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LINEAR EXPANSION OF ISOTROPIC BROWNIAN FLOWS

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Abstract:
We consider an isotropic Brownian flow on $\mathbb{R}^d$ for $d \geq 2$ with a positive Lyapunov exponent, and show that any nontrivial connected set almost surely contains points whose distance from the origin under the flow grows linearly with time. The speed is bounded below by a fixed constant, which may be computed from the covariance tensor of the flow. This complements earlier work, which showed that stochastic flows with bounded local characteristics and zero drift cannot grow at a rate faster than linear.

1. Introduction

A stochastic flow on a manifold M with respect to the probability space $(\Omega, \mathcal{F}, P)$ is a family of measurable maps $\phi_{st} : \Omega \times M \to M$ indexed by s and t in $\mathbb{R}$ or $\mathbb{R}^+$, which satisfy the following properties for every ω in Ω: $\phi_{st}(\omega, x)$ is jointly continuous in s, t and x; for every s and t $\phi_{st}(\omega, \cdot)$ is a homeomorphism; and for every x in M,

$$\phi_{ss}(\omega) = \mathrm{id}_M \text{ , and}$$
$$\phi_{st}(\omega, \phi_{rs}(\omega, x)) = \phi_{rt}(\omega, x).$$

One may think of ϕ_{st} as the map that carries a point at time s to its new position at time t, according to a random rule given by ω. A field of semimartingales $F(t, x)$ on $x \in \mathbb{R}^d$ for

$d \geq 2$ with bounded and Lipschitz local characteristics defines a stochastic flow on $\mathbb{R}^d$, by the equations

$$\phi_{st}(x) = x + \int_s^t F(du, \phi_{su}(x)), \text{ and}$$
$$\phi_{ts}(x) = \phi_{st}^{-1}(x)$$

for $s \leq t$. (We will sometimes write ϕ_t for ϕ_{0t}, where there seems no danger of confusion.) The filtration is denoted by $\mathcal{F}_t$, the σ-algebra generated by the set of random variables $\{F(s,x) : s \leq t, x \in \mathbb{R}^d\}$. Similar flows have been proposed as models for the spread of a passive tracer in complex fluid flows, such as the ocean [Dav91] and groundwater filtering through pores [Dag90]. The physical basis is the Kolmogorov model of turbulence (see, for instance, [GK95] and [Mol96]).

Isotropic Brownian flows (IBFs) were first described by K. Itô [Itô56] and A. Yaglom [Yag57]. This is a class of flows for which the image of any single point is a Brownian motion, and for which the covariance tensor between two different Brownian motions is an isotropic function of their positions. The local structure of these flows was studied in the 1980s by T. Harris [Har81] for the case of $\mathbb{R}^2$, and more generally by P. Baxendale and T. Harris [BH86] and by Y. Le Jan [Le 85], among others. Le Jan computed the Lyapunov exponents in [Le 85]; Baxendale performed a similar computation in [Bax83], and, as S. Molchanov has pointed out (personal communciation and [MR94]), the result in fact originated with E. Dynkin in the early 1960s, though this remained long unrecognized in the West. Le Jan and M. Cranston have gone a long way toward analyzing the growth of the local curvature, in [CL98] and [CL99].

In contrast to this fairly tractable local behavior, the global behavior of stochastic flows has largely eluded understanding. For the more "realistic" models for fluid flow, which admit a wider range of temporal correlations, even the two-point motion is as yet poorly understood. As a consequence, computer simulations have entrained considerable attention. R. Carmona, who has led the simulation efforts, was inspired by the data several years ago to a seemingly odd conjecture: the image of a compact set, he suggested, could expand linearly in time — the diameter is meant here — but no faster. (Carmona, personal communcation. The conjecture appears in section 5.2 of [CC99], where it is variously attributed as well to Ya. Sinai, and to M. Isichenko [Isi92].) What may be surprising here is presumably not the upper limit, but that such rapid growth should be attained at all. Without drift, the path of any point is a diffusion — or, in the more general models studied by Carmona, approximates a diffusion over long times — and so grows on average like the square-root of the time. A linear bound seems almost absurdly conservative. With a bounded drift the growth still can be no more than linear. But this glib answer ignores the fact that the maximum of a random function can be much larger than the image of a generic point. The square-root growth rate applies patently to the images of any finite set of starting points, but not to an infinite set. In their paper, Carmona and Cerou point out that such a discrepancy between the behavior of finite sets of points and that of continuous sets might be an obstacle to understanding the global behavior of stochastic flows through simulation.

