

Invariant and ergodic nonlinear expectations for G -diffusion processes*

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Abstract

In this paper we study the problems of invariant and ergodic expectations under G -expectation framework. In particular, the stochastic differential equations driven by G -Brownian motion (G -SDEs) have the unique invariant and ergodic expectations. Moreover, the invariant and ergodic expectations of G -SDEs are also sublinear expectations. However, the invariant expectations may not coincide with the ergodic expectations, which is different from the classical case.

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1 Introduction

Recently, Peng systemically established a time-consistent fully nonlinear expectation theory (see [11, 14, 15] and the references therein), which is an effective tool to study the problems of model uncertainty, nonlinear stochastic dynamical systems and fully nonlinear partial differential equations (PDEs). As a typical and important case, Peng introduced the G -expectation theory. In the G -expectation framework, the notion of G -Brownian motion and the corresponding stochastic calculus of Itô's type were also established. Moreover, Peng [14] and Gao [4] obtained the existence and uniqueness theorem of G -SDEs.

It is well known that invariant measure plays an important role in the theory of stochastic dynamical systems and ergodic theory. In particular, the invariant measure can be thought of as describing the long-term behaviour of a dynamical system, which has many important applications in, for example, PDEs and financial mathematics. So far, there are many papers in the literature which were devoted to studying the invariant measures of Markov processes, in both finite and infinite dimensional spaces (see [1] and the references therein).

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The aim of this paper is to study the asymptotic property of G -SDEs. First, we obtain the existence and uniqueness theorem of “invariant measures” for G -SDEs, which indeed is a sublinear expectations. The proof of the existence theorem is based on Daniell-Stone Theorem. It is important to point out that the standard techniques and results on invariant measures for Markov processes cannot be applied to deal with this problem because G -expectation is not a linear expectation. Under G -expectation framework, the invariant expectations of G -SDE is a family of probability measures. In particular, if the initial condition has the distribution equal to an invariant expectation, then the distribution of the solution to G -SDE is invariant in time as the classical case. Next, we study the ergodicity of G -SDEs. Under nonlinear case, the ergodic expectation of G -SDE may not be the corresponding invariant expectation. The proof of the existence theorem of ergodic expectation is based on the theory of ergodic backward differential equations driven by G -Brownian motion, which is obtained in [8] (see also [2, 9, 16, 17]).

The paper is organized as follows. In section 2, we present some notations and results which will be used in this paper. The existence and uniqueness theorem of invariant expectations of G -diffusion processes is established in section 3. In section 4, we shall study the relationships between invariant expectations and ergodic expectations under the G -expectation framework.

2 Preliminaries

The main purpose of this section is to recall some basic notions and results of G -expectation, which are needed in the sequel. The readers may refer to [5], [6], [12], [13], [14] for more details.

Definition 2.1. Let Ω be a given set and let \mathcal{H} be a vector lattice of real valued functions defined on Ω , namely $c \in \mathcal{H}$ for each constant c and $|X| \in \mathcal{H}$ if $X \in \mathcal{H}$. \mathcal{H} is considered as the space of random variables. A sublinear expectation $\hat{\mathbb{E}}$ on \mathcal{H} is a functional $\hat{\mathbb{E}} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

- (a) *Monotonicity:* If $X \geq Y$ then $\hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y]$;
- (b) *Constant preservation:* $\hat{\mathbb{E}}[c] = c$;
- (c) *Sub-additivity:* $\hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y]$;
- (d) *Positive homogeneity:* $\hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X]$ for each $\lambda \geq 0$.

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space. $X \in \mathcal{H}$ is called a random variable in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. We often call $Y = (Y_1, \dots, Y_d), Y_i \in \mathcal{H}$ a d -dimensional random vector in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.

Definition 2.2. Let X_1 and X_2 be two n -dimensional random vectors defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1 \stackrel{d}{=} X_2$, if $\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)]$, for all $\varphi \in C_{Lip}(\mathbb{R}^n)$, where $C_{Lip}(\mathbb{R}^n)$ is the space of real \mathbb{R} -valued Lipschitz continuous functions defined on \mathbb{R}^n .

Definition 2.3. In a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, a random vector $Y = (Y_1, \dots, Y_n), Y_i \in \mathcal{H}$, is said to be independent of another random vector $X = (X_1, \dots, X_m), X_i \in \mathcal{H}$ under $\hat{\mathbb{E}}[\cdot]$, denoted by $Y \perp X$, if for every test function $\varphi \in C_{Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\hat{\mathbb{E}}[\varphi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x, Y)]_{x=X}]$.

Definition 2.4. (G -normal distribution) A d -dimensional random vector $X = (X_1, \dots, X_d)$ in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called G -normally distributed if for each $a, b \geq 0$ we have

$$aX + b\bar{X} \stackrel{d}{=} \sqrt{a^2 + b^2}X,$$

where \bar{X} is an independent copy of X , i.e., $\bar{X} \stackrel{d}{=} X$ and $\bar{X} \perp X$. Here the letter G denotes the function

$$G(A) := \frac{1}{2} \hat{\mathbb{E}}[\langle AX, X \rangle] : \mathbb{S}_d \rightarrow \mathbb{R},$$

where \mathbb{S}_d denotes the collection of $d \times d$ symmetric matrices.

Let $\Omega = C_0([0, \infty); \mathbb{R}^d)$, the space of \mathbb{R}^d -valued continuous functions on $[0, \infty)$ with $\omega_0 = 0$, be endowed with the distance

$$\rho(\omega^1, \omega^2) := \sum_{N=1}^{\infty} 2^{-N} [(\max_{t \in [0, N]} |\omega_t^1 - \omega_t^2|) \wedge 1],$$

and $B = (B^i)_{i=1}^d$ be the canonical process. For each $T > 0$, denote

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1}, \dots, B_{t_n}) : n \geq 1, t_1, \dots, t_n \in [0, T], \varphi \in C_{Lip}(\mathbb{R}^{d \times n})\}, \quad L_{ip}(\Omega) := \bigcup_T L_{ip}(\Omega_T).$$

For any given monotonic and sublinear function $G : \mathbb{S}_d \rightarrow \mathbb{R}$, let $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}}, \hat{\mathbb{E}}_t)$ be the G -expectation space, where $G(A) = \frac{1}{2} \hat{\mathbb{E}}[\langle AB_1, B_1 \rangle] \leq \frac{1}{2} \bar{\sigma}^2 |A|$.

Denote by $L_G^p(\Omega)$ the completion of $L_{ip}(\Omega)$ under the norm $\|\xi\|_{L_G^p} := (\hat{\mathbb{E}}[|\xi|^p])^{1/p}$ for $p \geq 1$. Denis et al. [3] proved that the completions of $C_b(\Omega)$ (the set of bounded continuous function on Ω) and $L_{ip}(\Omega)$ under $\|\cdot\|_{L_G^p}$ are the same. Similarly, we can define $L_G^p(\Omega_T)$ for each $T > 0$.

Theorem 2.5 ([3, 7]). *There exists a weakly compact set $\mathcal{P} \subset \mathcal{M}_1(\Omega)$, the set of all probability measures on $(\Omega, \mathcal{B}(\Omega))$, such that*

$$\hat{\mathbb{E}}[\xi] = \sup_{P \in \mathcal{P}} E_P[\xi] \quad \text{for all } \xi \in L_G^1(\Omega).$$

\mathcal{P} is called a set that represents $\hat{\mathbb{E}}$.

