Electron. Commun. Probab. **20** (2015), no. 39, 1–6.
DOI: 10.1214/ECP.v20-3945
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

The maximal drawdown of the Brownian meander*

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Abstract

Motivated by evaluating the limiting distribution of randomly biased random walks on trees, we compute the exact value of a negative moment of the maximal drawdown of the standard Brownian meander.

Keywords: Brownian meander; Bessel process; maximal drawdown.
AMS MSC 2010: 60J65.
Submitted to ECP on November 25, 2014, final version accepted on February 12, 2015.

1 Introduction

Let $(X(t),\, t\in[0,\,1])$ be a random process. Its maximal drawdown on $[0,\,1]$ is defined by

$$X^{\#}(1) := \sup_{s\in[0,\,1]} [\,\overline{X}(s) - X(s)]\,,$$

where $\overline{X}(s) := \sup_{u\in[0,\,s]} X(u)$. There has been some recent research interest on the study of drawdowns from probabilistic point of view ([7], [8]) as well as applications in insurance and finance ([1], [2], [3], [10], [12]).

We are interested in the maximal drawdown $\mathfrak{m}^{\#}(1)$ of the standard Brownian meander $(\mathfrak{m}(t),\, t\in[0,\,1])$. Our motivation is the presence of the law of $\mathfrak{m}^{\#}(1)$ in the limiting distribution of randomly biased random walks on supercritical Galton–Watson trees ([4]); in particular, the value of $\mathbb{E}(\frac{1}{\mathfrak{m}^{\#}(1)})$ is the normalizing constant in the density function of this limiting distribution. The sole aim of the present note is to compute $\mathbb{E}(\frac{1}{\mathfrak{m}^{\#}(1)})$, which turns out to have a nice numerical value.

Let us first recall the definition of the Brownian meander. Let $W := (W(t),\, t\in[0,\,1])$ be a standard Brownian motion, and let $\mathfrak{g} := \sup\{t\le 1:\, W(t)=0\}$ be the last passage time at 0 before time 1. Since $\mathfrak{g}<1$ a.s., we can define

$$\mathfrak{m}(s) := \frac{|W(\mathfrak{g}+s(1-\mathfrak{g}))|}{(1-\mathfrak{g})^{1/2}}\,, \qquad s\in[0,\,1]\,.$$

The law of $(\mathfrak{m}(s),\, s\in[0,\,1])$ is called the law of the standard Brownian meander. For an account of general properties of the Brownian meander, see Yen and Yor [11].

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Theorem 1.1. *Let* $(\mathfrak{m}(s),\, s \in [0,\,1])$ *be a standard Brownian meander. We have*

$$\mathbb{E}\Big(\frac{1}{\sup_{s\in[0,\,1]}[\,\overline{\mathfrak{m}}(s) - \mathfrak{m}(s)]}\Big) = \Big(\frac{\pi}{2}\Big)^{1/2}, \tag{1.1}$$

where $\overline{\mathfrak{m}}(s) := \sup_{u\in[0,\,s]} \mathfrak{m}(u)$*.*

The theorem is proved in Section 2.

We are grateful to an anonymous referee for a careful reading of the manuscript and for many suggestions for improvements.

N.B. from the first-named coauthors: This note originates from a question we asked our teacher, **Professor Marc Yor (1949–2014)**, who passed away in January 2014, during the preparation of this note. He provided us, in November 2012, with the essential of the material in Section 2.

2 Proof

Let $R := (R(t),\, t \geq 0)$ be a three-dimensional Bessel process with $R(0) = 0$, i.e., the Euclidean modulus of a standard three-dimensional Brownian motion. The proof of Theorem 1.1 relies on an absolute continuity relation between $(\mathfrak{m}(s),\, s \in [0,\,1])$ and $(R(s),\, s \in [0,\,1])$, recalled as follows.

