(x)|\varphi(y) - \varphi(x)| \\
&\leq 4M^{-1/4}|\bar{g}(y) - \bar{g}(x)|\bar{g}(x)(|y-x|^{1/2} \wedge |y-x|)1_{[0,M]}(y).
\end{aligned}
$$

the Cauchy-Schwarz inequality and (4.13), (4.15) to see that

$$
\begin{aligned}
(4.17) \qquad & 2c_\beta \int_{R^1}\int_{R^1} \frac{|\bar{g}(y) - \bar{g}(x)|\varphi(y)\bar{g}(x)|\varphi(y) - \varphi(x)|}{|y-x|^{1+\beta}}\,dy\,dx \\
&\leq 8M^{-1/4}c_\beta \Big(\int_{R^1}\int_{R^1} \frac{|\bar{g}(y) - \bar{g}(x)|^2 1_{[0,M]}(y)}{|y-x|^{1+\beta}}\,dy\,dx\Big)^{1/2} \\
&\qquad \Big(\int_{R^1}\int_{R^1} \frac{\bar{g}^2(x)(|y-x|^2 \wedge |y-x|)1_{[0,M]}(y)}{|y-x|^{1+\beta}}\,dy\,dx\Big)^{1/2} \\
&\leq cM^{-1/4}\bar{\mathcal{E}}_\beta^{1/2}(\bar{g}, \bar{g})
\end{aligned}
$$

Putting this all together we find that

$$\alpha\mathcal{E}_\beta(f,f) = \int_{R^1}\int_{R^1} \frac{|\bar g(y)\varphi(y)-\bar g(x)\varphi(x)|^2}{|y-x|^{1+\beta}}\,dy\,dx \le \bar{\mathcal{E}}_\beta(\bar g,\bar g)+cM^{-1/4}\bar{\mathcal{E}}_\beta^{1/2}(\bar g,\bar g)+cM^{-1/2} \tag{4.18}$$

On the other hand,

$$\begin{aligned}\Big(\int_0^M \bar g^{2p}(x)dx\Big)^{1/p} &\le \Big(\alpha^p\int_{-\infty}^{\infty}|f(x)|^{2p}dx+\int_E|\bar g(x)|^{2p}dx\Big)^{1/p}\\ &\le \alpha\Big(\int_{-\infty}^{\infty}|f(x)|^{2p}dx\Big)^{1/p}+\Big(\int_E|\bar g(x)|^{2p}dx\Big)^{1/p}\end{aligned} \tag{4.19}$$

and from (4.11)

$$\begin{aligned}\Big(\int_E|\bar g(x)|^{2p}dx\Big)^{1/p} &\le \sup_{0\le x\le M}|\bar g(x)|^{2/q}\Big(\int_E|\bar g(x)|^{2}dx\Big)^{1/p}\\ &\le (2M^{-1/2})^{1/p}\sup_{0\le x\le M}|\bar g(x)|^{2/q}.\end{aligned} \tag{4.20}$$

Observe that since $\beta>1$

$$\begin{aligned}\sup_x|\bar g(x)| &\le \sum_{\lambda\in(\frac{2\pi}{M})Z^1}|\hat{\bar g}(k)|\frac{1}{M}\\ &\le \sum_{\lambda\in(\frac{2\pi}{M})Z^1}\frac{\sqrt{1+|k|^\beta}}{\sqrt{1+|k|^\beta}}|\hat{\bar g}(k)|\frac{1}{M}\\ &\le \Big(\sum_{\lambda\in(\frac{2\pi}{M})Z^1}\frac{1}{1+|k|^\beta}\frac{1}{M}\Big)^{1/2}\Big(\sum_{\lambda\in(\frac{2\pi}{M})Z^1}(1+|k|^\beta)|\hat{\bar g}(k)|^2\frac{1}{M}\Big)^{1/2}\\ &\le c\Big(1+\bar{\mathcal{E}}_\beta(\bar g,\bar g)\Big)^{1/2}.\end{aligned} \tag{4.21}$$

Therefore,

$$\Big(\int_E|\bar g(x)|^{2p}dx\Big)^{1/p} \le (2M^{-1/2})^{1/p}c\Big(1+\bar{\mathcal{E}}_\beta(\bar g,\bar g)\Big)^{1/q}. \tag{4.22}$$

Let $J=\sup_{g\in\mathcal{F}_\beta}\Big\{\lambda\|g\|_{2p}^2-\mathcal{E}_\beta(g,g)\Big\}$. By combining (4.18), (4.19), (4.22), we see

$$\begin{aligned}&\Big(\int_0^M \bar g^{2p}(x)dx\Big)^{1/p}-(1-\epsilon)\bar{\mathcal{E}}_\beta(\bar g,\bar g)\\ &\quad-\Big(cM^{-1/4}\Big(\bar{\mathcal{E}}_\beta(\bar g,\bar g)\Big)^{1/2}+(2M^{-1/2})^{1/p}c\Big(1+\bar{\mathcal{E}}_\beta(\bar g,\bar g)\Big)^{1/q}\Big)\\ &\le c(1-\epsilon)M^{-1/2}+\alpha\Big\{\Big(\int_{-\infty}^{\infty}|f(x)|^{2p}dx\Big)^{1/p}-(1-\epsilon)\mathcal{E}_\beta(f,f)\Big\}\end{aligned} \tag{4.23}$$

$$
\begin{aligned}
&\leq \; cM^{-1/2} + \sup_{f\in\mathcal{F}_\beta}\Big\{\Big(\int_{-\infty}^{\infty}|f(x)|^{2p}dx\Big)^{1/p} - (1-\epsilon)\mathcal{E}_\beta(f,f)\Big\}\\
&= \; cM^{-1/2} + \Big(\frac{1}{1-\epsilon}\Big)^{\frac{p-1}{p(\beta-1)+1}}J
\end{aligned}
$$

where the second inequality follows from the fact that $\alpha \leq 1$ and the final step from the substitution

$$f(x) = \Big(\frac{1}{1-\epsilon}\Big)^{p/2(p(\beta-1)+1)}h\Big(\Big(\frac{1}{1-\epsilon}\Big)^{p/(p(\beta-1)+1)}x\Big).$$

Since $q > 1$, there is a sufficiently large $M = M(\epsilon) > 0$ such that $M^{-1/2} \leq \epsilon$ and that

$$cM^{-1/4}x^{1/2} + (2M^{-1/2})^{1/p}c(1+x)^{1/q} \leq \epsilon(x+1)$$

for all $x \geq 0$. Note that the choice of M is independent of the function $g!$. For such M,

$$\Big(\int_0^M \bar{g}^{2p}(x)dx\Big)^{1/p} - \bar{\mathcal{E}}_\beta(\bar{g},\bar{g}) \leq \epsilon + \Big(\frac{1}{1-\epsilon}\Big)^{\frac{p-1}{p(\beta-1)+1}}J \tag{4.24}$$

This completes the proof of Lemma 7.

5 Large deviations for independent stable local times and the law of the iterated logarithm

Proof of Theorem 2: The upper bound for (1.13) follows exactly as in [3]. Given Lemma 11 and our proof of the lower bound in Theorem 1 the lower bound for (1.13) will follow exactly as in [3].

