

# Probability that the maximum of the reflected Brownian motion over a finite interval $[0, t]$ is achieved by its last zero before $t$

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## Abstract

We calculate the probability  $p_c$  that the maximum of a reflected Brownian motion  $U$  is achieved on a complete excursion, i.e.  $p_c := P(\bar{U}(t) = U^*(t))$  where  $\bar{U}(t)$  (respectively  $U^*(t)$ ) is the maximum of the process  $U$  over the time interval  $[0, t]$  (resp.  $[0, g(t)]$  where  $g(t)$  is the last zero of  $U$  before  $t$ ).

**Keywords:** Reflected Brownian motion ; Bessel process ; Brownian meander and Brownian bridge ; Brownian excursions ; Gamma function ; local score.

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## 1 Introduction

**1.1 Motivation.** The local score of a biological sequence is its "best" segment with respect to some scoring scheme (see e.g. [12] for more details) and the knowledge of its distribution is important (see e.g. [8], [9]). Let us briefly recall the mathematical setting while biological interpretations can be found in [5]. Let  $S_n := \epsilon_1 + \dots + \epsilon_n$  be the random walk generated by the sequence of the independent and identically distributed random variables  $(\epsilon_i, i \in \mathbb{N})$  that are centered with unit variance. The local score is the process:  $U_n := S_n - \min_{0 \leq i \leq n} S_i$ , where  $n \geq 0$ . The path of  $(U_n, n \in \mathbb{N})$  is a succession of 0 and excursions above 0. In [5], the authors only took into consideration *complete* excursions up to a fixed time  $n$  and so considered the maximum  $U_n^*$  of the heights of all the complete excursions up to time  $n$  instead of the maximum  $\bar{U}_n$  of the path until time  $n$ . They also introduced the random time  $\theta^*$  of the length of the segment that realizes  $U_n^*$ . Since it is easy to simulate  $(S_k, 0 \leq k \leq n)$ , for any  $n$  not too large, we get an approximation of the law of  $(U_n^*, \theta_n^*)$  for a given  $n$ . Simulations have shown that for an important proportion of sequences,  $\bar{U}_n$  is realized during the last incomplete excursion. As expected, the number of excursions naturally increases when the length of the sequence grows, however the proportion of sequences that achieve their maximum on a complete excursion remains strikingly constant. The main goal of this study is to explain these observations and to calculate this probability when  $n$  is large, see Proposition 1.2 below.

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**1.2 Link with the Brownian motion.** According to the functional convergence theorem of Donsker, the random walk  $(S_k, 0 \leq k \leq n)$  (resp.  $(U_k, 0 \leq k \leq n)$ ) normalized by the factor  $1/\sqrt{n}$  converges in distribution, as  $n \rightarrow \infty$ , to the Brownian motion (resp. the reflected Brownian motion). We prove (see Theorem 1.1 for a precise formulation) that the probability that the maximum of a reflected Brownian motion over a finite interval  $[0, t]$  is achieved on a complete excursion is around 30% and is thus independent of  $t$ . This result permits to answer to the two questions asked in the discrete setting, when  $n$  is large.

Let  $U$  be the reflected Brownian motion started at 0, i.e.  $U(t) = |B(t)|$  where  $(B(t), t \geq 0)$  is the standard one-dimensional Brownian motion started at 0. In Chabriac *et al.* [5], the authors have considered two maxima:  $\bar{U}(t)$  and  $U^*(t)$ , the first (resp. second) one being the maximum of  $U$  up to time  $t$  (resp. the last zero before  $t$ ), namely  $\bar{U}(t) := \max_{0 \leq s \leq t} U(s)$  and  $U^*(t) = \bar{U}(g(t))$ , where  $g(t) := \sup\{s \leq t, U(s) = 0\}$ . In [5], the density function of the pair  $(U^*(t), \theta^*(t))$  has been calculated where  $\theta^*(t)$  is the length of time segment that realizes  $U^*(t)$ , i.e. the first hitting time of level  $U^*(t)$  by the process  $(U(s), 0 \leq s \leq g(t))$ . Here we only deal with  $U^*(t)$  and  $\bar{U}(t)$ .

