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Finite time blowup of the stochastic shadow Gierer-Meinhardt System*

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Abstract

By choosing some special (random) initial data, we prove that with probability 1, stochastic shadow Gierer-Meinhardt system blows up in finite time in the pointwise sense. We also give a (random) upper bound for the blowup time and some estimates about this bound. By increasing the amplitude of initial data, we can get a blowup in any short time with a positive probability.

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1 Introduction

Inspired by the recent work [9] and [10], we study the blow up of the shadow Gierer-Meinhardt system with random migrations with the following form:

$$\begin{cases} \partial_t u = \Delta u - u + \frac{u^p}{\xi^q} & \text{in } \mathcal{O} \times (0,T), \\ \mathrm{d}\xi = \left(-\xi + \frac{\overline{u^r}}{\xi^s}\right)\mathrm{d}t + \xi \mathrm{d}B_t & \text{in } (0,T), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{O} \times (0,T), \\ u(0) = u_0 & \text{in } \mathcal{O}, \\ \xi(0) = \xi_0, \end{cases} \tag{1.1}$$

where $\xi \mathrm{d}B_t$ can be explained as random migrations and B_t is a one-dimensional standard Brownian motion. Due to the random effects, we need to introduce the sample space Ω and re-define

$$u(t,x,\omega) : \mathbb{R}^+ \times \mathcal{O} \times \Omega \to \mathbb{R}^+, \quad \xi(t,\omega) : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+ \setminus \{0\}.$$

The motivation for studying Eq. (1.1) can be found in [16], [15] and [8].

We shall study in this paper the blowup problem of Eq. (1.1) under quite general assumptions. When $p \geq r$ and $\frac{p-1}{r} > \frac{2}{n+2}$, we show that with probability 1, Eq. (1.1)

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blows up in the pointwise sense if we choose some suitable (random) initial data. We also give a (random) upper bound for the blow up time and consequently obtain a probabilistic estimate of this blow up.

To our knowledge, there are not many results for the blow up of stochastic systems. The work [1] proved that the 2nd moment of the solution of some nonlinear wave equations blows up, while [2] gave a nice criterion for the blowup of the pth moment for some stochastic reaction-diffusion equations. As pointed out in [2], the blowup of the pth moment even does not imply the pathwise blowup with a positive probability. [4] extended the result in [2] to the case of stochastic parabolic equations with delay. Most recently, Chow and Khasminski established an almost sure blowup result for a family of SDEs ([3]). [13] and [12] studied stochastic heat equations and showed that the noises can produce blowup with positive probability. In contrast, our blowup results depend on the choices of initial data, it is inspired by the deterministic work of [6], [10] and [9]. A special (random) data can, with probability 1, lead to a blowup of SPDEs solutions. By increasing the amplitude of the initial data, we can get the blowup in any short time with positive probability.

Both probabilistic and PDE's methods play important roles in our approach. Itô's formula in the proof of Lemma 2.2 below is the key point for finding a monotone stochastic process $\hat{\xi}(t)$, which paves the way to applying classical PDE techniques and estimating the upper bounds of blow up time. For the PDE's argument, we follow the approaches shown in [9] and [10].

The organization of the paper is as follows. In section 2, we introduce some notations and give some prerequisite lemmas. To show our approach more transparently, we prove a blowup theorem under some additional assumption in section 3. The 4th section removes the assumption and build the general blowup result by integral estimates.

2 Some auxiliary lemmas and a monotone stochastic process $\hat{\xi}(t)$

From now on, we assume $\mathcal{O} = B_1(0)$, the unit open ball in $\mathbb{R}^n$ with zero center. For notational simplicity, write $v(t,z) = e^t u(t,z)$ for all $t > 0$ and $z \in \bar{B}_1(0)$ and

$$K(t) = \frac{e^{-(p-1)t}}{\xi^q(t)}, \tag{2.1}$$

note that $K(t)$ depends on ξ. It is easy to check

$$\begin{cases} \partial_t v = \Delta v + K(t) v^p & \text{in } B_1(0), \\ \mathrm{d}\xi = \left(-\xi + e^{-rt}\frac{\overline{v^r}}{\xi^s}\right)\mathrm{d}t + \xi \mathrm{d}B_t, & \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } \{z = 1\}, \\ v(0) = u_0, & \\ \xi(0) = \xi_0, & \end{cases} \tag{2.2}$$

To study the blow up of Eq. (1.1) , we only need to study that of Eq. (2.2). So we shall concentrate on the blow up of v and ξ in the sequel.

Write

$$B_t^* = \sup_{0 \le s \le t} |B_s| \quad \forall t > 0,$$

it is well known ([7, p. 96]) that $\mathbb{P}(B_t^* \ge A) \le \frac{\sqrt{t}}{\sqrt{2\pi}} \frac{4}{A} \mathrm{e}^{-\frac{A^2}{2t}}$ for any $A > 0$ and $t > 0$. Hence,

$$\mathbb{P}(B_t^* < \infty) = 1 - \lim_{A \to \infty} \mathbb{P}(B_t^* \ge A) = 1 \qquad \forall\, t > 0.$$