For stochastic flows, this conjecture was largely proved by the present authors in 1998. In our earlier paper [CSS98] we showed that, when F is a martingale, the fastest point in the flow almost surely travels at most a bounded linear distance in time. That is, there is a constant K, a simple function of the bounds on the local characteristics, such that

$$\limsup_{T\to\infty} \sup_{\|x\|\leq 1} \sup_{0\leq t\leq T} \frac{\|\phi_t(x)\|}{T} \leq K \tag{1}$$

almost surely. Also, for any $\alpha > 1$ and $\gamma > 0$,

$$\sup_{T>1} \mathrm{E}\, \psi_{\gamma,2\alpha}\bigl(T^{-1} \sup_{\|x\|\le 1} \sup_{0\le t\le T} \|\phi_t(x)\|\bigr) < \infty, \tag{2}$$

where $\psi_{\gamma,\alpha}(z) = \exp\{\gamma z^2/(1 \vee \log z)^\alpha\}$. If F is not a martingale, but is allowed to have a drift component, the rate may go up to $T \log^\alpha T$, where $\alpha > 1$. We also gave an example of a flow which actually does exhibit linear growth of the fastest point. This example does have a drift in the Stratonovich sense built in, though, and it seems in many respects somewhat contrived. Here we repair this deficiency, by showing that linear growth may also be found in isotropic Brownian flows (IBFs). The main result of this paper is that a positive Lyapunov exponent alone is enough to guarantee the peculiar phenomenon of almost-sure linear growth. Theorem 1 states that if $\mathcal{X}$ is any nontrivial connected subset of $\mathbb{R}^d$ — that is, a set with more than one point — and ϕ_t an IBF with a positive Lyapunov exponent, then for any given vector there almost surely exist points which move with linear speed in that direction. That is, there is a minimum linear speed $a > 0$ such that with probability 1

$$\inf_{v\in S^{d-1}} \sup_{x\in\mathcal{X}} \liminf_{t\to\infty} \frac{\langle \phi_t(x), v\rangle}{t} \ge a.$$

In particular, volume preserving flows, which are of especial interest for applications to fluid dynamics, always have a positive Lyapunov exponent. Here we are faced with the surprising fact that while the image of a compact set has constant volume, individual points run out very far from the main body of the image. It is also known — see Baxendale and Harris [BH86] — that the lengths of curves grow exponentially quickly.

2. Definitions, and an outline of the method

We begin with the trivial observation that the "reason" why a single Brownian motion moves with square-root speed is the mutual cancellation of successive steps: to first order, there are about as many steps forward as backward. The maximum over a connected set, though, under an expanding (positive Lyapunov exponent) Brownian flow, is in a sense "sticky", somewhat like the passive transport mechanism of a cell membrane: not forcing the pace of motion, but inhibiting the randomly driven gains from being lost. The image of a connected set leaves pieces behind in its wake every time it backs up on itself, because of the homeomorphism property.

We repeat here the basic definitions and properties of IBFs from Baxendale and Harris [BH86] and Le Jan [Le 85]. An IBF is a stochastic flow such that for every $s_1 \le t_1 \le s_2 \le t_2 \le \cdots$ the random maps $\phi_{s_1 t_1}, \phi_{s_2 t_2}, \ldots$ are independent, and such that the covariance tensor is isotropic. That is,

$$b^{pq}(x) = \lim_{t\to 0} t^{-1}\, \mathrm{E}\bigl(\phi_t(y+x) - y - x\bigr)^p \bigl(\phi_t(y) - y\bigr)^q$$

(here the superscripts p and q represent choices of p-th and q-th coordinates) satisfies the identity $b(x) = U^* b(Ux) U$ for every real orthogonal matrix U. This means (see, for instance, [BH86]) that there are scalar functions B_L and B_N from $\mathbb{R}^+$ to $\mathbb{R}$ such that

$$\begin{aligned} b^{pq}(x) &= \bigl(B_L(\|x\|) - B_N(\|x\|)\bigr) x^p x^q/\|x\|^2 + B_N(\|x\|)\delta^{pq}, \quad x \ne 0,\\ b^{pq}(0) &= \delta^{pq} B_L(0) = \delta^{pq} B_N(0). \end{aligned} \tag{3}$$

Here b^{pq} is assumed to be not identically constant and to have continuous partial derivatives of orders up to 4. It is also true (remark (2.18) of [BH86]) that $|B_L(r)| < B_L(0)$ and $|B_N(r)| < B_N(0)$ for $r > 0$, and that these inequalities hold uniformly on the complement of any neighborhood of the origin.

