Electron. Commun. Probab. **20** (2015), no. 74, 1–10.
DOI: 10.1214/ECP.v20-4332
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

A note on the extremal process of the supercritical Gaussian Free Field*

Alberto Chiarini† Alessandra Cipriani‡ Rajat Subhra Hazra§

Abstract

We consider both the infinite-volume discrete Gaussian Free Field (DGFF) and the DGFF with zero boundary conditions outside a finite box in dimension larger or equal to 3. We show that the associated extremal process converges to a Poisson point process. The result follows from an application of the Stein-Chen method from [5].

Keywords: Extremal process; Gaussian free field; point processes; Poisson approximation; Stein–Chen method.
AMS MSC 2010: 60G15; 60G30; 60G55; 60G70; 82B41.
Submitted to ECP on May 27, 2015, final version accepted on October 14, 2015.
Supersedes arXiv:1505.05324.

1 Introduction

In this article we study the behavior of the extremal process of the DGFF in dimension larger or equal to 3. This extends the result presented in [9] in which the convergence of the rescaled maximum of the infinite-volume DGFF and the 0-boundary condition field was shown. It was proved there that the field belongs to the maximal domain of attraction of the Gumbel distribution; hence, a natural question that arises is that of describing more precisely its extremal points. In dimension 2, this was carried out by [6, 7] complementing a result of [8] on the convergence of the maximum; namely, the characterization of the limiting point process with a random mean measure yields as by-product an integral representation of the maximum. The extremes of the DGFF in dimension 2 have deep connections with those of Branching Brownian Motion ([1, 2, 3, 4]). These works showed that the limiting point process is a randomly shifted decorated Poisson point process, and we refer to [15] for structural details. In $d \geq 3$, one does not get a non-trivial decoration but instead a Poisson point process analogous to the extremal process of independent Gaussian random variables. To be more precise, we let $E := [0,\, 1]^d \times (-\infty,\, +\infty]$ and $V_N := [0,\, n-1]^d \cap \mathbb{Z}^d$ the hypercube of volume $N = n^d$. Let $(\varphi_\alpha)_{\alpha \in \mathbb{Z}^d}$ be the infinite-volume DGFF, that is a centered Gaussian field on the square lattice with covariance $g(\cdot,\, \cdot)$, where g is the Green's function of the simple random walk. We define the following sequence of point processes on E:

$$\eta_n(\cdot) := \sum_{\alpha \in V_N} \varepsilon_{\left(\frac{\alpha}{n},\, \frac{\varphi_\alpha - b_N}{a_N}\right)}(\cdot) \tag{1.1}$$

*The first author's research is supported by RTG 1845. The last author's research was supported by Cumulative Professional Development Allowance from Ministry of Human Resource Development, Government of India, and Department of Science and Technology, Inspire funds.

†TU Berlin, Germany. E-mail: chiarini@math.tu-berlin.de
‡Weierstrass Institute, Germany. E-mail: Alessandra.Cipriani@wias-berlin.de
§Indian Statistical Institute, Kolkata, India. E-mail: rajatmaths@gmail.com

where $\varepsilon_x(\cdot)$, $x \in E$, is the point measure that gives mass one to a set containing x and zero otherwise, and

$$b_N := \sqrt{g(0)}\left[\sqrt{2\log N} - \frac{\log\log N + \log(4\pi)}{2\sqrt{2\log N}}\right], \qquad a_N := g(0)(b_N)^{-1}. \tag{1.2}$$

Here $g(0)$ denotes the variance of the DGFF. Our main result is

Theorem 1.1. *For the sequence of point processes η_n defined in* (1.1) *we have that*

$$\eta_n \xrightarrow{d} \eta,$$

as $n \to +\infty$, where η is a Poisson random measure on E with intensity measure given by $\mathrm{d}\, t \otimes (\mathrm{e}^{-z}\,\mathrm{d}\, z)$ where $\mathrm{d}\, t \otimes \mathrm{d}\, z$ is the Lebesgue measure on E, and $\xrightarrow{d}$ is the convergence in distribution on $\mathcal{M}_p(E)$[1]*.*

The proof is based on the application of the two-moment method of [5] that allows us to compare the extremal process of the DGFF and a Poisson point process with the same mean measure. To prove that the two processes converge, we will exploit a classical theorem by Kallenberg.

It is natural then to consider also convergence for the DGFF $(\psi_\alpha)_{\alpha\in\mathbb{Z}^d}$ with zero boundary conditions outside V_N. For the sequences of point measures

$$\rho_n(\cdot) := \sum_{\alpha\in V_N} \varepsilon_{\left(\frac{\alpha}{n},\, \frac{\psi_\alpha - b_N}{a_N}\right)}(\cdot) \tag{1.3}$$

we establish the following Theorem:

Theorem 1.2. *For the sequence of point processes ρ_n defined in* (1.3) *we have that*

$$\rho_n \xrightarrow{d} \eta,$$

as $n \to +\infty$ in $\mathcal{M}_p(E)$, where η is as in Theorem 1.1.

The convergence is shown by reducing ourselves to check the conditions of Kallenberg's Theorem on the bulk of V_N, where we have a good control on the drift of the conditioned field, and then by showing that the process on the whole of V_N and on the bulk are close as n becomes large.

The outline of the paper is as follows. In Section 2 we will recall the definition of DGFF and the Stein-Chen method, while Section 3 and Section 4 are devoted to the proofs of Theorems 1.1 and 1.2 respectively.