Fact 2.1. (Imhof [5]) *Let* $(\mathfrak{m}(s),\, s \in [0,\,1])$ *be a standard Brownian meander. Let* $(R(s),\, s \in [0,\,1])$ *be a three-dimensional Bessel process with* $R(0) = 0$*. For any measurable and non-negative functional* F*, we have*

$$\mathbb{E}\Big[F(\mathfrak{m}(s),\, s \in [0,\,1])\Big] = \Big(\frac{\pi}{2}\Big)^{1/2} \mathbb{E}\Big[\frac{1}{R(1)}\, F(R(s),\, s \in [0,\,1])\Big]\,.$$

We now proceed to the proof of Theorem 1.1. Let

$$L := \mathbb{E}\Big(\frac{1}{\sup_{s\in[0,\,1]}[\,\overline{\mathfrak{m}}(s) - \mathfrak{m}(s)]}\Big)\,.$$

Write $\overline{R}(t) := \sup_{u\in[0,\,t]} R(u)$ for $t \geq 0$. By Fact 2.1,

$$\begin{aligned} L &= \Big(\frac{\pi}{2}\Big)^{1/2} \mathbb{E}\Big[\frac{1}{R(1)}\, \frac{1}{\sup_{s\in[0,\,1]}[\,\overline{R}(s) - R(s)]}\Big] \\ &= \Big(\frac{\pi}{2}\Big)^{1/2} \int_0^\infty \mathbb{E}\Big[\frac{1}{R(1)}\, \mathbf{1}_{\{\sup_{s\in[0,\,1]}[\,\overline{R}(s)-R(s)]<\frac{1}{a}\}}\Big]\, \mathrm{d}a\,, \end{aligned}$$

the last equality following from the Fubini–Tonelli theorem. By the scaling property, $\mathbb{E}[\frac{1}{R(1)}\, \mathbf{1}_{\{\sup_{s\in[0,\,1]}[\,\overline{R}(s)-R(s)]<\frac{1}{a}\}}] = \mathbb{E}[\frac{a}{R(a^2)}\, \mathbf{1}_{\{\sup_{u\in[0,\,a^2]}[\,\overline{R}(u)-R(u)]<1\}}]$ for all $a > 0$. So by means of a change of variables $b = a^2$, we obtain:

$$L = \Big(\frac{\pi}{8}\Big)^{1/2} \int_0^\infty \mathbb{E}\Big[\frac{1}{R(b)}\, \mathbf{1}_{\{\sup_{u\in[0,\,b]}[\,\overline{R}(u)-R(u)]<1\}}\Big]\, \mathrm{d}b\,.$$

Define, for any random process X,

$$\tau_1^X := \inf\{t \geq 0 :\, \overline{X}(t) - X(t) \geq 1\}\,,$$

with $\overline{X}(t) := \sup_{s\in[0,\,t]} X(s)$. For any $b > 0$, the event $\{\sup_{u\in[0,\,b]}[\,\overline{R}(u) - R(u)] < 1\}$ means $\{\tau_1^R > b\}$, so

$$L = \Big(\frac{\pi}{8}\Big)^{1/2} \int_0^\infty \mathbb{E}\Big[\frac{1}{R(b)}\, \mathbf{1}_{\{\tau_1^R > b\}}\Big]\, \mathrm{d}b = \Big(\frac{\pi}{8}\Big)^{1/2} \mathbb{E}\Big(\int_0^{\tau_1^R} \frac{1}{R(b)}\, \mathrm{d}b\Big),$$

the second identity following from the Fubini–Tonelli theorem. According to a relation between Bessel processes of dimensions three and four (Revuz and Yor [9], Proposition XI.1.11, applied to the parameters $p = q = 2$ and $\nu = \frac{1}{2}$),

$$R(t) = U\Big(\frac{1}{4}\int_0^t \frac{1}{R(b)}\,\mathrm{d}b\Big), \qquad t \ge 0\,,$$

where $U := (U(s),\, s \ge 0)$ is a four-dimensional *squared* Bessel process with $U(0) = 0$; in other words, U is the square of the Euclidean modulus of a standard four-dimensional Brownian motion.

Let us introduce the increasing functional $\sigma(t) := \frac{1}{4}\int_0^t \frac{1}{R(b)}\,\mathrm{d}b$, $t \ge 0$. We have $R = U \circ \sigma$, and

$$\begin{aligned} \tau_1^R &= \inf\{t \ge 0 :\, \overline{R}(t) - R(t) \ge 1\} \\ &= \inf\{t \ge 0 :\, \overline{U}(\sigma(t)) - U(\sigma(t)) \ge 1\} \\ &= \inf\{\sigma^{-1}(s) :\, s \ge 0 \text{ and } \overline{U}(s) - U(s) \ge 1\} \end{aligned}$$