It is clear that  $\bar{U}(t) = U^*(t)$  if and only if  $U^{**}(t) \leq \bar{U}(t)$ , where

$$U^{**}(t) := \max_{g(t) \leq s \leq t} U(s). \tag{1.1}$$

In that case, the maximum of  $U$  over  $[0, t]$  is the maximum of all the complete excursions of  $U$  which hold before  $t$ . We introduce the probability  $p_c$  that the maximum of  $U$  over  $[0, t]$  is achieved on a complete excursion:

$$p_c = \mathbb{P}(\bar{U}(t) = U^*(t)) = \mathbb{P}(U^*(t) > U^{**}(t)). \tag{1.2}$$

Let  $\psi$  be the logarithmic derivative of the Gamma function:

$$\psi(x) := \Gamma'(x)/\Gamma(x). \tag{1.3}$$

The main result of our study is

**Theorem 1.1.** *The probability  $p_c$  equals  $\psi(1/4) - \psi(1/2) + 1 + \pi/2 \approx 0.3069$ .*

**1.3 Back to discrete sequences.** We now go back to the setting of random walks introduced in paragraph 1.1. Let  $p_c^{(n)}$  be the probability that the maximum of  $(U_k, 0 \leq k \leq n)$  is achieved on a complete excursion, namely

$$p_c^{(n)} := P\left(\max_{0 \leq k \leq n} U_k = \max_{0 \leq k \leq g_n} U_k\right) \quad \text{where } g_n := \max\{k \leq n, U_k = 0\}. \tag{1.4}$$

**Proposition 1.2.**  *$p_c^{(n)}$  converges to  $p_c$  as  $n \rightarrow \infty$ .*

The convergence of  $p_c^{(n)}$  can be obtained from Theorem 3.3 in [5] and the fact that the event  $N$  defined by (4.23) in [5] is actually included in  $\left\{ \max_{0 \leq k \leq n} U_k = \max_{0 \leq k \leq g_n} U_k \right\}$ .

**1.4 Main steps of the proof.** We now consider the Brownian motion setting. The density function of  $\bar{U}(t)$  is known (see either Subsection 2.11 in [3] or Lemma 3.2 in [11]) and the one of  $U^*(t)$  has been calculated in [5]. Obviously, the knowledge of the distributions of  $\bar{U}(t)$  and  $U^*(t)$  is not sufficient to determine  $p_c$ . The trajectory of  $(U(s), 0 \leq s \leq t)$  naturally splits in two parts before and after the random time  $g(t)$  which is not a stopping time. Although  $(U(s), 0 \leq s \leq g(t))$  and  $(U(s), g(t) \leq s \leq t)$  are not independent, the scaling with  $g(t)$  leads to independence. Indeed,

$$\left( \frac{1}{\sqrt{g(t)}} U(sg(t)), 0 \leq s \leq 1 \right), \left( \frac{1}{\sqrt{t-g(t)}} |B(g(t) + s(t-g(t)))|, 0 \leq s \leq 1 \right), g(t) \tag{1.5}$$

are independent. Moreover each part of the above triplet has a known distribution. The process  $(g(t)^{-1/2}B(g(t)s), 0 \leq s \leq 1)$  is distributed as the Brownian bridge  $(b(s), 0 \leq s \leq 1)$ , (see e.g. [1]) and the second component in (1.5) is the Brownian meander denoted  $m$ . The scaling property of the Brownian motion implies that  $g(t)$  is distributed as  $tg(1)$  while the distribution of  $g(1)$  is the arcsine one (see again [1]):

$$\mathbb{P}(g(1) \in dx) = \frac{1}{\pi\sqrt{x(1-x)}} \mathbb{1}_{[0,1]}(x) dx. \tag{1.6}$$

Consequently,

$$(U^*(t), U^{**}(t)) \stackrel{(d)}{=} \left( \sqrt{tg(1)}b^*, \sqrt{t(1-g(1))} \max_{0 \leq u \leq 1} m(u) \right) \tag{1.7}$$

where  $b^* := \sup_{0 \leq s \leq 1} |b(s)|$ . Its distribution function is given by the Kolmogorov-Smirnov formula (see e.g. [10]):

$$\mathbb{P}(b^* > x) = 2 \sum_{k \geq 1} (-1)^{k-1} e^{-2k^2 x^2}, \quad x > 0. \tag{1.8}$$