2 Preliminaries

2.1 The DGFF

Let $d \geq 3$ and denote with $\|\cdot\|$ the ℓ_∞-norm on $\mathbb{Z}^d$. Let $\psi = (\psi_\alpha)_{\alpha\in\mathbb{Z}^d}$ be a discrete Gaussian Free Field with zero boundary conditions outside $\Lambda \subset \mathbb{Z}^d$. On the space $\Omega := \mathbb{R}^{\mathbb{Z}^d}$ endowed with its product topology, its law $\widetilde{\mathsf{P}}_\Lambda$ can be explicitly written as

$$\widetilde{\mathsf{P}}_\Lambda(\mathrm{d}\,\psi) = \frac{1}{Z_\Lambda}\exp\left(-\frac{1}{4d}\sum_{\alpha,\,\beta\in\mathbb{Z}^d\colon\|\alpha-\beta\|=1}(\psi_\alpha - \psi_\beta)^2\right)\prod_{\alpha\in\Lambda}\mathrm{d}\,\psi_\alpha\prod_{\alpha\in\mathbb{Z}^d\setminus\Lambda}\varepsilon_0(\psi_\alpha).$$

In other words $\psi_\alpha = 0$ $\widetilde{\mathsf{P}}_\Lambda$-a. s. if $\alpha \in \mathbb{Z}^d\setminus\Lambda$, and $(\psi_\alpha)_{\alpha\in\Lambda}$ is a multivariate Gaussian random variable with mean zero and covariance $(g_\Lambda(\alpha,\,\beta))_{\alpha,\,\beta\in\mathbb{Z}^d}$, where g_Λ is the Green's

[1] $\mathcal{M}_p(E)$ denotes the set of (Radon) point measures on E endowed with the topology of vague convergence.

function of the discrete Laplacian problem with Dirichlet boundary conditions outside Λ. For a thorough review on the model the reader can refer for example to [16]. It is known [10, Chapter 13] that the finite-volume measure ψ admits an infinite-volume limit as $\Lambda \uparrow \mathbb{Z}^d$ in the weak topology of probability measures. This field will be denoted as $\varphi = (\varphi_\alpha)_{\alpha \in \mathbb{Z}^d}$. It is a centered Gaussian field with covariance matrix $g(\alpha, \beta)$ for $\alpha, \beta \in \mathbb{Z}^d$. With a slight abuse of notation, we write $g(\alpha - \beta)$ for $g(0, \alpha - \beta)$ and also $g_\Lambda(\alpha) = g_\Lambda(\alpha, \alpha)$. g admits a so-called random walk representation: if $\mathbb{P}_\alpha$ denotes the law of a simple random walk S started at $\alpha \in \mathbb{Z}^d$, then

$$g(\alpha, \beta) = \mathbb{E}_\alpha \left[\sum_{n \geq 0} \mathbb{1}_{\{S_n = \beta\}} \right].$$

In particular this gives $g(0) < +\infty$ for $d \geq 3$. A comparison of the covariances in the infinite and finite-volume is possible in the *bulk* of V_N: for $\delta > 0$ this is defined as

$$V_N^\delta := \left\{ \alpha \in V_N : \|\alpha - \beta\| > \delta n, \forall \beta \in \mathbb{Z}^d \setminus V_N \right\}. \tag{2.1}$$

In order to compare covariances in the finite and infinite-volume field, we recall the following Lemma, whose proof is presented in [9, Lemma 7]).

Lemma 2.1. *For any $\delta > 0$ and $\alpha, \beta \in V_N^\delta$ one has*

$$g(\alpha, \beta) - C_d \left(\delta N^{1/d}\right)^{2-d} \leq g_{V_N}(\alpha, \beta) \leq g(\alpha, \beta). \tag{2.2}$$

In particular we have, $g_{V_N}(\alpha) = g(0)\left(1 + \mathrm{O}\left(N^{(2-d)/d}\right)\right)$ uniformly for $\alpha \in V_N^\delta$.

2.2 The Stein-Chen method

As main tool of this article we will use (and restate here) a theorem from [5]. Consider a sequence of Bernoulli random variables $(X_\alpha)_{\alpha \in \mathcal{I}}$ where $X_\alpha \sim Be(p_\alpha)$ and $\mathcal{I}$ is some index set. For each α we define a subset $B_\alpha \subseteq \mathcal{I}$ which we consider a "neighborhood" of dependence for the variable X_α, such that X_α is nearly independent from X_β if $\beta \in \mathcal{I} \setminus B_\alpha$. Set

$$b_1 := \sum_{\alpha \in \mathcal{I}} \sum_{\beta \in B_\alpha} p_\alpha p_\beta,$$

$$b_2 := \sum_{\alpha \in \mathcal{I}} \sum_{\alpha \neq \beta \in B_\alpha} \mathsf{E}\left[X_\alpha X_\beta\right],$$

$$b_3 := \sum_{\alpha \in \mathcal{I}} \mathsf{E}\left[\left|\mathsf{E}\left[X_\alpha - p_\alpha \mid \mathcal{H}_1\right]\right|\right]$$

where

$$\mathcal{H}_1 := \sigma\left(X_\beta : \beta \in \mathcal{I} \setminus B_\alpha\right).$$

Theorem 2.2 ([5, Theorem 2])**.** *Let $\mathcal{I}$ be an index set. Partition the index set $\mathcal{I}$ into disjoint non-empty sets $\mathcal{I}_1, \ldots, \mathcal{I}_k$. For any $\alpha \in \mathcal{I}$, let $(X_\alpha)_{\alpha \in \mathcal{I}}$ be a dependent Bernoulli process with parameter p_α. Let $(Y_\alpha)_{\alpha \in \mathcal{I}}$ be independent Poisson random variables with intensity p_α. Also let*

$$W_j := \sum_{\alpha \in \mathcal{I}_j} X_\alpha \quad \text{and} \quad Z_j := \sum_{\alpha \in \mathcal{I}_j} Y_\alpha \quad \text{and} \quad \lambda_j := \mathsf{E}[W_j] = \mathsf{E}[Z_j].$$

Then

$$\|\mathcal{L}(W_1, \ldots, W_k) - \mathcal{L}(Z_1, \ldots, Z_k)\|_{TV} \leq 2 \min\left\{1,\, 1.4 \left(\min_{1 \leq j \leq k} \lambda_j\right)^{-1/2}\right\} (2b_1 + 2b_2 + b_3) \tag{2.3}$$

where $\|\cdot\|_{TV}$ denotes the total variation distance and $\mathcal{L}(W_1, \ldots, W_k)$ denotes the joint law of these random variables.