which is $\sigma^{-1}(\tau_1^U)$. So $\tau_1^U = \sigma(\tau_1^R)$, i.e.,

$$\int_0^{\tau_1^R} \frac{1}{R(b)}\,\mathrm{d}b = 4\tau_1^U\,,$$

which implies that

$$L = (2\pi)^{1/2}\,\mathbb{E}(\tau_1^U)\,.$$

The Laplace transform of τ_1^U is determined by Lehoczky [6], from which, however, it does not seem obvious to deduce the value of $\mathbb{E}(\tau_1^U)$. Instead of using Lehoczky's result directly, we rather apply his method to compute $\mathbb{E}(\tau_1^U)$. By Itô's formula, $(U(t) - 4t,\, t \ge 0)$ is a continuous martingale, with quadratic variation $4\int_0^t U(s)\,\mathrm{d}s$; so applying the Dambis–Dubins–Schwarz theorem (Revuz and Yor [9], Theorem V.1.6) to $(U(t) - 4t,\, t \ge 0)$ yields the existence of a standard Brownian motion $B = (B(t),\, t \ge 0)$ such that

$$U(t) = 2B(\int_0^t U(s)\,\mathrm{d}s) + 4t\,, \qquad t \ge 0\,.$$

Taking $t := \tau_1^U$, we get

$$U(\tau_1^U) = 2B(\int_0^{\tau_1^U} U(s)\,\mathrm{d}s) + 4\tau_1^U\,.$$

We claim that

$$\mathbb{E}\Big[B(\int_0^{\tau_1^U} U(s)\,\mathrm{d}s)\Big] = 0\,. \tag{2.1}$$

Then $\mathbb{E}(\tau_1^U) = \frac{1}{4}\,\mathbb{E}[U(\tau_1^U)]$; hence

$$L = (2\pi)^{1/2}\,\mathbb{E}(\tau_1^U) = (\frac{\pi}{8})^{1/2}\,\mathbb{E}[U(\tau_1^U)]\,. \tag{2.2}$$

Let us admit (2.1) for the moment, and prove the theorem by computing $\mathbb{E}[U(\tau_1^U)]$ using Lehoczky [6]'s method; in fact, we determine the law of $U(\tau_1^U)$.

Lemma 2.2. *The law of $U(\tau_1^U)$ is given by*

$$\mathbb{P}\{U(\tau_1^U) > a\} = (a+1)\mathrm{e}^{-a}, \qquad \forall a > 0.$$

In particular,

$$\mathbb{E}[U(\tau_1^U)] = \int_0^\infty (a+1)\mathrm{e}^{-a}\,\mathrm{d}a = 2.$$

Since $L = (\frac{\pi}{8})^{1/2}\,\mathbb{E}[U(\tau_1^U)]$ (see (2.2)), this yields $L = (\frac{\pi}{2})^{1/2}$ as stated in Theorem 1.1.

The rest of the note is devoted to the proof of Lemma 2.2 and (2.1).

Proof of Lemma 2.2. Fix $b > 1$. We compute the probability $\mathbb{P}\{\overline{U}(\tau_1^U) > b\}$ which, due to the equality $\overline{U}(\tau_1^U) = U(\tau_1^U) + 1$, coincides with $\mathbb{P}\{U(\tau_1^U) > b - 1\}$. By applying the strong Markov property at time $\sigma_0^U := \inf\{t \ge 0 : U(t) = 1\}$, we see that the value of $\mathbb{P}\{\overline{U}(\tau_1^U) > b\}$ does not change if the squared Bessel process U starts at $U(0) = 1$. Indeed, observing that $\sigma_0^U \le \tau_1^U$, $U(\sigma_0^U) = 1$ and that $\overline{U}(\tau_1^U) = \sup_{s\in[\sigma_0^U,\,\tau_1^U]} U(s)$, we have

$$\mathbb{P}\{\overline{U}(\tau_1^U) > b\} = \mathbb{P}\Big\{\sup_{s\in[\sigma_0^U,\,\tau_1^U]} U(s) > b\Big\} = \mathbb{P}_1\{\overline{U}(\tau_1^U) > b\}\,,$$

the subscript 1 in $\mathbb{P}_1$ indicating the initial value of U. More generally, for $x \ge 0$, we write $\mathbb{P}_x(\bullet) := \mathbb{P}(\bullet\,|\,U(0) = x)$; so $\mathbb{P} = \mathbb{P}_0$.