Formula (1.7) permits to determine the law of  $(U^*(t), U^{**}(t))$ , once we know the distribution of  $\max_{0 \leq u \leq 1} m(u)$ . But by [2], for any bounded Borel function  $f$ ,

$$\mathbb{E}[f(m(u), 0 \leq u \leq 1)] = \sqrt{\frac{\pi}{2}} \mathbb{E} \left[ \frac{1}{R(1)} f(R(u), 0 \leq u \leq 1) \right], \tag{1.9}$$

where  $(R(u), 0 \leq u \leq 1)$  stands for a 3-dimensional Bessel started at 0. Due to the scaling property (1.7), we deduce that  $p_c$  does not depend on  $t$  and

$$p_c = \sqrt{\frac{\pi}{2}} \mathbb{E} \left[ F \left( b^* \sqrt{\frac{g(1)}{1-g(1)}} \right) \right] \tag{1.10}$$

where

$$F(x) := \mathbb{E} \left[ \frac{1}{R(1)} \mathbb{1}_{\{\max_{0 \leq u \leq 1} R(u) < x\}} \right]. \tag{1.11}$$

According to (1.10) we have first to determine the function  $F$  (see Lemma 2.1 below), second the distribution function of  $b^* \sqrt{\frac{g(1)}{1-g(1)}}$  (see Proposition 2.2) and third to calculate the expectation. The details are given in Section 2.

## 2 Proof of Theorem 1.1

**Lemma 2.1.** For any  $x > 0$ ,

$$F(x) = \sqrt{\frac{2}{\pi}} \sum_{k \in \mathbb{Z}} \left\{ e^{-2k^2 x^2} - e^{-(2k+1)^2 x^2 / 2} \right\} = \frac{4}{x} \sum_{k \geq 0} \exp \left\{ -\frac{(2k+1)^2 \pi^2}{2x^2} \right\}. \tag{2.1}$$

*Proof.* First, by [4, formula 1.1.8, p317],

$$\mathbb{P} \left( \max_{0 \leq u \leq 1} R(u) < y, R(1) \in dz \mid R(0) = x \right) = \frac{z}{x\sqrt{2\pi}} \times S \times \mathbb{1}_{\{y > x, z < y\}} dz \tag{2.2}$$

where

$$S = \sum_{k \in \mathbb{Z}} \left[ \exp \left\{ -\frac{(z-x+2ky)^2}{2} \right\} - \exp \left\{ -\frac{(z+x+2ky)^2}{2} \right\} \right].$$

A Taylor expansion of  $x \mapsto \exp \left\{ -\frac{(z \pm x + 2ky)^2}{2} \right\}$  at  $x = 0$  leads to

$$\mathbb{P} \left( \max_{0 \leq u \leq 1} R(u) < y, R(1) \in dz \mid R(0) = 0 \right) = \frac{2z}{\sqrt{2\pi}} \left[ \sum_{k \in \mathbb{Z}} (z + 2ky) \exp \left\{ -\frac{(z + 2ky)^2}{2} \right\} \right] \mathbb{1}_{\{z < y\}} dz.$$

As a consequence,

$$\begin{aligned} F(y) &= \frac{2}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \int_0^y (z + 2ky) \exp \left\{ -\frac{(z + 2ky)^2}{2} \right\} dz \\ &= \frac{2}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \left[ \exp \left\{ -2k^2 y^2 \right\} - \exp \left\{ -\frac{(2k+1)^2 y^2}{2} \right\} \right]. \end{aligned}$$