If $x, y \in \mathbb{R}^d$ then the difference process $z_t = \phi_t(x) - \phi_t(y)$ is a diffusion, with generator

$$\mathcal{L}f(z) = \sum_{p=1}^{d}\sum_{q=1}^{d}\big(b^{pq}(0) - b^{pq}(z)\big)\frac{\partial^2 f}{\partial z^p \partial z^q} \tag{4}$$

for $f \in C_b^2(\mathbb{R}^d)$. Because of the isotropy, the distance function $\rho_t = \|z_t\|$ is itself a one-dimensional diffusion. Its generator is

$$\mathcal{A}g(\rho) = \big(B_L(0) - B_L(\rho)\big)g''(\rho) + (d-1)\left(\frac{B_N(0) - B_N(\rho)}{\rho}\right)g'(\rho) \tag{5}$$

for $g \in C_b^2(\mathbb{R})$. The local behavior of the flow is determined by two nonnegative numbers, $\beta_L := -B_L''(0)$ and $\beta_N := -B_N''(0)$. The i-th Lyapunov exponent of the flow is then

$$\lambda_i := \frac{1}{2}\big[(d-i)\beta_N - i\beta_L\big]$$

for $i = 1, \ldots, d$.

We assume now that the IBF ϕ_t has a positive Lyapunov exponent. We begin with a connected set $\mathcal{X} \subset \mathbb{R}^d$, containing at least two points. Let v be a fixed unit vector, and assume without loss of generality that the diameter of $\mathcal{X}$ is at least 1. (If not, we simply wait until the diameter does reach 1, and begin the argument from there. By Corollary 3.16 of [BH86] this must occur within a finite time.)

The first aim is to show that $\sup_{x\in\mathcal{X}}\langle\phi_t(x), v\rangle$ grows linearly in time. After that, we need to see that some linear growth is actually achieved by a single (random) point for all times t. Let $x_t = \phi_t(x)$ and $y_t = \phi_t(y)$, until such a time as x_t and y_t have separated by a distance of 2 in the v direction. Suppose that x_t is in front; that is, $\langle x_t, v\rangle = \langle y_t, v\rangle + 2$. By continuity, there must be another point in the image of ϕ_t whose absolute separation from x_t is only 1. We drop the old point y, and henceforth let y_t be the image of this new point. The process continues until again the separation has grown to 2, and then we again pull up the straggler to a distance 1, and so on forever. Since the separation in the v direction is no greater than the absolute separation, every time the separation becomes 2, the sum $x_t + y_t$ gets a kick of size at least 1 in the positive v direction. Essentially, the only thing that needs to be proved is that these kicks come often enough; that is, that the number of kicks up to time T divided by T has a positive limes inferior. It follows that there are times at which this linear average speed from the origin is attained, and a supplemental argument shows that the points will not fall back too far in the intervening epoch, which means that the linear speed is indeed maintained forever. With a bit of additional care we may also guarantee that the point there actually exists a single point which has moved with this speed the entire time.

3. Hitting times for diffusions

We need to show that the time between kicks, in the procedure defined above, has finite expectation. Intuitively this seems straightforward: The positive Lyapunov exponent implies that points close enough together repel one another. Thus for any positive ϵ, excursions of the separation process $z_t = x_t - y_t$ into the sphere of radius ϵ around the origin have finite expected duration. Each time z_t returns to the unit sphere, it has a chance of reaching the target set $\langle z_t, v\rangle = \pm 2$, so it is reasonable to expect that this too will occur in finite expected time. That is, the only alternative to the diffusion z_t leaving the slab $-2 \le \langle z_t, v\rangle \le 2$ in finite expected time, as would a Brownian motion, is that it plummets into the origin; this latter behavior is excluded by the positive Lyapunov exponent.

