

Large cycles in random permutations related to the Heisenberg model*

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Dedicated to Svante Janson on the occasion of his 60'th birthday

Abstract

We study the weighted version of the interchange process where a permutation receives weight $\theta^{\#\text{cycles}}$. For $\theta = 2$ this is Tóth's representation of the quantum Heisenberg ferromagnet on the complete graph. We prove, for $\theta > 1$, that large cycles appear at 'low temperature'.

Keywords: Interchange process; Heisenberg model.

AMS MSC 2010: 60K35; 60B15.

Submitted to ECP on May 25, 2015, final version accepted on July 24, 2015.

Supersedes arXiv:1505.06716.

1 Introduction

The interchange process and related models of random permutations are interesting both for their beautiful mathematics, and for their relevance to quantum theoretical models for magnetization. The interchange process may be described as follows. Fix an integer n and put n labelled balls into n labelled boxes, ball i in box i . At each time $t = 1, 2, \dots$, select uniformly (independently) a pair of distinct boxes i, j and transpose the balls inside them. At a given time t , box i contains some ball $\pi_t(i)$, where π_t is a permutation of $1, 2, \dots, n$. Said otherwise, π_t is the composition of t independent, uniformly chosen transpositions.

Being a permutation, π_t can be written as a product of disjoint cycles. Schramm showed in [11], proving a conjecture of Aldous in [4], that if t is of the form $\lfloor cn \rfloor$ with $c > 1/2$, then with probability approaching 1 as $n \rightarrow \infty$, the largest cycle has size of order n (for $c < 1/2$ it is of order $\log n$). He also described the scaling limit of the cycles in terms of the Poisson–Dirichlet distribution.

In this paper we study random permutations which are biased towards having many small cycles. A precise definition is given in the next subsection, but roughly speaking we consider the weighted version of the interchange process where each permutation π receives a weight $\theta^{\ell(\pi)}$. Here $\theta \geq 1$ is fixed, and $\ell(\pi)$ is the total number of cycles in π . For $\theta = 1$ one recovers the interchange process. Our main result (Theorem 1.1) is that large cycles appear for $c > \theta/2$.

The model is motivated by considerations in statistical physics, where it provides a probabilistic representation of the (ferromagnetic) quantum Heisenberg model on the

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complete graph K_n . It is a notorious open problem to prove that the quantum Heisenberg ferromagnet on the lattice \mathbb{Z}^d , $d \geq 3$, can exhibit a nonzero magnetization.

1.1 Model and main result

We will in fact work in continuous time, and with a different time-scaling than described above. Let $G = K_n = (V, E)$ be the complete graph on the vertex set $V = \{1, \dots, n\}$, with edge set $E = \binom{V}{2}$. Let $\beta > 0$ and let $\mathbb{P}_1(\cdot)$ denote a probability measure governing a collection $\omega = (\omega_{xy} : xy \in E)$ of independent rate 1 Poisson processes on $[0, \beta]$, indexed by the edges. If ω_{xy} has an event at time $t \in [0, \beta]$ we write $(xy, t) \in \omega$. We think of such an event as a transposition of the vertices x, y at time t . The time-ordered product of these transpositions gives a permutation $\pi = \pi(\omega)$ of V . More precisely, if we write $(x_i y_i, t_i)$, $i = 1, \dots, N$, for the points of ω , indexed so that $0 < t_1 < \dots < t_N < \beta$, and write $\tau_i = (x_i, y_i)$ for the transposition of x_i and y_i , then we have that $\pi = \tau_N \cdots \tau_2 \tau_1$.

Let $\ell = \ell(\omega)$ denote the number of cycles in a disjoint-cycle decomposition of $\pi(\omega)$, including singletons. Let $\mathcal{C}_1(\pi), \dots, \mathcal{C}_\ell(\pi)$ denote the cycles ordered by decreasing size (breaking ties by any rule). For $\theta \geq 1$ we will consider the distribution of ω and $\pi(\omega)$ under the probability measure $\mathbb{P}_\theta(\cdot)$ given by

$$\frac{d\mathbb{P}_\theta}{d\mathbb{P}_1}(\omega) = \frac{\theta^{\ell(\omega)}}{Z}. \tag{1.1}$$

Here $Z = Z(\theta, \beta)$ is the appropriate normalization. (The same definition makes sense on a general finite graph G .)

For $\theta = 1$ our model is the continuous-time version of the interchange process, sped up by a factor $\binom{n}{2}$ compared to the introduction, viewed at time β . We take β of the form $\beta = \lambda/n$ where $\lambda > 0$ is constant.

Theorem 1.1. *Let $G = K_n$ and $\beta = \lambda/n$ with $\lambda > \theta > 1$. For each $\varepsilon > 0$ there is $\delta > 0$ such that, for n large enough,*

$$\mathbb{P}_\theta(|\mathcal{C}_1(\pi)| \geq \delta n) \geq 1 - \varepsilon.$$

Our proof of Theorem 1.1 relies on a colouring-lemma inspired by the approach of Bollobás, Grimmett and Janson [6] to the random-cluster model. Roughly speaking, if we sample ω and then colour each cycle red or white independently, with probability $1/\theta$ for red, then the conditional distribution of the red cycles is determined by an interchange process. See Lemma 2.2 for a precise statement. When $\lambda > \theta$, we can use the results of Schramm [11] to show that there are red cycles of order n .