3 Proof of Theorem 1.1: the infinite-volume case

Proof. We recall that $E = [0\,,1]^d \times (-\infty, +\infty]$ and $V_N = [0,\, n-1]^d \cap \mathbb{Z}^d$. To show the convergence of η_n to η, we will exploit Kallenberg's theorem [11, Theorem 4.7]. According to it, we need to verify the following conditions:

i) for any A, a bounded rectangle[2] in $[0,1]^d$, and $(x,y] \subset (-\infty,+\infty]$

$$\mathsf{E}[\eta_n(A\times(x,y])] \to \mathsf{E}[\eta(A\times(x,y])] = |A|(\mathrm{e}^{-x}-\mathrm{e}^{-y}).$$

We adopt the convention $\mathrm{e}^{-\infty}=0$ and the notation $|A|$ for the Lebesgue measure of A.

ii) For all $k\geq 1$, and $A_1,\, A_2,\, \dots,\, A_k$ disjoint rectangles in $[0,1]^d$ and $R_1,\, R_2,\, \dots,\, R_k$, each of which is a finite union of disjoint intervals of the type $(x,\, y] \subset (-\infty,+\infty]$,

$$\begin{aligned}&\mathsf{P}\left(\eta_n(A_1\times R_1)=0,\,\dots,\,\eta_n(A_k\times R_k)=0\right)\\ &\quad\to \mathsf{P}\left(\eta(A_1\times R_1)=0,\,\dots,\,\eta(A_k\times R_k)=0\right) = \exp\left(-\sum_{j=1}^{k}|A_j|\omega\left(R_j\right)\right) \qquad (3.1)\end{aligned}$$

where $\omega(\mathrm{d}\,z) := \mathrm{e}^{-z}\,\mathrm{d}\,z$.

Let us denote by $u_N(z) := a_N z + b_N$. The first condition follows by Mills ratio

$$\left(1-\frac{1}{t^2}\right)\frac{\mathrm{e}^{-t^2/2}}{\sqrt{2\pi}t} \leq \mathsf{P}\left(\mathcal{N}(0,\,1)>t\right) \leq \frac{\mathrm{e}^{-t^2/2}}{\sqrt{2\pi}t}, \quad t>0. \qquad (3.2)$$

More precisely

$$\begin{aligned}\mathsf{E}[\eta_n(A\times(x,y])] &= \sum_{\alpha\in nA\cap V_N}\mathsf{P}\left(\varphi_\alpha\in(u_N(x),u_N(y)]\right)\\ &\leq \sum_{\alpha\in nA\cap V_N}\left(\frac{\mathrm{e}^{-\frac{u_N(x)^2}{2g(0)}}}{\sqrt{2\pi}u_N(x)} - \frac{\mathrm{e}^{-\frac{u_N(y)^2}{2g(0)}}}{\sqrt{2\pi}u_N(y)}\left(1-\frac{1}{u_N(y)^2}\right)\right) \qquad (3.3)\\ &\leq |nA\cap V_N|\left(\frac{\mathrm{e}^{-x+\mathrm{o}(1)}}{N} - \frac{\mathrm{e}^{-y+\mathrm{o}(1)}}{N}\left(1-\frac{1}{2g(0)\log N(1+\mathrm{o}\,(1))}\right)\right)\\ &\to |A|(\mathrm{e}^{-x}-\mathrm{e}^{-y}). \qquad (3.4)\end{aligned}$$

Similarly, one can plug in (3.3) the reverse bounds of (3.2) to prove the lower bound, and thus condition i).

To show ii), we need a few more details. Let $k\geq 1$, $A_1,\,\dots,\,A_k$ and $R_1,\dots,R_k$ be as in the assumptions. Let us denote by $\mathcal{I}_j = nA_j\cap V_N$ and $\mathcal{I} = \mathcal{I}_1\cup\ldots\cup\mathcal{I}_k$. For $\alpha\in\mathcal{I}_j$ define

$$X_\alpha := \mathbb{1}_{\left\{\frac{\varphi_\alpha - b_N}{a_N}\in R_j\right\}}$$

and $p_\alpha := \mathsf{P}\left((\varphi_\alpha - b_N)/a_N \in R_j\right)$. Choose now a small $\epsilon>0$ and fix the neighborhood of dependence $B_\alpha := B\left(\alpha, (\log N)^{2+2\epsilon}\right)\cap\mathcal{I}$[3] for $\alpha\in\mathcal{I}$. Let $W_j := \sum_{\alpha\in\mathcal{I}_j}X_\alpha$ and Z_j be as in Theorem 2.2.

By the simple observation that

$$\mathsf{P}\left(\eta_n(A_1\times R_1)=0,\,\dots,\,\eta_n(A_k\times R_k)=0\right) = \mathsf{P}\left(W_1=0,\,\dots,\,W_k=0\right),$$

[2] A *bounded rectangle* has the form $J_1\times\cdots\times J_d$ with $J_i = [0,\,1]\cap(a_i,\,b_i]$, $a_i,\,b_i\in\mathbb{R}$ for all $1\leq i\leq d$.
[3] $B(x,r)$ denotes a ball of radius r centered at x

to prove the convergence (3.1), we can use Theorem 2.2 and show that the error bound on the RHS of (2.3) goes to 0.