The second equality in (2.1) is a direct consequence of the Poisson summation formula (see, e.g. [7, Chap. XIX p.630]:

$$\begin{aligned} &\frac{1}{\sqrt{2\pi t}} \sum_{k \in \mathbb{Z}} \left[ \exp \left\{ -\frac{1}{2t} (x_0 + 2k(\beta_0 - \alpha_0))^2 \right\} - \exp \left\{ -\frac{1}{2t} (2\beta_0 - x_0 + 2k(\beta_0 - \alpha_0))^2 \right\} \right] \\ &= \frac{1}{\beta_0 - \alpha_0} \sum_{k \geq 1} \left[ \left( \cos \left( \frac{k\pi x_0}{\beta_0 - \alpha_0} \right) - \cos \left( \frac{k\pi(2\beta_0 - x_0)}{\beta_0 - \alpha_0} \right) \right) \exp \left\{ -\frac{k^2 \pi^2 t}{2(\beta_0 - \alpha_0)^2} \right\} \right] \end{aligned}$$

applied with  $t = 1, x_0 = 0, \beta_0 = x/2$  and  $\alpha_0 = -x/2$ . □

**Proposition 2.2.** For any  $x > 0$ ,

$$\mathbb{P} \left( b^* \sqrt{\frac{g(1)}{1-g(1)}} > x \right) = \frac{2}{\pi} \int_0^\infty A(u) e^{-2x^2 u} \frac{du}{\sqrt{u}}, \tag{2.3}$$

where

$$A(u) := \sum_{k \geq 1} (-1)^{k-1} \frac{k}{k^2 + u}. \tag{2.4}$$

*Proof.* We introduce a cut-off  $0 < \varepsilon < 1$  and we define:

$$\varphi_\varepsilon(x) := \mathbb{P} \left( g(1) < 1 - \varepsilon, b^* \sqrt{\frac{g(1)}{1-g(1)}} > x \right). \tag{2.5}$$

Using the independence between  $g(1)$  and  $b^*$ , (1.6) and (1.8), we deduce:

$$\varphi_\varepsilon(x) = \frac{1}{\pi} \int_0^{1-\varepsilon} \frac{1}{\sqrt{y(1-y)}} \mathbb{P} \left( b^* > x \sqrt{\frac{1-y}{y}} \right) dy = \frac{2}{\pi} \sum_{k \geq 1} I_k(\varepsilon)$$

where  $I_k(\varepsilon) := (-1)^{k-1} \int_0^{1-\varepsilon} \frac{1}{\sqrt{y(1-y)}} \exp \left\{ -\frac{2k^2 x^2 (1-y)}{y} \right\} dy$ . The inversion of the sum and the integral is available since  $(1-y)/y \geq \varepsilon' > 0$  where  $\varepsilon' := \varepsilon/(1-\varepsilon)$ . Making the change of variables  $(1-y)/y = u/k^2$  leads to:

$$I_k(\varepsilon) = \frac{(-1)^{k-1}}{k} \int_{\varepsilon' k^2}^\infty \frac{1}{\sqrt{u(1+u/k^2)}} \exp \left\{ -2x^2 u \right\} du.$$

The identity  $\frac{1}{1+u/k^2} = 1 - \frac{u}{k^2(1+u/k^2)}$  allows to invert the sum and the integral. Finally we get:

$$\varphi_\varepsilon(x) = \frac{2}{\pi} \int_{\varepsilon'}^\infty \left( \sum_{\varepsilon'k^2 \leq u} \frac{(-1)^{k-1}}{k} \frac{1}{1+u/k^2} \right) \exp\{-2x^2u\} \frac{du}{\sqrt{u}} = \frac{2}{\pi} \int_{\varepsilon'}^\infty S_{n(\varepsilon',u)}(u) \frac{\exp\{-2x^2u\}}{\sqrt{u}} du$$

with  $n(\varepsilon', u) = \lfloor \sqrt{u/\varepsilon'} \rfloor$ ,  $S_n(u) := \sum_{k=1}^n (-1)^{k-1} \phi(1/k, u)$  and  $\phi(y, u) := y/(1+uy^2)$ .