With high probability there are no cycles having size of order n when λ is small enough, e.g. when $\lambda < e^{-1}$. This follows from [7, Theorem 6.1]. One would expect that there is a critical value $\lambda_c(\theta)$ such that there are cycles of order n for $\lambda > \lambda_c$ but not for $\lambda < \lambda_c$. Schramm’s result shows that $\lambda_c(1) = 1$. It follows from the work of Tóth [12] that $\lambda_c(2) = 2$. For other values of θ the existence of λ_c is not known, and Theorem 1.1 is the first result on the occurrence of large cycles in this generality.

Regarding other choices for G , the interchange process ($\theta = 1$) has been investigated on general graphs by Alon and Kozma [1], and on infinite trees by Angel [2] and by Hammond [8, 9]. In ongoing work, Kotecký, Miłoś and Ueltschi are investigating cases with $\theta \neq 1$ on the hypercube.

1.2 Relation to the Heisenberg model

For $\theta = 2$ the cycles in our model represent correlations in the (ferromagnetic, quantum) Heisenberg model, as shown by Tóth [13]. Here is a brief account, see the review [7] for more details.

The Heisenberg model on G is given by the Hamiltonian

$$H = -2 \sum_{xy \in E} (\sigma_x^{(1)} \sigma_y^{(1)} + \sigma_x^{(2)} \sigma_y^{(2)} + \sigma_x^{(3)} \sigma_y^{(3)}).$$

Here

$$\sigma^{(1)} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^{(2)} = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^{(3)} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are spin- $\frac{1}{2}$ Pauli matrices, and $\sigma_x^{(j)}$ acts on the Hilbert space $\mathcal{H}_V = \bigotimes_{x \in V} \mathbb{C}^2$ as $\sigma^{(j)} \otimes \text{Id}_{V \setminus \{x\}}$. Magnetic correlations between vertices $x, y \in G$ are given by the *correlation functions*

$$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle := \frac{\text{tr}(\sigma_x^{(3)} \sigma_y^{(3)} e^{-\beta H})}{\text{tr}(e^{-\beta H})}, \tag{1.2}$$

where $\text{tr}(\cdot)$ denotes the trace of a matrix. In this formulation the parameter $\beta > 0$ is usually called the *inverse temperature*. (It is the same β as in Section 1.1.)

Tóth’s representation expresses the correlations (1.2) probabilistically. Write $\{x \leftrightarrow y\}$ for the event that x and y belong to the same cycle in the permutation $\pi(\omega)$. Then we have, with $\theta = 2$:

$$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle = \frac{1}{4} \mathbb{P}_2(x \leftrightarrow y).$$

Thus the occurrence of large cycles in $\pi(\omega)$ corresponds, in physical terms, to magnetic ordering.

The quantum model also possesses other probabilistic representations. In the paper [12] Tóth studied a *lattice gas* representation and explicitly computed the free energy. (The same result was independently obtained by Penrose [10].) By standard arguments one may deduce from these results that the quantum Heisenberg model on $G = K_n$ undergoes a phase transition at $\beta = 2/n$, as mentioned above.

Outline and notation

We describe a graphical representation, and present the key colouring-lemma, in Section 2, followed by the proof of Theorem 1.1 in Section 3.

The abbreviation i.i.d. means independent and identically distributed. The identity permutation is denoted id . Unspecified limits are for $n \rightarrow \infty$. If $a_n/b_n \rightarrow 0$ then we write $a_n = o(b_n)$, if a_n/b_n is bounded above then we write $a_n = O(b_n)$. The indicator of an event A is written \mathbb{I}_A , and takes the value 1 if A happens, otherwise 0. Expectation with respect to \mathbb{P}_θ will be written \mathbb{E}_θ .

2 Colouring-lemma

The following graphical representation of a sample ω will be useful. We picture ω in $G \times [0, \beta]$, representing a point $(xy, t) \in \omega$ by a ‘cross’ as in Figure 1. The crosses decompose $V \times [0, \beta]$ into a collection $\mathcal{L}(\omega)$ of disjoint *loops*, obtained as follows. Starting at some point $(x, 0) \in V \times [0, \beta]$, we follow the interval $\{x\} \times [0, \beta]$ until we reach the first cross (if any), i.e. the first point $(xy, t) \in \omega$ for some y . At this point we jump to (y, t) and continue as before on the interval $\{y\} \times [t, \beta]$ until the next cross (if any). Eventually we reach a point (z, β) . We then continue the loop at $(z, 0)$, i.e. we apply periodic boundary conditions ‘vertically’. The loop is completed when we return to the starting point $(x, 0)$. Denote the loop by γ and identify γ with the union of all intervals of the form $\{y\} \times [s, t)$ that it traverses.