First we bound b_1 as follows. By definition of $R_1, R_2, \ldots, R_k$, there exists $z \in \mathbb{R}$ such that $R_j \subset (z, +\infty]$ for $1 \le j \le k$. Hence for any $1 \le j \le k$, for any $\alpha \in \mathcal{I}_j$ we have that

$$p_\alpha = \mathsf{P}\left(\frac{\varphi_\alpha - b_N}{a_N} \in R_j\right) \le \mathsf{P}(\varphi_\alpha > u_N(z)) \overset{(3.2)}{\le} \frac{\mathrm{e}^{-\frac{u_N(z)^2}{2g(0)}}}{\sqrt{2\pi} u_N(z)} \sqrt{g(0)}.$$

The bound is independent of α and j, therefore for some $C > 0$

$$b_1 \le CN(\log N)^{d(2+2\epsilon)} \mathrm{e}^{-2z} N^{-2} \to 0. \tag{3.5}$$

For b_2 note that it was shown in [9] that for $z \in \mathbb{R}$ and $\alpha \ne \beta \in V_N$

$$\mathsf{P}(\varphi_\alpha > u_N(z),\, \varphi_\beta > u_N(z)) \le \frac{(2-\kappa)^{3/2}}{\kappa^{1/2}} N^{-2/(2-\kappa)} \max\left\{\mathrm{e}^{-2z}\, \mathbb{1}_{\{z \le 0\}}, \mathrm{e}^{-2z/(2-\kappa)}\, \mathbb{1}_{\{z>0\}}\right\}. \tag{3.6}$$

Here we have introduced $\kappa := \mathbb{P}_0\left(\widetilde{H}_0 = +\infty\right) \in (0,1)$ and $\widetilde{H}_0 = \inf\{n \ge 1 : S_n = 0\}$. Observe that for any $1 \le j \le k$, $\alpha \in \mathcal{I}$ and $\beta \in B_\alpha$ one has

$$\mathsf{E}[X_\alpha X_\beta] \le \mathsf{P}(\varphi_\alpha > u_N(z), \varphi_\beta > u_N(z))$$

so that by (3.6) we can find some constant $C' > 0$ such that

$$b_2 \le C' N^{-\kappa/(2-\kappa)} (\log N)^{d(2+2\epsilon)} \max\left\{\mathrm{e}^{-2z}\, \mathbb{1}_{\{z \le 0\}}, \mathrm{e}^{-2z/(2-\kappa)}\, \mathbb{1}_{\{z>0\}}\right\} \to 0.$$

The error is similar to the estimate obtained in [9, Equation (8)]. Finally we need to handle b_3. From Section 2.2 we set for $\alpha \in \mathcal{I}$, $\mathcal{H}_1 := \sigma(X_\beta : \beta \in \mathcal{I} \setminus B_\alpha)$ and we define $\mathcal{H}_2 := \sigma(\varphi_\beta : \beta \in \mathcal{I} \setminus B_\alpha)$. We observe that

$$b_3 = \sum_{\alpha \in \mathcal{I}} \mathsf{E}\left[\left|\mathsf{E}\left[X_\alpha - p_\alpha \mid \mathcal{H}_1\right]\right|\right] \le \sum_{\alpha \in \mathcal{I}} \mathsf{E}\left[\left|\mathsf{E}\left[X_\alpha \mid \mathcal{H}_2\right] - p_\alpha\right|\right]$$

since $\mathcal{H}_1 \subseteq \mathcal{H}_2$ and using the tower property of the conditional expectation. Now denote by $U_\alpha := \mathbb{Z}^d \setminus (\mathcal{I} \setminus B_\alpha)$. Let us abbreviate $u_N(R_j) := \{u_N(y) : y \in R_j\}$. Then for $\alpha \in I_j$ and $1 \le j \le k$, by the Markov property of the DGFF [14, Lemma 1.2] we have that

$$\mathsf{E}\left[X_\alpha \mid \mathcal{H}_2\right] = \widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha \in u_N(R_j)) \qquad \mathsf{P}-a.\ s.$$

where $(\psi_\alpha)_{\alpha \in \mathbb{Z}^d}$ is a Gaussian Free Field with zero boundary conditions outside U_α and

$$\mu_\alpha = \sum_{\beta \in \mathcal{I} \setminus B_\alpha} \mathbb{P}_\alpha\left(H_{\mathcal{I}\setminus B_\alpha} < +\infty,\, S_{H_{\mathcal{I}\setminus B_\alpha}} = \beta\right) \varphi_\beta.$$

Here $H_\Lambda := \inf\{n \ge 0 : S_n \in \Lambda\}$, $\Lambda \subset \mathbb{Z}^d$. Now as in [9, Equation (10)] one can show, using the Markov property, that

$$\mathsf{Var}\left[\mu_\alpha\right] \le \sup_{\beta \in \mathcal{I}\setminus B_\alpha} g(\alpha, \beta) \le \frac{c}{(\log N)^{2(1+\epsilon)(d-2)}}$$

for some $c > 0$. Hence we get that there exists a constant $c' > 0$ (independent of α and j) such that

$$\mathsf{P}\left(|\mu_\alpha| > (u_N(z))^{-1-\epsilon}\right) \le c' \exp\left(-(\log N)^{(2d-5)(1+\epsilon)}\right). \tag{3.7}$$

Recalling that $R_j \subset (z, +\infty]$ for all $1 \le j \le k$, this immediately shows that for $d \ge 3$

$$\sum_{j=1}^{k} \sum_{\alpha \in \mathcal{I}_j} \mathsf{E}\left[\left|\widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha \in u_N(R_j)) - p_\alpha\right| \mathbb{1}_{\left\{|\mu_\alpha| > (u_N(z))^{-1-\epsilon}\right\}}\right] \to 0.$$

So to show that $b_3 \to 0$ we are left with proving

$$\sum_{j=1}^{k} \sum_{\alpha \in \mathcal{I}_j} \mathsf{E}\left[\left|\widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha \in u_N(R_j)) - p_\alpha\right| \mathbb{1}_{\left\{|\mu_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}}\right] \to 0. \tag{3.8}$$