Let $b_0 = 1 < b_1 < \cdots < b_n := b$ be a subdivision of $[1,\,b]$ such that $\max_{1\le i\le n}(b_i - b_{i-1}) \to 0$, $n \to \infty$. Consider the event $\{\overline{U}(\tau_1^U) > b\}$: since $U(0) = 1$, this means U hits position b before time τ_1^U; for all $i \in [1,\,n-1]\cap\mathbb{Z}$, starting from position b_i, U must hit b_{i+1} before hitting $b_i - 1$ (caution: not to be confused with b_{i-1}). More precisely, let $\sigma_i^U := \inf\{t \ge 0 : U(t) = b_i\}$ and let $U_i(s) := U(s + \sigma_i^U)$, $s \ge 0$; then

$$\{\overline{U}(\tau_1^U) > b\} \subset \bigcap_{i=1}^{n-1} \{U_i \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}\,.$$

By the strong Markov property, the events $\{U_i$ hits b_{i+1} before hitting $b_i - 1\}$, $1 \le i \le n-1$, are independent (caution : the processes $(U_i(s),\, s \ge 0)$, $1 \le i \le n-1$, are not independent). Hence

$$\mathbb{P}_1\{\overline{U}(\tau_1^U) > b\} \le \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}\,. \tag{2.3}$$

Conversely, let $\varepsilon > 0$, and if $\max_{1\le i\le n}(b_i - b_{i-1}) < \varepsilon$, then we also have

$$\mathbb{P}_1\{\overline{U}(\tau_{1+\varepsilon}^U) > b\} \ge \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\}\,,$$

with $\tau_{1+\varepsilon}^U := \inf\{t \ge 0 : \overline{U}(t) - U(t) \ge 1+\varepsilon\}$. By scaling, $\overline{U}(\tau_{1+\varepsilon}^U)$ has the same distribution as $(1+\varepsilon)\overline{U}(\tau_1^U)$. So, as long as $\max_{1\le i\le n}(b_i - b_{i-1}) < \varepsilon$, we have

$$\mathbb{P}_1\{\overline{U}(\tau_1^U) > b\} \le \prod_{i=1}^{n-1} \mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\} \le \mathbb{P}_1\{\overline{U}(\tau_1^U) > \frac{b}{1+\varepsilon}\}\,.$$

Since $\frac{1}{x}$ is a scale function for U, we have

$$\mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i - 1\} = \frac{\frac{1}{b_i-1} - \frac{1}{b_i}}{\frac{1}{b_i-1} - \frac{1}{b_{i+1}}} = 1 - \frac{\frac{1}{b_i} - \frac{1}{b_{i+1}}}{\frac{1}{b_i-1} - \frac{1}{b_{i+1}}}\,.$$

If $\lim_{n\to\infty}\max_{0\le i\le n-1}(b_{i+1} - b_i) = 0$, then for $n \to \infty$,

$$\begin{aligned}
\sum_{i=1}^{n-1} \frac{\frac{1}{b_i} - \frac{1}{b_{i+1}}}{\frac{1}{b_i-1} - \frac{1}{b_{i+1}}} &= \sum_{i=1}^{n-1} \frac{b_i - 1}{b_i}\,(b_{i+1} - b_i) + o(1) \\
&\to \int_1^b \frac{r-1}{r}\,\mathrm{d}r \\
&= b - 1 - \log b\,.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\prod_{i=1}^{n-1}\mathbb{P}_{b_i}\{U \text{ hits } b_{i+1} \text{ before hitting } b_i-1\} = \mathrm{e}^{-(b-1-\log b)} = b\,\mathrm{e}^{-(b-1)}\,.$$

Consequently,

$$\mathbb{P}\{\overline{U}(\tau_1^U) > b\} = b\,\mathrm{e}^{-(b-1)}, \qquad \forall b > 1.$$

We have already noted that $U(\tau_1^U) = \overline{U}(\tau_1^U) - 1$. This completes the proof of Lemma 2.2. □

Proof of (2.1). The Brownian motion B being the Dambis–Dubins–Schwarz Brownian motion associated with the continuous martingale $(U(t) - 4t,\, t \geq 0)$, it is a $(\mathscr{G}_r)_{r\geq 0}$-Brownian motion (Revuz and Yor [9], Theorem V.1.6), where, for $r \geq 0$,

$$\mathscr{G}_r := \mathscr{F}_{C(r)}, \qquad C(r) := A^{-1}(r), \qquad A(t) := \int_0^t U(s)\,\mathrm{d}s\,,$$

and A^{-1} denotes the inverse of A. [We mention that $\mathscr{F}_{C(r)}$ is well defined because $C(r)$ is an $(\mathscr{F}_t)_{t\geq 0}$-stopping time.] As such,

$$\int_0^{\tau_1^U} U(s)\,\mathrm{d}s = A(\tau_1^U)\,.$$

For all $r \geq 0$, $\{A(\tau_1^U) > r\} = \{\tau_1^U > C(r)\} \in \mathscr{F}_{C(r)} = \mathscr{G}_r$ (observing that τ_1^U is an $(\mathscr{F}_t)_{t\geq 0}$-stopping time), which means that $A(\tau_1^U)$ is a $(\mathscr{G}_r)_{r\geq 0}$-stopping time. If $A(\tau_1^U) = \int_0^{\tau_1^U} U(s)\,\mathrm{d}s$ has a finite expectation, then we are entitled to apply the (first) Wald identity to see that $\mathbb{E}[B(A(\tau_1^U))] = 0$ as claimed in (2.1).

