Central limit theorem under variance uncertainty*

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Abstract

We prove the central limit theorem (CLT) for a sequence of independent zero-mean random variables ξ_j, perturbed by predictable multiplicative factors λ_j with values in intervals $[\underline{\lambda}_j, \overline{\lambda}_j]$. It is assumed that the sequences $\underline{\lambda}_j$, $\overline{\lambda}_j$ are bounded and satisfy some stabilization condition. Under the classical Lindeberg condition we show that the CLT limit, corresponding to a "worst" sequence λ_j, is described by the solution v of one-dimensional G-heat equation. The main part of the proof follows Peng's approach to the CLT under sublinear expectations, and utilizes Hölder regularity properties of v. Under the lack of such properties, we use the technique of half-relaxed limits from the theory of viscosity solutions.

1 Introduction

Consider a sequence of independent one-dimensional random variables $(\xi_j)_{j=1}^{\infty}$ with zero means and finite variances $\sigma_j^2 = \mathsf{E}\xi_j^2 > 0$. Put $s_n^2 = \sum_{j=1}^n \sigma_j^2$, $\varepsilon > 0$ and assume that the Lindeberg condition

$$L_n(\varepsilon) = \frac{1}{s_n^2} \sum_{j=1}^n \mathsf{E}\left(\xi_j^2 I_{\{|\xi_j| > \varepsilon s_n\}}\right) \to 0, \quad n \to \infty \tag{1.1}$$

is satisfied. Then, by the classical central limit theorem (CLT), for any bounded continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ we have

$$\lim_{n\to\infty} \mathsf{E}f\left(\frac{1}{s_n} \sum_{j=1}^n \xi_j\right) = \mathsf{E}f(\zeta), \tag{1.2}$$

where ζ has the standard normal law.

In this paper we assume that the variance of ξ_j is not known exactly and may belong to an interval. Our goal is to obtain the "least upper bound" $\mathscr{L}$ for the quantity (1.2) under such model uncertainty. The result, as well as its proof, are similar to those obtained by Peng [13, 14] and the followers [11, 20, 8] under the nonlinear expectations theory paradigm. It appears that $\mathscr{L}$ can be described in terms of the solution v of a nonlinear

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parabolic equation, called G-heat equation. One of the objectives of the present paper is to show that this description also comes from a classical problem statement, and need not be linked to the nonlinear expectations theory.

To give a precise problem formulation, consider a filtered probability space

$$(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_j)_{j=0}^\infty)$$

and an adapted sequence $(\xi_j)_{j=1}^\infty$ of random variables such that $\mathsf{E}\xi_j = 0$, $\mathsf{E}\xi_j^2 = \sigma_j^2 \in (0, \infty)$ and ξ_j is independent from $\mathscr{F}_{j-1}$. Let $(\lambda_j)_{j=0}^\infty$ be an adapted sequence, whose elements λ_j take values in deterministic intervals $[\underline{\lambda}_j, \overline{\lambda}_j]$, $0 \leq \underline{\lambda}_j \leq \overline{\lambda}_j$. Considering the sequence $\eta_j = \lambda_{j-1}\xi_j$, one can regard the multipliers λ_{j-1} as a "predictable perturbation" of the original sequence ξ_j. The intervals $[\underline{\lambda}_{j-1}\sigma_j, \overline{\lambda}_{j-1}\sigma_j]$ indicate possible standard deviations of η_j.

Assumption 1.1. *The Lindeberg condition (1.1) is satisfied.*

Assumption 1.2. *The sequence $\overline{\lambda}_j$ is bounded by a constant Λ.*

Assumption 1.3. *The sequences $\underline{\lambda}_j$, $\overline{\lambda}_j$ satisfy the following stabilization condition:*

$$M_n = \sum_{j=0}^{n-1} \frac{\sigma_{j+1}^2}{s_n^2} \left(|\overline{\lambda}_j^2 - \overline{\lambda}^2| + |\underline{\lambda}_j^2 - \underline{\lambda}^2| \right) \to 0, \quad n \to \infty \tag{1.3}$$

for some $\overline{\lambda} \geq \underline{\lambda} \geq 0$.

Put $B_j = [\underline{\lambda}_j^2, \overline{\lambda}_j^2]$, $B = [\underline{\lambda}^2, \overline{\lambda}^2]$ and denote by

$$d_H(B_j, B) = \max\{|\overline{\lambda}_j^2 - \overline{\lambda}^2|, |\underline{\lambda}_j^2 - \underline{\lambda}^2|\}$$

the Hausdorff distance between these intervals (see, e.g., [1, Chapter 2]). Condition (1.3) is equivalent to the following one:

$$\sum_{j=0}^{n-1} \frac{\sigma_{j+1}^2}{s_n^2} d_H(B_j, B) \to 0, \quad n \to \infty. \tag{1.4}$$