We now focus on the term inside the summation. For this, first we write $R_j = \bigcup_{l=1}^m (w_l, r_l]$ with $-\infty < w_1 < r_1 < w_2 < \cdots < r_m \le +\infty$ for some $m \ge 1$. Hence, we can expand the difference in the absolute value of (3.8) as follows:

$$\begin{aligned}
&\left(p_\alpha - \widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha \in u_N(R_j))\right) \\
&= \sum_{l=1}^{m} \left(\mathsf{P}(\varphi_\alpha \in (u_N(w_l), u_N(r_l)]) - \widetilde{\mathsf{P}}_{U_\alpha}\left(\psi_\alpha + \mu_\alpha \in (u_N(w_l), u_N(r_l)]\right)\right) \\
&= \sum_{l=1}^{m} \left(\mathsf{P}(\varphi_\alpha > u_N(w_l)) - \widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(w_l))\right) \\
&- \sum_{l=1}^{m} \left(\mathsf{P}(\varphi_\alpha > u_N(r_l)) - \widetilde{\mathsf{P}}_{U_\alpha}\left(\psi_\alpha + \mu_\alpha > u_N(r_l)\right)\right)
\end{aligned} \tag{3.9}$$

(if $r_l = +\infty$ for some l, we conventionally set $\mathsf{P}(\varphi_\alpha > u_N(r_l)) = 0$ and similarly for the other summand). Using the triangular inequality in (3.8), it turns out that to finish it is enough to show that for an arbitrary $w \in \mathbb{R}$,

$$\sum_{\alpha \in \mathcal{I}} \mathsf{E}\left[\left|\widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(w)) - \mathsf{P}(\varphi_\alpha > u_N(w))\right| \mathbb{1}_{\left\{|\mu_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}}\right] \to 0. \tag{3.10}$$

For this, first we show that on $\mathcal{Q} := \left\{\mathsf{P}(\varphi_\alpha > u_N(w)) > \widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(w))\right\}$

$$T_{1,2} = \sum_{\alpha \in \mathcal{I}} \mathsf{E}\left[\left(\mathsf{P}(\varphi_\alpha > u_N(w)) - \widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(w))\right) \mathbb{1}_{\left\{|\mu_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}} \mathbb{1}_{\mathcal{Q}}\right] \to 0. \tag{3.11}$$

This follows from the same estimates of $T_{1,2}$ and Claim 6 of [9]. Indeed on $\mathcal{Q} \cap \left\{|\mu_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}$

$$\begin{aligned}
&\sum_{\alpha \in \mathcal{I}} \left(\mathsf{P}(\varphi_\alpha > u_N(w)) - \widetilde{\mathsf{P}}_{U_\alpha}\left(\psi_\alpha + \mu_\alpha > u_N(w)\right)\right) \\
&\le \sum_{\alpha \in \mathcal{I}} \frac{\sqrt{g(0)}\,\mathrm{e}^{-\frac{u_N(w)^2}{2g(0)}}}{\sqrt{2\pi}u_N(w)} \left(1 - (1 + \mathrm{o}(1)) \left(\frac{\sqrt{g_{U_\alpha}(\alpha)}u_N(w)\,\mathrm{e}^{\left(1 - \frac{g(0)}{g_{U_\alpha}(\alpha)}\right)\frac{u_N(w)^2}{2g(0)} + \mathrm{o}(1)}}{\sqrt{g(0)}u_N(w)(1 + \mathrm{o}(1))}\right)\right) \\
&\le CN \frac{\sqrt{g(0)}\,\mathrm{e}^{-\frac{u_N(w)^2}{2g(0)}}}{\sqrt{2\pi}u_N(w)} \mathrm{o}(1) = \mathrm{o}(1).
\end{aligned}$$

Similarly one can show that on the complementary event $\mathcal{Q}^{\mathrm{c}}$ (recall (3.11) for the definition of $\mathcal{Q}$)

$$T_{1,1} = \sum_{\alpha \in \mathcal{I}} \mathsf{E}\left[\left(\widetilde{\mathsf{P}}_{U_\alpha}(\psi_\alpha + \mu_\alpha > u_N(w)) - \mathsf{P}(\varphi_\alpha > u_N(w))\right) \mathbb{1}_{\left\{|\mu_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}} \mathbb{1}_{\mathcal{Q}^{\mathrm{c}}}\right] = \mathrm{o}(1).$$

This shows that $b_3 \to 0$. Hence from Theorem 2.2 it follows that

$$\left|\mathsf{P}(W_1 = 0, \ldots, W_k = 0) - \prod_{j=1}^{k} \mathsf{P}(Z_j = 0)\right| = \mathrm{o}(1),$$

having used the independence of the Z_j's. Notice that by definition Z_j is a Poisson random variable with intensity $\sum_{\alpha\in\mathcal{I}_j} \mathsf{P}\left((\varphi_\alpha - b_N)/a_N \in R_j\right)$. Decomposing R_j as a union of finite intervals and using Mills ratio, similarly to the argument leading to (3.4), one has

$$P(Z_j = 0) \to \exp(-|A_j|\omega(R_j))$$

(recall $\omega(R_j) = \int_{R_j} \mathrm{e}^{-z}\,\mathrm{d}\,z$). Hence it follows that

$$\prod_{j=1}^{k} \mathsf{P}(Z_j = 0) \to \exp\left(-\sum_{j=1}^{k} |A_j|\omega(R_j)\right), \tag{3.12}$$

which completes the proof of ii) and therefore of Theorem 1.1. □

4 Proof of Theorem 1.2: the finite-volume case

We will now show the theorem for the field with zero boundary conditions. As remarked in the Introduction, since on the bulk defined in (2.1) we have a good control on the conditioned field, we will first prove convergence therein, and then we will use a converging-together theorem to achieve the final limit. We will first need some notation used throughout the Section: first, we consider $(\psi_\alpha)_{\alpha\in V_N}$ with law $\widetilde{\mathsf{P}}_N := \widetilde{\mathsf{P}}_{V_N}$. We also use the shortcut $g_N(\cdot,\cdot) = g_{V_N}(\cdot,\cdot)$. We will need the notation $\mathcal{C}_K^+(E)$ for the set of positive, continuous and compactly supported functions on $E = [0,1]^d \times (-\infty, +\infty]$. We first begin with a lemma on the point process convergence on bulk. Define a point process on E by

$$\rho_n^\delta(\cdot) = \sum_{\alpha\in V_N^\delta} \varepsilon_{\left(\frac{\alpha}{n},\,\frac{\psi_\alpha - b_N}{a_N}\right)}(\cdot). \tag{4.1}$$

Lemma 4.1. *Let $\delta > 0$. On $\mathcal{M}_p(E)$, $\rho_n^\delta \xrightarrow{d} \rho^\delta$ where ρ^δ is a Poisson random measure with intensity $\mathrm{d}\,t_{|_{[\delta,1-\delta]^d}} \otimes (\mathrm{e}^{-x}\,\mathrm{d}\,x)$*[4].