Recall the permutation $\pi = \pi(\omega)$ defined in Section 1.1. The points z such that γ visits $(z, 0)$ are precisely $x, \pi(x), \pi^2(x), \dots$ (in the same order). Thus the loops γ are in

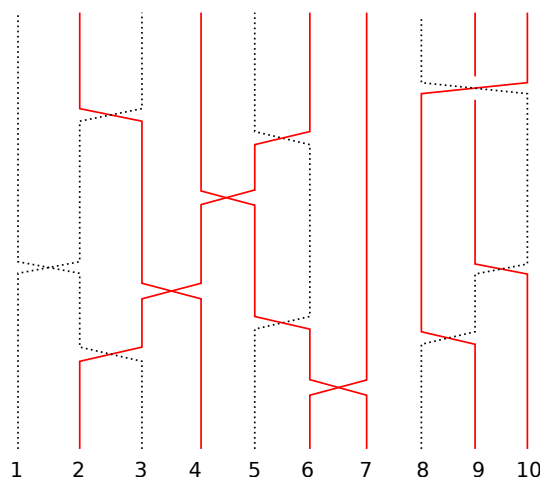


Figure 1: Loop-representation of a sample ω . The vertex set $V = \{1, \dots, 10\}$ is drawn on the horizontal axis, and ‘time’ is on the vertical axis. Transpositions $(xy, t) \in \omega$ are represented as crosses. The resulting loops are coloured red (solid red lines) or white (dotted lines). Here $\pi(\omega) = (1, 3)(2, 6, 7, 4)(9, 10)$, and there are two red loops corresponding to the cycles $(2, 6, 7, 4)$ and $(9, 10)$.

one-to-one correspondence with the cycles \mathcal{C} of π , and the total vertical length of a loop equals β times the size of the corresponding cycle.

Let $\theta > 1$ and $r = 1/\theta$. Given ω , colour each loop $\gamma \in \mathcal{L}(\omega)$ red or white, independently of each other, with probability r for red. Write R and W for the unions of the red and white loops, respectively; they are subsets of $V \times [0, \beta]$. See Figure 1. The points of ω (i.e. the crosses) now fall into three categories: *red*, *white* and *mixed*. Write ω_r , ω_w and ω_m for the red, white and mixed crosses, respectively. Thus $\omega = \omega_r \cup \omega_w \cup \omega_m$.

For $H \subseteq E \times [0, \beta]$ measurable, let $\mathbb{P}_1^H(\cdot)$ denote the law of the restriction $\omega \cap H$ of ω to H . Similarly, for measurable $S \subseteq V \times [0, \beta]$, let $\mathbb{P}_1^S(\cdot)$ be the restriction of $\mathbb{P}_1(\cdot)$ to the set

$$\{(xy, t) \in E \times [0, \beta] : (x, t) \in S \text{ and } (y, t) \in S\};$$

that is, the set of points in $E \times [0, \beta]$ ‘between’ points of S . The following is the key colouring-lemma.

Lemma 2.1. *Given R , the distribution of ω_r is $\mathbb{P}_1^R(\cdot)$, and ω_r is conditionally independent of ω_w .*

In words, conditional on the red set, the red crosses simply form a Poisson process. This means that (given R) the red cycles are obtained from a sample of the interchange process, in a way which will be made precise in Lemma 2.2. Lemma 2.1 holds for general graphs G , with the same proof.

One way to check Lemma 2.1 is to finely discretize the Poisson processes. We present instead a ‘clean’ proof, and the basic approach is as follows. We write a coloured loop-configuration as a pair (q, ω) , where $q \in \{r, w\}^V$ is a colouring of the vertices. We interpret q_x as the colour of the loop containing $(x, 0)$, thus we require the pair (q, ω) to be *consistent* in that $q_x = q_y$ whenever x, y belong to the same cycle. Write $\mathcal{C}(q)$ for the set of ω that are consistent with q . Note that, for $\omega \in \mathcal{C}(q)$, the red and white sets R, W are determined by the pair (q, ω_m) where ω_m are the mixed crosses as before. Indeed, deleting red or white crosses does not change R or W : compare Figure 2 with Figure 1. Thus we may write $R = R(q, \omega_m)$ and $W = W(q, \omega_m)$, and moreover there is some freedom in the choice of $\omega_r \cup \omega_w$. The only restriction on $\omega_r \cup \omega_w$ is that it is a

subset of

$$H = H(q, \omega_m) = \{(xy, t) : (x, t) \in R \text{ and } (y, t) \in R\} \\ \cup \{(xy, t) : (x, t) \in W \text{ and } (y, t) \in W\}.$$

We will proceed by conditioning on the mixed crosses ω_m . When integrating over the allowed choices for ω_r , a cancellation occurs in the factor $\theta^{\ell(\omega)}$ which removes the dependencies in R .