In the summability theory the transformation

$$t_n = \frac{p_1 a_1 + \cdots + p_n a_n}{p_1 + \cdots + p_n}, \quad p_n > 0$$

of a sequence $(a_i)_{i=1}^\infty$ is called a Riesz mean (see [16, Section 1.4], [4, Section 3.2]). By Assumption 1.3, the sequence $d_H(B_j, B)$ is summable to 0 by the Riesz method, determined by the sequence $p_i = \sigma_i^2$. Furthermore, the Lindeberg condition implies the Feller condition

$$\lim_{n \to \infty} \max_{1 \leq j \leq n} \frac{\sigma_j}{s_n} = 0 \tag{1.5}$$

(see, e.g., [3], Chapter 6, §28). In particular, $s_n \to \infty$. Hence, the Riesz summation method, defined above, is regular (see [16, Theorem 1.4.4]), and if $d_H(B_j, B) \to 0$, then the Assumption 1.3 is satisfied. We also mention a necessary and sufficient condition for (1.4) to hold true, given in [4] (Lemma 3.2.14). This result is applicable since $\sigma_n^2/s_n^2 \to 0$ by (1.5).

Note that from the identity

$$\sum_{j=0}^{n-1} \frac{\sigma_{j+1}^2}{s_n^2} = 1$$

it easily follows that

$$\lim_{n \to \infty} \sum_{j=0}^{n-1} \frac{\sigma_{j+1}^2}{s_n^2} \overline{\lambda}_j^2 = \overline{\lambda}^2, \quad \lim_{n \to \infty} \sum_{j=0}^{n-1} \frac{\sigma_{j+1}^2}{s_n^2} \underline{\lambda}_j^2 = \underline{\lambda}^2.$$

Denote by $\mathfrak{A}^n$ the set of adapted sequences $\lambda_0^n = (\lambda_j)_{j=0}^n$ with values in $[\underline{\lambda}_j, \overline{\lambda}_j]$. Our goal is to describe the quantity

$$\mathscr{L} = \lim_{n \to \infty} \sup_{\lambda_0^{n-1} \in \mathfrak{A}^{n-1}} \mathsf{E} f\left(\frac{1}{s_n} \sum_{j=0}^{n-1} \lambda_j \xi_{j+1} \right), \tag{1.6}$$

which can be loosely characterized as the least upper bound of (1.2) under variance uncertainty.

The main role in this description is played by the solution of the nonlinear parabolic equation

$$v_t + \frac{1}{2} \sup_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} \left(\lambda^2 v_{xx} \right) = v_t + \frac{1}{2} \left(\overline{\lambda}^2 v_{xx}^+ - \underline{\lambda}^2 v_{xx}^- \right) = 0, \quad (t, x) \in [0, 1) \times \mathbb{R}, \tag{1.7}$$

satisfying the boundary condition

$$v(1, x) = f(x), \quad x \in \mathbb{R}. \tag{1.8}$$

In the context of the CLT under sublinear expectations, equation (1.7) appeared in [13]. It was called G-heat equation in [12]. As is mentioned in [5], such equation arises in various applications in control theory, mechanics, combustion, biology, and finance. It is known also as a Barenblatt equation: see, e.g., [9].

One can obtain (1.7) by considering λ_j as a control sequence, writing down dynamic programming equations for discrete time finite horizon optimization problems

$$\sup_{\lambda_0^{n-1} \in \mathfrak{A}^{n-1}} \mathsf{E} f\left(\frac{1}{s_n} \sum_{j=0}^{n-1} \lambda_j \xi_{j+1} \right),$$

and passing to the limit as $n \to \infty$. This approach was proposed in [18] in the case of identically distributed (multidimensional) random variables ξ_j. However, in the present context, it seems that this method requires hypotheses, which are stronger than the Lindeberg condition. Thus, we follow Peng's approach, which takes equation (1.7) as a starting point, and utilizes a deep result on the existence of its solution in an appropriate Hölder class.

Put $Q = [0, 1] \times \mathbb{R}$,

$$\|h\|_{0;\mathbb{R}} = \sup_{x \in \mathbb{R}} |h(x)|, \quad \|g\|_{0;Q} = \sup_{(t,x) \in Q} |g(t, x)|,$$

$$[h]_{\alpha;\mathbb{R}} = \sup_{\substack{x_i \in \mathbb{R}, \\ x_1 \neq x_2}} \frac{|h(x_1) - h(x_2)|}{|x_1 - x_2|^\alpha}, \quad \alpha \in (0, 1],$$

$$[g]_{\alpha;Q} = \sup_{\substack{(t_i, x_i) \in Q, \\ (t_1, x_1) \neq (t_2, x_2)}} \frac{|g(t_1, x_1) - g(t_2, x_2)|}{(|t_1 - t_2|^{1/2} + |x_1 - x_2|)^\alpha}, \quad \alpha \in (0, 1],$$

and consider the Hölder spaces $C^{2+\alpha}(\mathbb{R})$, $C^{1+\alpha/2, 2+\alpha}(Q)$ with the norms

$$\|h\|_{C^{2+\alpha}(\mathbb{R})} = \|h\|_{0;\mathbb{R}} + \|h_x\|_{0;\mathbb{R}} + \|h_{xx}\|_{0;\mathbb{R}} + [h_{xx}]_{\alpha;\mathbb{R}},$$

$$\|g\|_{C^{1+\alpha/2,2+\alpha}(Q)} = \|g\|_{0;Q} + \|g_x\|_{0;Q} + \|g_t\|_{0;Q} + \|g_{xx}\|_{0;Q} + [g_t]_{\alpha;Q} + [g_{xx}]_{\alpha;Q}.$$

