

ESTIMATES FOR THE DERIVATIVE OF DIFFUSION SEMIGROUPS

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Abstract

Let $\{P_t\}_{t \geq 0}$ be the transition semigroup of a diffusion process. It is known that P_t sends continuous functions into differentiable functions so we can write $DP_t f$. But what happens with this derivative when $t \rightarrow 0$ and $P_0 f = f$ is only continuous? We give estimates for the supremum norm of the Fréchet derivative of the semigroups associated with the operators $\mathcal{A} + V$ and $\mathcal{A} + Z \cdot \nabla$ where \mathcal{A} is the generator of a diffusion process, V is a potential and Z is a vector field.

1 Introduction

Consider the following stochastic differential equation on \mathbb{R}^n

$$\begin{aligned} dX_t &= \mathbb{X}(X_t) dB_t + \mathbb{A}(X_t) dt, \\ X_0 &= x \in \mathbb{R}^n, \end{aligned}$$

for $t \geq 0$, where the first integral is an Itô stochastic integral and the second is a Riemann integral. Here $\{B_t\}_{t \geq 0}$ is Brownian motion on \mathbb{R}^m and the equality holds almost everywhere. The coefficients of this equation are the mapping $\mathbb{X}: \mathbb{R}^n \rightarrow \mathbb{L}(\mathbb{R}^m; \mathbb{R}^n)$ and the vector field $\mathbb{A}: \mathbb{R}^n \rightarrow \mathbb{R}^n$. Assume standard regularity conditions on these coefficients so that there exists a strong solution $\{X_t\}_{t \geq 0}$ to our equation. We write X_t^x for X_t when we want to make clear its dependence on the initial value x . It is known that under further assumptions on the coefficients of our equation, the mapping $x \mapsto X_t^x$ is differentiable (see for instance [1]).

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Assume coefficients \mathbb{X} and \mathbb{A} are smooth enough and consider the associated derivative equation

$$\begin{aligned} dV_t &= D\mathbb{X}(X_t)(V_t) dB_t + DA(X_t)(V_t) dt, \\ V_0 &= v \in \mathbb{R}^n, \end{aligned}$$

whose solution $\{V_t\}_{t \geq 0}$ is the derivative of the mapping $x \mapsto X_t^x$ at x in the direction v . We will assume that there exists a smooth map $\mathbb{Y}: \mathbb{R}^n \rightarrow \mathbb{L}(\mathbb{R}^n; \mathbb{R}^m)$ such that $\mathbb{Y}(x)$ is the right inverse of $\mathbb{X}(x)$. That is, $\mathbb{X}(x)\mathbb{Y}(x) = I_{\mathbb{R}^n}$ for all x in \mathbb{R}^n . We shall also assume that the process $\{\mathbb{Y}(X_t)(V_t)\}_{t \geq 0}$ belongs to $L^2([0, t])$ for each $t > 0$, that is, $\int_0^t |\mathbb{Y}(X_s)(V_s)|^2 ds < \infty$ and thus we can write $\int_0^t \mathbb{Y}(X_s)(V_s) dB_s$.

With all above assumptions, we have from [6], the following result (see Appendix for a proof)

Theorem 1 *For every $t > 0$ there exist positive constants k and a such that*

$$\mathbb{E} \left| \int_0^t \mathbb{Y}(X_s)(V_s) dB_s \right| \leq k\sqrt{e^{at} - 1}. \quad (1)$$

We are interested in small values for t . Thus, since $\sqrt{e^{at} - 1} = O(\sqrt{t})$ as $t \rightarrow 0$, we have that there exist a positive constant N such that $\sqrt{e^{at} - 1} \leq N\sqrt{t}$ for small t . Hence for sufficiently small t , we have the following estimate

$$\mathbb{E} \left| \int_0^t \mathbb{Y}(X_s)(V_s) dB_s \right| \leq c\sqrt{t}, \quad (2)$$

where c is a positive constant.