Proof. We will show i) and ii) of Page 4 (and from which we will borrow the notation starting from now).
i) We begin with an upper bound on $\widetilde{\mathsf{E}}_N\left[\rho_n^\delta(A\times(x,y])\right]$:

$$\begin{aligned}
&\sum_{\alpha\in nA\cap V_N^\delta} \widetilde{\mathsf{P}}_N(\psi_\alpha > u_N(x)) - \widetilde{\mathsf{P}}_N(\psi_\alpha > u_N(y)) \\
&\overset{(3.2)}{\leq} \sum_{\alpha\in nA\cap V_N^\delta} \frac{\mathrm{e}^{-\frac{u_N(x)^2}{2g_N(\alpha)}}}{\sqrt{2\pi}u_N(x)}\sqrt{g_N(\alpha)} - \frac{\mathrm{e}^{-\frac{u_N(y)^2}{2g_N(\alpha)}}}{\sqrt{2\pi}u_N(y)}\sqrt{g_N(\alpha)}\,(1+\mathrm{o}\,(1)) \\
&\overset{\text{Lemma 2.1}}{=} \sum_{\alpha\in nA\cap V_N^\delta} \frac{\mathrm{e}^{-\frac{u_N(x)^2}{2g(0)(1+c_n)}}}{\sqrt{2\pi}u_N(x)}\sqrt{g(0)}(1+c_n) - \frac{\mathrm{e}^{-\frac{u_N(y)^2}{2g(0)(1+c_n)}}}{\sqrt{2\pi}u_N(y)}\sqrt{g(0)}\,(1+c_n) \\
&\overset{n\to+\infty}{\longrightarrow} (\mathrm{e}^{-x} - \mathrm{e}^{-y})\left|A\cap[\delta,\,1-\delta]^d\right|.
\end{aligned} \tag{4.2}$$

We stress that in the second step the error term $c_n := \mathrm{O}\left(n^{2-d}\right)$ coming from Lemma 2.1 guarantees the convergence in the last line. The lower bound follows similarly.
ii) To show the second condition we again use Theorem 2.2. Let $A_1,\dots,A_k$ and $R_1,\dots,R_k$ be as in proof of Theorem 1.1. Let $\mathcal{I}_j := nA_j \cap V_N^\delta$ and $\mathcal{I} = \mathcal{I}_1 \cup\dots\cup \mathcal{I}_k$. For $\epsilon > 0$ we are setting $B_\alpha := B\left(\alpha,\,(\log N)^{2(1+\epsilon)}\right) \cap \mathcal{I}$. Note that, albeit slightly different, we are using the same notations for the neighborhood of dependence and the index sets of Section 3,

[4] $\mathrm{d}\,t_{|_{[\delta,1-\delta]^d}}$ is the restriction of the Lebesgue measure to $[\delta, 1-\delta]^d$.

but no confusion should arise. Observe that there exists $z \in \mathbb{R}$ such that for all $1 \le j \le k$, $R_j \subset (z, \infty]$; we have

$$p_\alpha = \widetilde{\mathsf{P}}_N\left(\frac{\psi_\alpha - b_N}{a_N} \in u_N(R_j)\right) \le \widetilde{\mathsf{P}}_N\left(\psi_\alpha > u_N(z)\right) \overset{(3.2)}{\le} \frac{\mathrm{e}^{-\frac{u_N(z)^2}{2g(0)}}}{\sqrt{2\pi}u_N(z)}\sqrt{g(0)}$$

where we have also used the fact that $g_N(\alpha) \le g(0)$. The bound on b_1 (cf. Theorem 2.2) follows exactly as in (3.5) and yields that, for some $C > 0$,

$$b_1 \le CN(\log N)^{d(2+2\epsilon)}\,\mathrm{e}^{-2z}\,N^{-2} \to 0.$$

The calculation of b_2 can be performed similarly using the covariance matrix of the vector $(\psi_\alpha,\, \psi_\beta)$, $\alpha \neq \beta \in V_N^\delta$ and Lemma 2.1. This gives that for some $C,\, C' > 0$ independent of $\alpha, \beta \in V_N^\delta$

$$\begin{aligned} b_2 &\le \sum_{\alpha\in\mathcal{I}}\sum_{\beta\in B_\alpha}\frac{C}{\log N}\exp\left(-\frac{u_N(z)^2}{g(0)+g(\alpha-\beta)}\left(1+\mathrm{O}\left(N^{(2-d)/d}\right)\right)\right) \\ &\le C' N^{-\kappa/(2-\kappa)}(\log N)^{2d(1+\epsilon)}\max\left\{\mathrm{e}^{-2z}\,\mathbb{1}_{\{z\le 0\}},\, \mathrm{e}^{-2z/(2-\kappa)}\,\mathbb{1}_{\{z>0\}}\right\} \to 0 \end{aligned}$$

(cf. [9, Equation (8)]). Note the estimate for b_2 is exactly same as in the infinite volume case.