Under the assumptions $f \in C^{2+\alpha}(\mathbb{R})$, $\alpha \in (0, 1]$; $\underline{\lambda} > 0$ the existence of a classical solution $v \in C^{1+\alpha'/2,2+\alpha'}(Q)$ (with some of $\alpha' \in (0, 1]$) of (1.7), (1.8) was proved by Krylov: see [10] (Theorem 1.1 or Theorem 5.3).

If $\underline{\lambda} = 0$ then only the existence of a viscosity solution is guaranteed. Let us recall this result along with related definitions. Put $Q^\circ = [0, 1) \times \mathbb{R}$ and assume that f is a bounded continuous function: $f \in C_b(\mathbb{R})$. A bounded upper semicontinuous (usc) function $v : Q \mapsto \mathbb{R}$ is called a *viscosity subsolution* of (1.7), (1.8) if

$$v(1, x) \leq f(x), \quad x \in \mathbb{R}, \tag{1.9}$$

and for any $(\bar{t}, \bar{x}) \in Q^\circ$ and any test function $\varphi \in C^2(\mathbb{R}^2)$ such that $(\bar{t}, \bar{x}) \in Q^\circ$ is a strict local maximum point of $v - \varphi$ on Q°, the inequality

$$-\varphi_t(\bar{t}, \bar{x}) - \frac{1}{2} \sup_{\lambda \in [\underline{\lambda}, \overline{\lambda}]} \left(\lambda^2 \varphi_{xx}(\bar{t}, \bar{x}) \right) \leq 0 \tag{1.10}$$

holds true. To define a *viscosity supersolution*, one should consider a bounded lower semicontinuous (lsc) function v, a strict local minimum point of $v - \varphi$, and reverse the inequalities (1.9), (1.10).

We will use the following comparison result. Consider a viscosity subsolution u and a viscosity supersolution w of (1.7), (1.8). Since we require (1.7) to be satisfied in the viscosity sense at the lower boundary of Q, by the accessibility theorem of [6], we have

$$u(0, x) = \limsup_{\substack{(t,y) \in (0,1) \times \mathbb{R}, \\ t \to 0, y \to x}} u(t, y); \quad w(0, x) = \liminf_{\substack{(t,y) \in (0,1) \times \mathbb{R}, \\ t \to 0, y \to x}} w(t, y)$$

and by the comparison result of [7] (Theorem 1) it follows that $u \leq w$ on Q.

A bounded *continuous* function $v : Q \mapsto \mathbb{R}$ is called a *viscosity solution* of (1.7), (1.8), if it is viscosity sub- and supersolution. The existence of a continuous viscosity solution of (1.7), (1.8) for $f \in C_b(\mathbb{R})$ is well known from the theory of optimal control. The stochastic control representation of such solution can be found in [19] (Chap. 4, Theorem 5.2).

Theorem 1.4. *Let f be a bounded continuous function, and let v be the continuous viscosity solution of (1.7), (1.8). Then, under Assumptions 1.1–1.3, we have $\mathscr{L} = v(0, 0)$.*

It is interesting to compare Theorem 1.4 with related results obtained in the framework of sublinear expectations theory. Besides the original result of Peng [13, 14], which is discussed in [18], we mention the papers [11, 20, 8], where the random variables were not assumed to be identically distributed. We will discuss only the result of [20], which extends [11]. The result of [8] concerns the multidimensional case.

Let us briefly describe the construction of a sublinear expectation space $(\Omega, \mathcal{H}, \widehat{\mathbb{E}})$, which allows to rewrite the expression (1.6) in terms of a sublinear expectation. This construction is, in fact, the same as in [18, Section 4], where some more details are given. Consider the space of sequences $\Omega = \{(y_i)_{i=1}^\infty : y_i \in \mathbb{R}\}$, and introduce the space of random variables $\mathcal{H}$ as follows: $\mathcal{H} = \cup_{n=1}^\infty \mathcal{H}_n$, where $\mathcal{H}_n$ is some linear space (we do not go into details) of functions $Y = \psi(y_1, \ldots, y_n)$ of n variables. Define the sublinear expectation by the formula

$$\widehat{\mathbb{E}}Y = \sup_{\lambda_0^{n-1} \in \mathfrak{A}_0^{n-1}} \mathsf{E}\psi(\lambda_0 \xi_1 / \sigma_1, \ldots, \lambda_{n-1} \xi_n / \sigma_n). \tag{1.11}$$

Let Y_i be the projection mappings: $Y_i(y) = y_i$. One can show that Y_n is independent from $(Y_1, \ldots, Y_{n-1})$ in the sense of sublinear expectations theory (see [15], Definition 3.10).