Let now $BC^r(\mathbb{R}^n)$ be the Banach space of bounded measurable functions on \mathbb{R}^n which are r -times continuously differentiable with bounded derivatives. The norm of this space is given by the supremum norm of the function plus the supremum norm of each of its r derivatives. In particular $B(\mathbb{R}^n)$ is the Banach space of bounded measurable functions on \mathbb{R}^n with supremum norm $\|f\|_\infty = \sup_{x \in \mathbb{R}^n} |f(x)|$. Suppose our diffusion process $\{X_t\}_{t \geq 0}$ has transition probabilities $P(t, x, \Gamma)$. Then this induces a semigroup of operators $\{P_t\}_{t \geq 0}$ as follows. For every $t \geq 0$ we define on $B(\mathbb{R}^n)$ the bounded linear operator

$$(P_t f)(x) = \int_{\mathbb{R}^n} f(y) P(t, x, dy) = \mathbb{E}(f(X_t^x)). \quad (3)$$

The semigroup $\{P_t\}_{t \geq 0}$ is a strongly continuous semigroup on $BC^0(\mathbb{R}^n)$. Denote by \mathcal{A} its infinitesimal generator.

It is known that $\{P_t\}_{t \geq 0}$ is a strong Feller semigroup, that is, P_t sends continuous functions into differentiable functions. In fact, under above assumptions, a formula for the derivative of $P_t f$ is known (see [4] or [5]).

Theorem 2 *If $f \in BC^2(\mathbb{R}^n)$ then the derivative of $P_t f: \mathbb{R}^n \rightarrow \mathbb{R}$ is given by*

$$D(P_t f)(x)(v) = \frac{1}{t} \mathbb{E} \left\{ f(X_t) \int_0^t \mathbb{Y}(X_s)(V_s) dB_s \right\}. \quad (4)$$

Higher derivatives in a more general setting are given in [4]. See also [2] for a general formula of this derivative in the context of a stochastic control system. Observe the mapping $f \mapsto \frac{1}{t} \mathbb{E}\{f(X_t) \int_0^t \mathbb{Y}(X_s)(V_s) dB_s\}$ defines a bounded linear functional on $BC^2(\mathbb{R}^n)$. Hence there exists a unique extension on $BC^0(\mathbb{R}^n)$. Since the expression of this linear functional does not depend on the derivatives of f , it has the same expression for any f in $BC^0(\mathbb{R}^n)$.

From last theorem we obtain

$$\begin{aligned} |DP_t f(x)(v)| &\leq \frac{1}{t} \|f\|_\infty \mathbb{E} \left| \int_0^t \mathbb{Y}(X_s)(V_s) dB_s \right| \\ &\leq \frac{1}{t} \|f\|_\infty k \sqrt{e^{at} - 1}. \end{aligned}$$

And hence for small t

$$\|DP_t f\|_\infty \leq \frac{c \|f\|_\infty}{\sqrt{t}}. \quad (5)$$

Observe that, as expected, our estimate goes to infinity as t approaches 0 since $P_0 f = f$ is not necessarily differentiable. Also the rate at which it goes to infinity is not faster than $\frac{1}{\sqrt{t}}$ does.

2 Potential

Let $V: \mathbb{R}^n \rightarrow \mathbb{R}$ be a bounded measurable function. We shall perturb the generator \mathcal{A} by adding to it the function V . We define the linear operator

$$\mathcal{A}^V = \mathcal{A} + V,$$

with the same domain as for \mathcal{A} . A semigroup $\{P_t^V\}_{t \geq 0}$ having \mathcal{A}^V as generator is given by the Feynman-Kac formula $P_t^V f = \mathbb{E}\{f(X_t) e^{\int_0^t V(X_u) du}\}$. We will find a similar estimate as (5) for $\|DP_t^V f\|_\infty$. We first derive a recursive formula that will help us calculate the derivative of $P_t^V f$. We have

$$\begin{aligned} P_t^V f &= P_t f + [P_{t-s} P_s^V f]_{s=0}^{s=t} \\ &= P_t f + \int_0^t \frac{\partial}{\partial s} (P_{t-s} P_s^V f) ds \\ &= P_t f + \int_0^t [-\mathcal{A}(P_{t-s} P_s^V f) + P_{t-s} ((\mathcal{A} + V) P_s^V f)] ds. \end{aligned}$$