Proof of Theorem 3: Using (1.15) and the scaling (1.12) we see that for any $h > 0$

$$\lim_{t\to\infty}\frac{1}{t}\log P\Big\{\int_{R^1}\prod_{j=1}^m (L^x_{j,1})^p dx \geq ht^{(mp-1)/\beta}\Big\} = -h^{\beta/(mp-1)}A_{\beta,m,p}. \tag{5.1}$$

Replacing t by $\log\log t$ we get

$$\lim_{t\to\infty}\frac{1}{\log\log t}\log P\Big\{\int_{R^1}\prod_{j=1}^m (L^x_{j,1})^p dx \geq h(\log\log t)^{(mp-1)/\beta}\Big\} = -h^{\beta/(mp-1)}A_{\beta,m,p}. \tag{5.2}$$

Let $t_k = \theta^k$ for $\theta > 1$. Using (5.2) and the scaling (1.12) we see that

$$\sum_{k=1}^\infty P\left(\int_{R^1}\prod_{j=1}^m (L^x_{j,t_k})^p dx \geq ct_k^{(mp(1-1/\beta)+1/\beta)}(\log\log t_k)^{(mp-1)/\beta}\right) < \infty \tag{5.3}$$

for any $c > A_{\beta,m,p}^{-(mp-1)/\beta}$. Borel-Cantelli and interpolation then give the upper bound in Theorem 3.

To prove the lower bound, write $s_k = k^{2k}$, $k \geq 1$ and notice that

$$L^x_{j,s_{k+1}} - L^x_{j,s_k} = L^{x-X_{j,s_k}}_{j,k,s_{k+1}-s_k} \tag{5.4}$$

where $L^x_{j,k,t}$ is the local time of $X_{j,k,t}$:

$$X_{j,k,t} = X_{j,s_k+t} - X_{j,s_k} \qquad t \geq 0, \quad k \geq 1, \quad 1 \leq j \leq m.$$

Hence, using $|a^{1/m} - b^{1/m}| \leq 2^{1/m}|a-b|^{1/m}$

$$\begin{aligned} &\Big| \| \prod_{j=1}^m [L^x_{j,s_{k+1}} - L^x_{j,s_k}] \|^{1/m}_{p,R^1} - \| \prod_{j=1}^m L^x_{j,k,s_{k+1}-s_k} \|^{1/m}_{p,R^1} \Big|^m \\ = \;& \Big| \| \prod_{j=1}^m L^{x-X_{j,s_k}}_{j,k,s_{k+1}-s_k} \|^{1/m}_{p,R^1} - \| \prod_{j=1}^m L^x_{j,k,s_{k+1}-s_k} \|^{1/m}_{p,R^1} \Big|^m \\ \leq \;& 2^{1/m} \| \prod_{j=1}^m L^{x-X_{j,s_k}}_{j,k,s_{k+1}-s_k} - \prod_{j=1}^m L^x_{j,k,s_{k+1}-s_k} \|_{p,R^1}. \end{aligned} \tag{5.5}$$

Then using $\prod_{j=1}^m a_j - \prod_{j=1}^m b_j = \sum_{j=1}^m (\prod_{i=1}^{j-1} a_i)(a_j - b_j)(\prod_{k=j+1}^m b_k)$ followed by Hölder's inequality we see that

$$\begin{aligned} &\Big| \| \prod_{j=1}^m [L^x_{j,s_{k+1}} - L^x_{j,s_k}] \|^{1/m}_{p,R^1} - \| \prod_{j=1}^m L^x_{j,k,s_{k+1}-s_k} \|^{1/m}_{p,R^1} \Big|^m \\ &\leq 2^{1/m} \sum_{j=1}^m \left(\prod_{i \neq j} \| L^x_{i,k,s_{k+1}-s_k} \|_{mp,R^1} \right) \; \| L^{x-X_{j,s_k}}_{j,k,s_{k+1}-s_k} - L^x_{j,k,s_{k+1}-s_k} \|_{mp,R^1}. \end{aligned} \tag{5.6}$$

By the already proven upper bound in Theorem 3, taking $m = 1$, replacing p by mp and using the abbreviation $\phi(s) = s^{mp(1-1/\beta)+1/\beta} (\log\log s)^{(mp-1)/\beta}$ we have

$$\| L^x_{i,k,s_{k+1}-s_k} \|_{mp,R^1} = O(\phi(s_{k+1}))^{1/mp} \quad a.s. \tag{5.7}$$

for each $1 \leq i \leq m$.

It follows from Lemma 10, after rescaling, that for any $\alpha > 0$

$$\begin{aligned} &\lim_{\delta \to 0^+} \limsup_{t \to \infty} \; (\log\log t)^{-1} \\ &\log P\Big\{ \sup_{|x-y| \leq \delta(t/\log\log t)^{1/\beta}} \| L^{y+\cdot}_t - L^{x+\cdot}_t \|_{mp,R^1} \geq \alpha t^{1-1/\beta} \log\log t^{1/\beta} \Big\} = -\infty. \end{aligned}$$

Hence by the Borel-Cantelli lemma

$$\begin{aligned} &\lim_{\delta \to 0^+} \limsup_{k \to \infty} s_{k+1}^{-(1-1/\beta)} \log\log s_{k+1}^{-1/\beta} \\ &\qquad \sup_{|x-y| \leq \delta(s_{k+1}/\log\log s_{k+1})^{1/\beta}} \| L^{y+\cdot}_{j,k,s_{k+1}-s_k} - L^{x+\cdot}_{j,k,s_{k+1}-s_k} \|_{mp,R^1} \\ &\qquad = 0 \quad a.s. \end{aligned} \tag{5.8}$$

However, it is easy to see that $n_{k+1} - n_k > n_k k^2$ as $k \to \infty$ so that by the scaling property of the stable process

$$P(|X_{n_k}| > \frac{(n_{k+1} - n_k)^{1/\beta}}{\log\log(n_{k+1} - n_k)}) \leq P(|X_1| > \frac{k^{2/\beta}}{\log k}) \leq C/k^{2-2\epsilon}$$

for any $\epsilon > 0$, since X_1 has $\beta - \epsilon$ moments. By the Borel-Cantelli lemma, with probability 1 the events

$$\{|(X_{1,n_k}, \ldots, X_{m,n_k})| \leq \frac{(n_{k+1} - n_k)^{1/\beta}}{\log\log(n_{k+1} - n_k)}\} \quad k = 1, 2, \cdots$$

eventually hold. Therefore, by (5.8)

$$\begin{aligned} \|L^{X_{j,s_k}+\cdot}_{j,k,s_{k+1}-s_k} - L^{+\cdot}_{j,k,s_{k+1}-s_k}\|_{mp,R^1} &= o(s_{k+1}^{(1-1/\beta)} \log\log s_{k+1}^{1/\beta}) \quad a.s. \\ &= o(\phi(s_{k+1}))^{1/mp} \quad a.s. \end{aligned} \tag{5.9}$$

Combining (5.6), (5.7) and (5.9), we reach the conclusion that

$$\begin{aligned} &\left|\Big(\int_{-\infty}^{\infty}\prod_{j=1}^m [L^x_{j,s_{k+1}} - L^x_{j,s_k}]^p dx\Big)^{1/mp} - \Big(\int_{-\infty}^{\infty}\prod_{j=1}^m (L^x_{j,k,s_{k+1}-s_k})^p dx\Big)^{1/mp}\right| \\ &= o(\phi(s_{k+1}))^{1/mp} \quad a.s. \ (k \to \infty). \end{aligned} \tag{5.10}$$

On the other hand, by (1.15) in Theorem 2, for any $\gamma < A_{\beta,m,p}^{-(mp-1)/\beta}$

$$\begin{aligned} &\sum_k P\left\{\int_{-\infty}^{\infty}\prod_{j=1}^m (L^x_{j,k,s_{k+1}-s_k})^p dx \geq \gamma\phi(s_{k+1})\right\} \\ = &\sum_k P\left\{\int_{-\infty}^{\infty}\Big(\prod_{j=1}^m L^x_{j,s_{k+1}-s_k}\Big)^p dx \geq \gamma\phi(s_{k+1})\right\} = \infty. \end{aligned}$$