By (1.11) we get the following representation for $\mathscr{L}$:

$$\mathscr{L} = \lim_{n\to\infty} \sup_{\lambda_0^{n-1}\in\mathfrak{A}^{n-1}} \mathsf{E}f\left(\frac{1}{s_n}\sum_{j=1}^n \lambda_{j-1}\xi_j\right) = \lim_{n\to\infty} \widehat{\mathbb{E}}f\left(\sum_{j=1}^n \frac{\sigma_j}{s_n}Y_j\right).$$

Let us apply Theorem 3.1 of [20] to the sequence Y_i. We have

$$\widehat{\mathbb{E}}(\pm Y_i) = \sup_{\lambda_{i-1}\in[\underline{\lambda}_{i-1},\overline{\lambda}_{i-1}]} \mathsf{E}(\pm\lambda_{i-1}\xi_i/\sigma_i) = 0,$$

$$\widehat{\mathbb{E}}Y_i^2 = \sup_{\lambda_{i-1}\in[\underline{\lambda}_{i-1},\overline{\lambda}_{i-1}]} \mathsf{E}(\lambda_{i-1}\xi_i/\sigma_i)^2 = \overline{\lambda}_{i-1}^2,$$

$$-\widehat{\mathbb{E}}(-Y_i^2) = -\sup_{\lambda_{i-1}\in[\underline{\lambda}_{i-1},\overline{\lambda}_{i-1}]} \mathsf{E}\left(-(\lambda_{i-1}\xi_i/\sigma_i)^2\right) = \underline{\lambda}_{i-1}^2.$$

Besides a condition, identical to Assumption 1.3, in [20] it is assumed that

$$\widehat{\mathbb{E}}|Y_i|^{2+\delta} = \sup_{\lambda_{i-1}\in[\underline{\lambda}_{i-1},\overline{\lambda}_{i-1}]} \mathsf{E}\left|\lambda_{i-1}\xi_i/\sigma_i\right|^{2+\delta} = \overline{\lambda}_{i-1}^{2+\delta}\mathsf{E}|\xi_i/\sigma_i|^{2+\delta} \le M, \tag{1.12}$$

$$\lim_{n\to\infty}\sum_{j=1}^n \left(\frac{\sigma_j}{s_n}\right)^{2+\delta} = 0 \tag{1.13}$$

for some $M > 0$, $\delta > 0$. Note that (1.12) was used in [20] in this form, although it was not clearly formulated (see condition (3) of Theorem 3.1 in [20]). The result of [20] tells us that $\mathscr{L} = \widehat{\mathbb{E}}f(Z)$, where Z is a G-normal random variable with

$$G(s) = \frac{1}{2}(s^+\overline{\lambda}^2 - s^-\underline{\lambda}^2).$$

By the characterization of the G-normal distribution (see, e.g., [15], Example 1.13) this is equivalent to the assertion of Theorem 1.4.

Thus, under the assumptions (1.12), (1.13) (instead of Assumptions 1.1, 1.2), Theorem 1.4 follows from the result of [20]. It is easy to see that (1.12) implies Assumption 1.2:

$$\overline{\lambda}_{i-1} = \left(\mathsf{E}(\overline{\lambda}_{i-1}^2\xi_i^2/\sigma_i^2)\right)^{1/2} \le \left(\mathsf{E}(\overline{\lambda}_{i-1}^{2+\delta}|\xi_i/\sigma_i|^{2+\delta})\right)^{1/(2+\delta)}$$

and (1.12), (1.13) imply that

$$\frac{1}{s_n^2}\sum_{j=1}^n \mathsf{E}\left(\overline{\lambda}_{j-1}^{2+\delta}\xi_j^2 I_{\{|\xi_j|>\varepsilon s_n\}}\right) \le \frac{1}{\varepsilon^\delta s_n^{2+\delta}}\sum_{j=1}^n \overline{\lambda}_{j-1}^{2+\delta}\mathsf{E}|\xi_j|^{2+\delta} \le \frac{M}{\varepsilon^\delta s_n^{2+\delta}}\sum_{j=1}^n \sigma_j^{2+\delta} \to 0, \quad n\to\infty.$$

The last condition is slightly weaker than Assumption 1.1, and coincides with the latter if $\liminf_{j\to\infty}\overline{\lambda}_j > 0$.

Note that if there is no model uncertainty: $\underline{\lambda}_j = \overline{\lambda}_j = 1$, then Theorem 1.4 reduces to the classical CLT, mentioned at the beginning of the present paper. This is not the case with the result of [20], since in this case the conditions (1.12), (1.13) are stronger then the Lindeberg condition. We also mention that [20] deals only with classical solutions of the G-heat equation, so the case $\underline{\lambda} = 0$ is, in fact, not considered there. However, the sublinear expectations theory is able to handle the degenerate case via perturbation methods, see [14] (the proof of Theorem 5.1), [8] (the proof of Theorem 3.1).