Hence

$$P_t^V f = P_t f + \int_0^t P_{t-s} (V P_s^V f) ds.$$

Now we use our formula for differentiation (4) to calculate the derivative of this semigroup. We have

$$\begin{aligned} DP_t^V f(x)(v) &= DP_t f(x)(v) + \int_0^t DP_{t-s}(VP_s^V f)(x)(v) ds \\ &= \frac{1}{t} \mathbb{E}\{f(X_t) \int_0^t \mathbb{Y}(X_s)(V_s) dB_s\} \\ &\quad + \int_0^t \frac{1}{t-s} \mathbb{E}\{V(X_{t-s}) P_s^V f(X_{t-s}) \int_0^{t-s} \mathbb{Y}(X_u)(V_u) dB_u\} ds. \end{aligned}$$

Then, by the Feynman-Kac formula and the Markov property we have

$$\begin{aligned} DP_t^V f(x)(v) &= \frac{1}{t} \mathbb{E}\{f(X_t) \int_0^t \mathbb{Y}(X_s)(V_s) dB_s\} \\ &\quad + \int_0^t \frac{1}{t-s} \mathbb{E}\{V(X_{t-s}) \mathbb{E}\{f(X_t) e^{\int_{t-s}^t V(X_u) du}\} \int_0^{t-s} \mathbb{Y}(X_u)(V_u) dB_u\} ds, \end{aligned}$$

from which we obtain

$$\begin{aligned} \|DP_t^V f\|_\infty &\leq \frac{1}{t} \|f\|_\infty \mathbb{E} \left| \int_0^t \mathbb{Y}(X_s)(V_s) dB_s \right| \\ &\quad + \|V\|_\infty \|f\|_\infty \int_0^t \frac{e^{s\|V\|_\infty}}{t-s} \mathbb{E} \left| \int_0^{t-s} \mathbb{Y}(X_u)(V_u) dB_u \right| ds. \end{aligned}$$

And hence for small t

$$\begin{aligned} \|DP_t^V f\|_\infty &\leq \frac{c\|f\|_\infty}{\sqrt{t}} + \|V\|_\infty \|f\|_\infty e^{t\|V\|_\infty} \int_0^t \frac{c}{\sqrt{t-s}} ds \\ &= \frac{c\|f\|_\infty}{\sqrt{t}} + 2c\sqrt{t} \|V\|_\infty \|f\|_\infty e^{t\|V\|_\infty}. \end{aligned}$$

Observe again that our estimate goes to infinity as $t \rightarrow 0$.

3 Bounded Smooth Drift

Let $Z : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a bounded smooth vector field. We shall consider another perturbation to the generator \mathcal{A} . This time we define the linear operator

$$\mathcal{A}^Z = \mathcal{A} + Z \cdot \nabla.$$

The existence of a semigroup $\{P_t^Z\}_{t \geq 0}$ having \mathcal{A}^Z as infinitesimal generator is guaranteed by the regularity of Z . Indeed, if we write $Z(x) = (Z^1(x), \dots, Z^n(x))$, then the operator \mathcal{A}^Z can be written as

$$\mathcal{A}^Z = \frac{1}{2} \sum_{i,j=1}^n (\mathbb{X}(x) \mathbb{X}(x)^*)^{ij} \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^n (\mathbb{A}^i(x) + Z^i(x)) \frac{\partial}{\partial x^i},$$

and this operator is the infinitesimal generator associated with the equation

$$\begin{aligned} dX_t &= \mathbb{X}(X_t) dB_t + [\mathbb{A}(X_t) + Z(X_t)] dt, \\ X_0 &= x \in \mathbb{R}^n. \end{aligned}$$

Thanks to the smoothness of Z , this equation yields a diffusion process $(X_t^{x,Z})_{t \in T}$ and hence the semigroup $P_t^Z f(x) = \mathbb{E}(f(X_t^{x,Z}))$. Then previous estimate applies also to $\|DP_t^Z f\|_\infty$. But we can do better because we can find the explicit dependence of the estimate upon Z as follows. As before, we first find a recursive formula for this semigroup.