Let \mathcal{P} be a weakly compact set that represents $\hat{\mathbb{E}}$. For this \mathcal{P} , we define the capacity

$$c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

A set $A \subset \mathcal{B}(\Omega)$ is polar if $c(A) = 0$. A property holds “quasi-surely” (q.s.) if it holds outside a polar set. In the following, we do not distinguish two random variables X and Y if $X = Y$ q.s..

Definition 2.6. *Let $M_G^0(0, T)$ be the collection of processes in the following form: for a given partition $\{t_0, \dots, t_N\} = \pi_T$ of $[0, T]$,*

$$\eta_t(\omega) = \sum_{j=0}^{N-1} \xi_j(\omega) \mathbf{1}_{[t_j, t_{j+1})}(t),$$

where $\xi_i \in L_{ip}(\Omega_{t_i})$, $i = 0, 1, 2, \dots, N - 1$. For each $p \geq 1$, denote by $M_G^p(0, T)$ the completion of $M_G^0(0, T)$ under the norm $\|\eta\|_{M_G^p} := (\hat{\mathbb{E}}[\int_0^T |\eta_s|^p ds])^{1/p}$.

For two processes $\eta \in M_G^2(0, T)$ and $\xi \in M_G^1(0, T)$, the G -Itô integrals $(\int_0^t \eta_s dB_s^i)_{0 \leq t \leq T}$ and $(\int_0^t \xi_s d\langle B^i, B^j \rangle_s)_{0 \leq t \leq T}$ are well defined, see Li-Peng [10] and Peng [14].

3 Invariant nonlinear expectation

In this section, we shall study the invariant expectations of G -diffusion processes. Let $G : \mathbb{S}_d \rightarrow \mathbb{R}$ be a given monotonic and sublinear function and $B_t = (B_t^i)_{i=1}^d$ be the corresponding d -dimensional G -Brownian motion. For a given integer $p \geq 1$, a real-valued function f defined on \mathbb{R}^n is said to be in $C_{p, Lip}(\mathbb{R}^n)$ if there exists a constant K_f

depending on f such that $|f(x) - f(x')| \leq K_f(1 + |x|^{p-1} + |x'|^{p-1})|x - x'|$. Consider the following type of G -SDEs (in this paper we always use Einstein convention): for each $t \geq 0$ and $\xi \in L_G^m(\Omega_t)$ with $m \geq 2$,

$$X_s^{t,\xi} = \xi + \int_t^s b(X_r^{t,\xi})dr + \int_t^s h_{ij}(X_r^{t,\xi})d\langle B^i, B^j \rangle_r + \int_t^s \sigma(X_r^{t,\xi})dB_r, \quad (3.1)$$

where $b, h_{ij} : \mathbb{R}^n \rightarrow \mathbb{R}^n, \sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times d}$ are deterministic continuous functions. In particular, denote $X^x = X^{0,x}$. Consider also the following assumptions:

(H1) There exists a constant $L > 0$ such that

$$|b(x) - b(x')| + \sum_{i,j} |h_{ij}(x) - h_{ij}(x')| + |\sigma(x) - \sigma(x')| \leq L|x - x'|.$$

(H2) $G((2p-1) \sum_{i=1}^n (\sigma_i(x) - \sigma_i(x'))^T (\sigma_i(x) - \sigma_i(x')) + 2(\langle x - x', h_{ij}(x) - h_{ij}(x') \rangle)_{i,j=1}^d + \langle x - x', b(x) - b(x') \rangle) \leq -\eta|x - x'|^2$ for some constants $\eta > 0$, where σ_i is the i -th row of σ .

We have the following estimates of G -SDEs which can be found in Chapter V in Peng [14].

Lemma 3.1. *Under assumption (H1), the G -SDE (3.1) has a unique solution $X^{t,\xi} \in M_G^2(t, T)$ for each $T > t$. Moreover, if $\xi, \xi' \in L_G^m(\Omega_t)$ with $m \geq 2$, then we have, for each $\delta \in [0, T - t]$,*

- (i) $\hat{\mathbb{E}}_t[\sup_{s \in [t, T]} |X_s^{t,\xi} - X_s^{t,\xi'}|^m] \leq C'|\xi - \xi'|^m;$
- (ii) $\hat{\mathbb{E}}_t[\sup_{s \in [t, T]} |X_s^{t,\xi}|^m] \leq C'(1 + |\xi|^m);$
- (iii) $\hat{\mathbb{E}}_t[\sup_{s \in [t, t+\delta]} |X_s^{t,\xi} - \xi|^m] \leq C'(1 + |\xi|^m)\delta^{m/2},$

where the constant C' depends on L, G, m, n and T .

The following result is important for our future discussion (see also [8]). Especially, the constant C is independent of T .

Lemma 3.2. *Under assumptions (H1) and (H2), if $\xi, \xi' \in L_G^{2p}(\Omega_t)$, then there exists a constant C depending on G, L, p, n and η , such that:*

- (i) $\hat{\mathbb{E}}_t[|X_s^{t,\xi} - X_s^{t,\xi'}|^{2p}] \leq \exp(-2\eta p(s - t))|\xi - \xi'|^{2p};$
- (ii) $\hat{\mathbb{E}}_t[|X_s^{t,\xi}|^{2p}] \leq C(1 + |\xi|^{2p}), \forall t > 0.$

Proof. To simplify presentation, we shall prove only the case when $n = d = 1$, as the higher dimensional case can be treated in the same way without difficulty. Set $C_s := \exp(2p\eta(s - t))$. Applying the G -Itô formula yields that

$$\begin{aligned} & C_s(X_s^{t,\xi} - X_s^{t,\xi'})^{2p} - |\xi - \xi'|^{2p} \\ &= 2p\eta \int_t^s C_r(X_r^{t,\xi} - X_r^{t,\xi'})^{2p} dr + 2p \int_t^s C_r(X_r^{t,\xi} - X_r^{t,\xi'})^{2p-1} (b(X_r^{t,\xi}) - b(X_r^{t,\xi'})) dr \\ &+ p \int_t^s \zeta_r d\langle B \rangle_r + 2p \int_t^s C_r(X_r^{t,\xi} - X_r^{t,\xi'})^{2p-1} (\sigma(X_r^{t,\xi}) - \sigma(X_r^{t,\xi'})) dB_r \\ &= 2p\eta \int_t^s C_r(X_r^{t,\xi} - X_r^{t,\xi'})^{2p} dr + 2p \int_t^s C_r(X_r^{t,\xi} - X_r^{t,\xi'})^{2p-1} (b(X_r^{t,\xi}) - b(X_r^{t,\xi'})) dr \\ &+ 2p \int_t^s G(\zeta_r) dr + 2p \int_t^s C_r(X_r^{t,\xi} - X_r^{t,\xi'})^{2p-1} (\sigma(X_r^{t,\xi}) - \sigma(X_r^{t,\xi'})) dB_r \\ &+ p \int_t^s \zeta_r d\langle B \rangle_r - 2p \int_t^s G(\zeta_r) dr, \end{aligned}$$

where

$$\zeta_r = C_r(X_r^{t,\xi} - X_r^{t,\xi'})^{2p-2}((2p-1)|\sigma(X_r^{t,\xi}) - \sigma(X_r^{t,\xi'})|^2 + 2(X_r^{t,\xi} - X_r^{t,\xi'})(h(X_r^{t,\xi}) - h(X_r^{t,\xi'}))).$$