2 Proof of Theorem 1.4

(i) We first consider the case $f \in C^{2+\alpha}(\mathbb{R})$, $\alpha > 0$ and $\underline{\lambda} > 0$. Put

$$X_{j+1} = X_j + \frac{\lambda_j}{s_n}\xi_{j+1}, \quad j = 0,\dots,n-1, \quad X_0 = 0; \quad t_j = \sum_{k=0}^{j}\frac{\sigma_k^2}{s_n^2}.$$

Since the solution v of (1.7), (1.8) belongs to $v \in C^{1+\alpha'/2,2+\alpha'}(Q)$, we can apply Taylor's formula:

$$v(1,X_n) - v(0,0) = \sum_{j=0}^{n-1}\left(v(t_{j+1},X_{j+1}) - v(t_j,X_{j+1}) + v(t_j,X_{j+1}) - v(t_j,X_j)\right)$$

$$= \sum_{j=0}^{n-1}\left(v_t(\widehat{t}_j,X_{j+1})(t_{j+1}-t_j) + v_x(t_j,X_j)(X_{j+1}-X_j) + \frac{1}{2}v_{xx}(t_j,\widehat{X}_j)(X_{j+1}-X_j)^2\right),$$

where $\widehat{t}_j = t_j + \beta(t_{j+1}-t_j)$, $\widehat{X}_j = X_j + \gamma(X_{j+1}-X_j)$, $\beta,\gamma \in [0,1]$. By the independence of X_j and ξ_{j+1} we conclude that $\mathsf{E}(v_x(t_j,X_j)(X_{j+1}-X_j)) = 0$. Thus,

$$\mathsf{E}v(1,X_n) - v(0,0) = \mathsf{E}\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}\left(v_t(\widehat{t}_j,X_{j+1}) + \frac{\lambda_j^2}{2}\frac{\xi_{j+1}^2}{\sigma_{j+1}^2}v_{xx}(t_j,\widehat{X}_j)\right) = J_n + I_n,$$

$$J_n = \mathsf{E}\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}\left(v_t(t_j,X_j) + \frac{\lambda_j^2}{2}\frac{\xi_{j+1}^2}{\sigma_{j+1}^2}v_{xx}(t_j,X_j)\right) = \mathsf{E}\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}\left(v_t + \frac{\lambda_j^2}{2}v_{xx}\right)(t_j,X_j),$$

$$I_n = \mathsf{E}\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}\left(v_t(\widehat{t}_j,X_{j+1}) - v_t(t_j,X_j) + \frac{\lambda_j^2}{2}\frac{\xi_{j+1}^2}{\sigma_{j+1}^2}(v_{xx}(t_j,\widehat{X}_j) - v_{xx}(t_j,X_j))\right).$$

We can rewrite J_n as $J_n^1 + J_n^2$, where

$$J_n^1 = \mathsf{E}\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}\left(v_t + \frac{1}{2}\left(\overline{\lambda}^2 v_{xx}^+ - \underline{\lambda}^2 v_{xx}^-\right)\right)(t_j,X_j),$$

$$J_n^2 = \frac{1}{2}\mathsf{E}\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}\left((\lambda_j^2 - \overline{\lambda}^2)v_{xx}^+ + (\underline{\lambda}^2 - \lambda_j^2)v_{xx}^-\right)(t_j,X_j).$$

From the definition of v we see that $J_n^1 = 0$. Furthermore, from the stabilization condition (1.3) it follows that

$$\sup_{\lambda_0^{n-1}\in\mathfrak{A}^{n-1}} J_n^2 \leq \frac{1}{2}\mathsf{E}\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}\left((\overline{\lambda}_j^2 - \overline{\lambda}^2)v_{xx}^+ + (\underline{\lambda}^2 - \underline{\lambda}_j^2)v_{xx}^-\right)(t_j,X_j) \leq CM_n \to 0,$$

$n \to \infty$, since the second derivative of v is uniformly bounded. On the other hand, choosing a sequence

$$\lambda_j = \overline{\lambda}_j I_{\{v_{xx}(t_j,X_j)>0\}} - \underline{\lambda}_j I_{\{v_{xx}(t_j,X_j)\leq 0\}}, \quad j \geq 1,$$

with an arbitrary λ_0, we get an opposite inequality

$$\sup_{\lambda_0^{n-1}\in\mathfrak{A}^{n-1}} J_n^2 \geq \frac{1}{2}\mathsf{E}\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}\left((\overline{\lambda}_j^2 - \overline{\lambda}^2)v_{xx}^+ + (\underline{\lambda}^2 - \underline{\lambda}_j^2)v_{xx}^-\right)(t_j,X_j) \geq -CM_n \to 0.$$

Combining all these results, we conclude that

$$\lim_{n\to\infty} \sup_{\lambda_0^{n-1}\in\mathfrak{A}^{n-1}} J_n = 0. \tag{2.1}$$