For every $t > 0$, denote $\mathcal{N}_t = \{\omega : B_t^* = \infty\}$, it is clear that $\mathbb{P}(\mathcal{N}_t) = 0$. Take $t = 1, 2, ...$, it is easy to see that $\mathcal{N}_t \subset \mathcal{N}_m$ for all $t \leq m$. Define $\mathcal{N} = \lim_{m\to\infty} \mathcal{N}_m$, we have $\mathbb{P}(\mathcal{N}) = \lim_{m\to\infty} \mathbb{P}(\mathcal{N}_m) = 0$. Hence, for all $\omega \in \Omega \setminus \mathcal{N}$, $B_t^*(\omega) < \infty$ for all $t > 0$. From the above observation, without loss of generality, we can assume that for all $\omega \in \Omega$,

$$B_t^*(\omega) < \infty \qquad \forall\, t > 0. \tag{2.3}$$

For all $x \in \mathbb{R}^n$, denote $z = |x|$. Consider the following isotropic function

$$\phi(z) = \begin{cases} z^{-\alpha}, & \delta \leq z \leq 1, \\ \delta^{-\alpha}(1 + \frac{\alpha}{2}) - \frac{\alpha}{2}\delta^{-\alpha-2}z^2, & 0 \leq z < \delta, \end{cases}$$

with some $\delta \in (0, 1)$ and

$$\alpha = \frac{2}{p-1}. \tag{2.4}$$

It is easy to check that for all $z \in (0, 1)$

$$\partial_z^2 \phi + \frac{n-1}{z}\partial_z \phi + \alpha n \phi^p \geq 0. \tag{2.5}$$

Take

$$v_0 = \gamma\phi$$

as the initial data of Eq. (2.2), where $\gamma > 0$ is some (random) number. This special choice of initial data is inspired by the deterministic work of [6], [10] and [9]. Since the initial data is isotropic in the space, then the solution $v(x, t)$ is also spatially isotropic for all $t > 0$. Hence, we denote the solution by $v(z, t)$ and Eq. (2.2) can be rewritten as

$$\begin{cases} \partial_t v = \partial_z^2 v + \frac{n-1}{z}\partial_z v + K(t)v^p & \text{in } B_1(0), \\ \mathrm{d}\xi = \left(-\xi + e^{-rt}\frac{\overline{v^r}}{\xi^s}\right)\mathrm{d}t + \xi \mathrm{d}B_t, & \\ \frac{\partial v}{\partial z} = 0 & \text{on } \{z = 1\}, \\ v(0) = v_0, & \\ \xi(0) = \xi_0, & \end{cases} \tag{2.6}$$

By a Banach fixed point argument as in [16, Lem. 2.1], Eq. (2.6) has a unique *local* solution. More precisely, for $\omega \in \Omega$ a.s., there exists some random time $T(\omega) > 0$, Eq. (2.6) has a unique solution $(u, \xi) \in C([0, T]; C(B_1(0), \mathbb{R}) \times \mathbb{R})$. The next lemma is about the properties of the solution.

Lemma 2.1. *Let v be the solution to Eq. (2.6) on $[0, T]$ ($T > 0$ is random), then the following statements hold:*

(i). $v(z, t) \geq \gamma$ for all $0 \leq t \leq T$ and $0 < z < 1$.
(ii). $\partial_z v(z, t) \leq 0$ for all $0 \leq t \leq T$ and all $0 < z < 1$.
(iii). For all $\beta \in (0, 1]$, we have $z^n v^\beta(z, t) \leq \overline{v^\beta}(t)$ for all $0 \leq t \leq T$ and all $0 < z < 1$.
(iv). $\partial_z v(\frac{1}{2}, t) \leq -C_0 2^{n-1}$ for all $0 \leq t \leq T$, where $C_0 > 0$ depends on γ.

Proof. The proofs are similar to those in [10, Lemma 2.1]. □

Define

$$\hat{\xi}(t) = e^{\frac{3t}{2} - B_t}\xi(t) \qquad t > 0,$$

we have the following lemma:

Lemma 2.2. *We have*

$$\hat{\xi}(t) \geq \hat{\xi}(s) \qquad t \geq s \geq 0. \tag{2.7}$$

Proof. By Itô's formula, we have

$$\begin{aligned} \mathrm{d}\hat{\xi}(t) &= \mathrm{d}\left(e^{\frac{3t}{2}-B_t}\xi(t)\right) \\ &= \xi(t)\left[\frac{3}{2}e^{\frac{3t}{2}-B_t}\mathrm{d}t - e^{\frac{3t}{2}-B_t}\mathrm{d}B_t + \frac{1}{2}e^{\frac{3t}{2}-B_t}\mathrm{d}t\right] \\ &\quad + e^{\frac{3t}{2}-B_t}\left[-\xi(t)\mathrm{d}t + e^{-rt}\frac{\overline{v^r}(t)}{\xi^s(t)}\mathrm{d}t + \xi(t)\mathrm{d}B_t\right] - e^{\frac{3t}{2}-B_t}\xi(t)\mathrm{d}t \\ &= e^{-rt+\frac{3t}{2}-B_t}\frac{\overline{v^r}(t)}{\xi^s(t)}\mathrm{d}t. \end{aligned} \tag{2.8}$$

Since $\overline{v^r}(t) \geq 0$ and $\xi(t) \geq 0$ for all $t \geq 0$, $\hat{\xi}(t)$ is an increasing function with respect to t. This completes the proof. $\square$

Since $\hat{\xi}(0) = \xi_0$, by Lemma 2.2 we have $\hat{\xi}(t) \geq \xi_0$ for all $t \geq 0$. For any $\lambda \in (1, \infty)$, define

$$t_\lambda = \inf\{t \geq 0 : \hat{\xi}(t) \geq \lambda\xi_0\} \tag{2.9}$$

with the convention $\inf\emptyset = \infty$. (t_λ is actually a stopping time). It is easy to see that $t_\lambda = \infty$ holds as long as $\hat{\xi}(t) < \lambda\xi_0$ for all $t > 0$. We clearly have

$$\xi_0 \leq \hat{\xi}(t) \leq \lambda\xi_0 \qquad t \in [0, t_\lambda]. \tag{2.10}$$

In (2.10), we define $\hat{\xi}(\infty) = \lim_{t\to\infty}\hat{\xi}(t)$ as $t_\lambda = \infty$.