Proof of Lemma 2.1. Let $X = X(R)$ be a bounded R -measurable random variable, and consider events $A = A(\omega_r)$ and $B = B(\omega_w)$ depending on ω_r and ω_w , respectively. We will give an expression for $\mathbb{E}_\theta[X \mathbb{1}_A \mathbb{1}_B]$ which will allow us to deduce the result.

Write $\ell_R(\omega_r)$ and $\ell_W(\omega_w)$ for the number of red and white loops, respectively, so that $\ell(\omega) = \ell_R(\omega_r) + \ell_W(\omega_w)$. For each colouring q , let $\Xi(q)$ be the set of possible ω_m for $\omega \in \mathcal{C}(q)$. Then ω is consistent with q if and only if it can be decomposed as a disjoint union $\omega = \xi \cup \zeta$ with $\xi \in \Xi(q)$ and $\zeta \subseteq H(q, \xi)$. We thus have that

$$\mathbb{E}_\theta[X \mathbb{1}_A \mathbb{1}_B] = \frac{1}{Z} \sum_{q \in \{r, w\}^V} \mathbb{E}_1[\mathbb{1}\{\omega \in \mathcal{C}(q)\} \theta^{\ell(\omega)} r^{\ell_R(\omega_r)} (1-r)^{\ell_W(\omega_w)} X \mathbb{1}_A \mathbb{1}_B] \\ = \frac{1}{Z} \sum_{q \in \{r, w\}^V} \mathbb{E}_1 \left[\sum_{\xi \subseteq \omega} \mathbb{1}\{\xi \in \Xi(q); \omega \setminus \xi \subseteq H\} (\theta-1)^{\ell_W(\omega_w)} X \mathbb{1}_A \mathbb{1}_B \right]. \tag{2.1}$$

The expectation on the right-hand-side of (2.1) can be written in the form

$$\int d\mathbb{P}_1(\omega) \sum_{\xi \subseteq \omega} \alpha(\xi; \omega \setminus \xi) = e^{\beta|E|} \int d\mathbb{P}_1(\xi) \int d\mathbb{P}_1(\zeta) \alpha(\xi; \zeta) \tag{2.2}$$

with

$$\alpha(\xi; \zeta) = \mathbb{1}\{\xi \in \Xi(q); \zeta \subseteq H(q, \xi)\} (\theta-1)^{\ell_W(\zeta_w)} X(R(q, \xi)) \mathbb{1}_A(\zeta_r) \mathbb{1}_B(\zeta_w).$$

We give a proof of the identity (2.2) below. From (2.2) we see that

$$\mathbb{E}_\theta[X \mathbb{1}_A \mathbb{1}_B] = \frac{e^{\beta|E|}}{Z} \sum_{q \in \{r, w\}^V} \int_{\Xi(q)} d\mathbb{P}_1(\xi) \int d\mathbb{P}_1(\zeta) \mathbb{1}\{\zeta \subseteq H\} (\theta-1)^{\ell_W(\zeta_w)} X \mathbb{1}_A \mathbb{1}_B.$$

Here, ξ plays the role of ω_m and $\zeta = \zeta_r \cup \zeta_w$ that of $\omega \setminus \omega_m$. Thus X is a function of $R = R(q, \xi)$, i.e. it is ξ -measurable, and A, B depend on ζ_r and ζ_w , respectively. It follows that

$$\int_{\Xi(q)} d\mathbb{P}_1(\xi) \int d\mathbb{P}_1(\zeta) \mathbb{1}\{\zeta \subseteq H\} (\theta-1)^{\ell_W(\zeta_w)} X \mathbb{1}_A \mathbb{1}_B \\ = \int_{\Xi(q)} d\mathbb{P}_1(\xi) X(R(q, \xi)) e^{-|H^c|} \mathbb{P}_1^R(A) \mathbb{E}_1^W[(\theta-1)^{\ell_W(\zeta_w)} \mathbb{1}_B(\zeta_w)],$$

where $H^c = (E \times [0, \beta]) \setminus H$. It follows that $\mathbb{P}_\theta(A | R) = \mathbb{P}_1^R(A)$, that $\mathbb{P}_\theta(B | R) = \mathbb{P}_{(\theta-1)}^W(B)$, and that A and B are conditionally independent, as claimed.

We now verify (2.2). Let U_1, U_2, \dots and U'_1, U'_2, \dots denote independent collections of i.i.d. uniform random variables on $E \times [0, \beta]$. The conditional distribution $\mathbb{P}_1(\cdot | |\omega| = m)$