Now consider $I_n = I_n^1 + I_n^2 + I_n^3$:

$$I_n^1 = \mathsf{E}\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}\left(v_t(\widehat{t}_j, X_{j+1}) - v_t(t_j, X_j)\right),$$

$$I_n^2 = \mathsf{E}\sum_{j=0}^{n-1}\frac{\xi_{j+1}^2}{s_n^2}\frac{\lambda_j^2}{2}\left(v_{xx}(t_j, \widehat{X}_j) - v_{xx}(t_j, X_j)\right)I_{\{|\xi_{j+1}|>\varepsilon s_n\}},$$

$$I_n^3 = \mathsf{E}\sum_{j=0}^{n-1}\frac{\xi_{j+1}^2}{s_n^2}\frac{\lambda_j^2}{2}\left(v_{xx}(t_j, \widehat{X}_j) - v_{xx}(t_j, X_j)\right)I_{\{|\xi_{j+1}|\leq\varepsilon s_n\}}.$$

By the Hölder continuity of v_t we have

$$|I_n^1| \leq C\mathsf{E}\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}\left(|\widehat{t}_j - t_j|^{\alpha'/2} + |X_{j+1} - X_j|^{\alpha'}\right)$$

$$\leq C\mathsf{E}\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}\left(\left(\frac{\sigma_{j+1}}{s_n}\right)^{\alpha'} + \left(\frac{\lambda_j|\xi_{j+1}|}{s_n}\right)^{\alpha'}\right).$$

Using the inequality $\mathsf{E}|\xi_{j+1}|^{\alpha'} \leq (\mathsf{E}\xi_{j+1}^2)^{\alpha'/2} = \sigma_{j+1}^{\alpha'}$, and the independence of λ_j and ξ_{j+1}, we obtain the estimate

$$|I_n^1| \leq C\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}(1 + \overline{\lambda}_j^{\alpha'})\left(\frac{\sigma_{j+1}}{s_n}\right)^{\alpha'} \leq C\left(\max_{1\leq j\leq n}\frac{\sigma_j}{s_n}\right)^{\alpha'}\left(1 + \Lambda^{\alpha'}\right).$$

From (1.5) it follows that $I_n^1 \to 0$.

Furthermore, since the sequence λ_j is bounded and the second derivative of v is uniformly bounded, by the Lindeberg condition we get

$$|I_n^2| \leq CL_n(\varepsilon) \to 0, \quad n \to \infty.$$

The last term I_n^3 is estimated with the use of the Hölder continuity property of v_{xx}:

$$|I_n^3| \leq C\mathsf{E}\sum_{j=0}^{n-1}\frac{\xi_{j+1}^2}{s_n^2}\frac{\lambda_j^2}{2}\left|\frac{\lambda_j|\xi_{j+1}|}{s_n}\right|^{\alpha'}I_{\{|\xi_{j+1}|\leq\varepsilon s_n\}} \leq C\Lambda^{2+\alpha'}\sum_{j=0}^{n-1}\frac{\sigma_{j+1}^2}{s_n^2}\varepsilon^{\alpha'} = C\Lambda^{2+\alpha'}\varepsilon^{\alpha'}.$$

Therefore,

$$\lim_{n\to\infty} \sup_{\lambda_0^{n-1}\in\mathfrak{A}^{n-1}} |I_n| = 0. \tag{2.2}$$

From (2.1) and (2.2) it follows that

$$\mathscr{L} = \lim_{n\to\infty} \sup_{\lambda_0^{n-1}\in\mathfrak{A}^{n-1}} \mathsf{E}f(X_n) = \lim_{n\to\infty} \sup_{\lambda_0^{n-1}\in\mathfrak{A}^{n-1}} \mathsf{E}v(1, X_n) = v(0,0).$$

So, we have proved the theorem in the case $f \in C^{2+\alpha}$, $\underline{\lambda} > 0$.

(ii) Now assume that $\underline{\lambda} = 0$. Put

$$X_n^\varepsilon = \frac{1}{s_n}\sum_{j=0}^{n-1}(\lambda_j^2 + \varepsilon^2)^{1/2}\xi_{j+1}, \quad \mathscr{L}^\varepsilon = \lim_{n\to\infty} \sup_{\lambda_0^{n-1}\in\mathfrak{A}^{n-1}} \mathsf{E}f(X_n^\varepsilon).$$

The intervals $[\underline{\mu}_j, \overline{\mu}_j] = [(\underline{\lambda}_j^2 + \varepsilon^2)^{1/2}, (\overline{\lambda}_j^2 + \varepsilon^2)^{1/2}]$ stabilize to $[\varepsilon, (\overline{\lambda}^2 + \varepsilon^2)^{1/2}]$ in the sense of Assumption 1.3:

$$\sum_{j=0}^{n-1} \frac{\sigma_{j+1}^2}{s_n^2} \left(|\overline{\mu}_j^2 - (\overline{\lambda}^2 + \varepsilon^2)| + |\underline{\mu}_j^2 - \varepsilon^2| \right) \to 0, \quad n \to \infty.$$