Let $\theta : \Omega \to (0, \infty)$ be a positive random variable. From (2.3), we clearly have

$$B^*_{\theta(\omega)}(\omega) < \infty \qquad \forall\, \omega \in \Omega, \tag{2.11}$$

for notational simplicity, we shall suppress the variable ω and write it as B^*_θ. Recall the definition of $K(t)$ in (2.1), we have

$$(\lambda\xi_0)^{-q}\exp\left(-(p-1)\theta - qB^*_\theta\right) \leq K(t) \leq \xi_0^{-q}\exp\left(\frac{3}{2}q\theta + qB^*_\theta\right),\ t \in [0, \theta]. \tag{2.12}$$

Indeed, it is easy to see that $K(t) = \hat{\xi}(t)^{-q}\exp\left(-(p-1)t + \frac{3}{2}qt - qB_t\right)$ holds. By (2.10), we have

$$(\lambda\xi_0)^{-q}\exp\left(-(p-1)t + \frac{3}{2}qt - qB_t\right) \leq K(t) \leq \xi_0^{-q}\exp\left(-(p-1)t + \frac{3}{2}qt - qB_t\right),$$

which immediately implies (2.12), as desired. For the further usage, we denote

$$T_b \ \text{ the blowup time of the solution } \ v(z,t), \tag{2.13}$$

$$K_\theta = (\lambda\xi_0)^{-q}\exp\left(-(p-1)\theta - qB^*_\theta\right). \tag{2.14}$$

3 Pointwise blow up as $t_\lambda \geq \theta$

Let $\theta \in (0, \infty)$ be some strictly positive random variable as in the previous section. Recall the definition of t_λ in (2.9) with $\lambda \in (1, \infty)$ being some fixed number, under the assumption $t_\lambda \geq \theta$, we shall prove the next two theorems, whose proofs also partly give the main idea of our approach. The first theorem gives an upper bound of the blow up time in the pointwise sense, while the second claims that the upper bound of the blowup time is larger than θ as $t_\lambda \geq \theta$, which means that the blow up could happen after the time θ.

Note that the quantities below such as τ and T_b are random variables, we should write them as $\tau(\omega)$ and $T_b(\omega)$ more precisely. For notational simplicity, we shall suppress the argument ω if no confusions arise.

Theorem 3.1. *Let $\lambda > 1$ and let $\theta \in (0, \infty)$ be some random number. If $t_\lambda \geq \theta$, choose γ such that $\gamma^{p-1}K_\theta > \frac{4n}{p-1}$, then we have*

$$T_b \leq \frac{2\delta^2}{\gamma^{p-1}K_\theta(p-1)}\left(1+\frac{\alpha}{2}\right)^{-p+1} < \frac{\delta^2}{2n}\left(1+\frac{\alpha}{2}\right)^{-p+1}. \tag{3.1}$$

Proof. By (2.12), we have

$$K(t) \geq K_\theta, \qquad t \in [0, \theta].$$

By (2.6) and the above inequality, we have

$$\begin{cases} \partial_t v \geq \partial_z^2 v + \frac{n-1}{z}\partial_z v + K_\theta v^p & \text{in } B_1(0) \times (0,\theta), \\ \partial_z v = 0 & \text{on } \{z=1\} \times (0,\theta), \\ v(0) = v_0 & \text{in } B_1(0). \end{cases}$$

Now consider another equation

$$\begin{cases} \partial_t w = \partial_z^2 w + \frac{n-1}{z}\partial_z w + K_\theta w^p & \text{in } B_1(0) \times (0,\theta), \\ \partial_z w = 0 & \text{on } \{z=1\} \times (0,\theta), \\ w(0) = v_0 & \text{in } B_1(0). \end{cases} \tag{3.2}$$

By comparison principle, we have

$$v(z,t) \geq w(z,t), \qquad (z,t) \in B_1(0) \times [0,\theta].$$

Writing $\rho = \partial_t w - \frac{K_\theta}{2} w^p$, by (3.2) we have

$$\begin{aligned} \partial_t \rho &= \Delta\rho + \frac{K_\theta}{2}p(p-1)w^{p-1}|\nabla w|^2 + \frac{K_\theta}{2}pw^{p-1}\Delta w + \frac{K_\theta}{2}pw^{p-1}\partial_t w \\ &\geq \Delta\rho + \frac{K_\theta}{2}pw^{p-1}\Delta w + \frac{K_\theta}{2}pw^{p-1}\partial_t w \\ &= \Delta\rho + K_\theta p w^{p-1}\rho. \end{aligned}$$

It is straightforward to check that for all $z \in B_1(0)$,

$$\rho(z,0) = \gamma\left[\partial_z^2\phi(z) + \frac{n-1}{z}\partial_z\phi(z) + \frac{K_\theta}{2}\gamma^{p-1}\phi^p(z)\right].$$

Under the condition in the theorem, (2.5) holds and thus the term in the square bracket is positive. Therefore,

$$\rho(z,0) \geq 0, \quad z \in B_1(0).$$

It is easy to check $\partial_z\rho = 0$ for $(z,t) \in \{z=1\} \times [0,\theta]$. Hence, the maximum principle gives $\rho(z,t) \geq 0$ for $(z,t) \in B_1(0) \times [0,\theta]$. That is $\partial_t w - \frac{K_\theta}{2}w^p \geq 0$ for all $(z,t) \in B_1(0) \times [0,\theta]$, which implies

$$w(z,t) \geq \left[\frac{1}{v_0^{-p+1}(z) - \frac{K_\theta(p-1)t}{2}}\right]^{\frac{1}{p-1}}. \tag{3.3}$$

By the form of $v_0(z) = \gamma\phi(z)$, for every $z \in (0,1)$ the term on the right hand side (3.3) blows up at $t = \tau(z)$ with

$$\tau(z) := \begin{cases} \frac{2}{K_\theta(p-1)}\gamma^{-p+1}\left[1 + \frac{1-\left(\frac{z}{\delta}\right)^2}{2}\alpha\right]^{-p+1}\delta^2 & z \in [0,\delta], \\ \frac{2}{K_\theta(p-1)}\gamma^{-p+1}z^2 & z \in (\delta, 1), \end{cases}$$

where we have used the relation $\alpha(p-1) = 2$ (see (2.4)). It is easy to see that $\tau(z)$ is an increasing function and $\tau(0) = \frac{2\delta^2}{\gamma^{p-1}K_\theta(p-1)}\left(1 + \frac{\alpha}{2}\right)^{-p+1}$, thus we get the desired bound for T_b. □