If we were concerned with the total distance $\|z_t\|$, the problem would be straightforward. This is a one-dimensional diffusion, so can be described by a speed function and a scale measure. These may be readily computed — see [BH86] — and tell us directly what points communicate with one another, and how long it takes. The separation in one fixed direction, however, $\langle x_t - y_t, v\rangle$, is not a Markov process, and so the question of how long it takes to reach a given level cannot be resolved quite so simply.

Let ϕ_t be an IBF in $\mathbb{R}^d$, let x, y be any two distinct points in $\mathbb{R}^d$, $x_t := \phi_t(x)$ and $y_t := \phi_t(y)$. Also let $z_t = x_t - y_t$. For a fixed unit vector v, we define

$$\tau_v(x,y) = \inf\bigl\{t : \bigl|\langle z_t, v\rangle\bigr| \geq 2\bigr\}.$$

Lemma 3.1. *If the IBF (ϕ_t) has a positive Lyapunov exponent, then for μ sufficiently small*

$$\sup_{\|x-y\|=1} \mathrm{E}\, e^{\mu^2 \tau_v(x,y)}$$

is finite. In particular, a_γ given by (8) is positive for γ sufficiently large, and

$$\sup_{\|x-y\|=1} \mathrm{E}\,\tau_v(x,y) \leq \inf\left\{\frac{4\gamma}{a_\gamma} \;:\; a_\gamma > 0\,,\, \gamma > 0.1\right\}.$$

Also,

$$\inf_{\|x-y\|=1} \mathrm{E}\,\tau_v(x,y) \geq \frac{3}{4B_N(0)}.$$

Proof. Let x and y be chosen, with $\|x-y\| = \|z_0\| = 1$. By isotropy, we may assume that v is the direction of the first coordinate axis. Define $f_1 : \mathbb{R}^d \to \mathbb{R}^+$ by $f_1(z) = \langle z, v\rangle^2$, and define $f_2 : \mathbb{R}^+ \to \mathbb{R}$ by

$$f_2(r) = \begin{cases} \log r & \text{if } 0 < r \leq 1; \\ \frac{1}{2}(r-1)(r-2)^3(-5r+3) & \text{if } 1 < r \leq 2; \\ 0 & \text{if } r > 2. \end{cases}$$

It is easy to check that f_2 is in the class C^2. Let γ be a positive parameter, and

$$f(z) := \gamma f_1(z) + f_2(\|z\|).$$

A simple calculation shows that $\tau := \tau_v(x,y)$ is the first time that $f(z_t) \geq 4\gamma$, as long as $\gamma > 0.1$. To prove that this happens in finite expected time, we will show that for γ sufficiently large $\mathcal{L}f$ is positive and bounded away from zero.

Using the formula (4) we see that

$$\begin{aligned} \mathcal{L}f_1(z) &= 2\bigl(b^{11}(0) - b^{11}(z)\bigr) \\ &= 2\Bigl(B_N(0) - \bigl(\alpha B_L(\|z\|) + (1-\alpha)B_N(\|z\|)\bigr)\Bigr), \end{aligned}$$

where $\alpha = \langle z, v\rangle^2/\|z\|^2$. The important thing to observe is that $\mathcal{L}f_1$ is continuous, nonnegative, and bounded away from zero on the complement of any neighborhood of the origin. It is also bounded above by $4B_N(0)$.

For the other piece of f, (5) yields

$$\mathcal{A}f_2(\rho) = \begin{cases} \rho^{-2}\bigl\{-(B_L(0) - B_L(\rho)) + (d-1)(B_N(0) - B_N(\rho))\bigr\} & \text{if } 0 < \rho < 1; \\ \quad 0 & \text{if } \rho \geq 2. \end{cases}$$

For $\rho \in [1,2]$, $\mathcal{A}f_2$ is bounded.