Note that  $\left| \frac{\partial \phi}{\partial y} \right| \leq 1$ , then, considering  $n = 2m$  and  $n = 2m + 1$  and using the mean value inequality we obtain:

$$|S_{2m}(u)| = \left| \sum_{k'=1}^m \phi\left(\frac{1}{2k'-1}, u\right) - \phi\left(\frac{1}{2k'}, u\right) \right| \leq \sum_{k'=1}^m \left| \frac{1}{2k'-1} - \frac{1}{2k'} \right| \leq \sum_{k'=1}^\infty \frac{1}{2k'(2k'-1)} < \infty. \tag{2.6}$$

Similarly,  $|S_{2m+1}(u)| < \infty$ . Since  $\varepsilon' \rightarrow 0$  and  $n(\varepsilon', u) \rightarrow 0$  as  $\varepsilon$  goes to 0, then, identity (2.3) is a direct consequence of the Lebesgue dominated convergence theorem.  $\square$

Lemma 2.1 and Proposition 2.2 allow to obtain a new integral form for  $p_c$ .

**Lemma 2.3.** *One has*

$$p_c = 8 \int_0^\infty \frac{uA(u^2)}{\sinh(2\pi u)} du.$$

*Proof.* We deduce easily from (2.4) that

$$A(u) = \sum_{k' \geq 1} \phi\left(\frac{1}{2k'-1}, u\right) - \phi\left(\frac{1}{2k'}, u\right),$$

where  $\phi(y, u) := y/(1+uy^2)$ . Then inequality (2.6) implies that  $\sup_{u \geq 0} A(u) < \infty$ . By Lemma 2.1, Proposition 2.2, the definition (2.4) of  $A$  and the Fubini theorem, we get

$$p_c = \sqrt{\frac{\pi}{2}} \frac{32}{\pi} \sum_{k \geq 0} \int_0^\infty \sqrt{u} A(u) \left( \int_0^\infty \exp\left\{-\frac{(2k+1)^2 \pi^2}{2x^2} - 2x^2 u\right\} dx \right) du.$$

But making  $s = 2x^2u$  and letting  $z = 2(2k+1)\pi\sqrt{u}$ , we get:

$$\int_0^\infty \exp\left\{-\frac{(2k+1)^2 \pi^2}{2x^2} - 2x^2 u\right\} dx = \frac{1}{\sqrt{2u}} \left(\frac{z}{2}\right)^{1/2} K_{1/2}(z)$$

where (cf [13, Formula (15) p183])

$$K_\nu(x) = \frac{1}{2} \left(\frac{x}{2}\right)^\nu \int_0^\infty \exp\left\{-s - \frac{x^2}{4s}\right\} ds.$$

Recall that  $K_\nu = K_{-\nu}$  [13, Formula (8) p79] and  $K_{-1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$  [13, Formula (13) p80]. It follows that

$$p_c = 8 \sum_{k \geq 0} \int_0^\infty A(u) e^{-2(2k+1)\pi\sqrt{u}} du = 4 \int_0^\infty \frac{A(u)}{\sinh(2\pi\sqrt{u})} du = 8 \int_0^\infty \frac{vA(v^2)}{\sinh(2\pi v)} dv.$$

$\square$

We now focus on the function  $A$ . Our method is based on the crucial fact that  $A$  can be expressed with the function  $\psi$  defined by (1.3).

**Lemma 2.4.** 1. We have:

$$A(u) = \frac{1}{4} \left[ \psi \left( \frac{i\sqrt{u}}{2} \right) + \psi \left( -\frac{i\sqrt{u}}{2} \right) - \psi \left( \frac{1+i\sqrt{u}}{2} \right) - \psi \left( \frac{1-i\sqrt{u}}{2} \right) \right], \quad u \geq 0. \tag{2.7}$$

2. There exists  $a, b > 0$  such that

$$|\psi(z)| \leq a + b|z|^2, \quad \forall z \in \mathbb{C}, \quad |Im z| \geq 1. \tag{2.8}$$

Consequently,  $p_c = I_2 - I_1$ , where

$$I_k := 2 \int_0^\infty \frac{vF_k(v)}{\sinh(2\pi v)} dv, \quad k = 1, 2 \tag{2.9}$$

and  $F_1(v) := \psi \left( \frac{1+iv}{2} \right) + \psi \left( \frac{1-iv}{2} \right)$ ,  $F_2(v) := \psi \left( \frac{iv}{2} \right) + \psi \left( -\frac{iv}{2} \right)$ .