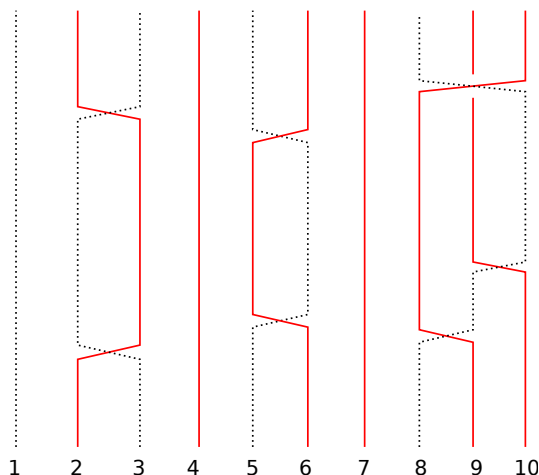


Figure 2: The configuration $\tilde{\omega} = \omega_m$ corresponding to ω of Figure 1. Here $\tilde{\varphi}$ is the permutation $(9, 10)$ of $R_0 = \{2, 4, 6, 7, 9, 10\}$.

coincides with the law of $\{U_1, \dots, U_m\}$. Thus

$$\begin{aligned} \int d\mathbb{P}_1(\omega) \sum_{\xi \subseteq \omega} \alpha(\xi; \omega \setminus \xi) &= \sum_{m \geq 0} \mathbb{P}_1(|\omega| = m) \mathbb{E} \left[\sum_{\xi \subseteq \{U_1, \dots, U_m\}} \alpha(\xi; \{U_1, \dots, U_m\} \setminus \xi) \right] \\ &= \sum_{m \geq 0} \frac{e^{-\beta|E|} (\beta|E|)^m}{m!} \sum_{k=0}^m \binom{m}{k} \mathbb{E} [\alpha(\{U_1, \dots, U_k\}; \{U_{k+1}, \dots, U_m\})] \\ &= e^{-\beta|E|} \sum_{k \geq 0} \sum_{\ell \geq 0} \frac{(\beta|E|)^k}{k!} \frac{(\beta|E|)^\ell}{\ell!} \mathbb{E} [\alpha(\{U_1, \dots, U_k\}; \{U'_1, \dots, U'_\ell\})] \\ &= e^{\beta|E|} \int d\mathbb{P}_1(\xi) \int d\mathbb{P}_1(\zeta) \alpha(\xi; \zeta), \quad \text{as claimed.} \quad \square \end{aligned}$$

As noted above, R is the same if we remove all red and all white crosses, that is if we replace $\omega = \omega_r \cup \omega_w \cup \omega_m$ with $\tilde{\omega} = \omega_m$. We now introduce some notation, see Figure 2 again. Let

$$R_0 = \{x \in V : (x, 0) \in R\} \subseteq V$$

be the set of red vertices at time $t = 0$. These are the elements of the red cycles of $\pi(\omega)$. For each $x \in R_0$, the trajectory of x formed by following the vertical lines and crosses in $\tilde{\omega}$ in the time interval $[0, \beta]$ resembles a ‘crooked line’. We write $\tilde{h}_t(x) \in V$ for the location of this line at time t . We write $h_t(x)$ for the corresponding location obtained using ω . We take both these functions to be right-continuous in t .

The functions $h_t(\cdot)$ and $\tilde{h}_t(\cdot)$ will differ in general, due to the red crosses. Our present goal is to describe their relationship precisely. Of particular importance are the functions

$$\varphi(\cdot) := h_\beta(\cdot) \quad \text{and} \quad \tilde{\varphi}(\cdot) := \tilde{h}_\beta(\cdot). \tag{2.3}$$

These are permutations of R_0 , and φ is precisely the restriction of $\pi(\omega)$ to R_0 . Hence the cycles of φ are the red cycles of π . Clearly $\tilde{\varphi}$ is R -measurable.

Let $\xi = (\xi_{xy} : x, y \in R_0, x \neq y)$ be a collection of independent rate 1 Poisson processes on $[0, \beta]$, independent of everything else. We interpret the points $(xy, t) \in \xi$ as transpositions of the vertices x, y as before, and let σ_t be the time-ordered product of these transpositions up to time t . Thus σ_t is a sample of the interchange process in the set R_0 . We will use Lemma 2.1 to prove the following.

Lemma 2.2. *Given R , the conditional distribution of $(h_t(\cdot) : t \in [0, \beta])$ coincides with that of $(\tilde{h}_t \circ \sigma_t : t \in [0, \beta])$. In particular,*

$$\varphi \stackrel{(d)}{=} \tilde{\varphi} \circ \sigma_\beta.$$

In words, the conditional distribution of the red cycles of $\pi(\omega)$, given the union R of the red loops, is given by the interchange process σ_β on R_0 , composed with a ‘twist’ $\tilde{\varphi}$ (which is a function of R).

Proof of Lemma 2.2. The key observation is that, thanks to Lemma 2.1 and the symmetry of the complete graph, ω_r has the same (conditional) distribution as the collection of points of the form

$$(\tilde{h}_t(x)\tilde{h}_t(y), t) \text{ for } (xy, t) \in \xi. \tag{2.4}$$

For simpler notation we identify ω_r with the collection of points in (2.4). With this identification, we can prove the statement of the lemma with equality (not just in distribution).