Following [BH86], we may write

$$B_L(\rho) = B_L(0) - \frac{1}{2}\beta_L\rho^2 + O(\rho^4), \tag{6}$$

$$B_N(\rho) = B_N(0) - \frac{1}{2}\beta_N\rho^2 + O(\rho^4) \tag{7}$$

as $\rho \to 0$, which means that for small ρ,

$$\begin{aligned}\mathcal{A}f_2(\rho) &= \frac{1}{2}\big((d-1)\beta_N - \beta_L\big) + O(\rho^2)\\ &= \lambda_1 + O(\rho^2).\end{aligned}$$

We have assumed that λ_1 (the largest Lyapunov exponent) is positive. This means that the lower bound on $\mathcal{L}f(z)$

$$a_\gamma := \inf\big\{2\gamma\left(B_N(0) - \max\big\{B_L(\rho), B_N(\rho)\big\}\right) + \mathcal{A}f_2(\rho)\ :\ \rho > 0\big\} \tag{8}$$

is positive for γ sufficiently large.

Now, it is in the nature of a generator (see chapter 6 of Stroock and Varadhan [SV79]) that the stochastic process

$$M_t = f(z_t) - \int_0^t \mathcal{L}f(z_s)ds \tag{9}$$

is a local martingale. This means that $M_{t\wedge\tau}$ is a submartingale, since it is bounded above (cf. problem 5.19 of Karatzas and Shreve [KS88]). For any positive T,

$$\begin{aligned}f(z_0) &\le \mathrm{E}\, f(z_{\tau\wedge T}) - \mathrm{E}\int_0^{\tau\wedge T}\mathcal{L}f(z_s)ds\\ &\le 4\gamma - a_\gamma\,\mathrm{E}\big[\tau\wedge T\big].\end{aligned}$$

In the last line we use the fact that for $s \le \tau$, $f(z_s) \le 4\gamma$ (since τ is the first time that this level is crossed), and $\mathcal{L}f(z_s) \ge a_\gamma$. By the monotone convergence theorem, this yields the bound

$$\mathrm{E}\,\tau \le \frac{4\gamma - f(z_0)}{a_\gamma} \le \frac{4\gamma}{a_\gamma}. \tag{10}$$

Now fix $\gamma > 0.1$, so that a_γ is positive. To bound the exponential moment of τ, we need to consider the behavior of $e^{-\mu f(z)}$ for μ a positive parameter, to be determined later. The generator $\mathcal{L}$ applied to this function yields

$$e^{\mu f(z)}\mathcal{L}e^{-\mu f(z)} = -\mu\mathcal{L}f(z) + \mu^2\sum_{i,j=1}^{d}\big(b^{ij}(0) - b^{ij}(z)\big)\frac{\partial f(z)}{\partial z_i}\frac{\partial f(z)}{\partial z_j}.$$

We want to show that the extra term on the right-hand side is bounded on the set of z such that $|\langle z, v\rangle| \le 2$. The crucial components are those arising from $f_2(\|z\|)$ for $\|z\|$ close to 0. There we have

$$\frac{\partial f_2(\|z\|)}{\partial z_i}\frac{\partial f_2(\|z\|)}{\partial z_j} = \frac{z_i z_j}{\|z\|^4}.$$

The asymptotic behavior of $b^{ij}(z)$ then implies that the product

$$\big|b^{ij}(0) - b^{ij}(z)\big| \cdot \left|\frac{\partial f_2(z)}{\partial z_i}\frac{\partial f_2(z)}{\partial z_j}\right|$$

is bounded. The partials of f_1 are themselves bounded, so raise no further difficulties. The upshot is that for μ sufficiently small (but positive), $e^{\mu f(z)}\mathcal{L}e^{-\mu f(z)} \le -\mu^2$ for all z with $\big|\langle z,v\rangle\big| \le 2$. Defining

$$V_t := \exp\bigl\{-\mu f(z_t) + \mu^2 t\bigr\},$$

by the Stroock-Varadhan theory mentioned earlier,

$$\tilde{V}_t := V_t - \int_0^t \bigl\{e^{\mu^2 s}\mathcal{L}e^{-\mu f(z_s)} + e^{-\mu f(z_s)}\frac{\partial}{\partial s}e^{\mu^2 s}\bigr\}ds \quad = V_t - \int_0^t V_s\bigl(e^{\mu f(z_s)}\mathcal{L}e^{-\mu f(z_s)} + \mu^2\bigr)ds$$

is a local martingale. For $s \le \tau$ the integrand is negative, which means that $V_{t\wedge\tau}$ is a supermartingale. Thus for all positive t, since $f(z)$ is bounded above by $1+4\gamma$ whenever $|\langle z,v\rangle| \le 2$, and $\|z_0\| = 1$,

$$\begin{aligned}
1 &\ge e^{-\mu f(z_0)}\\
&\ge \mathrm{E}\, V_{t\wedge\tau}\\
&\ge e^{-\mu(4\gamma+1)}\,\mathrm{E}\, e^{\mu^2 t\wedge\tau}.
\end{aligned}$$