Note that $\int_t^s \zeta_r d\langle B \rangle_r - 2 \int_t^s G(\zeta_r) dr \leq 0$ and (H2), then we obtain

$$C_s(X_s^{t,\xi} - X_s^{t,\xi'})^{2p} - |\xi - \xi'|^{2p} \leq 2p \int_t^s C_r(X_r^{t,\xi} - X_r^{t,\xi'})^{2p-1}(\sigma(X_r^{t,\xi}) - \sigma(X_r^{t,\xi'}))dB_r. \tag{3.2}$$

On the other hand, by Lemma 3.1,

$$\hat{\mathbb{E}}\left[\left(\int_t^T |X_r^{t,\xi} - X_r^{t,\xi'}|^{4p} dr\right)^{1/2}\right] \leq \sqrt{T} \hat{\mathbb{E}}\left[\sup_{r \in [t, T]} |X_r^{t,\xi} - X_r^{t,\xi'}|^{2p}\right] \leq \sqrt{T} C' \hat{\mathbb{E}}[|\xi - \xi'|^{2p}].$$

Then the right side of inequality (3.2) is a G -martingale. Thus we conclude that

$$\hat{\mathbb{E}}_t[C_s |X_s^{t,\xi} - X_s^{t,\xi'}|^{2p}] \leq |\xi - \xi'|^{2p}.$$

Consequently,

$$\hat{\mathbb{E}}_t[|X_s^{t,\xi} - X_s^{t,\xi'}|^{2p}] \leq \exp(-2p\eta(s-t))|\xi - \xi'|^{2p}.$$

By a similar analysis as Lemma 4.1 of [8], we can also obtain the second inequality, which completes the proof. \square

Theorem 3.3. Assume (H1) and (H2) hold. Then for each $f \in C_{2p, Lip}(\mathbb{R}^n)$, there exists a constant $\bar{\lambda}^f$ such that

$$\lim_{t \rightarrow \infty} \hat{\mathbb{E}}[f(X_t^x)] = \bar{\lambda}^f, \quad \forall x \in \mathbb{R}^n.$$

In particular, for each t , there exists a constant C_1 depending on G, η, L, K_f, n and p such that

$$|\bar{\lambda}^f - \hat{\mathbb{E}}[f(X_t^x)]| \leq C_1(1 + |x|^{2p}) \exp(-\eta t).$$

Proof. For a fixed x and each $f \in C_{2p, Lip}(\mathbb{R}^n)$, from Lemma 3.2, we can find some constant \bar{C} depending on C and K_f such that

$$\hat{\mathbb{E}}[|f(X_t^x)|] \leq |f(0)| + \bar{C} \hat{\mathbb{E}}[|X_t^x|^{2p}] \leq \bar{C}(1 + |x|^{2p}).$$

Then there exists a sequence $T_n \rightarrow \infty$ such that $\hat{\mathbb{E}}[f(X_{T_n}^x)] \rightarrow \bar{\lambda}^f$ for some constant $\bar{\lambda}^f$. From the uniqueness of solutions to G -SDEs, we obtain $X_s^x = X_s^{t, X_t^x}$ with $s \geq t$. Note that $\hat{\mathbb{E}}[f(X_{t'}^x)] = \hat{\mathbb{E}}[f(X_t^{t-t', x})]$ for each t and t' with $t' \leq t$, then we have

$$\begin{aligned} |\hat{\mathbb{E}}[f(X_t^x)] - \hat{\mathbb{E}}[f(X_{t'}^x)]| &= |\hat{\mathbb{E}}[f(X_t^{t-t', X_{t-t'}^x})] - \hat{\mathbb{E}}[f(X_t^{t-t', x})]| \\ &\leq K_f \hat{\mathbb{E}}[(1 + |X_t^{t-t', X_{t-t'}^x}|^{2p-1} + |X_t^{t-t', x}|^{2p-1}) |X_t^{t-t', X_{t-t'}^x} - X_t^{t-t', x}|]. \end{aligned}$$

Applying Hölder's inequality and Lemma 3.2, we obtain that

$$\begin{aligned} &|\hat{\mathbb{E}}[f(X_t^x)] - \hat{\mathbb{E}}[f(X_{t'}^x)]| \\ &\leq K_f \hat{\mathbb{E}}[(1 + |X_t^{t-t', X_{t-t'}^x}|^{2p-1} + |X_t^{t-t', x}|^{2p-1})^{\frac{2p}{2p-1}}]^{\frac{2p-1}{2p}} \hat{\mathbb{E}}[|X_t^{t-t', X_{t-t'}^x} - X_t^{t-t', x}|^{2p}]^{\frac{1}{2p}} \\ &\leq C_1 \hat{\mathbb{E}}[1 + |X_t^{t-t', X_{t-t'}^x}|^{2p} + |X_t^{t-t', x}|^{2p}]^{\frac{2p-1}{2p}} \hat{\mathbb{E}}[|X_{t-t'}^x|^{2p} + |x|^{2p}]^{\frac{1}{2p}} \exp(-\eta t') \\ &\leq C_1(1 + |x|^{2p})^{\frac{1}{2p}} \hat{\mathbb{E}}[\hat{\mathbb{E}}_{t-t'}[1 + |X_t^{t-t', X_{t-t'}^x}|^{2p} + |X_t^{t-t', x}|^{2p}]]^{\frac{2p-1}{2p}} \exp(-\eta t') \\ &\leq C_1(1 + |x|^{2p})^{\frac{1}{2p}} \hat{\mathbb{E}}[1 + |X_{t-t'}^x|^{2p} + |x|^{2p}]^{\frac{2p-1}{2p}} \exp(-\eta t') \\ &\leq C_1(1 + |x|^{2p}) \exp(-\eta t'), \end{aligned}$$

where the constant C_1 depending on p, G, η, n, L , and K_f may vary from line to line.

Consequently, for each t , we get

$$|\bar{\lambda}^f - \hat{\mathbb{E}}[f(X_t^x)]| = \lim_{\rho \rightarrow \infty} |\hat{\mathbb{E}}[f(X_\rho^x)] - \hat{\mathbb{E}}[f(X_t^x)]| \leq C_1(1 + |x|^{2p}) \exp(-\eta t),$$

which derives that

$$\bar{\lambda}^f = \lim_{t \rightarrow \infty} \hat{\mathbb{E}}[f(X_t^x)].$$

For each $x, x' \in \mathbb{R}^n$, applying Lemma 3.2 (i) yields that

$$\begin{aligned} & \lim_{t \rightarrow \infty} |\hat{\mathbb{E}}[f(X_t^x)] - \hat{\mathbb{E}}[f(X_t^{x'})]| \\ & \leq \lim_{t \rightarrow \infty} \hat{\mathbb{E}}[|f(X_t^x) - f(X_t^{x'})|] \\ & \leq K_f \lim_{t \rightarrow \infty} \hat{\mathbb{E}}[(1 + |X_t^x|^{2p-1} + |X_t^{x'}|^{2p-1})|X_t^x - X_t^{x'}|] \\ & \leq K_f \lim_{t \rightarrow \infty} \hat{\mathbb{E}}[(1 + |X_t^x|^{2p-1} + |X_t^{x'}|^{2p-1})^{\frac{2p}{2p-1}}]^{\frac{2p-1}{2p}} \hat{\mathbb{E}}[|X_t^x - X_t^{x'}|^{2p}]^{\frac{1}{2p}} \\ & \leq C_1 \lim_{t \rightarrow \infty} (1 + |x|^{2p} + |x'|^{2p}) \exp(-\eta t) = 0, \end{aligned}$$

which completes the proof. □

The following result is a direct consequence of Theorem 3.3.