We will now pass to b_3. We repeat our choice of $\mathcal{H}_1 = \sigma\left(X_\beta : \beta \in \mathcal{I} \setminus B_\alpha\right)$ and $\mathcal{H}_2 = \sigma\left(\psi_\beta : \beta \in \mathcal{I}\setminus B_\alpha\right)$ so that b_3 becomes

$$\sum_{j=1}^{k}\sum_{\alpha\in\mathcal{I}_j}\widetilde{\mathsf{E}}_N\left[\left|\widetilde{\mathsf{E}}_N\left[X_\alpha - p_\alpha|\mathcal{H}_1\right]\right|\right] \le \sum_{j=1}^{k}\sum_{\alpha\in\mathcal{I}_j}\widetilde{\mathsf{E}}_N\left[\left|\widetilde{\mathsf{E}}_N\left[X_\alpha|\mathcal{H}_2\right] - p_\alpha\right|\right].$$

We define $U_\alpha := V_N \setminus (\mathcal{I}\setminus B_\alpha)$. By the Markov property of the DGFF

$$\widetilde{\mathsf{E}}_N\left[X_\alpha \mid \mathcal{H}_2\right] = \widetilde{\mathsf{P}}_{U_\alpha}(\xi_\alpha + h_\alpha \in u_N(R_j)) \quad \widetilde{\mathsf{P}}_N - a.\ s. \tag{4.3}$$

for $(\xi_\alpha)_{\alpha\in\mathbb{Z}^d}$ a DGFF with law $\widetilde{\mathsf{P}}_{U_\alpha}$ and $(h_\alpha)_{\alpha\in\mathbb{Z}^d}$ is independent of ξ. From [9, Equation (28)] we can see that, for any $\alpha \in V_N^\delta$ and N large enough such that $B\left(\alpha,\, (\log N)^{2(1+\epsilon)}\right) \subsetneq V_N$,

$$\begin{aligned} \mathsf{Var}\left[h_\alpha\right] &= \sum_{\beta\in\mathcal{I}\setminus B_\alpha}\mathbb{P}_\alpha\left(H_{\mathcal{I}\setminus B_\alpha} < +\infty,\, S_{H_{\mathcal{I}\setminus B_\alpha}} = \beta\right) g_N(\alpha,\, \beta) \\ &\le \sup_{\beta\in\mathcal{I}\setminus B_\alpha} g_N(\alpha,\, \beta) \le \frac{c}{(\log N)^{2(1+\epsilon)(d-2)}}. \end{aligned}$$

This yields

$$\sum_{j=1}^{k}\sum_{\alpha\in\mathcal{I}_j}\widetilde{\mathsf{E}}_N\left[\left|\widetilde{\mathsf{P}}_{U_\alpha}(\xi_\alpha + h_\alpha) > u_N(R_j)) - p_\alpha\right|\mathbb{1}_{\left\{|h_\alpha| > (u_N(z))^{-1-\epsilon}\right\}}\right] \to 0. \tag{4.4}$$

It then suffices to show

$$\sum_{j=1}^{k}\sum_{\alpha\in\mathcal{I}_j}\widetilde{\mathsf{E}}_N\left[\left|\widetilde{\mathsf{P}}_{U_\alpha}(\xi_\alpha + h_\alpha) > u_N(R_j)) - p_\alpha\right|\mathbb{1}_{\left\{|h_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}}\right] \to 0. \tag{4.5}$$

One sees that the breaking up (3.9) can be performed also here replacing φ_α and ψ_α (with their laws) with ψ_α and ξ_α (with their laws) respectively, and μ_α with h_α. Accordingly, it is enough to show that

$$\sum_{\alpha\in\mathcal{I}}\widetilde{\mathsf{E}}_N\left[\left|\widetilde{\mathsf{P}}_{U_\alpha}(\xi_\alpha + h_\alpha > u_N(w)) - \widetilde{\mathsf{P}}_N(\psi_\alpha > u_N(w))\right|\mathbb{1}_{\left\{|h_\alpha| \le (u_N(z))^{-1-\epsilon}\right\}}\right] \to 0 \tag{4.6}$$

for all $w \in \mathbb{R}$. To this aim, we choose for any $w \in \mathbb{R}$ the event

$$\mathcal{Q}' := \left\{ \widetilde{\mathsf{P}}_N(\psi_\alpha > u_N(w)) > \widetilde{\mathsf{P}}_{U_\alpha}(\xi_\alpha + h_\alpha > u_N(w)) \right\}$$

and we proceed as in (3.11) with the help of Lemma 2.1 to show (4.6). Given this, the convergence $b_3 \to 0$ is finally ensured. Hence we can conclude that

$$\|\mathcal{L}(W_1, \ldots, W_k) - \mathcal{L}(Z_1, \ldots, Z_k)\|_{TV} \to 0$$

where Z_j are i. i. d. Poisson of mean p_α. By Mills ratio, as in (4.2) we see that

$$\mathsf{P}(Z_j = 0) \to \exp\left(-\left|A_j \cap [\delta,\, 1-\delta]^d\right| \omega(R_j)\right).$$

From this it follows that the two conditions i) and ii) of Kallenberg's Theorem are satisfied, and thus we obtain the convergence to a Poisson point process with mean measure given in i). □

Proof of Theorem 1.2. $\mathcal{M}_p(E)$ is a Polish space with metric d_p:

$$\mathrm{d}_p(\mu,\, \mu') = \sum_{i \geq 1} \frac{\min\{|\mu(f_i) - \mu'(f_i)|\,,\, 1\}}{2^i}, \quad \mu,\, \mu' \in \mathcal{M}_p(E)$$

for a sequence of functions $f_i \in \mathcal{C}_K^+(E)$ (cf. [12, Section 3.3]). Therefore we are in the condition to use a converging-together theorem [13, Theorem 3.5], namely to prove that $\rho_n \stackrel{d}{\to} \eta$ it is enough to show the following:

(a) $\rho_n^\delta \stackrel{d}{\to} \rho^\delta$, as $n \to +\infty$.

(b) $\rho^\delta \stackrel{d}{\to} \eta$ as $\delta \to 0$.