*Proof.* Formula (2.7) and inequality (2.8) are a direct consequence of identity (3), p 15 in [6], i.e.

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n \geq 1} \frac{z}{n(z+n)} = -\gamma - \frac{1}{z} + \left( \sum_{n \geq 1} \frac{1}{n^2} \right) z - \sum_{n \geq 1} \frac{z^2}{n^2(z+n)} \tag{2.10}$$

with  $\gamma := \lim_{m \rightarrow \infty} \left( \sum_{k=1}^m \frac{1}{k} - \ln m \right)$ . □

Due to the form of the functions  $F_1$  and  $F_2$ , the integrals  $I_1$  and  $I_2$  can be viewed as integrals over a straight line in the plane. More precisely, we have:

**Lemma 2.5.**  $I_1$  and  $I_2$  can be written as:

$$I_1 = -8i \int_{\Delta_{1/2}} \frac{z - 1/2}{\sin(4\pi z)} \psi(z) dz, \quad I_2 = -8i \int_{\Delta_0} \frac{z\psi(z+1)}{\sin(4\pi z)} dz \tag{2.11}$$

where, for any  $a \in \mathbb{R}$ ,  $\Delta_a$  is the line with parametrization by  $z = a + it$ ,  $t \in \mathbb{R}$ .

*Proof.* 1) We note that  $\sin(2\pi iv) = i \sinh(2\pi v)$ . Thus

$$\begin{aligned} 2 \int_0^\infty \frac{v}{\sinh(2\pi v)} \psi \left( \frac{1+iv}{2} \right) dv &= \frac{8}{i} \int_0^\infty \frac{\frac{1+iv}{2} - \frac{1}{2}}{\sin \left( 4\pi i \left( \frac{1+iv}{2} \right) \right)} \psi \left( \frac{1+iv}{2} \right) d \left( \frac{1+iv}{2} \right) \\ &= -8i \int_{\Delta'_{1/2}} \frac{z - 1/2}{\sin(4\pi z)} \psi(z) dz \end{aligned}$$

where  $\Delta'_a$  is the half-line:  $z = a + it$ ,  $t \geq 0$ . Similarly,

$$2 \int_0^\infty \frac{v}{\sinh(2\pi v)} \psi \left( \frac{1-iv}{2} \right) dv = -8i \int_{\Delta''_{1/2}} \frac{z - 1/2}{\sin(4\pi z)} \psi(z) dz$$

where  $\Delta''_a := \{z = a + it, t \leq 0\}$  and  $a \in \mathbb{R}$ . This implies the value of  $I_1$  given by (2.11).

2) Formula (2.10) tells us that  $\hat{\psi}(z) := \psi(z) + \frac{1}{z}$  has no singularity at  $z = 0$ . Thus we study

$$2 \int_0^\infty \frac{v}{\sinh(2\pi v)} \hat{\psi}\left(\frac{iv}{2}\right) dv = \frac{8}{i} \int_0^\infty \frac{iv/2}{\sin(4\pi \frac{iv}{2})} \hat{\psi}\left(\frac{iv}{2}\right) d\left(\frac{iv}{2}\right) = -8i \int_{\Delta'_0} \frac{z}{\sin(4\pi z)} \hat{\psi}(z) dz$$

and similarly

$$2 \int_0^\infty \frac{v}{\sinh(2\pi v)} \hat{\psi}\left(-\frac{iv}{2}\right) dv = 2 \int_0^\infty \frac{iv}{\sin(2\pi iv)} \hat{\psi}\left(\frac{iv}{2}\right) dv = -8i \int_{\Delta''_0} \frac{z}{\sin(4\pi z)} \hat{\psi}(z) dz.$$

The identity (formula (8) p 16 in [6]) :

$$\hat{\psi}(z) = \psi(z) + \frac{1}{z} = \psi(z+1) \tag{2.12}$$

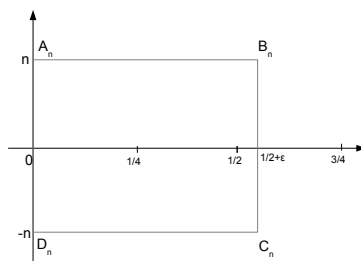
implies  $\psi\left(\frac{iv}{2}\right) + \psi\left(-\frac{iv}{2}\right) = \hat{\psi}\left(\frac{iv}{2}\right) + \hat{\psi}\left(-\frac{iv}{2}\right)$  and finally (2.11). □

We show in the following lemma that  $I_1$  and  $I_2$  are integrals over the vertical line.