Corollary 3.2. *Assume that $\theta \le \theta_0$ a.s. with $\theta_0 > 0$ being some constant and that $\gamma > 0$ is some (sufficiently large) deterministic number, then we have*

$$\mathbb{P}\left(T_b \le \frac{\delta^2}{2n}\left(1 + \frac{\alpha}{2}\right)^{-p+1}\right) \ge 1 - \frac{\sqrt{\theta_0}}{\sqrt{2\pi}}\frac{4}{A_0}\mathrm{e}^{-\frac{A_0^2}{2\theta_0}} \tag{3.4}$$

with $A_0 = \frac{1}{q}\ln\frac{(p-1)\gamma^{p-1}}{4n(\lambda\xi_0)^q} - \frac{p-1}{q}\theta_0$.

Proof. By Theorem 3.1, it suffices to prove that

$$\mathbb{P}\left(\gamma^{p-1}K_\theta > \frac{4n}{p-1}\right) \ge 1 - \frac{\sqrt{\theta_0}}{\sqrt{2\pi}}\frac{4}{A_0}\mathrm{e}^{-\frac{A_0^2}{2\theta_0}}. \tag{3.5}$$

Since K_θ is an decreasing function of θ and $\theta \le \theta_0$ a.s., we have

$$\begin{aligned} \mathbb{P}\left(\gamma^{p-1}K_\theta > \frac{4n}{p-1}\right) &\ge \mathbb{P}\left(\gamma^{p-1}K_{\theta_0} > \frac{4n}{p-1}\right) \\ &= 1 - \mathbb{P}\left(B^*_{\theta_0} \ge \frac{1}{q}\ln\frac{(p-1)\gamma^{p-1}}{4n(\lambda\xi_0)^q} - \frac{p-1}{q}\theta_0\right) \\ &\ge 1 - \frac{\sqrt{\theta_0}}{\sqrt{2\pi}}\frac{4}{A_0}\mathrm{e}^{-\frac{A_0^2}{2\theta_0}} \end{aligned} \tag{3.6}$$

with $A_0 = \frac{1}{q}\ln\frac{(p-1)\gamma^{p-1}}{4n(\lambda\xi_0)^q} - \frac{p-1}{q}\theta_0$. □

Corollary 3.3. *Assume that the conditions in Theorem 3.1 hold. Let $\gamma \to \infty$ a.s., then we have*

$$T_b \to 0, \quad a.s..$$

Proof. By Theorem 3.1, we have

$$T_b \le \frac{2\delta^2}{\gamma^{p-1}K_\theta(p-1)}\left(1 + \frac{\alpha}{2}\right)^{-p+1}.$$

As $\gamma \to \infty$ a.s., we get $\frac{2\delta^2}{\gamma^{p-1}K_\theta(p-1)}\left(1 + \frac{\alpha}{2}\right)^{-p+1}$ a.s. and thus $T_b \to 0$ $a.s.$. □

4 General pointwise blow up result

Recall that T_b is the blowup time of $v(z,t)$ and the K_θ is defined in (2.14), in this section, we shall prove the following blow up theorem:

Theorem 4.1. *Let $\lambda > 1$ and let $p \ge r$ and $\frac{p-1}{r} > \frac{2}{n+2}$. We have the following two statements:*

(i) In the case $t_\lambda \geq 1$*, choose* $\gamma > 0$ *such that* $\gamma^{p-1}K_1 > \frac{4n}{p-1}$*, we have*

$$T_b \leq \frac{2\delta^2}{\gamma^{p-1}K_1(p-1)}\left(1+\frac{\alpha}{2}\right)^{-p+1}. \tag{4.1}$$

(ii) In the case $t_\lambda \leq 1$*, there exists some* $\hat{\theta} \in (0,1]$ *such that as long as* $\gamma^{p-1}K_{\hat{\theta}} > \frac{4n}{p-1}$*, we have*

$$T_b \leq \frac{2\delta^2}{\gamma^{p-1}K_{\hat{\theta}}(p-1)}\left(1+\frac{\alpha}{2}\right)^{-p+1}. \tag{4.2}$$

By the same argument as showing Corollary 3.3, we immediately get the following corollary.

Corollary 4.2. *Assume that the conditions in Theorem 4.1 hold. Let* $\gamma \to \infty$ *a.s., then we have*

$$T_b \to 0, \quad a.s..$$

Let $\beta \in (0,1]$ be some number to be determined later. Denote

$$h(t) = \overline{v^\beta}(t) \qquad t > 0.$$

For ease of notation, we define

$$h_1(t) = \frac{1}{|B_1(0)|}\int_{B_R(0)} v^\beta(z,t)\mathrm{d}z, \quad h_2(t) = \frac{1}{|B_1(0)|}\int_{B_1(0)\setminus B_R(0)} v^\beta(z,t)\mathrm{d}z,$$

where $R \in (0,1)$ is some number to be determined later. We also define the following stochastic quantity:

$$h_\lambda^* = h(0) + \beta(\lambda-1)\lambda^{-q}\gamma^{\beta+p-1-r}\exp\left(-pt_\lambda - (s+q+1)(\frac{3t_\lambda}{2} + B^*_{t_\lambda})\right)\xi_0^{s-q+1}, \tag{4.3}$$

it will frequently appear in the arguments below. It is easy to see

$$h_\lambda^* \leq h(0) + \beta(\lambda-1)\lambda^{-q}\gamma^{\beta+p-1-r}\xi_0^{s-q+1}. \tag{4.4}$$