By monotone convergence, it follows that

$$\mathrm{E}\, e^{\mu^2\tau} \le e^{\mu(4\gamma+1)}.$$

To obtain the lower bound, we simply apply the same procedure to f_1 alone. Since f_1 and $\mathcal{L}f_1$ are bounded up to time τ,

$$M_t = f_1(z_{t\wedge\tau}) - \int_0^{t\wedge\tau}\mathcal{L}f_1(z_s)ds$$

is a martingale. This means that

$$\begin{aligned}
\mathrm{E}\,\tau\cdot 4B_N(0) &\ge \mathrm{E}\bigl[\tau\sup\bigl\{\mathcal{L}f_1(z)\ :\ |\langle z,v\rangle| \le 2\bigr\}\bigr]\\
&\ge \mathrm{E}\int_0^\tau \mathcal{L}f_1(z_s)ds\\
&= \mathrm{E}\, f_1(z_\tau) - \mathrm{E}\, M_0\\
&\ge 3,
\end{aligned}$$

since $M_0 \le 1$ and $f_1(z_\tau) = 4$.

□

4. The main result

Theorem 1. *Let ϕ_t be an IBF on $\mathbb{R}^d$ with a positive Lyapunov exponent, and $\mathcal{X}$ a connected subset of $\mathbb{R}^d$ containing at least two points. Then*

$$\inf_{v\in S^{d-1}}\sup_{x\in\mathcal{X}}\liminf_{t\to\infty}\frac{\langle\phi_t(x),v\rangle}{t} \ge \sup\left\{\frac{a_\gamma}{4\gamma} : \gamma > 0.1\right\} > 0.$$

almost surely.

Proof. We assume without loss of generality that the diameter of $\mathcal{X}$ is at least 1, and choose x_0 and y_0 in $\mathcal{X}$ with $\|x_0 - y_0\| = 1$. We may also assume, perhaps taking a subset, that $\mathcal{X}$ is compact.

Choose a unit vector v. We define a random sequence of points $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ in $\mathcal{X}$, and an increasing sequence of stopping times $0 = T_0 < T_1 < T_2 < \cdots$, which will represent the times at which the points have spread “sufficiently” in the v direction. Leaving these for the moment undefined, we fix the definitions

$$\begin{aligned} x_t &= \phi_t(x_i) \quad \text{for } t \in [T_i, T_{i+1}), \\ y_t &= \phi_t(y_i) \quad \text{for } t \in [T_i, T_{i+1}), \\ Z_t &= \lim_{s \uparrow t} \bigl(x_s - y_s \bigr) . \end{aligned}$$

We also define a sequence of compact connected subsets $\mathcal{X} = \mathcal{X}_0 \supset \mathcal{X}_1 \supset \mathcal{X}_2 \supset \cdots$. The idea is that we wait until one of the points has fallen behind by 2 units in the v direction, and then use the connectedness to pull it up again to a distance 1. At the times T_i the pair that we are following gets a “kick” of at least one unit in the positive v direction. If the number of kicks up to time T is linear in T, then there will be some point that has travelled a long way in that direction. This still falls short of the goal, though, since this is a random point, and could be changing at every time T_i. This is why we use $\mathcal{X}_i$ to keep track of points that have kept up this pace the whole time.