(c) For every $\epsilon > 0$,

$$\lim_{\delta \to 0} \lim_{n \to +\infty} \widetilde{\mathsf{P}}_N\left(\mathrm{d}_p\left(\rho_n, \rho_n^\delta\right) > \epsilon\right) = 0. \tag{4.7}$$

Note that by Lemma 4.1, (a) is satisfied. For $f \in \mathcal{C}_K^+(E)$, the Laplace functional of ρ^δ is given by (cf. [12, Prop. 3.6])

$$\Psi_\delta(f) := \mathsf{E}\left[\exp\left(-\rho^\delta(f)\right)\right] = \exp\left(-\int_E \left(1 - \mathrm{e}^{-f(t,x)}\right) \mathrm{d}\,t_{|_{[\delta,1-\delta]^d}}\, \mathrm{e}^{-x}\, \mathrm{d}\,x\right).$$

Hence by the dominated convergence theorem we can exchange limit and expectation as $\delta \to 0$ to obtain that

$$\Psi_\delta(f) \to \exp\left(-\int_E \left(1 - \mathrm{e}^{-f(t,x)}\right) \mathrm{d}\,t\, \mathrm{e}^{-x}\, \mathrm{d}\,x\right)$$

and the right hand side is the Laplace functional of η at f. This shows (b).

Hence to complete the proof it is enough to show (4.7). Thanks to the definition of the metric d_p it suffices to prove that for $f \in \mathcal{C}_K^+(E)$ and for $\epsilon > 0$

$$\limsup_{\delta \to 0} \lim_{n \to +\infty} \widetilde{\mathsf{P}}_N\left(\left|\rho_n(f) - \rho_n^\delta(f)\right| > \epsilon\right) = 0.$$

Without loss of generality assume that the support of f is contained in $[0,1]^d \times [z_0,\, +\infty)$ for some $z_0 \in \mathbb{R}$. Choosing n large enough such that $u_N(z_0) > 0$ and $g_N(\alpha) \leq g(0)$, we obtain that

$$\begin{aligned}\widetilde{\mathsf{E}}_N\left[\left|\rho_n(f)-\rho_n^\delta(f)\right|\right] &= \widetilde{\mathsf{E}}_N\left[\sum_{\alpha\in V_N\setminus V_N^\delta} f\left(\frac{\alpha}{n},\frac{\psi_\alpha-b_N}{a_N}\right)\mathbb{1}_{\left\{\frac{\psi_\alpha-b_N}{a_N}>z_0\right\}}\right]\\ &\le \sup_{z\in E}|f(z)|\sum_{\alpha\in V_N\setminus V_N^\delta}\widetilde{\mathsf{P}}_N\left(\frac{\psi_\alpha-b_N}{a_N}>z_0\right)\overset{(3.2)}{\le} C\sum_{\alpha\in V_N\setminus V_N^\delta}\frac{\mathrm{e}^{-u_N(z_0)^2/g(0)}}{\sqrt{2\pi}u_N(z_0)}\sqrt{g(0)}\\ &\le C'\left(1-(1-2\delta)^d\right)\mathrm{e}^{-z_0}\end{aligned}$$

as $n\to+\infty$ for some positive constants C, C'. Now letting $\delta\to 0$ the result follows and this completes the proof.

□

References

[1] E. Aïdékon, J. Berestycki, É. Brunet, and Z. Shi, *Branching Brownian motion seen from its tip*, Probability Theory and Related Fields **157** (2013), no. 1-2, 405–451 (English). MR-3101852

[2] L.-P. Arguin, A. Bovier, and N. Kistler, *Genealogy of extremal particles of branching Brownian motion*, Communications on Pure and Applied Mathematics **64** (2011), no. 12, 1647–1676. MR-2838339

[3] Louis-Pierre Arguin, Anton Bovier, and Nicola Kistler, *Poissonian statistics in the extremal process of branching brownian motion*, Ann. Appl. Probab. **22** (2012), no. 4, 1693–1711. MR-2985174

[4] ———, *The extremal process of branching Brownian motion*, Probability Theory and Related Fields **157** (2013), no. 3-4, 535–574 (English). MR-3129797

[5] R. Arratia, L. Goldstein, and L. Gordon, *Two Moments Suffice for Poisson Approximations: The Chen-Stein Method*, Ann. Probab. **17** (1989), no. 1, 9–25. MR-0972770

[6] M. Biskup and O. Louidor, *Extreme local extrema of two-dimensional discrete Gaussian free field*, ArXiv e-prints (2013), `http://adsabs.harvard.edu/abs/2013arXiv1306.2602B`.

[7] ———, *Conformal symmetries in the extremal process of two-dimensional discrete Gaussian Free Field*, ArXiv e-prints (2014), `http://adsabs.harvard.edu/abs/2014arXiv1410.4676B`.

[8] M. Bramson, J. Ding, and O. Zeitouni, *Convergence in law of the maximum of the two-dimensional discrete Gaussian free field*, ArXiv e-prints (2013), `http://arxiv.org/abs/1301.6669v4`.

[9] Alberto Chiarini, Alessandra Cipriani, and Rajat Subhra Hazra, *Extremes of the supercritical Gaussian Free Field*, arXiv preprint (2015).

[10] H. O. Georgii, *Gibbs measures and phase transitions*, de Gruyter, Berlin, 1988. MR-0956646

[11] O. Kallenberg, *Random measures*, Akademie-Verlag, 1983. MR-0818219

[12] S.I. Resnick, *Extreme values, regular variation, and point processes*, Applied probability : a series of the applied probability trust, Springer, 1987. MR-0900810

[13] ———, *Heavy-tail phenomena: Probabilistic and statistical modeling*, Heavy-tail Phenomena: Probabilistic and Statistical Modeling, no. Bd. 10, Springer, 2007. MR-2271424

[14] Pierre-François Rodriguez and Alain-Sol Sznitman, *Phase transition and level-set percolation for the gaussian free field*, Communications in Mathematical Physics **320** (2013), no. 2, 571–601 (English). MR-3053773

[15] Eliran Subag and Ofer Zeitouni, *Freezing and Decorated Poisson Point Processes*, Communications in Mathematical Physics **337** (2015), no. 1, 55–92 (English). MR-3324155

[16] A.S. Sznitman, *Topics in Occupation Times and Gaussian Free Fields*, Zurich Lectures in Advanced Mathematics, American Mathematical Society, 2012. MR-2932978

Acknowledgments. The authors thank the anonymous referee for the suggestions which improved the presentation of the article.