Lemma 4.3. *Let* $\lambda > 1$ *and* $p \geq r$*. Assume* $t_\lambda < \infty$*. Choose* $\beta \in (0,1]$ *such that* $p+\beta-1 \geq r$*, then we have*

$$h(t_\lambda) \geq h_\lambda^*.$$

Proof. We have

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t} = \frac{\beta(1-\beta)}{|B_1(0)|}\int_{B_1(0)} v^{\beta-2}(x,t)|\nabla v(x,t)|^2\mathrm{d}x + \beta K(t)\overline{v^{\beta+p-1}}(t) \geq \beta K(t)\overline{v^{\beta+p-1}}(t),$$

where the last inequality is by $\beta \in (0,1]$. Since $p \geq r$ and $\beta \in (0,1]$ are such that $p+\beta-1 \geq r$, by Lemma 2.1 (i), we have $v(z,t) \geq \gamma$ for all $t > 0$ and $0 < z < 1$ and thus $\overline{v^{\beta+p-1}}(t) \geq \gamma^{\beta+p-1-r}\overline{v^r}(t)$. Hence,

$$\frac{\mathrm{d}h(t)}{\mathrm{d}t} \geq \beta\gamma^{\beta+p-1-r}K(t)\overline{v^r}(t).$$

On the other hand, by (2.8), we have

$$\overline{v^r}(t) = e^{rt-\frac{3t}{2}+B_t}\xi^s(t)\frac{\mathrm{d}\hat{\xi}(t)}{\mathrm{d}t}.$$

Hence, by (2.1), (2.10) and the above relations, we have

$$\begin{aligned}\frac{\mathrm{d}h(t)}{\mathrm{d}t} &\geq \beta\gamma^{\beta+p-1-r}K(t)e^{rt-\frac{3t}{2}+B_t}\xi^s(t)\frac{\mathrm{d}\hat{\xi}(t)}{\mathrm{d}t}\\ &\geq \beta\gamma^{\beta+p-1-r}\lambda^{-q}\xi_0^{s-q}e^{-pt}e^{-(s+q+1)(\frac{3t}{2}+B_t^*)}\frac{\mathrm{d}\hat{\xi}(t)}{\mathrm{d}t}\\ &\geq \beta\gamma^{\beta+p-1-r}\lambda^{-q}\xi_0^{s-q}\exp\left(-pt_\lambda-(s+q+1)(\frac{3t_\lambda}{2}+B_{t_\lambda}^*)\right)\frac{\mathrm{d}\hat{\xi}(t)}{\mathrm{d}t}\end{aligned}$$

for all $t\in[0,t_\lambda]$. By the definition of t_λ and Lemma 2.2, we immediately get the desired inequality. □

Stimulated from the previous lemma, we define

$$\hat{t}_\lambda=\inf\{t\geq 0: h(t)\geq h_\lambda^*\}, \tag{4.5}$$

it is clear that $\hat{t}_\lambda\leq t_\lambda$ and

$$h(t)\leq h_\lambda^*, \qquad t\in[0,\hat{t}_\lambda]. \tag{4.6}$$

Denote $f(z,t)=z^{n-1}\partial_z v(z,t)$, it is easy to check

$$\mathcal{L}f=\partial_t f-\partial_z^2 f+\frac{n-1}{z}\partial_z f-pK(t)v^{p-1}f=0.$$

The proof of the next lemma has some similarity to that of [5, Lemma 2.2].

Lemma 4.4. *Let* $\lambda>1$. *Let* $k\in(1,p)$, $\beta\in(0,1]$ *and* $\ell\geq\frac{k}{\beta}$. *Assume* $t_\lambda<\infty$. *As* $\varepsilon\leq\varepsilon^*$ *with*

$$\varepsilon^*=\min\left\{\alpha(1+\frac{\alpha}{2})^{-k}h^\ell(0),\ \ 2^{-\ell n+n}C_0\gamma^{\beta\ell-k},\ \frac{(p-k)\gamma^{p+\ell-k}}{2k(\lambda\xi_0)^q}\exp\left(-(p-1)t_\lambda-qB_{t_\lambda}^*\right)\right\}, \tag{4.7}$$

we have

$$v(z,t)\leq\left(\frac{2h^\ell(t)}{\varepsilon(k-1)}\right)^{\frac{1}{k-1}}z^{-\frac{2}{k-1}}, \qquad \forall\ t\in[0,\hat{t}_\lambda]\ \ \forall\ z\in(0,\frac{1}{2}]. \tag{4.8}$$

Proof. Denote $\eta(z,t)=f(z,t)+\varepsilon z^n\frac{v^k(z,t)}{h^\ell(t)}$ with $f(z,t)=z^{n-1}\partial_z v(z,t)$ and $\varepsilon>0$ some number to be determined later and $\ell\geq\frac{k}{\beta}$, we prove the lemma in the following three steps.

Step 1: Property of $\eta(z,t)$. By (i), (iii) and (iv) of Lemma 2.1 and the relation $\ell\geq\frac{k}{\beta}$, we further have

$$\begin{aligned}\eta(\frac{1}{2},t)&\leq -C_0+\varepsilon(\frac{1}{2})^n\left(\frac{v^\beta(\frac{1}{2},t)}{h(t)}\right)^\ell v^{k-\beta\ell}(\frac{1}{2},t)\\ &\leq -C_0+\varepsilon(\frac{1}{2})^n\left(\frac{v^\beta(\frac{1}{2},t)}{h(t)}\right)^\ell\gamma^{k-\beta\ell}\\ &\leq -C_0+\varepsilon 2^{n\ell-n}\gamma^{k-\beta\ell}.\end{aligned} \tag{4.9}$$