Corollary 3.4. *For each $f \in C_{2p, Lip}(\mathbb{R}^n)$, we get*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \hat{\mathbb{E}}[f(X_t^x)] dt = \bar{\lambda}^f, \quad \forall x \in \mathbb{R}^n.$$

From the nonlinear Feynman-Kac formula in [14], we obtain $u^f(t, x) = \hat{\mathbb{E}}[f(X_t^x)]$ which is the unique viscosity solution to the following fully nonlinear PDE.

$$\begin{cases} \partial_t u^f - G(H(D_x^2 u^f, D_x u^f, x)) - \langle b(x), D_x u^f \rangle = 0, & (t, x) \in (0, \infty) \times \mathbb{R}^n, \\ u^f(0, x) = f(x). \end{cases} \quad (3.3)$$

where

$$H_{ij}(D_x^2 u^f, D_x u^f, x) = \langle D_x^2 u^f \sigma_i(x), \sigma_j(x) \rangle + 2 \langle D_x u^f, h_{ij}(x) \rangle.$$

Then by Lemma 3.3, we get the following large time behaviour of solution to the fully nonlinear parabolic PDE (3.3).

Corollary 3.5. *For each $f \in C_{2p, Lip}(\mathbb{R}^n)$, we have for any $x \in \mathbb{R}^n$,*

$$\lim_{T \rightarrow \infty} u^f(T, x) = \bar{\lambda}^f \text{ and } |u^f(T, x) - \bar{\lambda}^f| \leq C_1(1 + |x|^{2p}) \exp(-\eta T).$$

We define the function $\bar{\Lambda} : C_{2p, Lip}(\mathbb{R}^n) \mapsto \mathbb{R}$ by

$$\bar{\Lambda}[f] = \bar{\lambda}^f.$$

Lemma 3.6. *Assume (H1) and (H2) hold. Then $\bar{\Lambda}$ is a sublinear expectation on $(\mathbb{R}^n, C_{2p, Lip}(\mathbb{R}^n))$, i.e.,*

- (a) *If $f_1 \geq f_2$, then $\bar{\Lambda}[f_1] \geq \bar{\Lambda}[f_2]$;*
- (b) *$\bar{\Lambda}[c] = c$ for any constant c ;*
- (c) *$\bar{\Lambda}[f_1 + f_2] \leq \bar{\Lambda}[f_1] + \bar{\Lambda}[f_2]$;*

(d) $\bar{\Lambda}[\lambda f] = \lambda \bar{\Lambda}[f]$ for each $\lambda \geq 0$.

Proof. The proof is immediate from Theorem 3.3 and the definition of G -expectation. \square

Lemma 3.7. For each sequence $\{f_i\}_{i=1}^\infty \subset C_{2p-1,Lip}(\mathbb{R}^n)$ satisfying $f_i \downarrow 0$, we have $\bar{\Lambda}[f_i] \downarrow 0$.

Proof. For each fixed $N > 0$,

$$f_i(x) \leq k_i^N + f_1(x)\mathbf{1}_{\{|x|>N\}} \leq k_i^N + \frac{f_1(x)|x|}{N} \text{ for every } x \in \mathbb{R}^n,$$

where $k_i^N = \max_{|x| \leq N} f_i(x)$. Then we have,

$$\hat{\mathbb{E}}[f_i(X_t^x)] \leq k_i^N + \frac{1}{N} \hat{\mathbb{E}}[f_1(X_t^x)|X_t^x|].$$

Applying Lemma 3.2, there exists a constant C_1 depending on G, f_1, p, n and η such that,

$$\hat{\mathbb{E}}[f_1(X_t^x)|X_t^x|] \leq \bar{C} \hat{\mathbb{E}}[|f_1(0)X_t^x| + |X_t^x|^{2p}] \leq C_1(1 + |x|^{2p}).$$

Consequently,

$$\bar{\Lambda}[f_i] = \lim_{t \rightarrow \infty} \hat{\mathbb{E}}[f_i(X_t^x)] \leq k_i^N + \frac{C_1(1 + |x|^{2p})}{N}.$$

It follows from $f_i \downarrow 0$ and Dini's theorem that $k_i^N \downarrow 0$. Thus we have $\lim_{i \rightarrow \infty} \bar{\Lambda}[f_i] \leq \frac{C_1(1 + |x|^{2p})}{N}$. Since N can be arbitrarily large, we get $\bar{\Lambda}[f_i] \downarrow 0$. \square

Remark 3.8. From the above proof, in general we cannot get the result for $\{f_i\}_{i=1}^\infty \subset C_{2p,Lip}(\mathbb{R}^n)$.

Theorem 3.9. Suppose (H1) and (H2) hold. Then there exists a family of weakly compact probability measures $\{m_\theta\}_{\theta \in \hat{\Theta}}$ defined on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that

$$\bar{\lambda}^f = \sup_{\theta \in \hat{\Theta}} \int_{\mathbb{R}^n} f(x)m_\theta(dx), \quad \forall f \in C_{2p-1,Lip}(\mathbb{R}^n).$$

Proof. By the representation theorem (Theorem 2.1 of Chapter 1 in [14]), for the sub-linear expectation $\bar{\Lambda}[f]$ defined on $(\mathbb{R}^n, C_{2p-1,Lip}(\mathbb{R}^n))$, there exists a family of linear expectations $\{M_\theta\}_{\theta \in \hat{\Theta}}$ on $(\mathbb{R}^n, C_{2p-1,Lip}(\mathbb{R}^n))$ such that

$$\bar{\Lambda}[f] = \sup_{\theta \in \hat{\Theta}} M_\theta[f], \quad \forall f \in C_{2p-1,Lip}(\mathbb{R}^n).$$

By Lemma 3.7, for each sequence $\{f_i\}_{i=1}^\infty$ in $C_{2p-1,Lip}(\mathbb{R}^n)$ such that $f_i \downarrow 0$ on \mathbb{R}^n , we have $\bar{\Lambda}[f_i] \downarrow 0$. Thus $M_\theta[f_i] \downarrow 0$ for each $\theta \in \hat{\Theta}$. It follows from the Daniell-Stone Theorem that, for each $\theta \in \hat{\Theta}$, there exists a unique probability measure $m_\theta(\cdot)$ on $(\mathbb{R}^n, \sigma(C_{2p-1,Lip}(\mathbb{R}^n))) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$, such that $M_\theta[f] = \int_{\mathbb{R}^n} f(x)m_\theta(dx)$.