$$\begin{aligned} P_t^Z f &= P_t f + [P_{t-s} P_s^Z f]_{s=0}^{s=t} \\ &= P_t f + \int_0^t \frac{\partial}{\partial s} (P_{t-s} P_s^Z f) ds \\ &= P_t f + \int_0^t [-\mathcal{A}(P_{t-s} P_s^Z f) + P_{t-s} ((\mathcal{A} + Z \cdot \nabla) P_s^Z f)] ds. \end{aligned}$$

Hence

$$P_t^Z f = P_t f + \int_0^t P_{t-s} (Z \cdot \nabla P_s^Z f) ds. \quad (6)$$

We can now calculate its derivative as follows

$$\begin{aligned} DP_t^Z f(x)(v) &= DP_t f(x)(v) + \int_0^t DP_{t-s} (Z \cdot \nabla P_s^Z f)(x)(v) ds \\ &= \frac{1}{t} \mathbb{E} \{ f(X_t) \int_0^t \mathbb{Y}(X_s)(V_s) dB_s \} \\ &\quad + \int_0^t \frac{1}{t-s} \mathbb{E} \{ Z(X_{t-s}) \cdot \nabla P_s^Z f(X_{t-s}) \int_0^{t-s} \mathbb{Y}(X_u)(V_u) dB_u \} ds. \end{aligned}$$

We now find an estimate for the supremum norm of this derivative. Taking modulus we obtain

$$\begin{aligned} |DP_t^Z f(x)(v)| &\leq \frac{1}{t} \|f\|_\infty \mathbb{E} \left| \int_0^t \mathbb{Y}(X_s)(V_s) dB_s \right| \\ &\quad + \int_0^t \frac{1}{t-s} \mathbb{E} |DP_s^Z f(X_{t-s})(Z(X_{t-s}))| \left| \int_0^{t-s} \mathbb{Y}(X_u)(V_u) dB_u \right| ds. \end{aligned}$$

Hence for small t

$$\|DP_t^Z f\|_\infty \leq \frac{c}{\sqrt{t}} \|f\|_\infty + c \|Z\|_\infty \int_0^t \frac{\|DP_s^Z f\|_\infty}{\sqrt{t-s}} ds.$$

We now solve this inequality. If we iterate once we obtain

$$\begin{aligned} \|DP_t^Z f\|_\infty &\leq \frac{c}{\sqrt{t}} \|f\|_\infty + c^2 \|Z\|_\infty \|f\|_\infty \int_0^t \frac{ds}{\sqrt{s(t-s)}} \\ &\quad + c^2 \|Z\|_\infty^2 \int_0^t \int_0^s \frac{\|DP_u^Z f\|_\infty}{\sqrt{(t-s)(s-u)}} du ds. \end{aligned}$$

By Fubini's theorem, the double integral becomes

$$\int_0^t \|DP_u^Z f\|_\infty \int_u^t \frac{ds}{\sqrt{(t-s)(s-u)}} du,$$

and then we observe that

$$\int_u^t \frac{ds}{\sqrt{(t-s)(s-u)}} = 2 \tan^{-1} \sqrt{\frac{s-u}{t-s}} \Big|_u^t = \pi.$$

The case $u = 0$ solves also the first integral. Hence our inequality reduces to

$$\|DP_t^Z f\|_\infty \leq \frac{c}{\sqrt{t}} \|f\|_\infty + c^2 \pi \|Z\|_\infty \|f\|_\infty + c^2 \pi \|Z\|_\infty^2 \int_0^t \|DP_u^Z f\|_\infty du.$$

We now apply Gronwall's inequality. After some simplifications (extending the integral up to infinity) we finally obtain the estimate

$$\|DP_t^Z f\|_\infty \leq \frac{c}{\sqrt{t}} \|f\|_\infty + 2c^2 \pi \|Z\|_\infty \|f\|_\infty e^{t(c^2 \pi \|Z\|_\infty)} \quad (7)$$

As expected, our estimate goes to infinity as $t \rightarrow 0$ since $P_0^Z f = f$ is not necessarily differentiable.