**Lemma 2.6.** *Let  $0 < \varepsilon < 1/4$ , then*

$$I_1 = -8i \int_{\Delta_{1/2+\varepsilon}} \frac{z-1/2}{\sin(4\pi z)} \psi(z) dz, \quad I_2 = -8i \int_{\Delta_{1/2+\varepsilon}} \frac{z\psi(z+1)}{\sin(4\pi z)} dz + \psi(1/4) - 2\psi(1/2). \tag{2.13}$$

*Proof.* We only deal with  $I_2$ , the proof related to  $I_1$  is similar and easier. The quantity  $\sin(4\pi z)$  cancels at  $z = k/4$  for every  $k \in \mathbb{Z}$ , the zeros are simple. From (2.10), we deduce that  $h(z) := z\psi(z+1)/\sin(4\pi z)$  is meromorphic in  $\{z \in \mathbb{C}; -1/4 < \text{Re}z < 3/4\}$  with poles at  $1/4$  and  $1/2$ . We introduce the contour defined in Figure 2.



Then the residue theorem gives

$$\int_{C_n B_n} h(z) dz - \int_{D_n A_n} h(z) dz + \int_{D_n C_n} h(z) dz - \int_{B_n A_n} h(z) dz = 2i\pi \{Res(h, 1/4) + Res(h, 1/2)\}. \tag{2.14}$$

The residual at  $1/4$  is given by

$$Res(h, 1/4) = \lim_{z \rightarrow 1/4} \left\{ h(z)(z - 1/4) \right\} = \frac{1}{4} \psi(5/4) \left( \lim_{z \rightarrow 1/4} \frac{z - 1/4}{\sin(4\pi z) - \sin(4\pi 1/4)} \right) = -\frac{1}{16\pi} \psi(5/4)$$

Using (2.12) with  $z = 1/4$ , we get  $Res(h, 1/4) = -\frac{1}{4\pi} \left( 1 + \frac{1}{4} \psi(1/4) \right)$ . Similarly,  $Res(h, 1/2) = \frac{1}{4\pi} (1 + \psi(1/2)/2)$ .

Now, we let  $n$  goes to infinity. Inequality (2.8) implies

$$\lim_{n \rightarrow \infty} \int_{A_n B_n} h(z) dz = \lim_{n \rightarrow \infty} \int_{C_n D_n} h(z) dz = 0.$$

Indeed, it follows from the parametrization of  $A_n B_n$  of the type  $z = in + t$ , inequality  $|in + t| \leq n + 1$  is valid for  $0 \leq t \leq \varepsilon + 1/2$ , and

$$\begin{aligned} |\sin(4\pi(in + t))|^2 &= \sinh(4\pi n)^2 \cos(4\pi t)^2 + \cosh(4\pi n)^2 \sin(4\pi t)^2 \\ &\geq \min \{ \sinh(4\pi n)^2, \cosh(4\pi n)^2 \}. \end{aligned}$$

We proceed analogously on  $C_n D_n$ . As a consequence, letting  $n \rightarrow \infty$  in (2.14), we get

$$\begin{aligned} I_2 &= -8i \left\{ \int_{\Delta_{1/2+\varepsilon}} h(z) dz - 2i\pi \left[ -\frac{1}{4\pi} (1 + \psi(1/4)/4) + \frac{1}{4\pi} (1 + \psi(1/2)/2) \right] \right\} \\ &= -8i \int_{\Delta_{1/2+\varepsilon}} h(z) dz + \psi(1/4) - 2\psi(1/2). \end{aligned}$$