It remains to prove that $\mathbb{E}[A(\tau_1^U)] < \infty$.

Recall that U is the square of the Euclidean modulus of an $\mathbb{R}^4$-valued Brownian motion. By considering only the first coordinate of this Brownian motion, say β, we have

$$\mathbb{P}\Big\{\sup_{s\in[0,\,a]} U(s) < a^{1-\varepsilon}\Big\} \leq \mathbb{P}\Big\{\sup_{s\in[0,\,a]} |\beta(s)| < a^{(1-\varepsilon)/2}\Big\} = \mathbb{P}\Big\{\sup_{s\in[0,\,1]} |\beta(s)| < a^{-\varepsilon/2}\Big\}\,;$$

so by the small ball probability for Brownian motion, we obtain:

$$\mathbb{P}\Big\{\sup_{s\in[0,\,a]} U(s) < a^{1-\varepsilon}\Big\} \leq \exp(-c_1\,a^{\varepsilon})\,,$$

for all $a \geq 1$ et all $\varepsilon \in (0,\,1)$, with some constant $c_1 = c_1(\varepsilon) > 0$. On the event $\{\sup_{s\in[0,\,a]} U(s) \geq a^{1-\varepsilon}\}$, if $\tau_1^U > a$, then for all $i \in [1,\, a^{1-\varepsilon} - 1] \cap \mathbb{Z}$, the squared Bessel process U, starting from i, must first hit position $i+1$ before hitting $i-1$ (which, for each i, can be realized with probability $\leq 1 - c_2$, where $c_2 \in (0,\,1)$ is a constant that does not depend on i, nor on a). Accordingly,[1]

$$\mathbb{P}\Big\{\sup_{s\in[0,\,a]} U(s) \geq a^{1-\varepsilon},\ \tau_1^U > a\Big\} \leq (1-c_2)^{\lfloor a^{1-\varepsilon}-1\rfloor} \leq \exp(-c_3\,a^{1-\varepsilon})\,,$$

with some constant $c_3 > 0$, uniformly in $a \geq 2$. We have thus proved that for all $a \geq 2$ and all $\varepsilon \in (0,\,1)$,

$$\mathbb{P}\{\tau_1^U > a\} \leq \exp(-c_3\,a^{1-\varepsilon}) + \exp(-c_1\,a^{\varepsilon}).$$

Taking $\varepsilon := \frac{1}{2}$, we see that there exists a constant $c_4 > 0$ such that

$$\mathbb{P}\{\tau_1^U > a\} \leq \exp(-c_4\,a^{1/2}), \qquad \forall a \geq 2.$$

[1] This is the special case $b_i := i$ of the argument we have used to obtain (2.3).

On the other hand, U being a squared Bessel process, we have, for all $a > 0$ and all $b \geq a^2$,

$$\mathbb{P}\{A(a) \geq b\} = \mathbb{P}\{A(1) \geq \frac{b}{a^2}\} \leq \mathbb{P}\Big\{\sup_{s\in[0,\,1]} U(s) \geq \frac{b}{a^2}\Big\} \leq \mathrm{e}^{-c_5\, b/a^2}\,,$$

for some constant $c_5 > 0$. Hence, for $b \geq a^2$ and $a \geq 2$,

$$\mathbb{P}\{A(\tau_1^U) \geq b\} \leq \mathbb{P}\{\tau_1^U > a\} + \mathbb{P}\{A(a) \geq b\} \leq \exp(-c_4\, a^{1/2}) + \mathrm{e}^{-c_5\, b/a^2}\,.$$

Taking $a := b^{2/5}$ gives that

$$\mathbb{P}\{A(\tau_1^U) \geq b\} \leq \exp(-c_6\, b^{1/5})\,,$$

for some constant $c_6 > 0$ and all $b \geq 4$. In particular, $\mathbb{E}[A(\tau_1^U)] < \infty$ as desired. □

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