As $t=0$, for all $z\in(0,\delta)$, by the relation $\alpha+2=p\alpha>k\alpha$, we have

$$\begin{aligned}\eta(z,0)&\leq\left[-\alpha\delta^{-\alpha-2}+\varepsilon(1+\frac{\alpha}{2})^k\frac{1}{h^\ell(0)}\delta^{-\alpha k}\right]z^n\\ &\leq\left[-\alpha+\varepsilon(1+\frac{\alpha}{2})^k\frac{1}{h^\ell(0)}\right]\delta^{-\alpha-2}z^n.\end{aligned} \tag{4.10}$$

For all $z \in (\delta, 1)$, by the relation $\alpha + 2 = p\alpha > k\alpha$ again, we have

$$\eta(z,0) \le \left[-\alpha + \frac{\varepsilon}{h^\ell(0)}\right] z^{n-\alpha-2}. \tag{4.11}$$

Hence, collecting (4.9)-(4.11), as long as $\varepsilon \le \min\left\{\alpha(1+\frac{\alpha}{2})^{-k} h^\ell(0),\ \ 2^{-\ell n+n} C_0 \gamma^{\beta\ell - k}\right\}$, we have

$$\eta(z,0) \le 0, \ \ z \in (0,1); \qquad \eta(\frac{1}{2}, t) \le 0, \ \ t \in (0, \hat{t}_\lambda). \tag{4.12}$$

Step 2: Observe

$$\begin{aligned}
\mathcal{L}\eta &= \mathcal{L}\left(\varepsilon z^n \frac{v^k}{h^\ell}\right) \\
&= -2\varepsilon k z^{n-1} \frac{v^{k-1}}{h^\ell} \partial_z v - \varepsilon(p-k) e^{-(p-1)t} \frac{z^n}{\xi^q} \frac{v^{p-1+k}}{h^\ell} - \varepsilon k(k-1) z^n \frac{v^{k-2}}{h^\ell} (\partial_z v)^2 \\
&\quad - \varepsilon\beta(1-\beta)\ell z^n \frac{v^k}{h^{\ell+1}} \overline{v^{\beta-2}|\nabla v|^2} - \varepsilon\ell\beta e^{-(p-1)t} \frac{z^n}{\xi^q} \frac{v^k}{h^{\ell+1}} \overline{v^{\beta+p-1}} \\
&\le -2\varepsilon k z^{n-1} \frac{v^{k-1}}{h^\ell} \partial_z v - \varepsilon(p-k) e^{-(p-1)t} \frac{z^n}{\xi^q} \frac{v^{p-1+k}}{h^\ell} \\
&= -2\varepsilon k \frac{v^{k-1}}{h^\ell} \eta + \frac{\varepsilon z^n v^k}{h^{2\ell}} \left[2\varepsilon k v^{k-1} - (p-k) e^{-(p-1)t} \frac{h^\ell v^{p-1}}{\xi^q}\right].
\end{aligned}$$

Recall that $v(t) \ge \gamma$ for all $t \ge 0$ from Lemma 2.1 and that $\xi_0 \le \hat{\xi}(t) \le \lambda\xi_0$ for all $t \in [0, t_\lambda]$ where $\hat{\xi}(t) = e^{\frac{3t}{2} - B_t} \xi(t)$, we have

$$\begin{aligned}
e^{-(p-1)t} \frac{h^\ell(t) v^{p-k}(t)}{\xi^q(t)} &= e^{-(p-1)t} \frac{h^\ell(t) v^{p-k}(t)}{\hat{\xi}^q(t) e^{-\frac{q}{2}t + qB_t}} \ge e^{-(p-1)t} \frac{\gamma^{p+\ell-k}}{\hat{\xi}^q(t) e^{-\frac{q}{2}t + qB_t}} \\
&\ge e^{-(p-1)t} \frac{\gamma^{p+\ell-k}}{(\lambda\xi_0)^q e^{-\frac{q}{2}t + qB_t}} \ge \frac{e^{-(p-1)t_\lambda - qB^*_{t_\lambda}} \gamma^{p+\ell-k}}{(\lambda\xi_0)^q}, \qquad t \in [0, t_\lambda].
\end{aligned}$$

Hence, as long as $\varepsilon \le \frac{(p-k)e^{-(p-1)t_\lambda - qB^*_{t_\lambda}} \gamma^{p+\ell-k}}{2k(\lambda\xi_0)^q}$, we have

$$\mathcal{L}\eta \le -2\varepsilon k \frac{v^{k-1}}{h^\ell} \eta. \tag{4.13}$$

Step 3: Choose $\varepsilon \le \varepsilon^*$ with ε^* being defined as (4.7), then (4.13), (4.12) all hold. Hence,

$$\begin{cases}
\mathcal{L}\eta \le -2\varepsilon k \frac{v^{k-1}}{h^\ell} \eta, & 0 < z < \frac{1}{2}, \quad 0 < t < \hat{t}_\lambda, \\
\eta(z,0) \le 0, & 0 < z < \frac{1}{2}, \\
\eta(0,t) \le 0, & 0 < t < \hat{t}_\lambda, \\
\eta(\frac{1}{2}, t) \le 0, & 0 < t < \hat{t}_\lambda,
\end{cases} \tag{4.14}$$

by maximum principle, we immediately get

$$\eta(z,t) \le 0$$

for all $0 < t < \hat{t}_\lambda$ and $0 < z < \frac{1}{2}$, which implies

$$v(z,t) \le \left(\frac{2h^\ell(t)}{\varepsilon(k-1)}\right)^{\frac{1}{k-1}} z^{-\frac{2}{k-1}}, \qquad \forall\ t \in [0, \hat{t}_\lambda]\ \ \forall\ z \in (0, \frac{1}{2}]. \tag{4.15}$$

□

Lemma 4.5. *Assume* $t_\lambda \le 1$. *For* $R \in (0,1)$, *we have*

$$|\partial_z v(z,t)| \le C_1, \qquad \forall\, z \in [R,1] \quad \forall\, t \in [0,\hat{t}_\lambda], \tag{4.16}$$

where C_1 *is some number depending on* R *and* γ.