4 Bounded Uniformly Continuous Drift

We now find a similar estimate when $Z: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is only bounded and uniformly continuous. We look again at the operator $\mathcal{A}^Z = \mathcal{A} + Z \cdot \nabla$. The problem here is that in this case we do not have the semigroup $\{P_t^Z f\}_{t \geq 0}$ since the stochastic equation with the added nonsmooth drift Z might not have a strong solution. So we cannot even talk about its derivative. To solve this problem we proceed by approximation.

4.1 Existence of Semigroup

Since $Z \in BC^0(\mathbb{R}^n; \mathbb{R}^n)$ is uniformly continuous, and $BC^\infty(\mathbb{R}^n; \mathbb{R}^n)$ is dense in $BC^0(\mathbb{R}^n; \mathbb{R}^n)$, there exists a sequence $\{Z_i\}_{i=1}^\infty$ in $BC^\infty(\mathbb{R}^n; \mathbb{R}^n)$ such that Z_i converges to Z uniformly. Thus, for every $i \in \mathbf{N}$, we have the semigroup $\{P_t^{Z_i}\}_{t \geq 0}$ since our stochastic equation with the added smooth drift Z_i has a strong solution.

For every $t \geq 0$ and $f \in BC^0(\mathbb{R}^n)$ fixed, the sequence of functions $\{P_t^{Z_i} f\}_{i=1}^\infty$ is a Cauchy sequence in the Banach space $BC^0(\mathbb{R}^n)$. We will prove this fact later. Let us denote its limit by $P_t^Z f$. All properties required for $P_t^Z f$ are inherited from those of the semigroup $P_t^{Z_i} f$. Indeed by simply writing $P_t^Z f = \lim_{i \rightarrow \infty} P_t^{Z_i} f$ and using an interchange of limits we can prove

1. $f \mapsto P_t^Z f$ is a bounded linear operator.
2. $t \mapsto P_t^Z f$ is a contraction semigroup of operators.
3. $\{P_t^Z\}_{t \geq 0}$ has generator $\mathcal{A}^Z = \mathcal{A} + Z \cdot \nabla$.

Now, suppose for a moment that the sequence of derivatives $\{DP_t^{Z_i} f\}_{i=1}^\infty$ converges uniformly. Then we would have $D(\lim_{i \rightarrow \infty} P_t^{Z_i} f) = \lim_{i \rightarrow \infty} DP_t^{Z_i} f$. This proves that $P_t^Z f$ is differentiable. Then, for any $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that if $i \geq N$, $\|DP_t^Z f\|_\infty \leq \|DP_t^{Z_i} f\|_\infty + \epsilon$ and therefore our estimate also applies to $\|DP_t^Z f\|_\infty$.

4.2 Uniform Convergence of Derivatives

We now prove that the sequence $\{DP_t^{Z_i} f\}_{i=1}^\infty$ converges uniformly. We use again our recursive formula (6) to obtain

$$P_t^{Z_i} f - P_t^{Z_j} f = \int_0^t P_{t-s}(Z_i \cdot \nabla P_s^{Z_i} f - Z_j \cdot \nabla P_s^{Z_j} f) ds.$$

We now use our formula for differentiation (4). After differentiating and taking modulus we obtain

$$\begin{aligned} & |D(P_t^{Z_i} f - P_t^{Z_j} f)(x)(v)| \\ & \leq \int_0^t \frac{1}{t-s} \mathbb{E}\{|DP_s^{Z_i} f(X_{t-s})(Z_i(X_{t-s})) - DP_s^{Z_j} f(X_{t-s})(Z_j(X_{t-s}))| \\ & \quad | \int_0^{t-s} \mathbb{Y}(X_u)(V_u) dB_u|\} ds. \end{aligned}$$