Let $\bar{\mathcal{P}} = \{m_\theta : \theta \in \hat{\Theta}\}$ be the family of all probability measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that

$$\int_{\mathbb{R}^n} f(x)m_\theta(dx) \leq \bar{\Lambda}[f], \quad \forall f \in C_{2p-1,Lip}(\mathbb{R}^n).$$

Then from the above result, we obtain that

$$\bar{\Lambda}[f] = \sup_{\theta \in \bar{\mathcal{P}}} \int_{\mathbb{R}^n} f(x)m_\theta(dx), \quad \forall f \in C_{2p-1,Lip}(\mathbb{R}^n).$$

Now we prove that $\bar{\mathcal{P}}$ is weakly compact. Set $f_i(x) = (|x| - i)^+ \wedge 1$, it is easy to check that $f_i \in C_{2p-1, Lip}(\mathbb{R}^n)$ and $f_i \downarrow 0$. Then by Lemma 3.7, we obtain

$$\sup_{\theta \in \bar{\Theta}} m_\theta(\{|x| \geq i + 1\}) \leq \bar{\Lambda}[f_i] \downarrow 0.$$

Thus $\bar{\mathcal{P}}$ is tight. Let $m_{\theta_i}, i \geq 1$, converge weakly to m . Then by the definition of weak convergence, we can get for any $f \in C_{2p-1, Lip}(\mathbb{R}^n), N > 0, M > 0$,

$$\int_{\mathbb{R}^n} (f(x) \wedge N) \vee (-M) m(dx) \leq \bar{\Lambda}[(f \wedge N) \vee (-M)].$$

Note that $f \vee (-M) - (f \wedge N) \vee (-M) \downarrow 0$ as $N \uparrow \infty$, then by Lemma 3.7, we can get

$$0 \leq \bar{\Lambda}[f \vee (-M)] - \bar{\Lambda}[(f \wedge N) \vee (-M)] \leq \bar{\Lambda}[f \vee (-M) - (f \wedge N) \vee (-M)] \downarrow 0.$$

Thus by the monotone convergence theorem under m , we obtain

$$\int_{\mathbb{R}^n} f(x) \vee (-M) m(dx) \leq \bar{\Lambda}[f \vee (-M)],$$

which implies $\int_{\mathbb{R}^n} f(x) \vee (-M) m(dx) \in \mathbb{R}$. Similarly, we can get $\int_{\mathbb{R}^n} f(x) m(dx) \leq \bar{\Lambda}[f]$. Thus $m \in \bar{\mathcal{P}}$, which completes the proof. \square

In the classical case, i.e., $\bar{\Lambda}[\cdot]$ is a linear expectation, it is easy to check that $\bar{\Theta}$ consists of a single element θ_0 . In particular, the probability measure m_{θ_0} is the unique invariant measure for the diffusion process X . Under the G -expectation framework, we can also give the following definition.

Definition 3.10. A sublinear expectation $\tilde{\mathbb{E}}$ on $(\mathbb{R}^n, C_{2p, Lip}(\mathbb{R}^n))$ is said to be an invariant expectation for the G -diffusion process X if

$$\tilde{\mathbb{E}}[\hat{\mathbb{E}}[f(X_t^x)]] = \tilde{\mathbb{E}}[f(x)] \text{ for each } f \in C_{2p, Lip}(\mathbb{R}^n) \text{ and } t \geq 0.$$

The family of probability measures that represents $\tilde{\mathbb{E}}$ on $(\mathbb{R}^n, C_{2p-1, Lip}(\mathbb{R}^n))$ is called invariant for the G -diffusion process X .

Remark 3.11. For the invariant expectation $\tilde{\mathbb{E}}[\cdot]$, it corresponds to the family of probability measures, which can be explained as the uncertainty of the initial distribution. Given this uncertainty of the initial distribution, the left-hand side of the equality in the above definition can be explained as the uncertainty of the distribution of X_t . Thus under the invariant expectation $\tilde{\mathbb{E}}[\cdot]$, the distribution uncertainty of the G -diffusion process X is invariant in time.

Theorem 3.12. Assume (H1) and (H2) hold. Then there exists a unique invariant expectation $\tilde{\mathbb{E}}$ for the G -diffusion process X . Moreover, for each $f \in C_{2p, Lip}(\mathbb{R}^n)$, we have

$$\tilde{\mathbb{E}}[f] = \bar{\Lambda}[f].$$

Proof. Existence: Denote $\bar{f}(x) := \hat{\mathbb{E}}[f(X_t^x)]$. By Lemma 3.2 and Theorem 3.3, we can find some constant C_1 such that

$$\begin{aligned} |\bar{f}(x) - \bar{f}(x')| &\leq |\hat{\mathbb{E}}[f(X_t^x)] - \hat{\mathbb{E}}[f(X_t^{x'})]| \\ &\leq K_f \hat{\mathbb{E}}[(1 + |X_t^x|^{2p-1} + |X_t^{x'}|^{2p-1}) |X_t^x - X_t^{x'}|] \\ &\leq C_1 \hat{\mathbb{E}}[(1 + |X_t^x|^{2p-1} + |X_t^{x'}|^{2p-1})^{\frac{2p}{2p-1}}]^{\frac{2p-1}{2p}} \hat{\mathbb{E}}[|X_t^x - X_t^{x'}|^{2p}]^{\frac{1}{2p}} \\ &\leq C_1 \exp(-\eta t) (1 + |x|^{2p-1} + |x'|^{2p-1}) |x - x'|. \end{aligned}$$

Thus $\bar{f}(x) \in C_{2p,Lip}(\mathbb{R}^n)$. From Theorem 3.3 and Lemma A.3 of [8], we get

$$\begin{aligned} \bar{\Lambda}[f] &= \lim_{s \rightarrow \infty} \hat{\mathbb{E}}[\bar{f}(X_s^x)] = \lim_{s \rightarrow \infty} \hat{\mathbb{E}}[\hat{\mathbb{E}}[f(X_t^x)]_{x=X_s^x}] \\ &= \lim_{s \rightarrow \infty} \hat{\mathbb{E}}[\hat{\mathbb{E}}[f(X_{s+t}^{s,x})]_{x=X_s^x}] \\ &= \lim_{s \rightarrow \infty} \hat{\mathbb{E}}[\hat{\mathbb{E}}[f(X_{s+t}^{s,X_s^x})]] \\ &= \lim_{s \rightarrow \infty} \hat{\mathbb{E}}[f(X_{t+s}^x)] \\ &= \bar{\Lambda}[f], \end{aligned}$$

which concludes that $\bar{\Lambda}$ is an invariant expectation for the G -diffusion process X .

Uniqueness: Assume $\bar{\Lambda}$ is also an invariant expectation for the G -diffusion process X . Then for each $f \in C_{2p,Lip}(\mathbb{R}^n)$ and $t \geq 0$, we obtain

$$\bar{\Lambda}[f] = \bar{\Lambda}[\hat{\mathbb{E}}[f(X_t^x)]].$$

By Theorem 3.3, there exists a constant C_1 such that

$$|\bar{\Lambda}[f] - \hat{\mathbb{E}}[f(X_t^x)]| \leq C_1(1 + |x|^{2p}) \exp(-\eta t).$$

Consequently, we derive that

$$|\bar{\Lambda}[f] - \bar{\Lambda}[f]| \leq \lim_{t \rightarrow \infty} |\bar{\Lambda}[\bar{\Lambda}[f]] - \bar{\Lambda}[\hat{\mathbb{E}}[f(X_t^x)]]| \leq C_1 \lim_{t \rightarrow \infty} \exp(-\eta t) \bar{\Lambda}[(1 + |x|^{2p})] = 0,$$

and this completes the proof. \square

Theorem 3.13. Assume (H1)-(H2) hold and $\tilde{\mathbb{E}}$ is a sublinear expectation on $(\mathbb{R}^n, C_{2p,Lip}(\mathbb{R}^n))$. If there exists a point $t_0 > 0$ such that,

$$\tilde{\mathbb{E}}[\hat{\mathbb{E}}[f(X_{t_0}^x)]] = \tilde{\mathbb{E}}[f(x)], \quad \forall f \in C_{2p,Lip}(\mathbb{R}^n),$$

then $\tilde{\mathbb{E}}$ is the unique invariant expectation for X .