Remark 4.6. In the lemma, we assume $t_\lambda \le 1$, the 1 here can be replaced by any other positive number. It seems that the assumption $t_\lambda \le 1$ is necessary for getting the bound C_1 which only depends on R.

Proof. Writing $w(z,t) = \partial_z v(z,t)$, by Eq. (2.6) we have

$$\partial_t w = \partial_z^2 w + \frac{n-1}{z}\partial_z w + \left(pK(t)v^{p-1} - \frac{n-1}{z^2}\right) w. \tag{4.17}$$

By (iii) of Lemma 2.1 and (4.6), we have

$$v^\beta(z,t) \le z^{-n} h(t) \le z^{-n} h^*_\lambda, \qquad \forall\, t \in [0,\hat{t}_\lambda].$$

This and (4.4) further implies

$$v^\beta(z,t) \le R^{-n}\left[h(0) + \beta(\lambda-1)\gamma^{\beta+p-1-r}\lambda^{-q}\xi_0^{s-q+1}\right], \qquad \forall\, t \in [0,\hat{t}_\lambda],\ z \in [R,1]. \tag{4.18}$$

Since $\hat{t}_\lambda \le t_\lambda \le 1$, by (2.12), we have

$$K(t) \le \xi_0^{-q}\exp\left(\frac{3}{2}q + qB_1^*\right), \qquad t \in [0,\hat{t}_\lambda]. \tag{4.19}$$

We can extend Eq. (4.17) from the time interval $[0,\hat{t}_\lambda]$ to $[0,1]$ by

$$K(t)v^{p-1}(z,t) = K(\hat{t}_\lambda)v^{p-1}(z,\hat{t}_\lambda), \qquad \forall\, z \in [R,1)\ \ \forall\, t \in [\hat{t}_\lambda, 1].$$

Now Eq. (4.17) with $(z,t) \in [R,1) \times [0,1]$ has uniformly bounded coefficients.

On the other hand, as $t = 0$, it is easy to check

$$|\partial_z \phi(z)| = \alpha z^{-\alpha-1},\ \ z \in [\delta,1]; \qquad |\partial_z \phi(z)| = \alpha\delta^{-\alpha-2} z,\ \ z \in [0,\delta].$$

Indeed, if $R \ge \delta$, then the first identity above implies

$$|\partial_z v_0(z)| \le \alpha\gamma R^{-\alpha-1}, \qquad z \in [R,1]; \tag{4.20}$$

if $R < \delta$, then the second identity above implies (4.20) as well. Hence,

$$|\partial_z v_0(z)| \le \alpha\gamma R^{-\alpha-1}, \qquad z \in [0,1].$$

So, by parabolic regularity ([11]), we immediately get the desired inequality. □

Lemma 4.7. *Assume* $t_\lambda \le 1$. *Let* $p \ge r$ *and* $\frac{p-1}{r} > \frac{2}{n+2}$. *Let* $\beta \in (0,1]$ *be such that* $p+\beta-1 \ge r$ *holds. For any* $R \in (0,1)$, *we have*

$$\hat{t}_\lambda \ge \frac{h^*_\lambda - h(0) - n\left(\frac{2(h^*_\lambda)^\ell}{\varepsilon^*(k-1)}\right)^{\frac{1}{k-1}} \frac{R^{n-\frac{2\beta}{k-1}}}{n-\frac{2\beta}{k-1}}}{L(\beta, C_1, \lambda, \gamma, p, q, R)}, \tag{4.21}$$

where $k \in (1,p)$, $\ell \ge \frac{k}{\beta}$, ε^* *is defined by* (4.7), *and*

$$\begin{aligned} & L(\beta, C_1, \lambda, \gamma, p, q, R) \\ := \quad & C_1 n\beta R^{n-1}\gamma^{\beta-1} + C_1^2\beta(1-\beta)\gamma^{\beta-2} + \beta\xi_0^{-q}\exp\left(\frac{3}{2}qt_\lambda + qB^*_{t_\lambda}\right)\left(\frac{h^*_\lambda}{R^n}\right)^{\frac{\beta+p-1}{\beta}} \end{aligned}$$

with C_1 *being the number in Lemma 4.5 (which depends on* R*).*

Remark 4.8. We can tune the number R such that the right hand of (4.21) is strictly large than 0 and make the claim $\hat{t}_\lambda > \hat{\theta} > 0$ be true.

Proof. Recall that $h(t) = h_1(t) + h_2(t)$ with

$$h_1(t) = \frac{1}{|B_1(0)|}\int_{B_R(0)} v^\beta(x,t)\mathrm{d}x, \qquad h_2(t) = \frac{1}{|B_1(0)|}\int_{B_1(0)\setminus B_R(0)} v^\beta(x,t)\mathrm{d}x$$

with R being some number to be chosen. By Lemma 4.4, we have

$$v(z,t) \le \left(\frac{2h^\ell(t)}{\varepsilon^*(k-1)}\right)^{\frac{1}{k-1}} z^{-\frac{2}{k-1}}, \qquad \forall\, t\in[0,\hat{t}_\lambda] \;\; \forall\, z\in(0,\frac{1}{2}]. \tag{4.22}$$

Since $\frac{p-1}{r} > \frac{2}{n+2}$, we have $n(p-1) > 2(r+1-p)$. Thanks to the condition $p+\beta-1 \ge r$ with $\beta\in(0,1]$, there exists some $\beta\in(0,1]$ so that $n(p-1) > 2\beta \ge 2(r+1-p)$. Therefore, we can choose some $k\in(1,p)$ so that

$$n(k-1) > 2\beta.$$

Hence, for any $t\in[0,\hat{t}_\lambda]$, by (4.22) and (4.6), we have

$$h_1(t) \le n\left(\frac{2h^\ell(t)}{\varepsilon^*(k-1)}\right)^{\frac{\beta}{k-1}} \int_0^R z^{n-1-\frac{2\beta}{k-1}}\mathrm{d}z \le n\left(\frac{2(h^*_\lambda)^\ell}{\varepsilon^*(k-1)}\right)^{\frac{\beta}{k-1}} \frac{R^{n-\frac{2\beta}{k-1}}}{n-\frac{2\beta}{k-1}}. \tag{4.23}$$