Thus

$$\begin{aligned} \|D(P_t^{Z_i} f - P_t^{Z_j} f)\|_\infty & \leq \|Z_i - Z_j\|_\infty \int_0^t \frac{k}{t-s} \sqrt{e^{a(t-s)} - 1} \|DP_s^{Z_i} f\|_\infty ds \\ & \quad + \|Z_j\|_\infty \int_0^t \frac{k}{t-s} \sqrt{e^{a(t-s)} - 1} \|DP_s^{Z_i} f - DP_s^{Z_j} f\|_\infty ds. \end{aligned}$$

And hence for small t

$$\begin{aligned} \|D(P_t^{Z_i} f - P_t^{Z_j} f)\|_\infty & \leq \|Z_i - Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}} \|DP_s^{Z_i} f\|_\infty ds \\ & \quad + \|Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}} \|DP_s^{Z_i} f - DP_s^{Z_j} f\|_\infty ds. \end{aligned}$$

Now write estimate (7) as

$$\|DP_t^{Z_i} f\|_\infty \leq \frac{A}{\sqrt{t}} + B e^{Ct},$$

for some positive constants A, B and C depending on $\|f\|_\infty$ and $\|Z_i\|_\infty$. Thus, substituting this in our last estimate gives

$$\begin{aligned} \|D(P_t^{Z_i} f - P_t^{Z_j} f)\|_\infty & \leq \|Z_i - Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}} \frac{A}{\sqrt{s}} ds \\ & \quad + \|Z_i - Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}} B e^{Cs} ds \\ & \quad + \|Z_j\|_\infty \int_0^t \frac{k}{\sqrt{t-s}} \|DP_s^{Z_i} f - DP_s^{Z_j} f\|_\infty ds. \end{aligned}$$

The first two integrals are bounded if we allow t to move within a finite interval $(0, T]$. Thus there exist positive constants M_1 and M_2 such that

$$\begin{aligned} \|D(P_t^{Z_i} f - P_t^{Z_j} f)\|_\infty & \leq M_1 \|Z_i - Z_j\|_\infty \\ & \quad + M_2 \int_0^t \frac{1}{\sqrt{t-s}} \|DP_s^{Z_i} f - DP_s^{Z_j} f\|_\infty ds. \end{aligned}$$

Now iterate this to obtain

$$\begin{aligned} \|D(P_t^{Z_i} f - P_t^{Z_j} f)\|_\infty &\leq M_1 \|Z_i - Z_j\|_\infty \\ &\quad + M_2 M_1 \|Z_i - Z_j\|_\infty \int_0^t \frac{1}{\sqrt{t-s}} ds \\ &\quad + M_2^2 \int_0^t \int_0^s \frac{\|D(P_u^{Z_i} f - P_u^{Z_j} f)\|_\infty}{\sqrt{(t-s)(s-u)}} du ds. \end{aligned}$$

As before the double integral reduces to

$$\pi M_2^2 \int_0^t \|D(P_u^{Z_i} f - P_u^{Z_j} f)\|_\infty du.$$

Collecting constants into new constants M and N we arrive at

$$\|D(P_t^{Z_i} f - P_t^{Z_j} f)\|_\infty \leq M \|Z_i - Z_j\|_\infty + N \int_0^t \|D(P_u^{Z_i} f - P_u^{Z_j} f)\|_\infty du.$$

Now we apply again Gronwall's inequality to obtain

$$\|D(P_t^{Z_i} f - P_t^{Z_j} f)\|_\infty \leq M \|Z_i - Z_j\|_\infty + NM \|Z_i - Z_j\|_\infty \int_0^t e^{N(t-u)} du.$$

The right hand side goes to zero as $\|Z_i - Z_j\|_\infty \rightarrow 0$. This proves the sequence $\{DP_t^{Z_i}\}_{i=1}^\infty$ is uniformly convergent.