Then by the Borel-Cantelli lemma and the independence of the sequence

$$\int_{-\infty}^{\infty}\prod_{j=1}^m (L^x_{j,k,s_{k+1}-s_k})^p dx, \quad k = 1, 2, \cdots$$

we have

$$\limsup_{k\to\infty} \frac{1}{\phi(s_{k+1})}\int_{-\infty}^{\infty}\prod_{j=1}^m (L^x_{j,k,s_{k+1}-s_k})^p dx \geq A_{\beta,m,p}^{-(mp-1)/\beta} \quad a.s. \tag{5.11}$$

In view of (5.10),

$$\limsup_{k\to\infty} \frac{1}{\phi(s_{k+1})}\int_{-\infty}^{\infty}\prod_{j=1}^m [L^x_{j,s_{k+1}} - L^x_{j,s_k}]^p dx \geq A_{\beta,m,p}^{-(mp-1)/\beta} \quad a.s. \tag{5.12}$$

Note that

$$\int_{-\infty}^{\infty}\prod_{j=1}^m (L^x_{j,s_{k+1}})^p dx \geq \int_{-\infty}^{\infty}\prod_{j=1}^m [L^x_{j,s_{k+1}} - L^x_{j,s_k}]^p dx, \quad \forall k \geq 1.$$

Hence,

$$\limsup_{t\to\infty} \frac{1}{\phi(s_{k+1})}\int_{-\infty}^{\infty}\prod_{j=1}^m (L^x_{j,t})^p dx \geq A_{\beta,m,p}^{-(mp-1)/\beta} \quad a.s.$$

which finishes the proof of Theorem 3.

6 Exponential moments for local times

Let L^x_t denote the local time for X in R^1, and $\bar{L}^x_t$ denote the local time for X_* in T_M. We use $\|\cdot\|_{p,R^1}, \|\cdot\|_{p,T_M}$ to denote the norms in $L^p(R^1, dx)$ and $L^p(T_M, dx)$ respectively.

Lemma 8 *Let $p > 1$. For the symmetric stable process in R^1 of index $\beta > 1$ and any $\gamma < \infty$*

$$E\left(\exp\left(\gamma\|L^{\cdot}_1\|_{p,R^1}\right)\right) < \infty \tag{6.1}$$

and for some $\zeta > 0$

$$\sup_{x,y} E\left(\exp\left(\gamma\frac{\|L^{y+\cdot}_1 - L^{x+\cdot}_1\|_{p,R^1}}{|y-x|^\zeta}\right)\right) < \infty. \tag{6.2}$$

Furthermore

$$\lim_{\gamma\to 0}\sup_{x,y} E\left(\exp\left(\gamma\frac{\|L^{y+\cdot}_1 - L^{x+\cdot}_1\|_{p,R^1}}{|y-x|^\zeta}\right)\right) = 1. \tag{6.3}$$

Similar results hold for the symmetric stable process in T_M of index $\beta > 1$ with $\|L^{\cdot}_1\|_{p,R^1}$, $\|L^{y+\cdot}_1 - L^{x+\cdot}_1\|_{p,R^1}$ replaced by $\|\bar{L}^{\cdot}_1\|_{p,T_M}$, $\|\bar{L}^{y+\cdot}_1 - \bar{L}^{x+\cdot}_1\|_{p,T_M}$.

Proof of Lemma 8: We note that the tail estimate in [9] implies a slightly stronger result than (6.1). The direct proof of (6.1) given here serves as a warmup for (6.2) and (6.3). We will first assume that p is an integer greater than or equal to 2. Recall the notation

$$L^x_{t,\epsilon} = \int_0^t f_\epsilon(X_s - x)\, ds. \tag{6.4}$$

For any integer m we have

$$E\left(\prod_{j=1}^m (L^{x_j}_{t,\epsilon})^p\right) = E\left(\int_{R^{mp}}\int_{[0,1]^{mp}}\prod_{j=1}^m\prod_{k=1}^p e^{i\lambda_{j,k}(x_j - X_{t_{j,k}})}\widehat{f}(\epsilon\lambda_{j,k})\, dt_{j,k}\, d\lambda_{j,k}\right) \tag{6.5}$$

$$= \int_{R^{mp}}\int_{[0,1]^{mp}}\prod_{j=1}^m e^{i(\sum_{k=1}^p \lambda_{j,k})x_j} E\left(\prod_{j=1}^m\prod_{k=1}^p e^{-i\lambda_{j,k}X_{t_{j,k}}}\right)\prod_{j=1}^m\prod_{k=1}^p \widehat{f}(\epsilon\lambda_{j,k})\, dt_{j,k}\, d\lambda_{j,k}.$$

Using the Fourier inversion formula in the form

$$\int_{R^m} e^{i\sum_{j=1}^m(\sum_{k=1}^{p-1}\lambda_{j,k})x_j}\int_{R^m} e^{i\sum_{j=1}^m \lambda_{j,p}x_j} F(\lambda_{j,p})\prod_{j=1}^m d\lambda_{j,p}\, dx_j = F(-\sum_{k=1}^{p-1}\lambda_{j,k}) \tag{6.6}$$

we have that

$$E\left(\|L^{\cdot}_{1,\epsilon}\|^{pm}_{p,R^1}\right) = \int_{R^m} E\left(\prod_{j=1}^m (L^{x_j}_{t,\epsilon})^p\right)\prod_{j=1}^m dx_j \tag{6.7}$$

$$= \int_{R^{m(p-1)}}\int_{[0,1]^{mp}} E\left(\prod_{j=1}^m\prod_{k=1}^p e^{-i\lambda_{j,k}X_{t_{j,k}}}\right)\prod_{j=1}^m\prod_{k=1}^p \widehat{f}(\epsilon\lambda_{j,k})\, dt_{j,k}\prod_{j=1}^m\prod_{k=1}^{p-1} d\lambda_{j,k}$$

where in the last line $\lambda_{j,p} = -\sum_{k=1}^{p-1} \lambda_{j,k}$. To evaluate the expectation, for each bijection π of $\{1 \le n \le mp\}$ onto $\{(j,k)\,;\, 1 \le j \le m\,,\, 1 \le k \le p\}$ we let

$$D_\pi = \{t_{j,k}\,;\, t_{\pi(1)} < t_{\pi(2)} < \cdots < t_{\pi(mp)} < 1\} \tag{6.8}$$

and

$$u_{\pi,n} = \sum_{l=n}^{mp} \lambda_{\pi(l)}. \tag{6.9}$$

We use $\mathcal{C}$ to denote the set of such bijections π. Then

$$\begin{aligned} &E\left(\prod_{j=1}^{m}\prod_{k=1}^{p} e^{-i\lambda_{j,k}X_{t_{j,k}}}\right) \\ &= \sum_{\pi\in\mathcal{C}} E\left(e^{-i\sum_{n=1}^{mp}\lambda_{\pi(n)}X_{t_{\pi(n)}}}\right) \\ &= \sum_{\pi\in\mathcal{C}} E\left(e^{-i\sum_{n=1}^{mp}u_{\pi,n}(X_{t_{\pi(n)}}-X_{t_{\pi(n-1)}})}\right) \\ &= \sum_{\pi\in\mathcal{C}} e^{-\sum_{n=1}^{mp}|u_{\pi,n}|^\beta(t_{\pi(n)}-t_{\pi(n-1)})}. \end{aligned} \tag{6.10}$$