By part (i) of the proof, we infer that $\mathscr{L}^\varepsilon = v^\varepsilon(0,0)$, where v^ε satisfies

$$v_t^\varepsilon + \frac{1}{2} \left((\overline{\lambda}^2 + \varepsilon^2)(v_{xx}^\varepsilon)^+ - \varepsilon^2 (v_{xx}^\varepsilon)^- \right) = 0, \quad x \in Q^\circ; \quad v^\varepsilon(1,x) = f(x), \quad x \in \mathbb{R} \qquad (2.3)$$

in the classical sense. Let v be the continuous viscosity solution of the limiting problem

$$v_t + \frac{1}{2}\overline{\lambda}^2 v_{xx}^+ = 0, \quad x \in Q^\circ; \quad v(1,x) = f(x), \quad x \in \mathbb{R}. \qquad (2.4)$$

The desired result is a consequence of the relations

$$\mathscr{L} := \lim_{n\to\infty} \sup_{\lambda_0^{n-1} \in \mathfrak{A}^{n-1}} \mathsf{E} f(X_n) = \lim_{\varepsilon\to 0} \mathscr{L}^\varepsilon, \quad v(0,0) = \lim_{\varepsilon\to 0} v^\varepsilon(0,0), \qquad (2.5)$$

which we are going to prove.

Since we still assume that $f \in C^{2+\alpha}(\mathbb{R})$, this function is uniformly Lipschitz continuous. Put $\psi_\varepsilon(\lambda) = (\lambda^2 + \varepsilon^2)^{1/2} - \lambda$. We have

$$|\mathsf{E} f(X_n^\varepsilon) - \mathsf{E} f(X_n)| \leq C\mathsf{E}|X_n^\varepsilon - X_n| \leq \frac{C}{s_n} \left(\mathsf{E} \left(\sum_{j=0}^{n-1} \psi_\varepsilon(\lambda_j)\xi_{j+1} \right)^2 \right)^{1/2}$$

$$= C \left(\mathsf{E} \sum_{j=0}^{n-1} \frac{\sigma_{j+1}^2}{s_n^2} \psi_\varepsilon^2(\lambda_j) \right)^{1/2} \leq C\varepsilon,$$

since $\sup_{\lambda \geq 0} \psi_\varepsilon(\lambda) = \varepsilon$. Thus,

$$\mathscr{L}^\varepsilon - C\varepsilon \leq \liminf_{n\to\infty} \sup_{\lambda_0^{n-1} \in \mathfrak{A}^{n-1}} \mathsf{E} f(X_n) \leq \limsup_{n\to\infty} \sup_{\lambda_0^{n-1} \in \mathfrak{A}^{n-1}} \mathsf{E} f(X_n) \leq \mathscr{L}^\varepsilon + C\varepsilon, \qquad (2.6)$$

$$\limsup_{\varepsilon\to 0} \mathscr{L}^\varepsilon \leq \liminf_{n\to\infty} \sup_{\lambda_0^{n-1} \in \mathfrak{A}^{n-1}} \mathsf{E} f(X_n) \leq \limsup_{n\to\infty} \sup_{\lambda_0^{n-1} \in \mathfrak{A}^{n-1}} \mathsf{E} f(X_n) \leq \liminf_{\varepsilon\to 0} \mathscr{L}^\varepsilon.$$

These estimates imply the first equality in (2.5).

Furthermore, define the *half-relaxed* (or *weak*) limits of v^ε by

$$\underline{v}(t,x) = \liminf_{\substack{(s,y)\to(t,x), \\ \varepsilon\to 0}} v^\varepsilon(s,y), \quad \overline{v}(t,x) = \limsup_{\substack{(s,y)\to(t,x), \\ \varepsilon\to 0}} v^\varepsilon(s,y), \quad (t,x) \in Q.$$

The function $\overline{v}$ (resp., $\underline{v}$) is usc (resp., lsc): see [2] (Chap. 5, Lemma 1.5).

Take $\varphi \in C^2(\mathbb{R}^2)$ and assume that $\overline{z} = (\overline{t}, \overline{x}) \in Q$ is a strict local maximum point of $\overline{v} - \varphi$ on Q. Then there exist sequences $\varepsilon_k \to 0$, $z_k = (t_k, x_k) \in Q$ such that $z_k \to \overline{z}$, $v^{\varepsilon_k}(z_k) \to \overline{v}(\overline{z})$, and z_k is a local maximum point of $v^{\varepsilon_k} - \varphi$ on Q: see [2] (Chap. 5, Lemma 1.6).

If $t \in [0,1)$, then $t_k \in [0,1)$ for sufficiently large k and

$$-\varphi_t(z_k) - \sup_{\lambda \in [\varepsilon_k, \overline{\lambda}+\varepsilon_k]} (\lambda^2 \varphi_{xx}(z_k)) \leq 0,$$

since v^{ε_k} is a viscosity solution of (2.3). Passing to the limit as $\varepsilon_k \to 0$, we get the inequality

$$-\varphi_t(\overline{z}) - \sup_{\lambda \in [0,\overline{\lambda}]} (\lambda^2 \varphi_{xx}(\overline{z})) \leq 0, \tag{2.7}$$

which means that $\overline{v}$ is a viscosity subsolution of (2.4) on Q°.