Some further notation is required. Let $R_t = h_t(R_0) = \tilde{h}_t(R_0)$ be the set of red vertices at time t . Let $t_1 < t_2 < \dots$ denote the sequence of times at which there are mixed crosses (elements of $\omega_m = \tilde{\omega}$). Also set $t_0 = 0$. Then R_t is constant for $t_{k-1} \leq t < t_k$, for $k \geq 1$. Moreover, there is a unique $a_k \in R_{t_{k-1}}$ and a unique $b_k \notin R_{t_{k-1}}$ such that $R_{t_k} = \psi_k(R_{t_{k-1}})$, where

$$\psi_k(x) = \begin{cases} x, & \text{if } x \neq a_k, \\ b_k, & \text{if } x = a_k. \end{cases}$$

Then for $t_k \leq t < t_{k+1}$ we have that

$$\tilde{h}_t = \psi_k \circ \psi_{k-1} \circ \dots \circ \psi_1.$$

Now, for all $k \geq 0$ let

$$t_k < s_1^{(k)} < s_2^{(k)} < \dots < s_{m_k}^{(k)} < t_{k+1}$$

denote the times of events (transpositions) in ξ , and for $1 \leq q \leq m_k$ let $\tau_q^{(k)} = (x_q^{(k)}, y_q^{(k)})$ denote the corresponding transposition. Here $x_q^{(k)}, y_q^{(k)} \in R_{t_k}$.

With this notation in hand, we can turn to the proof. For $t = 0$ we have $\tilde{h}_0 = \text{id}$, $\sigma_0 = \text{id}$ and $h_0 = \text{id}$, so clearly the claim holds then. In fact, for $t < t_1$ we have that $\tilde{h}_t = \text{id}$ and that $h_t = \sigma_t$, so the claim holds for such t (by Lemma 2.1). Since the functions involved only change at times t_k and $s_q^{(k)}$, we can proceed by induction.

Assume that the claim holds for some $t > 0$, i.e. assume that

$$h_t = \tilde{h}_t \circ \sigma_t. \tag{2.5}$$

Let t' be the time of the next ‘event’. That is, either $t' = t_k$ for some k , or $t' = s_q^{(k)}$ for some k, q . It suffices to show that $h_{t'} = \tilde{h}_{t'} \circ \sigma_{t'}$ holds in both cases.

First case: $t' = t_k$. Then h_{t_k} is obtained by applying ψ_k , thus

$$h_{t_k} = \psi_k \circ h_t = (\psi_k \circ \tilde{h}_t) \circ \sigma_t = \tilde{h}_{t_k} \circ \sigma_t = \tilde{h}_{t_k} \circ \sigma_{t_k}.$$

Here we used (2.5) and the fact that $\sigma_{t_k} = \sigma_t$. Thus $h_{t'} = \tilde{h}_{t'} \circ \sigma_{t'}$ holds in this case.

Second case: $t' = s_q^{(k)}$. Now we have that $\tilde{h}_{t'} = \tilde{h}_t$, and that

$$\sigma_{t'} = \tau_q^{(k)} \circ \sigma_t = (x_q^{(k)}, y_q^{(k)}) \circ \sigma_t.$$

By the identification of ω_r with (2.4), we obtain $h_{t'}$ by transposing $\tilde{h}_t(x_q^{(k)})$ and $\tilde{h}_t(y_q^{(k)})$. That is,

$$h_{t'}(z) = \begin{cases} \tilde{h}_t(y_q^{(k)}), & \text{if } h_t(z) = \tilde{h}_t(x_q^{(k)}), \\ \tilde{h}_t(x_q^{(k)}), & \text{if } h_t(z) = \tilde{h}_t(y_q^{(k)}), \\ h_t(z), & \text{otherwise.} \end{cases}$$

Using that $h_t = \tilde{h}_t \circ \sigma_t$, we can rewrite this as $h_{t'}(z) = \tilde{h}_t(\tau_q^{(k)}(\sigma_t(z)))$, as required. \square

3 Large cycles

In order to use Lemma 2.2 to analyze the cycle structure of π , we will first need results on the cycle structure of $\tilde{\varphi} \circ \sigma_t$, where σ_t is given by the interchange process and $\tilde{\varphi}$ is a non-random permutation.

3.1 Random and non-random transpositions

The following result will be obtained using small modifications of Lemmas 2.1–2.3 of [11]. (A similar result can be obtained by a small modification of Theorem 1 of [3].) Here σ_t denotes a sample of the interchange process ($\theta = 1$ in (1.1)) on $1, \dots, n$, viewed at time t , and $\tilde{\varphi}$ is a deterministic permutation of $1, \dots, n$. In this subsection we write \mathbb{P} for \mathbb{P}_1 .