We will bound this by dropping the last factor in which $\lambda_{j,p}$ appears for each j. To be more precise, let $\nu_{\pi,j}$ be the unique n such that $u_{\pi,n} - u_{\pi,n+1} = \lambda_{j,p}$ and set $V_\pi = \{\nu_{\pi,j}\,,\, 1 \le j \le m\}$. Combining the above we have uniformly in $\epsilon > 0$

$$\begin{aligned} &E\left(\|L^{\cdot}_{1,\epsilon}\|_{p,R^1}^{pm}\right) \\ &\le \sum_{\pi\in\mathcal{C}} \int_{R^{m(p-1)}}\int_{D_\pi} e^{-\sum_{n\in V_\pi^c}|u_{\pi,n}|^\beta(t_{\pi(n)}-t_{\pi(n-1)})} \prod_{n=1}^{mp} dt_{\pi(n)} \prod_{j=1}^{m}\prod_{k=1}^{p-1} d\lambda_{j,k} \\ &\le c^{mp}\sum_{\pi\in\mathcal{C}}\int_{D_\pi}\prod_{n\in V_\pi^c}\frac{1}{|t_{\pi(n)}-t_{\pi(n-1)}|^{1/\beta}}\prod_{n=1}^{mp} dt_{\pi(n)} \\ &\le c^{mp}(mp)!/\Gamma(m(p-1)(1-1/\beta)+m) \le c^{mp}((mp)!)^{(p-1)/p\beta}. \end{aligned} \tag{6.11}$$

Since this is true for any integer m we can use uniform integrability to obtain

$$E\left(\|L^{\cdot}_{1}\|_{p,R^1}^{pm}\right) \le c^{mp}((mp)!)^{(p-1)/p\beta}. \tag{6.12}$$

Then for any integer n

$$E\left(\|L^{\cdot}_{1}\|_{p,R^1}^{n}\right) \le \left\{E\left(\|L^{\cdot}_{1}\|_{p,R^1}^{pn}\right)\right\}^{1/p} \le c^n(n!)^{(p-1)/p\beta}. \tag{6.13}$$

This immediately gives (6.1). To obtain (6.2) and (6.3) we begin with p an even integer and note that if we replace $L^{\cdot}_{1,\epsilon}$ in (6.7) by $L^{y+\cdot}_{1,\epsilon} - L^{x+\cdot}_{1,\epsilon}$, then in the last line of (6.7) we will have an extra factor of $\prod_{j=1}^{m}\prod_{k=1}^{p}\left(e^{i\lambda_{j,k}y} - e^{i\lambda_{j,k}x}\right)$. Using the bound

$$|e^{i\lambda_{j,k}y} - e^{i\lambda_{j,k}x}| \le 2|\lambda_{j,k}|^\zeta|y-x|^\zeta \tag{6.14}$$

which is valid for any $0 \leq \zeta \leq 1$ and proceeding in a manner similar to (6.11) leads to (6.2) and (6.3).

If $q > 1$ is not an integer, then for some integer $p \geq 1$ we have that $q = \alpha p + (1-\alpha)(p+1)$ for some $0 < \alpha < 1$. By Hölder's inequality, for any g we have

$$\|g\|_{q,R^1}^q = \int_{R^1} |g(x)|^{\alpha p+(1-\alpha)(p+1)}\, dx \leq \|g\|_{p,R^1}^{\alpha p} \|g\|_{p+1,R^1}^{(1-\alpha)(p+1)} \tag{6.15}$$

so that

$$\|g\|_{q,R^1} \leq \|g\|_{p,R^1}^{\alpha p/q} \|g\|_{p+1,R^1}^{(1-\alpha)(p+1)/q} \leq \|g\|_{p,R^1} + \|g\|_{p+1,R^1} \tag{6.16}$$

The general case of $q > 2$ then follows from case of integral $q \geq 2$. Finally, if $1 < q < 2$, since $\|L_1^{\cdot}\|_{1,R^1} = 1$ we obtain (6.1) using (6.16). For (6.2) and (6.3), if $q = \alpha + 2(1-\alpha)$ for some $0 < \alpha < 1$, we have by (6.15) and the fact that $\|L_{1,\epsilon}^{y+\cdot} - L_{1,\epsilon}^{x+\cdot}\|_{1,R^1} \leq 2$ that

$$\|L_{1,\epsilon}^{y+\cdot} - L_{1,\epsilon}^{x+\cdot}\|_{q,R^1} \leq 2^{\alpha/q} \|L_{1,\epsilon}^{y+\cdot} - L_{1,\epsilon}^{x+\cdot}\|_{2,R^1}^{2(1-\alpha)/q} \tag{6.17}$$

and (6.2)-(6.3) for $1 < q < 2$ now follows from the case of $q = 2$.

We now turn to the analogue of (6.1) for the symmetric stable process in T_M. We have as above for integer m, p

$$\begin{aligned} E\left(\prod_{j=1}^m (\bar{L}_t^{x_j})^p\right) &= E\left(M^{-mp} \sum_{\lambda_\cdot \in (\frac{2\pi}{M} Z^1)^{mp}} \int_{[0,1]^{mp}} \prod_{j=1}^m \prod_{k=1}^p e^{i\lambda_{j,k}(x_j - X_{t_{j,k}})}\, dt_{j,k}\right) \\ &= M^{-mp} \sum_{\lambda_\cdot \in (\frac{2\pi}{M} Z^1)^{mp}} \prod_{j=1}^m e^{i(\sum_{k=1}^p \lambda_{j,k})x_j} E\left(\prod_{j=1}^m \prod_{k=1}^p e^{-i\lambda_{j,k} X_{t_{j,k}}}\right) \prod_{j=1}^m \prod_{k=1}^p dt_{j,k} \end{aligned} \tag{6.18}$$

so that as before

$$\begin{aligned} E\left(\|\bar{L}_1^{\cdot}\|_{p,T_M}^{pm}\right) &= E\left(\int_{T_M^m} \prod_{j=1}^m (\bar{L}_t^{x_j})^p\, dx_j\right) \\ &= M^{-m(p-1)} \sum_{\lambda_\cdot \in (\frac{2\pi}{M} Z^1)^{m(p-1)}} \int_{[0,1]^{mp}} E\left(\prod_{j=1}^m \prod_{k=1}^p e^{-i\lambda_{j,k} X_{t_{j,k}}}\right) \prod_{j=1}^m \prod_{k=1}^p dt_{j,k} \end{aligned} \tag{6.19}$$

where in the last line $\lambda_{j,p} = -\sum_{k=1}^{p-1} \lambda_{j,k}$. Again as above this leads to

$$\begin{aligned} &E\left(\|\bar{L}_1^{\cdot}\|_{p,T_M}^{pm}\right) \\ &\leq \sum_{\pi \in \mathcal{C}} \int_{D_\pi} \left(M^{-m(p-1)} \sum_{\lambda_\cdot \in (\frac{2\pi}{M} Z^1)^{m(p-1)}} e^{-\sum_{n \in V_\pi^c} |u_{\pi,n}|^\beta (t_{\pi(n)} - t_{\pi(n-1)})}\right) \prod_{n=1}^{mp} dt_{\pi(n)} \\ &\leq c^{mp} \sum_{\pi \in \mathcal{C}} \int_{D_\pi} \left(\prod_{n \in V_\pi^c} \left\{1 + \frac{1}{|t_{\pi(n)} - t_{\pi(n-1)}|^{1/\beta}}\right\}\right) \prod_{n=1}^{mp} dt_{\pi(n)} \end{aligned} \tag{6.20}$$

and as before this leads to the analogue of (6.1) for the symmetric stable process in T_M. The analogues of (6.2)-(6.3) follows similarly. This completes the roof of Lemma 8.