Now we consider $h_2(t)$, by (2.2), it is easy to see

$$\begin{aligned}\frac{d}{dt}h_2(t) &= -n\beta R^{n-1}v^{\beta-1}(R,t)v_z(R,t) - \frac{\beta(\beta-1)}{|B_1(0)|}\int_{B_1(0)\setminus B_R(0)} v^{\beta-2}|\nabla v|^2dx \\ &\quad +\beta K(t)\frac{1}{|B_1(0)|}\int_{B_1(0)\setminus B_R(0)} v^{\beta+p-1}dx.\end{aligned}$$

By (i) of Lemma 2.1 and Lemma 4.5, we have $v^{\beta-1}\le\gamma^{\beta-1}$, $v^{\beta-2}\le\gamma^{\beta-2}$ and

$$\begin{aligned}\left|\frac{d}{dt}h_2(t)\right| &\le n\beta R^{n-1}\gamma^{\beta-1}(R,t)|v_z(R,t)| + \frac{\beta(1-\beta)}{|B_1(0)|}\gamma^{\beta-2}\int_{B_1(0)\setminus B_R(0)}|\nabla v|^2dx \\ &\quad +\beta\frac{e^{-(p-1)t}}{\xi^q(t)}\frac{1}{|B_1(0)|}\int_{B_1(0)\setminus B_R(0)} v^{\beta+p-1}dx.\end{aligned}$$

By (iii) of Lemma 2.1, we have $v^\beta(z,t)\le\frac{h(t)}{z^n}$. This and (4.6) further give

$$\begin{aligned}\left|\frac{d}{dt}h_2(t)\right| &\le C_1 n\beta R^{n-1}\gamma^{\beta-1} + C_1^2\beta(1-\beta)\gamma^{\beta-2} \\ &\quad +\beta e^{-(p-1)t}\left(\hat{\xi}(t)e^{-\frac{3}{2}t+B_t}\right)^{-q}\frac{1}{|B_1(0)|}\int_{B_1(0)\setminus B_R(0)}\left(\frac{h(t)}{z^n}\right)^{\frac{\beta+p-1}{\beta}}dx \\ &\le C_1 n\beta R^{n-1}\gamma^{\beta-1} + C_1^2\beta(1-\beta)\gamma^{\beta-2} \\ &\quad +\beta\xi_0^{-q}\exp\left(\frac{3}{2}qt_\lambda + qB^*_{t_\lambda}\right)\left(\frac{h^*_\lambda}{R^n}\right)^{\frac{\beta+p-1}{\beta}} \\ &:= L(\beta,C_1,\lambda,\gamma,p,q,R)\end{aligned} \tag{4.24}$$

for all $t\in[0,\hat{t}_\lambda]$.

By the definition of $\hat{t}_\lambda$, (4.23) and (4.24), we have

$$\begin{aligned} h_\lambda^* - h(0) &\le h(\hat{t}_\lambda) - h(0) \le h_1(\hat{t}_\lambda) + h_2(\hat{t}_\lambda) - h_2(0) \\ &\le n\left(\frac{2(h_\lambda^*)^\ell}{\varepsilon^*(k-1)}\right)^{\frac{1}{k-1}} \frac{R^{n-\frac{2\beta}{k-1}}}{n-\frac{2\beta}{k-1}} + \int_0^{\hat{t}_\lambda} \left|\frac{\mathrm{d}}{\mathrm{d}s} h_2(s)\right| \mathrm{d}s \\ &\le n\left(\frac{2(h_\lambda^*)^\ell}{\varepsilon^*(k-1)}\right)^{\frac{1}{k-1}} \frac{R^{n-\frac{2\beta}{k-1}}}{n-\frac{2\beta}{k-1}} + \hat{t}_\lambda L(\beta, C_1, \lambda, \gamma, p, q, R). \end{aligned}$$

This immediately implies the desired inequality. □

Proof of Theorem 4.1. To prove the theorem, we shall consider the two cases: (i) the case $t_\lambda \ge 1$ and (ii) the case $t_\lambda < 1$.

(i) $t_\lambda \ge 1$. Take $\theta = 1$ in Section 3, we immediately get the desired estimate by Theorem 3.1.

(ii) $t_\lambda < 1$. By (4.3), it is easy to see that if $t_\lambda < 1$ we have

$$h_\lambda^* \ge h(0) + \beta(\lambda-1)\lambda^{-q}\gamma^{p+\beta-1-r} \exp\left(-p - (s+q+1)(\frac{3}{2} + B_1^*)\right) \xi_0^{s-q+1}. \tag{4.25}$$

Recalling (4.4) as below:

$$h_\lambda^* \le h(0) + \beta(\lambda-1)\gamma^{p+\beta-1-r}\lambda^{-q}\xi_0^{s-q+1}. \tag{4.26}$$

The estimate (4.21), together with (4.25) and (4.26), implies that there exists some $R \in (0,1)$ (which can be tuned according to $p, q, \lambda, B_1^*, s, \lambda, \gamma, \beta, \xi_0$) and some some $\hat{\theta}$ (depending on $\beta, p, q, \lambda, \gamma, B_1^*, s, R$) such that $\hat{t}_\lambda \ge \hat{\theta} > 0$. $\hat{\theta} \in (0,1)$ is obvious. Since $t_\lambda \ge \hat{t}_\lambda$, we have $t_\lambda \ge \hat{\theta}$. Now we can use Theorem 3.1 to get the desired result. □

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