Proposition 3.1. *Let $\lambda > 1$ and $t = \lambda/n$. For each $\varepsilon > 0$ there is $\delta > 0$ and $n_0(\lambda, \varepsilon, \delta)$ such that for $n \geq n_0$ we have*

$$\mathbb{P}(|\mathcal{C}_1(\tilde{\varphi} \circ \sigma_t)| \geq \delta n) \geq 1 - \varepsilon.$$

Proof. First note that, since $\tilde{\varphi} \circ \sigma_t$ and $\sigma_t \circ \tilde{\varphi}$ are conjugate, it is equivalent to consider the largest cycle in the process $(\sigma_t \circ \tilde{\varphi})_{t \geq 0}$ which starts with the permutation $\tilde{\varphi}$ at time $t = 0$. We associate with $\tilde{\varphi}$ a graph \tilde{G} whose connected components coincide (as sets) with the cycles of $\tilde{\varphi}$. One way to do this is to decompose each cycle of $\tilde{\varphi}$ as:

$$(x_1, x_2, \dots, x_m) = (x_1, x_2)(x_2, x_3) \cdots (x_{m-1}, x_m)$$

and let the edges of \tilde{G} be the pairs $\{x_i, x_{i+1}\}$ obtained in this way. For $t \geq 0$ we let G_t be the (multi-)graph obtained by representing each new transposition that appears in the process $(\sigma_t)_{t \geq 0}$ by an edge, and we let \tilde{G}_t be the (multi-)graph obtained by superimposing G_t on \tilde{G} . Note that G_t has the distribution of an Erdős–Rényi graph $\mathcal{G}(n, p)$ with $p = 1 - e^{-t}$. Also note that each cycle of $\sigma_t \circ \tilde{\varphi}$ is contained in some connected component of \tilde{G}_t .

We have that Lemmas 2.1–2.3 of [11] hold in this situation, with the graph G_t replaced by \tilde{G}_t . Indeed, one need only check the proof of Lemma 2.2, since Lemmas 2.1 and 2.3 do not refer to the graph. We provide an outline of the arguments.

As in Lemma 2.1 of [11], each time we apply a new transposition in σ_t , the probability that it splits an existing cycle so that at least one of the resulting cycles has size $\leq k$ is at most $2k/(n - 1)$. Let $V_{\tilde{G}}^t(k) \subseteq V$ be the union of all components of \tilde{G}_t of size at least k , and let $V_X^t(k) \subseteq V$ be the union of all the cycles of $\sigma_t \circ \tilde{\varphi}$ of size at least k . As in Lemma 2.2 of [11] we have that

$$\mathbb{E}|V_{\tilde{G}}^t(k) \setminus V_X^t(k)| \leq t \binom{n}{2} \frac{4k^2}{n-1}. \tag{3.1}$$

(Recall that our process is sped up by a factor $\binom{n}{2}$ compared to [11].) This is because each cycle of size $< k$ which lies in a component of size $\geq k$ can be associated with a transposition which split an existing cycle so that at least one resulting cycle had size $\leq k$

(and this can be done so that at most two cycles get mapped to the same transposition). Here we use the fact that at time $t = 0$ the components are equal to the cycles.

Set $t_1 = \lambda/n$ and let $t_0 \leq t_1$ with $t_1 - t_0 = o(n^{-1})$. From (3.1) we see that, at time t_0 , only a tiny fraction of vertices in components $\geq 2n^{1/4}$ lie in cycles $< 2n^{1/4}$:

$$\mathbb{E}|V_{\tilde{G}}^{t_0}(2n^{1/4}) \setminus V_X^{t_0}(2n^{1/4})| \leq 32\lambda n^{1/2}.$$

Let $z(\lambda) > 0$ be the survival probability of a Galton-Watson process with Poisson(λ) offspring distribution, and set $\delta' = z(\lambda)/4$. Applying the Erdős-Rényi theorem to $G_{t_0} \subseteq \tilde{G}_{t_0}$ we find that

$$\begin{aligned} \mathbb{P}(|V_X^{t_0}(2n^{1/4})| \leq \delta'n) &\leq \mathbb{P}(|V_{\tilde{G}}^{t_0}(2n^{1/4}) \setminus V_X^{t_0}(2n^{1/4})| > \delta'n) + o(1) \\ &\leq \frac{32\lambda n^{1/2}}{\delta'n} + o(1). \end{aligned} \tag{3.2}$$

Here the $o(1)$ is uniform in $\tilde{\varphi}$.

Lemma 2.3 of [11] applies in our setting, with adjusted time-scaling. Let j be such that $n^{1/4} \leq 2^j \leq 2n^{1/4}$, write $\rho = 2^j/n$, set

$$\Delta = \binom{n}{2}^{-1} [2^6(\delta')^{-1}\rho^{-1} \log_2(\rho^{-1})] = O(n^{-5/4} \log n), \tag{3.3}$$

and let $t_0 = t_1 - \Delta$. The Lemma tells us that a large fraction of the vertices that lie in cycles of size $\geq 2n^{1/4}$ at time t_0 will lie in cycles of size of the order n at time t_1 . More precisely, there is a constant $C > 0$ such that for all $\varepsilon' \in (0, 1/8)$ we have

$$\begin{aligned} \mathbb{P}\left(|V_X^{t_0}(2n^{1/4}) \setminus V_X^{t_1}(\varepsilon'\delta'n)| > \delta'n \mid |V_X^{t_0}(2n^{1/4})| > \delta'n\right) \\ \leq C \cdot (\delta')^{-2}\varepsilon' |\log(\varepsilon'\delta')|. \end{aligned} \tag{3.4}$$