Consider the Young's function $\psi(x)=\exp(x)-1$, and let $\|\cdot\|_\psi$ denotes the Orlicz space norm with respect to E. By (5.1.9) of [8] we have that for any finite set of non-negative random variables $Y_1,\ldots,Y_n$

$$\| \sup_{k\le n} Y_k \|_\psi \le c\psi^{-1}(n) \sup_{k\le n} \| Y_k \|_\psi. \tag{6.21}$$

Let D denote the set of dyadic numbers in $[0,1]$. Thus $D=\cup_m D_m$ where D_m is the set of numbers in $[0,1]$ of the form $i/2^m$ for some integer i. The next Lemma follows from a standard chaining argument, see e.g. [12], Chapter 11, which also contains historical references.

Lemma 9 *Let $\{Z_t,\, t\in D\subseteq[0,1]\}$ be a Banach space valued stochastic process such that for some finite constants $c,\zeta>0$*

$$\| |Z_t-Z_s| \|_\psi \le c|t-s|^\zeta, \quad \forall s,t\in D. \tag{6.22}$$

Then for any $\zeta'<\zeta$ that

$$\| \sup_{\substack{s,t\in D\\ s\neq t}} |Z_t-Z_s|/|s-t|^{\zeta'} \|_\psi \le 2\sum_{m=0}^\infty C2^{\zeta'(m+1)}2^{-m\zeta}<\infty. \tag{6.23}$$

Lemma 10 *Let $p>1$. For the symmetric stable process in R^1 of index $\beta>1$, for some $\zeta'>0$*

$$\Big\| \sup_{x\neq y} \frac{\|L_1^{y+\cdot}-L_1^{x+\cdot}\|_{p,R^1}}{|x-y|^{\zeta'}} \Big\|_\psi <\infty. \tag{6.24}$$

Furthermore, for any $\alpha>0$

$$\lim_{\delta\to0}\sup_{t\ge1}\frac{1}{t}\log P\left(\sup_{|x-y|\le\delta}\|L_t^{y+\cdot}-L_t^{x+\cdot}\|_{p,R^1}>\alpha t\right)=-\infty. \tag{6.25}$$

Similar results hold for the symmetric stable process in T_M of index $\beta>1$ with $\|L_t^{y+\cdot}-L_t^{x+\cdot}\|_{p,R^1}$ replaced by $\|\bar L_t^{y+\cdot}-\bar L_t^{x+\cdot}\|_{p,T_M}$.

Proof of Lemma 10: Using (6.3) we see that for some $c<\infty$ and all x,y

$$\| \|L_1^{y+\cdot}-L_1^{x+\cdot}\|_{p,R^1} \|_\psi \le c|x-y|^\zeta. \tag{6.26}$$

Using Lemma 9 we see that

$$\Big\| \sup_{\substack{x\neq y\\ x,y\in D}} \frac{\|L_1^{y+\cdot}-L_1^{x+\cdot}\|_{p,R^1}}{|x-y|^{\zeta'}} \Big\|_\psi <\infty. \tag{6.27}$$

But since, using Fatou's Lemma and the continuity of local time

$$\sup_{\substack{x\neq y\\ x,y\in[0,1]}} \frac{\|L_1^{y+\cdot}-L_1^{x+\cdot}\|_{p,R^1}}{|x-y|^{\zeta'}} = \sup_{\substack{x\neq y\\ x,y\in D}} \frac{\|L_1^{y+\cdot}-L_1^{x+\cdot}\|_{p,R^1}}{|x-y|^{\zeta'}} \tag{6.28}$$

we get

$$\Big\| \sup_{\substack{x \neq y \\ x,y \in [0,1]}} \frac{\|L_1^{y+\cdot} - L_1^{x+\cdot}\|_{p,R^1}}{|x-y|^{\zeta'}} \Big\|_\psi < \infty. \tag{6.29}$$

Note that for any z

$$\|L_1^{z+y+\cdot} - L_1^{z+x+\cdot}\|_{p,R^1} = \|L_1^{y+\cdot} - L_1^{x+\cdot}\|_{p,R^1} \tag{6.30}$$

hence

$$\sup_{0<|x-y|\leq 1} \frac{\|L_1^{y+\cdot} - L_1^{x+\cdot}\|_{p,R^1}}{|x-y|^{\zeta'}} = \sup_{\substack{x \neq y \\ x,y \in [0,1]}} \frac{\|L_1^{y+\cdot} - L_1^{x+\cdot}\|_{p,R^1}}{|x-y|^{\zeta'}}. \tag{6.31}$$

Using (6.1) for $|x-y| \geq 1$, (6.24) then follows.

By the scaling (1.12), for any $t > 0$

$$\sup_{x \neq y} \frac{||L_t^{\cdot+x} - L_t^{\cdot+y}||_{p,R^1}}{|x-y|^{\zeta'}} \overset{d}{=} t^{1+\frac{1}{p\beta}-\frac{1+\zeta'}{\beta}} \sup_{x \neq y} \frac{||L_1^{\cdot+x} - L_1^{\cdot+y}||_{p,R^1}}{|x-y|^{\zeta'}}.$$

Hence by (6.24)

$$\sup_{1\leq t\leq 2} \Big\| \sup_{x \neq y} \frac{\|L_t^{y+\cdot} - L_t^{x+\cdot}\|_{p,R^1}}{|x-y|^{\zeta'}} \Big\|_\psi \leq K < \infty, \tag{6.32}$$

so that

$$\sup_{1\leq t\leq 2} E\left\{ \exp\Big(\sup_{x \neq y} \frac{\|L_t^{y+\cdot} - L_t^{x+\cdot}\|_{p,R^1}}{K|x-y|^{\zeta'}} \Big) \right\} \leq 2. \tag{6.33}$$

Using the additivity of local times and the Markov property we then have that for any $t \geq 1$

$$E\left\{ \exp\Big(\sup_{x \neq y} \frac{\|L_t^{y+\cdot} - L_t^{x+\cdot}\|_{p,R^1}}{K|x-y|^{\zeta'}} \Big) \right\} \leq 2^t. \tag{6.34}$$

Hence by Chebycheff

$$\begin{aligned} &\sup_{t\geq 1} P\left(\sup_{|x-y|\leq\delta} \|L_t^{y+\cdot} - L_t^{x+\cdot}\|_{p,R^1} > \alpha t \right) \\ &\leq \sup_{t\geq 1} P\left(\sup_{|x-y|\leq\delta} \frac{\|L_t^{y+\cdot} - L_t^{x+\cdot}\|_{p,R^1}}{K|x-y|^{\zeta'}} > \frac{\alpha t}{K\delta^{\zeta'}} \right) \leq e^{-\alpha t K^{-1}\delta^{->\zeta'}} 2^t. \end{aligned} \tag{6.35}$$

Our lemma then follows.

For the next lemma let $\mathcal{R}_a L_t^{\cdot}$ denote the restriction of L_t^x to $x \in [-a,a]$.