Suppose that $T_{i-1}, x_{i-1}, y_{i-1}$, and $\mathcal{X}_{i-1}$ are defined, with

$$\|\phi_{T_{i-1}}(x_{i-1}) - \phi_{T_{i-1}}(y_{i-1})\| = 1,$$

and now define T_i to be $\inf\{t > T_{i-1} : |\langle Z_t, v\rangle| = 2\}$. By Lemma 3.1 this is finite almost surely. Assume first that $\langle Z_{T_i}, v\rangle = +2$, and consider that the map

$$\psi : z \mapsto \bigl\|\phi_{T_i}(x_{i-1}) - \phi_{T_i}(z)\bigr\|$$

is continuous from $\mathcal{X}_{i-1}$ to $\mathbb{R}$. (Here we make use of the fact that $\phi_t(z)$ is — almost surely — a continuous function of z.) Also, $\psi(x_{i-1}) = 0$ while $\psi(y_{i-1}) \geq 2$. Let $\mathcal{X}_i$ be the connected component of $\psi^{-1}([0,1])$ which contains x_{i-1}. Since $\mathcal{X}_{i-1}$ is connected, we may find a point $y_i \in \mathcal{X}_i$ such that $\psi(y_i) = 1$. We then define x_i to be equal to x_{i-1}.

If $\langle Z_{T_i}, v\rangle = -2$, we make the same construction, reversing the roles of x and y. Observe that, in either case,

$$\langle \phi_{T_i}(z), v\rangle \geq \max\bigl\{\langle x_{T_i}, v\rangle, \langle y_{T_i}, v\rangle\bigr\} - 2 \tag{11}$$

for all $z \in \mathcal{X}_i$. Then $\|x_{T_i} - y_{T_i}\| = 1$, providing the correct starting condition for the next round.

Now define σ_t to be the sum of those kicks that have occurred up to time t; that is,

$$\sigma_t = \sum_{T_i \leq t} \lim_{s \uparrow T_i} \bigl(x_{T_i} + y_{T_i} - x_s - y_s \bigr).$$

Since each kick has size at least one in the v direction, $\langle \sigma_t, v\rangle \geq \#\{i \geq 1 : T_i \leq t\}$. The homogeneity of the process implies that $x_t + y_t - \sigma_t$ is the sum of two Brownian motions, which in turn implies that

$$\liminf_{n\to\infty} T_n^{-1} \langle x_{T_n} + y_{T_n}, v\rangle = \liminf_{n\to\infty} \frac{\langle \sigma_{T_n}, v\rangle}{T_n} \geq \liminf_{n\to\infty} \frac{n}{T_n} \tag{12}$$

almost surely.

Trivially, the discrete-time process

$$\left(T_n - \sum_{i=1}^{n} \mathrm{E}\bigl[T_i - T_{i-1} \bigm| \mathcal{F}_{T_{i-1}} \bigr] \, , \, \mathcal{F}_{T_n} \right)_{n=1}^{\infty}$$

is a martingale. By the results of Lemma 3.1 and the strong Markov property for the diffusion z_t, the differences have bounded second moments, which allows us to apply the strong law of large numbers for martingales — for instance, Theorem 2.19 of [HH80]. Thus

$$\lim_{n\to\infty} n^{-1}T_n - n^{-1}\sum_{i=1}^{n} \mathrm{E}\big[T_i - T_{i-1} \,\big|\, \mathcal{F}_{T_{i-1}}\big] = 0.$$

almost surely. This implies in turn that for any $\gamma > .1$ with $a_\gamma > 0$,

$$\limsup_{n\to\infty} n^{-1}T_n = \limsup_{n\to\infty} n^{-1}\sum_{i=1}^{n} \mathrm{E}\big[T_i - T_{i-1} \,\big|\, \mathcal{F}_{T_{i-1}}\big] \le \frac{4\gamma}{a_\gamma} \tag{13}$$

and

$$\liminf_{n\to\infty} n^{-1}T_n \ge \frac{3}{4B_N(0)} \tag{14}$$

almost surely. By equation (12),

$$\liminf_{n\to\infty} T_n^{-1}\langle x_{T_n} + y_{T_n}, v\rangle \ge \frac{a_\gamma}{4\gamma}.$$

By (11), if $z \in \mathcal{X}_i$ then

$$\langle \phi_{T_i}(z), v\rangle \ge \langle x_{T_i}, v\rangle - 2.$$

Since $(\mathcal{X}_i)$ is a decreasing sequence of nonempty compact sets, their intersection is nonempty. If z is a point in $\bigcap_{i=1}^{\infty} \mathcal{X}_i$, then we certainly have

$$\limsup_{t\to\infty} t^{-1}\langle \phi_t(z), v\rangle \ge \liminf_{n\to\infty} T_n^{-1}\langle \phi_{T_n}(z), v\rangle \ge \liminf_{n\to\infty} T_n^{-1}\big(\langle x_{T_n}, v\rangle - 2\big) \ge \frac{a_\gamma}{8\gamma}.$$