Let $\overline{t} = 1$. If there are infinitely many $t_k < 1$, then we again obtain (2.7) as above. Moreover, we can change the test function φ to $\widehat{\varphi} = \varphi + c(1-t)$, $c > 0$ since $(1, \overline{x})$ is still a strict local maximum point of $\overline{v} - \widehat{\varphi}$. Substituting $\widehat{\varphi}$ in (2.7), we get a contradiction:

$$c - \varphi_t(\overline{z}) - \sup_{\lambda \in [0,\overline{\lambda}]} (\lambda^2 \varphi_{xx}(\overline{z})) \leq 0, \quad \text{for any } c > 0.$$

Thus, for sufficiently large k, we have $v^{\varepsilon_k}(z_k) = f(x_k)$ and $\overline{v}(\overline{z}) = \lim_{k \to \infty} f(x_k) = f(\overline{x})$.

We have proved that $\overline{v}$ is a viscosity subsolution of (2.4). Similarly, one can prove that $\underline{v}$ is a viscosity supersolution of (2.4). By the comparison result of [7], mentioned in Section 1, we have $\overline{v} \leq \underline{v}$ on Q. The converse inequality $\overline{v} \geq \underline{v}$ is immediate from the definition. We infer that $v = \overline{v} = \underline{v}$ is a continuous viscosity solution of (2.4), and the second equality in (2.5) holds true:

$$v(0,0) \leq \liminf_{\varepsilon \to 0} v^\varepsilon(0,0) \leq \limsup_{\varepsilon \to 0} v^\varepsilon(0,0) \leq v(0,0).$$

This completes the proof of Theorem 1.4 in the case $\underline{\lambda} = 0$.

(iii) It remains to consider the case $f \in C_b(\mathbb{R})$. It is not difficult to show that there exists a function $f^\varepsilon \in C^\infty(\mathbb{R})$ such that $|f(x) - f^\varepsilon(x)| \leq \varepsilon$: see, e.g., [17]. Furthermore, consider a function $\chi \in C^\infty$,

$$\chi(x) = 1, \quad |x| \leq 1; \quad \chi(x) = 0, \quad |x| \geq 2,$$

and put $g^\varepsilon(x) = \chi(\varepsilon^{1/2} x) f^\varepsilon(x)$. We have

$$|\mathsf{E}f(X_n) - \mathsf{E}g^\varepsilon(X_n)| \leq |\mathsf{E}f(X_n) - \mathsf{E}f^\varepsilon(X_n)| + |\mathsf{E}f^\varepsilon(X_n) - \mathsf{E}g^\varepsilon(X_n)|$$

$$\leq \varepsilon + C\mathsf{P}(\varepsilon^{1/2}|X_n| \geq 1) \leq \varepsilon + C\varepsilon\mathsf{E}X_n^2 \leq \varepsilon + C\varepsilon \sum_{j=0}^{n-1} \frac{\overline{\lambda}_j^2 \sigma_{j+1}^2}{s_n^2} = (1 + C\Lambda^2)\varepsilon.$$

From this estimate we obtain the inequalities of the form (2.6) with

$$\mathscr{L}^\varepsilon = \lim_{n \to \infty} \sup_{\lambda_0^{n-1} \in \mathfrak{A}^{n-1}} \mathsf{E}g^\varepsilon(X_n).$$

Just mentioned inequalities imply that

$$\mathscr{L} := \lim_{n \to \infty} \sup_{\lambda_0^{n-1} \in \mathfrak{A}^{n-1}} \mathsf{E}f(X_n) = \lim_{\varepsilon \to 0} \mathscr{L}^\varepsilon. \tag{2.8}$$

Denote by V^ε, the viscosity solution of (1.7), (1.8), corresponding to the terminal condition g^ε instead of f. Since $g^\varepsilon \in C^{2+\alpha}(\mathbb{R})$, we have

$$\mathscr{L}^\varepsilon = V^\varepsilon(0,0) \tag{2.9}$$

by the result, already proved.

Finally, note, that the convergence $g^\varepsilon(x) = \chi(\varepsilon^{1/2} x) f^\varepsilon(x) \to f(x)$, $\varepsilon \to 0$ is uniform on compact sets. It follows that

$$\liminf_{\substack{y \to x \\ \varepsilon \to 0}} g^\varepsilon(y) = \limsup_{\substack{y \to x \\ \varepsilon \to 0}} g^\varepsilon(y) = f(x).$$

Using this fact, by the method of half-relaxed limits, applied above, it is easy to prove that

$$\lim_{\varepsilon \to 0} V^\varepsilon(0,0) = v(0,0). \tag{2.10}$$

From (2.8)–(2.10) we conclude that $\mathscr{L} = v(0,0)$. The proof of Theorem 1.4 is complete.

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