4.3 Uniform Convergence of Semigroups

We finally prove that the sequence of functions $\{P_t^{Z_i} f\}_{i=1}^\infty$ is a Cauchy sequence. From our recursive formula (6) we obtain

$$\begin{aligned} \|P_t^{Z_i} f - P_t^{Z_j} f\|_\infty &\leq \int_0^t \|Z_i \cdot \nabla P_s^{Z_i} f - Z_j \cdot \nabla P_s^{Z_j} f\|_\infty ds \\ &\leq \int_0^t \|Z_i - Z_j\|_\infty \|DP_s^{Z_i} f\|_\infty ds \\ &\quad + \int_0^t \|Z_j\|_\infty \|DP_s^{Z_i} f - DP_s^{Z_j} f\|_\infty ds \\ &\leq \|Z_i - Z_j\|_\infty \int_0^t k\sqrt{e^{as} - 1} ds \\ &\quad + \|Z_j\|_\infty \int_0^t \|DP_s^{Z_i} f - DP_s^{Z_j} f\|_\infty ds. \end{aligned}$$

The first integral is bounded so the first part goes to zero as $\|Z_i - Z_j\|_\infty$ approaches 0. The second part also goes to zero since we just proved the sequence of derivatives is a Cauchy sequence. Hence $\{P_t^{Z_i} f\}_{i=1}^\infty$ is uniformly convergent.

Thus, with this approximating procedure we found a semigroup for the operator $\mathcal{A} + Z \cdot \nabla$ for Z bounded and uniformly continuous and we proved it is differentiable and that our estimate also applies to its derivative.

Observe that the boundedness and uniform continuity of Z are required in order to ensure the existence of a sequence of smooth vector fields Z_i uniformly convergent to Z . Alternative assumptions on Z that guarantee the existence of such approximating sequence may be used.

Appendix

We here give a proof of inequality (1) which is taken from [6]. By using the Cauchy-Schwarz inequality and then the isometric property, we have

$$\begin{aligned} \mathbb{E} \left| \int_0^t \mathbb{Y}(x_s)(v_s) dB_s \right| &\leq \left(\mathbb{E} \left| \int_0^t \mathbb{Y}(x_s)(v_s) dB_s \right|^2 \right)^{1/2} \\ &= \left(\mathbb{E} \int_0^t \|\mathbb{Y}(x_s)(v_s)\|^2 ds \right)^{1/2} \\ &\leq \|\mathbb{Y}\|_\infty^2 \left(\int_0^t \mathbb{E} \|v_s\|^2 ds \right)^{1/2}. \end{aligned} \quad (8)$$

We will estimate the right-hand side of the last inequality. Itô's formula applied to the function $f(\cdot) = \|\cdot\|^2: \mathbb{R}^n \rightarrow \mathbb{R}$ and the semimartingale $(v_t)_{t \geq 0}$ yields

$$\|v_s\|^2 = \|v_0\|^2 + 2 \int_0^s v_u dv_u + \sum_{i=1}^n \int_0^s d\langle v^i \rangle_u.$$

Therefore

$$\begin{aligned} \mathbb{E} \|v_s\|^2 &= \|v_0\|^2 + \mathbb{E} \langle v \rangle_s \\ &= \|v_0\|^2 + \mathbb{E} \int_0^s [D\mathbb{X}(x_u)(v_u)] [D\mathbb{X}(x_u)(v_u)]^* du \\ &= \|v_0\|^2 + \mathbb{E} \int_0^s \|D\mathbb{X}(x_u)(v_u)\|^2 du \\ &\leq \|v_0\|^2 + \|D\mathbb{X}\|_\infty^2 \int_0^s \mathbb{E} \|v_u\|^2 du. \end{aligned}$$

We now use Gronwall's inequality to obtain

$$\mathbb{E} \|v_s\|^2 \leq \|v_0\|^2 e^{\|D\mathbb{X}\|_\infty s}.$$

Integrating from 0 to t yields

$$\int_0^t \mathbb{E} \|v_s\|^2 ds \leq \frac{\|v_0\|^2}{\|D\mathbb{X}\|_\infty} (e^{\|D\mathbb{X}\|_\infty t} - 1),$$

and thus, substituting in (8) we obtain

$$\mathbb{E} \left| \int_0^t \mathbb{Y}(x_s)(v_s) dB_s \right| \leq \|\mathbb{Y}\|_\infty^2 \frac{\|v_0\|}{\|D\mathbb{X}\|_\infty^{1/2}} \sqrt{e^{\|D\mathbb{X}\|_\infty t} - 1}.$$

This proves inequality (1).

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