Lemma 11 *Let $p > 1$. For the symmetric stable process in R^1 of index $\beta > 1$, for any $0 < a < \infty$ we can find a compact $K \subseteq L^p([-a,a])$ such that*

$$\lim_{\gamma\to\infty} \limsup_{t\to\infty} \frac{1}{t} \log P\left(t^{-1}\mathcal{R}_a L_t^{\cdot} \notin \gamma K \right) = -\infty. \tag{6.36}$$

For the symmetric stable process in T_M of index $\beta > 1$, we can find a compact $K \subseteq L^p(T_M)$ such that

$$\lim_{\gamma\to\infty} \limsup_{t\to\infty} \frac{1}{t} \log P\left(t^{-1} L_t^{\cdot} \notin \gamma K \right) = -\infty. \tag{6.37}$$

Proof of Lemma 11: By (6.25), for any $k \geq 1$, there is a $\delta_k > 0$ such that

$$\sup_{t\geq 1} \frac{1}{t} \log P\{\sup_{|h|\leq \delta_k} ||L_t^{\cdot+h} - L_t^{\cdot}||_{p,R^1} \geq \frac{1}{k} t\} \leq -k$$

Similarly we can take $N > 0$ such that

$$\sup_{t\geq 1} \frac{1}{t} \log P\{||L_t^{\cdot}||_{p,R^1} \geq Nt\} \leq -1.$$

If

$$A_{N,\delta_k} = \{f \mid \|f\|_{p,R^1} \leq N\} \bigcap_{k=1}^{\infty} \Big\{f \mid \sup_{|h|\leq \delta_k} \|f(x+h) - f(x)\|_{p,R^1} \leq \frac{1}{k}\Big\}$$

we take to be the closure of $\mathcal{R}_a A_{N,\delta_k}$ in $L^p([-a,a])$. By Lemma 15, K is a compact subset of $L^p([-a,a])$ and we have for any $\gamma \geq 1$

$$\begin{aligned}
(6.38) \qquad & P\{t^{-1}\mathcal{R}_a L_t^{\cdot} \notin \gamma K\} \\
& \leq P\{||L_t^{\cdot}||_{p,R^1} \geq N\gamma t\} + \sum_{k=1}^{\infty} P\{\sup_{|h|\leq \delta_k} ||L_t^{\cdot+h} - L_t^{\cdot}||_{p,R^1} \geq \frac{1}{k}\gamma t\} \\
& \leq \Big[1 + (1 - e^{-\gamma})^{-1}\Big] e^{-t\gamma}.
\end{aligned}$$

Lemma 11 follows immediately.

7 Random walks

We begin by studying exponential integrabilty for local times of random walks. We will use the notation

$$\widetilde{l}_n^x = b(n)^{1-1/p} n^{-1}\, l_n^x. \tag{7.1}$$

Lemma 12 *Let $p > 1$. For any $\gamma < \infty$*

$$\sup_n E\left(\exp\left(\gamma \|\widetilde{l}_n^{\cdot}\|_{p,Z^1}\right)\right) < \infty \tag{7.2}$$

and

$$\lim_{\gamma\to 0} \sup_n E\left(\exp\left(\gamma \|\widetilde{l}_n^{\cdot}\|_{p,Z^1}\right)\right) = 1. \tag{7.3}$$

For some $\zeta > 0$

$$\sup_{n,y} E\left(\exp\left(\gamma \frac{\|\widetilde{l}_n^{\cdot} - \widetilde{l}_n^{y+\cdot}\|_{p,R^1}}{|y/b(n)|^{\zeta}}\right)\right) < \infty \tag{7.4}$$

and

$$\lim_{\gamma\to 0} \sup_{n,y} E\left(\exp\left(\gamma \frac{\|\widetilde{l}_n^{\cdot} - \widetilde{l}_n^{y+\cdot}\|_{p,R^1}}{|y/b(n)|^{\zeta}}\right)\right) = 1. \tag{7.5}$$

Proof of Lemma 12: Assume first that $p > 1$ is an integer. We have, using Fourier inversion as before,

$$\begin{aligned}
(7.6)\qquad \|\widetilde{l}_n\|^p_{p,Z^1} &= \frac{b(n)^{p-1}}{n^p(2\pi)^p}\sum_x \sum_{n_1,\ldots,n_p=0}^{n}\Big[\int_{[-\pi,\pi]^p} e^{i\sum_{j=1}^p \lambda_j\cdot(S_{n_j}-x)}\prod_{j=1}^p d\lambda_j\Big] \\
&= \frac{b(n)^{p-1}}{n^p(2\pi)^{p-1}}\sum_{n_1,\ldots,n_p=0}^{n}\Big[\int_{[-\pi,\pi]^{(p-1)}} e^{i\sum_{j=1}^p \lambda_j\cdot S_{n_j}}\prod_{j=1}^{p-1} d\lambda_j\Big]
\end{aligned}$$

where from now on $\lambda_p = -\sum_{j=1}^{p-1}\lambda_j$ and we work modulo $\pm\pi$. Then by scaling we have

$$\begin{aligned}
(7.7)\qquad &\|\widetilde{l}_n\|^p_{p,Z^1} \\
&= \frac{1}{n^p(2\pi)^{p-1}}\sum_{n_1,\ldots,n_p=0}^{n}\Big[\int_{b(n)[-\pi,\pi]^{(p-1)}} e^{i\sum_{j=1}^p \lambda_j\cdot S_{n_j}/b(n)}\prod_{j=1}^{p-1} d\lambda_j\Big].
\end{aligned}$$

Let $\phi(u) = E(e^{iu\cdot S_1})$. We recall from (5.14) of [14] that for any $\zeta > 0$ we can find a $c > 0$ such that

$$(7.8)\qquad |\phi(u/b(n))| \le e^{-c|u|^{\beta-\zeta}/n}, \qquad 1 \le |u| \le \pi b(n).$$

Hence for any $s \le n$

$$\begin{aligned}
(7.9)\qquad &\int_{R^1} |\phi(u/b(n))|^s |u|^a\,du \\
&\le C + \int_{R^1} e^{-c\frac{s}{n}|u|^{\beta-\zeta}}|u|^a\,du \\
&\le C + C\left(\frac{s}{n}\right)^{-(1+a)/(\beta-\zeta)} \le C\left(\frac{s}{n}\right)^{-(1+a)/(\beta-\zeta)}.
\end{aligned}$$

The proof of (7.2) is then completed by following the proof of Lemma 8 and (7.3) follows similarly.

For (7.4) we see as in the derivation of (7.7) that when $p > 1$ is an integer.

$$\begin{aligned}
(7.10)\qquad \|\widetilde{l}_n - \widetilde{l}_n^{\cdot+y}\|^p_{p,Z^1} &= \frac{1}{n^p(2\pi)^{p-1}}\sum_{n_1,\ldots,n_p=0}^{n} \\
&\Big[\int_{b(n)[-\pi,\pi]^{(p-1)}} e^{i\sum_{j=1}^p \lambda_j\cdot S_{n_j}/b(n)}\prod_{j=1}^{p}(1-e^{i\lambda_j\cdot y/b(n)})\prod_{j=1}^{p-1} d\lambda_j\Big].
\end{aligned}$$

Using $|1-e^{i\lambda_j\cdot y/b(n)}| \le |\lambda_j\cdot y/b(n)|^\zeta$ and (7.9) the proof of Lemma 12 is then completed by following the proof of Lemma 8.

Set

$$(7.11)\qquad \widehat{l}^x_n = b(n)n^{-1}\, l_n^{[b(n)x]}.$$

Lemma 13 *Let $p > 1$. For any $\gamma < \infty$*

$$(7.12)\qquad \sup_n E\left(\exp\left(\gamma\|\widehat{l}_n\|_{p,R^1}\right)\right) < \infty$$

and

$$\lim_{\gamma\to 0}\sup_n E\left(\exp\left(\gamma\|\widehat{l}_n^{\cdot}\|_{p,R^1}\right)\right)=1 \tag{7.13}$$

For any $0<\zeta\le 1/p$

$$\sup_{n,y} E\left(\exp\left(\gamma\frac{\|\widehat{l}_n^{\cdot}-\widehat{l}_n^{y+\cdot}\|_{p,R^1}}{|y|^\zeta}\right)\right)<\infty \tag{7.14}$$

and

$$\lim_{\gamma\to 0}\sup_{n,y} E\left(\exp\left(\gamma\frac{\|\widehat{l}_n^{\cdot}-\widehat{l}_n^{y+\cdot}\|_{p,R^1}}{|y|^\zeta}\right)\right)=1. \tag{7.15}$$