Letting $\delta = \varepsilon'\delta'$ for $\varepsilon' > 0$ small enough we find using (3.2) and (3.4) that, for large enough n ,

$$\begin{aligned} \mathbb{P}(V_X^{t_1}(\delta n) = \emptyset) &\leq \mathbb{P}(|V_X^{t_0}(2n^{1/4}) \setminus V_X^{t_1}(\varepsilon'\delta'n)| > \delta'n) \\ &\quad + \mathbb{P}(|V_X^{t_0}(2n^{1/4})| \leq \delta'n) \\ &\leq \varepsilon, \quad \text{as claimed.} \end{aligned} \quad \square$$

3.2 Proof of Theorem 1.1

Fix $\lambda > \theta$ and let $\beta = \lambda/n$. It suffices to prove that, for any sequence $\delta = \delta_n \rightarrow 0$, we have that $\mathbb{P}(|\mathcal{C}_1(\pi)| < \delta n) \rightarrow 0$, where $\mathbb{P} = \mathbb{P}_\theta$.

Let $N = |R_0|$ denote the number of red vertices, and let $\Lambda = \lambda N/n$ so that the auxiliary process σ_t of Lemma 2.2 is an interchange process on N points viewed at time Λ/N . Let $\mathcal{C}_1^r = \mathcal{C}_1(\tilde{\varphi} \circ \sigma_{\Lambda/N})$ be the largest red cycle. Pick α such that $\frac{1}{\lambda} < \alpha < r$. Since \mathcal{C}_1 is red with probability r ,

$$\begin{aligned} r\mathbb{P}(|\mathcal{C}_1| < \delta n) &\leq \mathbb{P}(N \geq \alpha n, |\mathcal{C}_1^r| < \delta n) + \mathbb{P}(N < \alpha n, |\mathcal{C}_1| < \delta n) \\ &\leq \mathbb{E}[\mathbb{1}\{\Lambda \geq \alpha\lambda\} \mathbb{P}(|\mathcal{C}_1^r| < (\lambda\delta)n \mid R)] \\ &\quad + \mathbb{P}(N < \alpha n, |\mathcal{C}_1| < \delta n). \end{aligned} \tag{3.5}$$

When $\Lambda \geq \alpha\lambda > 1$, Prop. 3.1 implies that $\mathbb{P}(|\mathcal{C}_1^r| < (\lambda\delta)n \mid R) \rightarrow 0$. The dominated-convergence-theorem therefore implies that the first term on the right-hand-side of (3.5) converges to 0.

To bound the second term, let \mathbb{P}' , \mathbb{E}' and \mathbb{V}' denote the conditional probability, expectation and variance, respectively, given the cycle sizes $|\mathcal{C}_1|, |\mathcal{C}_2|, \dots$. Then $\mathbb{E}'(N) = \sum_{i \geq 1} r|\mathcal{C}_i| = rn$, and

$$\mathbb{V}'(N) = \sum_{i \geq 1} r(1-r)|\mathcal{C}_i|^2 \leq |\mathcal{C}_1| \sum_{i \geq 1} |\mathcal{C}_i| = |\mathcal{C}_1|n.$$

Using Chebyshev's inequality:

$$\begin{aligned} \mathbb{P}(N < \alpha n, |\mathcal{C}_1| < \delta n) &= \mathbb{E}[\mathbb{I}\{|\mathcal{C}_1| < \delta n\} \mathbb{P}'(N < \alpha n)] \\ &\leq \mathbb{E}\left[\mathbb{I}\{|\mathcal{C}_1| < \delta n\} \frac{|\mathcal{C}_1|n}{n^2(r-\alpha)^2}\right] \\ &\leq \frac{\delta}{(r-\alpha)^2} \rightarrow 0. \end{aligned}$$

This proves the result. □

Remark 3.2. Our proof can straightforwardly be extended to cases where the cycles \mathcal{C} (or loops γ) receive potentially different weights. Indeed, consider the probability measure $\mathbb{P}(\cdot)$ given by

$$\frac{d\mathbb{P}}{d\mathbb{P}_1}(\omega) = \frac{1}{Z} \prod_{\gamma \in \mathcal{L}(\omega)} \theta(\gamma).$$

The conclusion of Theorem 1.1 holds under the assumption that there is a constant $\theta < \lambda$ such that $1 \leq \theta(\cdot) \leq \theta$ uniformly.

Such weights occur for example in the Heisenberg model in the presence of an external field $h > 0$, where $\theta(\gamma) = 2 \cosh(h|\gamma|) \leq 2 \cosh(h\lambda)$. See e.g. [5, 7].

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Acknowledgments. The author wishes to thank Svante Janson for helpful discussions about Theorem 1.1, Balint Tóth for pointing out the reference [12], and the anonymous referee for helpful comments.

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