Proof. Denote $\bar{f}(x) := \hat{\mathbb{E}}[f(X_{t_0}^x)]$. Then using the same method as in the proof of Theorem 3.12, we have

$$\tilde{\mathbb{E}}[\bar{f}(x)] = \tilde{\mathbb{E}}[\hat{\mathbb{E}}[\bar{f}(X_{t_0}^x)]] = \tilde{\mathbb{E}}[\hat{\mathbb{E}}[\hat{\mathbb{E}}[f(X_{t_0}^x)]_{x=X_{t_0}^x}]] = \tilde{\mathbb{E}}[\hat{\mathbb{E}}[f(X_{2t_0}^x)]].$$

In a similar way, we obtain for each integer $n \geq 1$,

$$\tilde{\mathbb{E}}[f(x)] = \tilde{\mathbb{E}}[\hat{\mathbb{E}}[f(X_{nt_0}^x)]].$$

Then by Theorem 3.3, we get

$$\tilde{\mathbb{E}}[f(x)] = \lim_{n \rightarrow \infty} \tilde{\mathbb{E}}[\hat{\mathbb{E}}[f(X_{nt_0}^x)]] = \bar{\Lambda}^f,$$

which is the desired result. \square

Now we give some examples of invariant expectations.

Example 3.14. Assume that $b(0) = h_{ij}(0) = \sigma(0) = 0$, then it is easy to check that $X_t^0 = 0$. Then by Lemma 3.2, we obtain $\hat{\mathbb{E}}[|X_t^x|] \leq \exp(-\eta t)|x|$ for each $t \geq 0$. In particular, we obtain that

$$\bar{\Lambda}[f] = \lim_{t \rightarrow \infty} \hat{\mathbb{E}}[f(X_t^0)] = f(0), \quad \forall f \in C_{2p-1,Lip}(\mathbb{R}^n).$$

Thus

$$\bar{\Lambda}[f] = \int_{\mathbb{R}^n} f(x) \delta_0(dx), \quad \forall f \in C_{2p-1,Lip}(\mathbb{R}^n),$$

where δ_0 is the Dirac measure.

Consider the following Ornstein-Uhlenbeck process driven by G -Brownian motion: for each $x \in \mathbb{R}^d$,

$$Y_t^x = x - \alpha \int_0^t Y_s^x ds + B_t, \tag{3.4}$$

where $\alpha > 0$ is a given constant. It is obvious that assumption (H2) holds for each $p \geq 1$ in this case.

Lemma 3.15. *The invariant expectation for G -Ornstein-Uhlenbeck process Y is the G -normal distribution of $\sqrt{\frac{1}{2\alpha}}B_1$.*

Proof. From the G -Itô formula, we get

$$Y_t^x = \exp(-\alpha t)x + \exp(-\alpha t) \int_0^t \exp(\alpha s)dB_s, \text{ for all } t \geq 0.$$

For each integer N , denote $t_i^N = \frac{it}{N}$ with $0 \leq i \leq N$ and $h_s^N := \exp(\alpha t_i^N) \mathbf{1}_{[t_i^N, t_{i+1}^N)}(s)$. Then it is obvious that

$$\lim_{N \rightarrow \infty} \hat{\mathbb{E}} \left[\int_0^t |\exp(\alpha s) - h_s^N|^2 ds \right] = 0.$$

Thus $\| \int_0^t \exp(\alpha s)dB_s - \int_0^t h_s^N dB_s \|_{L_G^2} \rightarrow 0$ as $N \rightarrow \infty$.

Note that $\int_0^t h_s^N dB_s = \sum_{i=0}^{N-1} \exp(\alpha t_i^N) (B_{t_{i+1}^N} - B_{t_i^N})$. Then we get $\int_0^t h_s^N dB_s$ and

$\sqrt{\sum_{i=0}^N \exp(2\alpha t_i^N) (t_{i+1}^N - t_i^N) B_1}$ are identically distributed. Consequently, for each $p \geq 1$ and $f \in C_{p,Lip}(\mathbb{R}^d)$,

$$\begin{aligned} \hat{\mathbb{E}}[f(\int_0^t \exp(\alpha s)dB_s)] &= \lim_{N \rightarrow \infty} \hat{\mathbb{E}}[f(\int_0^t h_s^N dB_s)] = \lim_{N \rightarrow \infty} \hat{\mathbb{E}}[f(\sqrt{\sum_{i=0}^N \exp(2\alpha t_i^N) (t_{i+1}^N - t_i^N) B_1})] \\ &= \hat{\mathbb{E}}[f(\sqrt{\int_0^t \exp(2\alpha s) ds} B_1)] \\ &= \hat{\mathbb{E}}[f(\sqrt{\frac{1}{2\alpha} (\exp(2\alpha t) - 1)} B_1)]. \end{aligned}$$

Thus, for each $p \geq 1$ and $f \in C_{p,Lip}(\mathbb{R}^d)$, we have

$$\hat{\mathbb{E}}[f(\exp(-\alpha t) \int_0^t \exp(\alpha s)dB_s)] = \hat{\mathbb{E}}[f(\sqrt{\frac{1}{2\alpha} (1 - \exp(-2\alpha t))} B_1)].$$

Applying Lemma 3.2 yields that

$$\begin{aligned} \lim_{t \rightarrow \infty} \hat{\mathbb{E}}[f(Y_t^0)] &= \lim_{t \rightarrow \infty} \hat{\mathbb{E}}[f(\exp(-\alpha t) \int_0^t \exp(\alpha s)dB_s)] = \lim_{t \rightarrow \infty} \hat{\mathbb{E}}[f(\sqrt{\frac{1}{2\alpha} (1 - \exp(-2\alpha t))} B_1)] \\ &= \hat{\mathbb{E}}[f(\sqrt{\frac{1}{2\alpha}} B_1)]. \end{aligned}$$

Thus by Theorem 3.12, we obtain

$$\bar{\Lambda}[f] = \hat{\mathbb{E}}[f(\sqrt{\frac{1}{2\alpha}} B_1)],$$

which is the desired result. □

Example 3.16. Suppose B is a 1-dimensional G -Brownian motion. For each $x \in \mathbb{R}$, let

$$Y_t^x = x + \int_0^t (m - Y_s^x) ds + B_t + \langle B \rangle_t,$$

where m is a given constant. From the G -Itô formula, we get

$$Y_t^x = \exp(-t)x + m(1 - \exp(-t)) + \int_0^t \exp(s-t) dB_s + \int_0^t \exp(s-t) d\langle B \rangle_s, \text{ for all } t \geq 0.$$

By a similar analysis as in Lemma 3.15, we obtain that $\int_0^t \exp(s-t) dB_s + \int_0^t \exp(s-t) d\langle B \rangle_s$ and $\sqrt{\frac{1}{2}(1 - \exp(-2t))} B_1 + (1 - \exp(-t)) \langle B \rangle_1$ are identically distributed. Then for each $p \geq 1$ and $f \in C_{p,Lip}(\mathbb{R})$, we have

$$\hat{\mathbb{E}}[f(Y_t^x)] = \hat{\mathbb{E}}[f(m + \sqrt{\frac{1}{2}} B_1 + \langle B \rangle_1)].$$

Next we shall consider the following G -diffusion process: for each $x \in \mathbb{R}$,

$$Y_t^x = x - \alpha \int_0^t Y_s^x d\langle B \rangle_s + B_t, \tag{3.5}$$

where $\alpha > 0$ is a given constant. Applying the G -Itô formula, we get

$$Y_t^x = \exp(-\alpha \langle B \rangle_t) x + \exp(-\alpha \langle B \rangle_t) \int_0^t \exp(\alpha \langle B \rangle_s) dB_s, \text{ for all } t \geq 0.$$

From Theorems 3.3, 3.13 and Lemma 3.15, we have the following.