Furthermore, for some $\zeta'>0$

$$\sup_n\Big\|\sup_{x\neq y}\frac{\|\widehat{l}_n^{y+\cdot}-\widehat{l}_n^{x+\cdot}\|_{p,R^1}}{|x-y|^{\zeta'}}\Big\|_\psi<\infty. \tag{7.16}$$

Proof of Lemma 13: We have

$$\begin{aligned}\|\widehat{l}_n^{\cdot}\|_{p,R^1} &= \left\{\sum_{j\in Z^1}\int_{\{x:\,[b(n)x]=j\}}(\,\widehat{l}_n^{x}\,)^p\,dx\right\}^{1/p}\\ &=\left\{\sum_{j\in Z^1}\int_{\{x:\,[b(n)x]=j\}}(\,b(n)n^{-1}\,l_n^j\,)^p\,dx\right\}^{1/p}=\|\widetilde{l}_n^{\cdot}\|_{p,Z^1}\end{aligned} \tag{7.17}$$

so that (7.12) is simply (7.2). (7.13) follows similarly. (7.14) and (7.15) are more subtle. Without loss of generality we can assume that $y>0$.

$$\|\widehat{l}_n^{\cdot}-\widehat{l}_n^{y+\cdot}\|_{p,R^1}=\left\{\sum_{j\in Z^1}\int_{\{x:\,[b(n)x]=j\}}(\,\widehat{l}_n^{x}-\widehat{l}_n^{x+y}\,)^p\,dx\right\} \tag{7.18}$$

and

$$\begin{aligned}&\int_{\{x:\,[b(n)x]=j\}}(\,\widehat{l}_n^{x}-\widehat{l}_n^{x+y}\,)^p\,dx\\ &=\int_{\{x:\,[b(n)x]=j\}}(\,b(n)n^{-1}\,(l_n^j-l_n^{[b(n)(x+y)]})\,)^p\,dx.\end{aligned} \tag{7.19}$$

Let $u=b(n)x-[b(n)x]$, $v=b(n)y-[b(n)y]$. Then $0\le u,v\le 1$ and $[b(n)(x+y)]=[b(n)x]+[b(n)y]$ if $u<1-v$ while $[b(n)(x+y)]=[b(n)x]+[b(n)y]+1$ if $u\ge 1-v$. Thus

$$\begin{aligned}&\int_{\{x:\,[b(n)x]=j\}}(\,b(n)n^{-1}\,(l_n^j-l_n^{[b(n)(x+y)]})\,)^p\,dx\\ &=(\,b(n)n^{-1}\,(l_n^j-l_n^{j+[b(n)y]})\,)^p(1-v)/b(n)\\ &\quad+(\,b(n)n^{-1}\,(l_n^j-l_n^{j+[b(n)y]+1})\,)^p v/b(n)\\ &=(\,(\widetilde{l}_n^j-\widetilde{l}_n^{j+[b(n)y]})\,)^p(1-v)+(\,(\widetilde{l}_n^j-\widetilde{l}_n^{j+[b(n)y]+1})\,)^p v.\end{aligned} \tag{7.20}$$

Thus by (7.18)-(7.20) for any $\zeta \leq 1/p$

$$
\begin{aligned}
(7.21) \qquad & \|\widehat{l}^{\cdot}_n - \widehat{l}^{y+\cdot}_n\|_{p,R^1} \\
&\leq (1-v)^{1/p}\|\widetilde{l}^{\cdot}_n - \widetilde{l}^{[b(n)y]+\cdot}_n\|_{p,Z^1} + v^{1/p}\|\widetilde{l}^{\cdot}_n - \widetilde{l}^{[b(n)y]+1+\cdot}_n\|_{p,Z^1} \\
&\leq (1-v)^{\zeta}\|\widetilde{l}^{\cdot}_n - \widetilde{l}^{[b(n)y]+\cdot}_n\|_{p,Z^1} + v^{\zeta}\|\widetilde{l}^{\cdot}_n - \widetilde{l}^{[b(n)y]+1+\cdot}_n\|_{p,Z^1}.
\end{aligned}
$$

Note that if $[b(n)y] = 0$ then the first term on the right is 0 and $v = b(n)y$ so that we have the bound

$$
(7.22) \qquad \|\widehat{l}^{\cdot}_n - \widehat{l}^{y+\cdot}_n\|_{p,R^1} \leq (b(n)y)^{\zeta}\|\widetilde{l}^{\cdot}_n - \widetilde{l}^{1+\cdot}_n\|_{p,Z^1} = y^{\zeta}\frac{\|\widetilde{l}^{\cdot}_n - \widetilde{l}^{1+\cdot}_n\|_{p,Z^1}}{(1/b(n))^{\zeta}}.
$$

If $[b(n)y] \geq 1$ we obtain

$$
\begin{aligned}
(7.23) \qquad & \|\widehat{l}^{\cdot}_n - \widehat{l}^{y+\cdot}_n\|_{p,R^1} \\
&\leq \frac{(1-v)^{\zeta}[b(n)y]^{\zeta}}{b(n)^{\zeta}}\frac{\|\widetilde{l}^{\cdot}_n - \widetilde{l}^{[b(n)y]+\cdot}_n\|_{p,Z^1}}{([b(n)y]/b(n))^{\zeta}} \\
&\qquad + \frac{v^{\zeta}([b(n)y]+1)^{\zeta}}{b(n)^{\zeta}}\frac{\|\widetilde{l}^{\cdot}_n - \widetilde{l}^{[b(n)y]+1+\cdot}_n\|_{p,Z^1}}{(([b(n)y]+1)/b(n))^{\zeta}}.
\end{aligned}
$$

Since

$$
(1-v)[b(n)y] + v([b(n)y]+1) = b(n)y
$$

we see from (7.23) that

$$
\begin{aligned}
(7.24) \qquad & \frac{\|\widehat{l}^{\cdot}_n - \widehat{l}^{y+\cdot}_n\|_{p,R^1}}{y^{\zeta}} \\
&\leq \frac{\|\widetilde{l}^{\cdot}_n - \widetilde{l}^{[b(n)y]+\cdot}_n\|_{p,Z^1}}{([b(n)y]/b(n))^{\zeta}} + \frac{\|\widetilde{l}^{\cdot}_n - \widetilde{l}^{[b(n)y]+1+\cdot}_n\|_{p,Z^1}}{(([b(n)y]+1)/b(n))^{\zeta}}.
\end{aligned}
$$

Using (7.4) then completes the proof of (7.14). (7.15) follows similarly.

(7.16) follows from the proof of Lemma 10, using the fact that $\widehat{l}^{x}_n$ is right continuous in x.

Lemma 14 *Let $p > 1$. Then*

$$
(7.25) \qquad \widehat{l}^{\cdot}_n \xrightarrow{d} L^{\cdot}_1
$$

as $L^p(R^1)$ valued random variables. In particular,

$$
(7.26) \qquad \|\widetilde{l}^{\cdot}_n\|_{p,Z^1} \xrightarrow{d} \|L^{\cdot}_1\|_{p,R^1}.
$$