We would like to turn this into a statement about the limes inferior as $t \to \infty$. To do this, we use the obvious bound

$$\liminf_{t\to\infty} t^{-1}\langle \phi_t(z), v\rangle \ge \liminf_{n\to\infty} T_n^{-1}\langle x_{T_n}, v\rangle - \limsup_{n\to\infty} T_n^{-1} \sup_{T_n < t < T_{n+1}} \big\|\phi_t(z) - x_{T_n}\big\|.$$

This means that we need to show that the second term on the right is zero almost surely. Let $c = 3/4B_N(0)$ and ϵ positive. By definition of $\mathcal{X}_n$, the point $\phi_{T_n}(z)$ must be within one unit of either x_{T_n} or y_{T_n}. Since these are only one unit apart, the event

$$\Big\{T_n^{-1} \sup_{T_n < t < T_{n+1}} \big\|\phi_t(z) - x_{T_n}\big\| \ge \epsilon\Big\} \tag{15}$$

is contained in the union of A_n, B_n, and $C_{n,\epsilon}$, where

$$\begin{aligned} A_n &= \big\{T_n \le cn/2\big\}, \\ B_n &= \big\{T_{n+1} - T_n > \sqrt{n}\big\}, \\ C_{n,\epsilon} &= \big\{ \sup_{y:\|y - x_{T_n}\| \le 2} \ \sup_{T_n \le t \le T_n + \sqrt{n}} \|\phi_{T_n,t}(y) - x_{T_n}\| \ge \epsilon cn/2\big\}. \end{aligned}$$

We already know by (14) that A_n almost surely occurs only finitely often. By Lemma 3.1 there is a uniform constant k such that

$$\mathrm{E}\, e^{\mu^2 (T_{n+1} - T_n)} < k,$$

which means that $\mathrm{P}(B_n) \le k e^{-\mu^2\sqrt{n}}$. By the Borel-Cantelli Lemma, B_n occurs only finitely often, with probability 1. Finally, from (2) we know that there exists a K such that for all $T \ge 1$,

$$\mathrm{E}\exp\big\{T^{-1} \sup_{0\le t\le T} \sup_{\|z\|\le 2} \|\phi_t(z)\|\big\} \le K.$$

By isotropy and the strong Markov property (since T_n is a stopping time and x_{T_n} is $\mathcal{F}_{T_n}$-measurable), this implies that

$$\mathrm{P}(C_{n,\epsilon}) \le K e^{-\epsilon c\sqrt{n}/2}.$$

So again, by Borel-Cantelli, we see that $C_{n,\epsilon}$ almost surely occurs only finitely often. This means that the event (15) also occurs only finitely often, so that

$$\limsup_{n\to\infty} T_n^{-1} \sup_{T_n<t<T_{n+1}} \left\|\phi_t(z) - x_{T_n}\right\| < \epsilon$$

almost surely. Since this holds for every positive ϵ, the limit must in fact be zero.
What is true almost surely for any fixed v is also true almost surely for a dense countable subset $\mathcal{S}$ of S^{d-1}. If $\epsilon > 0$ and $v \in S^{d-1}$, there is $v^* \in \mathcal{S}$ with $\|v - v^*\| < \epsilon$. There almost surely exists a random $x \in \mathcal{X}$ such that

$$\begin{aligned}\liminf_{t\to\infty} t^{-1}\langle\phi_t(x), v\rangle &\ge \liminf_{t\to\infty} t^{-1}\Big(\langle\phi_t(x), v^*\rangle - \epsilon\|\phi_t(x)\|\Big)\\ &\ge \frac{a_\gamma}{4\gamma} - \epsilon \sup_{x'\in\mathcal{X}} \limsup_{t\to\infty} t^{-1}\|\phi_t(x')\|.\end{aligned}$$

By (1) we know then that the coefficient of ϵ is bounded almost surely by a constant, which completes the proof. □

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