Corollary 3.17. Given a sublinear space $(\mathbb{R}, C_{p,Lip}(\mathbb{R}), \tilde{\mathbb{E}})$ and denote $\zeta(x) = x$ for $x \in \mathbb{R}$, then $\tilde{\mathbb{E}}$ is the invariant expectation for G -process Y^x if and only if for some point $t > 0$ and $x \in \mathbb{R}$, $\exp(-\alpha \langle B \rangle_t) \zeta + \exp(-\alpha \langle B \rangle_t) \int_0^t \exp(\alpha \langle B \rangle_s) dB_s$ and ζ are identically distributed, where $(B_t)_{t \geq 0}$ is independent from ζ .

4 Ergodic nonlinear expectation

In this section, we shall only consider non-degenerate G -Brownian motion, i.e., there exists some constant $\underline{\sigma}^2 > 0$ such that, for any $A \geq B$

$$G(A) - G(B) \geq \frac{1}{2} \underline{\sigma}^2 \text{tr}[A - B].$$

We begin with the following lemma, which is essentially from [8].

Lemma 4.1. Assume (H1) and (H2) hold. Then for each $f \in C_{2p,Lip}(\mathbb{R}^n)$, the following fully nonlinear ergodic PDE:

$$G(H(D_x^2 v, D_x v, x)) + \langle b(x), D_x v \rangle + f(x) = \lambda^f, \tag{4.1}$$

has a solution $(v, \lambda^f) \in C_{2p,Lip}(\mathbb{R}^n) \times \mathbb{R}$, where

$$H_{ij}(D_x^2 v, D_x v, x) = \langle D_x^2 v \sigma_i(x), \sigma_j(x) \rangle + 2 \langle D_x v, h_{ij}(x) \rangle.$$

Moreover, if $(\bar{v}, \bar{\lambda}) \in C_{2p,Lip}(\mathbb{R}^n) \times \mathbb{R}$ is also a solution to equation (4.1), then we have

$$\bar{\lambda} = \lambda^f = \lim_{T \rightarrow \infty} \frac{1}{T} \hat{\mathbb{E}} \left[\int_0^T f(X_s^x) ds \right], \quad \forall x \in \mathbb{R}^n.$$

Proof. The proof is immediate from Lemma 3.2, Theorems 5.4 and 5.5 of [8]. □

Denote a mapping $\Lambda : C_{2p,Lip}(\mathbb{R}^n) \mapsto \mathbb{R}$ by

$$\Lambda[f] = \lambda^f.$$

By a similar analysis as in Lemma 3.3, it is easy to check that Λ is a sublinear expectation on $(\mathbb{R}^n, C_{2p,Lip}(\mathbb{R}^n))$.

Lemma 4.2. Assume (H1) and (H2) hold. Then we obtain

- (a) If $f_1 \geq f_2$, then $\Lambda[f_1] \geq \Lambda[f_2]$;
- (b) $\Lambda[c] = c$ for each constant c ;
- (c) $\Lambda[f_1 + f_2] \leq \Lambda[f_1] + \Lambda[f_2]$;
- (d) $\Lambda[\lambda f] = \lambda \Lambda[f]$ for each $\lambda \geq 0$.

In addition, we have the following result.

Theorem 4.3. Assume (H1) and (H2) hold. Then there exists a family of weakly compact probability measures $\{m_\theta\}_{\theta \in \Theta}$ defined on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ such that

$$\Lambda[f] = \sup_{\theta \in \Theta} \int_{\mathbb{R}^n} f(x) m_\theta(dx), \quad \forall f \in C_{2p-1,Lip}(\mathbb{R}^n).$$

Proof. The proof is similar to that of Theorem 3.9. □

Definition 4.4. A sublinear expectation $\tilde{\mathbb{E}}$ on $(\mathbb{R}^n, C_{2p,Lip}(\mathbb{R}^n))$ is said to be an ergodic expectation for the G -diffusion process X if

$$\tilde{\mathbb{E}}[f] = \lim_{T \rightarrow \infty} \frac{1}{T} \hat{\mathbb{E}} \left[\int_0^T f(X_s^x) ds \right], \quad \forall f \in C_{2p,Lip}(\mathbb{R}^n).$$

The family of probability measures that represents $\tilde{\mathbb{E}}$ is called ergodic for the G -diffusion process X .

Proposition 4.5. Let (H1) and (H2) hold. Then for each $v \in C_{2p-1,Lip}(\mathbb{R}^n)$ with $\partial_{x_i} v \in C_{2p-2,Lip}(\mathbb{R}^n)$ and $\partial_{x_i x_j}^2 v \in C_{2p-3,Lip}(\mathbb{R}^n)$, we have

$$\Lambda[-G(H(D_x^2 v, D_x v, x)) - \langle b(x), D_x v \rangle] = \sup_{\theta \in \Theta} \int_{\mathbb{R}^n} [-G(H(D_x^2 v, D_x v, x)) - \langle b(x), D_x v \rangle] m_\theta(dx) = 0.$$

Proof. Taking $f = -G(H(D_x^2 v, D_x v, x)) - \langle b(x), D_x v \rangle$, by equation (4.1), we obtain $\lambda^f = 0$ and the proof is complete. □

Example 4.6. Assume that $b(0) = h_{ij}(0) = \sigma(0) = 0$, then we obtain that

$$\Lambda[f] = \lim_{T \rightarrow \infty} \frac{1}{T} \hat{\mathbb{E}} \left[\int_0^T f(X_t^0) dt \right] = f(0), \quad \forall f \in C_{2p-1,Lip}(\mathbb{R}^n).$$

Thus

$$\Lambda[f] = \bar{\Lambda}[f] = \int_{\mathbb{R}^n} f(x) \delta_0(dx), \quad \forall f \in C_{2p-1,Lip}(\mathbb{R}^n).$$

Note that $\hat{\mathbb{E}}[\int_0^T f(X_s^x) ds] \leq \int_0^T \hat{\mathbb{E}}[f(X_s^x)] ds$. Then it follows from Corollary 3.4 that $\lambda^f \leq \bar{\lambda}^f$ and $\Theta \subset \bar{\Theta}$. In the classical case, it is obvious that $\Lambda = \bar{\Lambda}$. In particular, if $\bar{\Theta}$ only has a single element, it is easy to check that $\lambda^f = \bar{\lambda}^f$. However, in general we cannot get $\Lambda = \bar{\Lambda}$ under G -framework.