Proof of Lemma 14: Let us first show that the measures induced by the sequence $\widehat{l}^{\cdot}_n$ on $L^p(R^1)$ are tight. To do this, let K be the closure of the precompact set

$$
A = \bigcap_{k=1}^{\infty}\Big\{f;\ f \equiv 0 \text{ outside } [-a,a],\ ||f||_p \leq M \text{ and } \sup_{|h|\leq\delta_k}||f(\cdot+h) - f(\cdot)||_p \leq \frac{1}{k}\Big\}
$$

with $a, \delta_1, \delta_2, \ldots$ to be chosen. Then for any $\epsilon > 0$, using (1.18) with $a < \infty$ sufficiently large and Lemma 13 as in the proof of Lemma 11 with $\delta_k \to 0$ sufficiently rapidly

$$
\begin{aligned}
P\{\widehat{l}_n^{\cdot} \notin K\} &\leq P\{||\widehat{l}_n^{\cdot}||_p \geq M\} + P\{\max_{k\leq n}|S(k)| \geq ab(n)\} \\
&+ \sum_{k=1}^{\infty} P\{\sup_{|h|\leq\delta_k} ||\widehat{l}_n^{\cdot+h} - \widehat{l}_n^{\cdot}||_k \geq \frac{1}{k}\} \leq \epsilon
\end{aligned} \tag{7.27}
$$

which establishes tightness. Hence by Prohorov's criterion every subsequence $\widehat{l}_{n_j}^{\cdot}$ has a subsequence which converges in distribution. It only remains to identify the limit with the measure induced by $L_1^{\cdot}$ on $L^p(R^1)$. To this end it suffices to show that

$$
\int_{-\infty}^{\infty} f(x)\widehat{l}_n^x dx \xrightarrow{d} \int_{-\infty}^{\infty} f(x)L_1^x dx \tag{7.28}
$$

for each $f \in \mathcal{S}(R^1)$. But for such f

$$
\begin{aligned}
&\int_{-\infty}^{\infty} f(x)\widehat{l}_n^x dx = \frac{1}{n}\int_{-\infty}^{\infty} f\Big(\frac{x}{b(n)}\Big) l_n^{[x]} dx \\
&= \frac{1}{n}\Big(O(\frac{n}{b(n)}) + \sum_{x\in Z^1} f\Big(\frac{x}{b(n)}\Big) l_n^x\Big) = o(1) + \frac{1}{n}\sum_{k=1}^{n} f\Big(\frac{S(k)}{b(n)}\Big) \\
&\xrightarrow{d} \int_{-\infty}^{\infty} f(x)L_1^x dx
\end{aligned} \tag{7.29}
$$

by Skorohod's generalization of Donsker's invariance principle to random walks in the domain of attraction of a stable process, [17]. Finally, (7.26) follows from (7.17) and (7.25). This completes the proof of our lemma.

Proof of Theorem 4: Fix t large and let $t_n = [tn/\nu_n]$ and $\gamma_n = [n/t_n]$. Using the additivity of local times and the Markov property as before,

$$
\begin{aligned}
&E\exp\Big\{\lambda\frac{\nu_n b(n/\nu_n)^{1-1/p}}{n}\Big(\sum_x (l_n^x)^p\Big)^{1/p}\Big\} \\
&\leq \Big(E\exp\Big\{\lambda\frac{\nu_n b(n/\nu_n)^{1-1/p}}{n}\Big(\sum_x (l_{t_n}^x)^p\Big)^{1/p}\Big\}\Big)^{\gamma_n+1} \\
&= \Big(E\exp\Big\{\lambda\Big(\frac{t_n}{n/\nu_n}\Big)\Big(\frac{b(n/\nu_n)}{b(t_n)}\Big)^{1-1/p}\|\widetilde{l}_{t_n}^{\cdot}\|_{p,Z^1}\Big\}\Big)^{\gamma_n+1}
\end{aligned} \tag{7.30}
$$

Using (7.26), (7.2), and the regular variation of $b(n)$,

$$
\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{\nu_n}\log E\exp\Big\{\lambda\frac{\nu_n b(n/\nu_n)^{1-1/p}}{n}\Big(\sum_x (l_n^x)^p\Big)^{1/p}\Big\} \\
&\leq \frac{1}{t}\log E\exp\Big\{t^{\frac{\beta p-(p-1)}{\beta p}}||L_1^{\cdot}||_p\Big\}
\end{aligned} \tag{7.31}
$$

Letting $t \to \infty$ gives the upper bound.

For the lower bound, as in [3] it suffices to show that for any bounded continuous function f on R^1 and positive sequence $\{\nu_n\}$ satisfying (1.20),

$$\liminf_{n\to\infty} \nu_n^{-1} \log E \exp\left\{\frac{\nu_n}{n}\sum_{k=1}^{n} f\Big(S_k/b(n/\nu_n)\Big)\right\} \geq \sup_{g\in\mathcal{F}_\beta}\Big\{(g,fg)_2 - \mathcal{E}_\beta(g,g)\Big\}. \tag{7.32}$$

This follows along the lines of the proof of Theorem 4.1 in [3], noting that from [11], p. 661,

$$\lim_{n\to\infty}\sup_{x\in Z^1}\left|b(n)P_n(x) - p_1\left(\frac{x}{b(n)}\right)\right| = 0 \tag{7.33}$$

where $P_n(x)$ is the probability function for S_n, and, as before, $p_1(x)$ is the density for X_1. This completes the proof of Theorem 4. Theorem 5 then follows as in the continuous case.

8 Appendix

Let $\mathcal{R}_a f(x)$ denote the restriction of a function $f(x)$ to $x \in [-a,a]$.

Lemma 15 *Let $p > 1$. For any $N < \infty$, and $\delta_k \to 0$ let*

$$\begin{aligned} A_{N,\{\delta_k\}} = {} & \{f \in L^p(R^1) \,|\, \|f\|_{p,R^1} \leq N\} \bigcap \\ & \bigcap_{k=1}^{\infty}\Big\{f \in L^p(R^1) \,|\, \sup_{|h|\leq\delta_k}\|f(x+h)-f(x)\|_{p,R^1} \leq \frac{1}{k}\Big\} \end{aligned} \tag{8.1}$$

Then for any $a < \infty$, $\mathcal{R}_a A_{N,\{\delta_k\}}$ is precompact in $L^p([-a,a])$.

Proof of Lemma 15: This follows easily from Theorem IV.8.21 of [6], but we provide a short self-contained proof. Let $h \in C_0^\infty(R^1)$ be positive, symmetric, supported in $[-1,1]$ with $\int h(x)\,dx = 1$. Set $h_\epsilon(x) = \epsilon^{-1}h(x/\epsilon)$. Let $f_1, f_2, \ldots$ be a sequence in A. We must show that some subsequence converges in $L^p([-a,a])$. For each k let

$$f_{j,k}(x) = h_{\delta_k} * f_j(x) = \int h(y) f_j(x-\delta_k y)\,dy. \tag{8.2}$$

From the definition of A it follows that

$$\|f_{j,k}\|_{p,R^1} \leq N \qquad \text{and} \qquad \|f_{j,k} - f_j\|_{p,R^1} \leq \frac{1}{k}. \tag{8.3}$$

On the other hand, it is easy to see that for each fixed k, the sequence $f_{1,k}, f_{2,k}, \ldots$ is uniformly bounded and equicontinuous. Hence by Ascoli's Lemma we can find a subsequence $f_{j_{n,k},k}\,;\, n = 1,2,\ldots$ which converges uniformly on $[-a,a]$. Hence they converge in $L^p([-a,a])$. In particular we can assume that $j_{n,k} \geq k$ and that $\|f_{j_{n,k},k} - f_{j_{m,k},k}\|_{p,[-a,a]} \leq \frac{1}{k}$ for all m,n. By (8.3), $\|f_{j_{n,k}} - f_{j_{m,k}}\|_{p,[-a,a]} \leq \frac{3}{k}$ for all m,n. We can also assume that $j_{n,k+1}\,;\, n = 1,2,\ldots$ is a subsequence of $j_{n,k}\,;\, n = 1,2,\ldots$. Thus we have $\|f_{j_{n,n}} - f_{j_{m,m}}\|_{p,[-a,a]} \leq \frac{3}{m}$ for all $n > m$ so that $f_{j_{n,n}}\,;\, n = 1,2,\ldots$ converges in $L^p([-a,a])$.

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