□

Bringing together Lemmas 2.4, 2.6 leads to

$$p_c = -8i \int_{\Delta_{1/2+\varepsilon}} \frac{1}{\sin(4\pi z)} \left( z\psi(z+1) - (z - \frac{1}{2})\psi(z) \right) dz + \psi(1/4) - 2\psi(1/2).$$

Setting  $z = 1/2 + u$  and using identity (2.12) with  $u + 1/2$  instead of  $z$  gives:

$$p_c = -8i \int_{\Delta_\varepsilon} h^+(u) du + \psi(1/4) - 2\psi(1/2), \tag{2.15}$$

where  $h^\pm(z) := \frac{1}{\sin(4\pi z)} \left( 1 + \frac{1}{2}\psi(\frac{1}{2} \pm z) \right)$ .

We are not able to calculate directly  $\int_{\Delta_\varepsilon} h^+(u) du$ , however it is possible for  $\int_{\Delta_\varepsilon} h^+(u) du \pm \int_{\Delta_\varepsilon} h^-(u) du$ .

**Lemma 2.7.** For any  $\varepsilon \in ]0, 1/4[$ ,

$$\int_{\Delta_\varepsilon} h^+(u) du + \int_{\Delta_\varepsilon} h^-(u) du = \frac{i}{2} (1 + \psi(1/2)/2) \tag{2.16}$$

$$\int_{\Delta_\varepsilon} h^+(u) du - \int_{\Delta_\varepsilon} h^-(u) du = \frac{\pi}{2} \int_{\Delta_\varepsilon} \frac{\tan(\pi z)}{\sin(4\pi z)} dz. \tag{2.17}$$

*Proof.* 1) We begin proving (2.16). The function  $h^+$  is meromorphic in  $\{z \in \mathbb{C}; -1/4 < \text{Re}z < 1/4\}$  with unique pole  $z = 0$ . Then the residue theorem yields

$$\int_{\Delta_\varepsilon} h^+(z) dz = \int_{\Delta_{-\varepsilon}} h^+(z) dz + 2i\pi \text{Res}(h^+, 0) = - \int_{\Delta_\varepsilon} h^-(z) dz + \frac{i}{2} (1 + \psi(1/2)/2).$$

2) Formula (2.17) is a direct consequence of formula 11 p16 in [6]:  $\psi(\frac{1}{2} + z) - \psi(\frac{1}{2} - z) = \pi \tan(\pi z)$ . □

It is easy to deduce from (2.16) and (2.17) that:

$$2 \int_{\Delta_\varepsilon} h^+(z) dz = \frac{\pi}{2} \int_{\Delta_\varepsilon} \frac{\tan(\pi z)}{\sin(4\pi z)} dz + \frac{i}{2} (1 + \psi(1/2)/2).$$



Relation (2.15) implies directly that  $p_c$  equals  $\psi(1/4) - \psi(1/2) + 2 - 2\pi i\alpha(\varepsilon)$ , where  $\alpha(\varepsilon) := \int_{\Delta_\varepsilon} \frac{\tan(\pi z)}{\sin(4\pi z)} dz$ . Since the real number  $p_c$  does not depend on  $\varepsilon$ , letting  $\varepsilon \rightarrow 0$  leads to:

$$p_c = \psi(1/4) - \psi(1/2) + 2 - 2\pi i\alpha(0+), \quad (2.18)$$

where  $\alpha(0+) = \int_{\Delta_0} \frac{\tan(\pi z)}{\sin(4\pi z)} dz$ . This integral is easy to calculate.

$$\alpha(0+) = i \int_{\mathbb{R}} \frac{\tanh(\pi x)}{\sinh(4\pi x)} dx = \frac{i}{4} \int_{\mathbb{R}} \frac{dx}{\cosh^2(\pi x) \cosh(2\pi x)}.$$

We make the change of variable  $u = \tanh(\pi x)$ :

$$\alpha(0+) = \frac{i}{2\pi} \int_0^1 \frac{1-u^2}{1+u^2} du = \frac{i}{2\pi} \left( -1 + 2 \int_0^1 \frac{du}{1+u^2} \right) = \frac{i}{2\pi} \left( -1 + \frac{\pi}{2} \right).$$

Theorem 1.1 follows from (2.18) and the above result.

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