Example 4.7. Assuming $d = 1$ and $0 < \underline{\sigma}^2 < \bar{\sigma}^2 = 1$. Consider the following G -Ornstein-Uhlenbeck process: for each $x \in \mathbb{R}$,

$$Y_t^x = x - \frac{1}{2} \int_0^t Y_s^x ds + B_t. \tag{4.2}$$

Note that $Y_t^x = \exp(-\frac{1}{2}t)x + \exp(-\frac{1}{2}t) \int_0^t \exp(\frac{1}{2}s) dB_s$. By Proposition 4.5 and taking $v(x) = \frac{1}{2}x^4$, we have

$$\Lambda[x^4 - G(6x^2)] = \Lambda[x^4 - 3x^2] = 0.$$

It follows from Lemma 3.15 that $\bar{\Lambda}[x^4 - 3x^2] = \hat{\mathbb{E}}[B_1^4 - 3B_1^2]$. Denote by E_σ the linear expectation corresponding to the normally distributed density function $N(0, \sigma^2)$ with $\underline{\sigma}^2 \leq \sigma^2 \leq 1$. Then for each $p \geq 1$ and $f \in C_{p,Lip}(\mathbb{R})$,

$$\hat{\mathbb{E}}[f(B_1)] \geq \sup_{\underline{\sigma}^2 \leq \sigma^2 \leq 1} E_\sigma[f(B_1)].$$

From the definition of G -expectation, we obtain that $\hat{\mathbb{E}}[B_1^4 - 3B_1^2] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[(x + B_1 - B_{\frac{1}{2}})^4 - 3(x + B_1 - B_{\frac{1}{2}})^2]_{x=B_{\frac{1}{2}}}]$. Set $g(x) = \hat{\mathbb{E}}[(x + B_1 - B_{\frac{1}{2}})^4 - 3(x + B_1 - B_{\frac{1}{2}})^2]$ and $g_1(x) = E_1[(x + B_1 - B_{\frac{1}{2}})^4 - 3(x + B_1 - B_{\frac{1}{2}})^2]$, $g_2(x) = E_{\underline{\sigma}}[(x + B_1 - B_{\frac{1}{2}})^4 - 3(x + B_1 - B_{\frac{1}{2}})^2]$. It is obvious that $g(x) \geq g_1 \vee g_2(x)$. After a direct calculation, we obtain

$$g_1(x) = x^4 - \frac{3}{4}, \quad g_2(x) = x^4 + 3(\underline{\sigma}^2 - 1)x^2 + \frac{3}{4}\underline{\sigma}^4 - \frac{3}{2}\underline{\sigma}^2.$$

Consequently,

$$g_1 \vee g_2(x) = g_1(x)\mathbf{1}_{|x| > \frac{\sqrt{1-\underline{\sigma}^2}}{2}} + g_2(x)\mathbf{1}_{|x| \leq \frac{\sqrt{1-\underline{\sigma}^2}}{2}}.$$

Then we have

$$\begin{aligned} E_1[g_1 \vee g_2(B_{\frac{1}{2}})] &= E_1[B_{\frac{1}{2}}^4 - \frac{3}{4}\mathbf{1}_{|B_{\frac{1}{2}}| > \frac{\sqrt{1-\underline{\sigma}^2}}{2}} + (3(\underline{\sigma}^2 - 1)B_{\frac{1}{2}}^2 + \frac{3}{4}\underline{\sigma}^4 - \frac{3}{2}\underline{\sigma}^2)\mathbf{1}_{|B_{\frac{1}{2}}| \leq \frac{\sqrt{1-\underline{\sigma}^2}}{2}}] \\ &= 3E_1[[\frac{1}{4}(1 - \underline{\sigma}^2)^2 - (1 - \underline{\sigma}^2)B_{\frac{1}{2}}^2]\mathbf{1}_{|B_{\frac{1}{2}}| \leq \frac{\sqrt{1-\underline{\sigma}^2}}{2}}] \\ &\geq 3E_1[[\frac{1}{4}(1 - \underline{\sigma}^2)^2 - (1 - \underline{\sigma}^2)B_{\frac{1}{2}}^2]\mathbf{1}_{|B_{\frac{1}{2}}| \leq \frac{\sqrt{1-\underline{\sigma}^2}}{4}}] \\ &\geq \frac{9}{16}(1 - \underline{\sigma}^2)^2 E_1[\mathbf{1}_{|B_{\frac{1}{2}}| \leq \frac{\sqrt{1-\underline{\sigma}^2}}{4}}] > 0. \end{aligned}$$

Thus we get $\hat{\mathbb{E}}[B_1^4 - 3B_1^2] \geq E_1[g_1 \vee g_2(B_{\frac{1}{2}})] > 0$ and $\bar{\Lambda}[x^4 - 3x^2] \neq \Lambda[x^4 - 3x^2]$.

Example 4.8. Assuming $d = 1$ and $0 < \underline{\sigma}^2 < \bar{\sigma}^2 = 1$. Let us consider equation (3.5) with $\alpha = \frac{1}{2}$. Under each linear expectation E_σ with $\underline{\sigma}^2 \leq \sigma^2 \leq 1$, it is easy to check that the invariant expectation of equation (3.5) is the standard normal distribution E_1 . However, we claim that the invariant expectation of equation (3.5) cannot be the normal distribution E_1 . Otherwise, the ergodic expectation of equation (3.5) is also the normal distribution E_1 . Therefore, by Proposition 4.5 and taking $v(x) = x^2$, we have

$$\Lambda[-G(2-2x^2)] = E_1[-G(2-2B_1^2)] = E_1[(\underline{\sigma}^2(1-B_1^2)^- - (1-B_1^2)^+)] = (\underline{\sigma}^2-1)E_1[(1-B_1^2)^+] \neq 0,$$

which is a contradiction.

Remark 4.9. Assume $d = 1$ and $b(x) = -x$, $h(x) = 0$ and $\sigma = 1$. Then consider the following equation:

$$\begin{cases} \partial_t u - G(D_x^2 u) + x D_x u = 0, & (t, x) \in (0, \infty) \times \mathbb{R}, \\ u(0, x) = f(x). \end{cases} \tag{4.3}$$

Denote $\bar{u}(t, x) := \int_0^t u(s, x) ds = \int_0^t \hat{\mathbb{E}}[f(X_s^x)] ds$. Assume $u(s, x)$ is a smooth function. Then

$$\partial_t \bar{u}(t, x) = u(t, x), \quad \partial_x \bar{u}(t, x) = \int_0^t \partial_x u(s, x) ds, \quad \partial_{xx}^2 \bar{u}(t, x) = \int_0^t \partial_{xx}^2 u(s, x) ds.$$

In the linear case, i.e., $G(a) = \frac{1}{2}a$, it is easy to check that

$$\partial_t \bar{u} - \frac{1}{2} D_x^2 \bar{u} + x D_x \bar{u} + f = 0.$$

Then by the ergodic theory, we obtain

$$\bar{\Lambda}[f] = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T E[f(X_s^x)] ds = \lim_{T \rightarrow \infty} \frac{\bar{u}(T, x)}{T} = \Lambda[f].$$

However, under the nonlinear expectation framework, there is no such relationship for fully nonlinear PDE (4.3).

Remark 4.10. In the linear expectation case, ergodic theory and related problems are connected with the invariant expectation. However, from the above results, this relationship may not hold true under the nonlinear expectation framework. Thus we should study nonlinear ergodic problems via ergodic expectation Λ instead of invariant expectation $\bar{\Lambda}$. In particular, [8] obtained the links between ergodic expectation and large time behaviour of solutions to fully nonlinear PDEs.

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