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Uniqueness of the infinite homogeneous cluster in the 1-2 model

Zhongyang Li*

Abstract

A 1-2 model configuration is a subset of edges of the hexagonal lattice such that each vertex is incident to one or two edges. We prove that for any translation-invariant Gibbs measure of 1-2 model, almost surely the infinite homogeneous cluster is unique.

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1 Introduction

A 1-2 model configuration is a choice of subset of edges of the hexagonal lattice such that each vertex is incident to one or two edges. An example of 1-2 model configurations is shown in Figure 1.

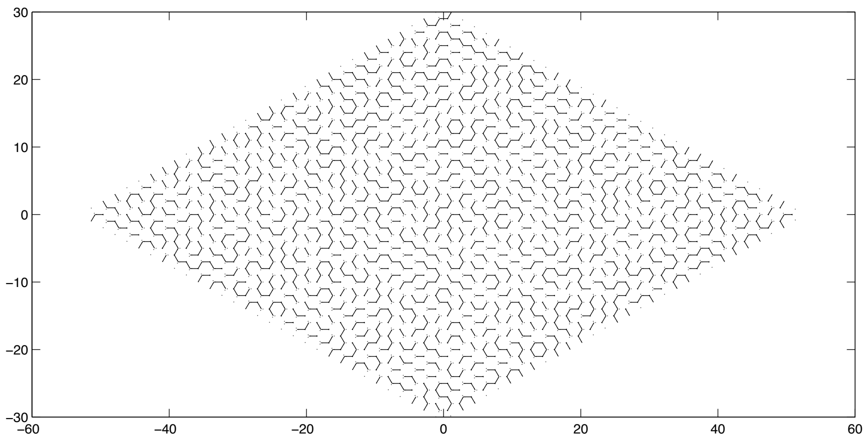

Figure 1: 1-2 model configuration

The uniform 1-2 model (not-all-equal relation) was studied by computer scientists Schwartz and Bruck [7]. They computed the partition function (total number of configurations) of the 1-2 model on a finite graph by the holographic algorithm [8]. In [4], we

*University of Cambridge, UK. E-mail: z.li@statslab.cam.ac.uk

studied a generalized holographic algorithm, which could compute the local statistics of more general vertex models, including the non-uniform 1-2 model. A new approach to solve the 1-2 model is explored in [5] by constructing a measure-preserving correspondence between 1-2 model configurations on the hexagonal lattice and perfect matchings [3] on a decorated lattice. Using a large torus to approximate the infinite planer graphs, we constructed in [5] an explicit translation-invariant, parameter-dependent measure for 1-2 model configurations on the planar hexagonal lattice, and proved that the 1-2 model percolates under certain parameters, i.e., it is almost surely the case that there exists an infinite homogenous cluster for some parameter values, while for some other parameter values, it is almost surely the case that there exists no infinite homogeneous cluster. See [2] for an introduction of the percolation theory. In this paper, we prove that for any translation-invariant measure of 1-2 model configurations, almost surely there is at most one infinite homogeneous cluster.

Let $\mathbb{H} = (V, E)$ be the hexagonal lattice embedded into the whole plane. Let $v \in V$ be a vertex of $\mathbb{H}$. There is a one-to-one correspondence between configurations at v (subsets of incident edges of v) and the set of all 3-digit binary numbers. Namely, since v has 3 incident edges, we may assume that the horizontal edge of each vertex corresponds to the right digit, and a one-to-one correspondence between incident edges and digits can be constructed by moving counter-clockwise around a vertex and right-to-left along the digits. If an edge is included in the configuration, then the corresponding digit takes the value "1"; otherwise the corresponding digit takes the value "0". See Figure 2 for examples of such a correspondence.

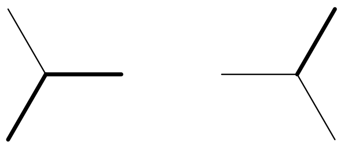

Figure 2: local configurations and binary numbers

The weight function at a vertex is an assignment of a nonnegative number to each configuration at the vertex. For a 1-2 model configuration, since we require that each vertex can have only one or two incident edges, the weights for configurations $\{000\}$ and $\{111\}$ are 0. Moreover, throughout this paper we assume that the weight of the configurations at each vertex is

$$\begin{array}{cccccccc} 000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\ 0 & a & b & c & c & b & a & 0 \end{array}, \tag{1.1}$$

where $a, b, c > 0$ are arbitrary positive numbers. Given the weights of configurations as in (1.1), we say that an edge is a-type (resp. b-type, c-type) if it is the unique present edge in the configuration $\{001\}$ (resp. $\{010\}$, $\{100\}$). Given the correspondence of edges with the digits described previously, an edge is a-type if and only if it is horizontal; and starting from an a-type edge, moving counter-clockwise around a vertex, we meet the b-edge and the c-edge in cyclic order.

A Gibbs measure μ for the 1-2 model on $\mathbb{H}$ is a probability measure on the sample space of all possible 1-2 model configurations (denote the sample space by Ω), such that for any finite subgraph $\Lambda \subset \mathbb{H}$, and any fixed configuration ω_{Λ^c} on the complement graph Λ^c, the probability of a configuration ω_Λ on Λ, conditional on ω_{Λ^c}, is proportional to the product of configuration weights at each vertex of Λ. Namely,

$$\mu(\omega_\Lambda | \omega_{\Lambda^c}) \propto \prod_{v \in \Lambda} w(\omega_\Lambda|_v)$$

where $\omega_\Lambda|_v$ is the configuration of ω_Λ restricted at the vertex v, i.e. $\omega_\Lambda|_v$ is one of the six possible 1-2 model configurations $\{001\}, \{010\}, \{011\}, \{100\}, \{101\}, \{110\}$, and $w(\cdot)$ is the weight function at a vertex.

A homogeneous cluster of a 1-2 model configuration, is a connected subset of vertices of $\mathbb{H}$, in which each vertex has the same configuration, i.e., one of $\{001\}$, $\{010\}$, $\{011\}$, $\{100\}$, $\{101\}$, $\{110\}$. See Figure 3 for examples of homogeneous $\{101\}$ clusters.

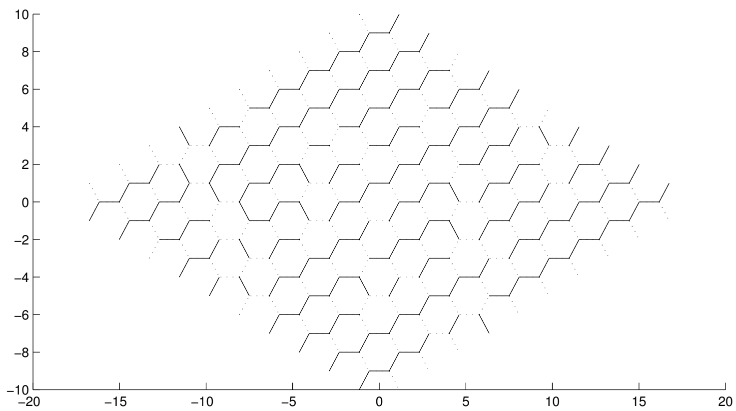

Figure 3: homogeneous $\{101\}$ clusters

A homogenous cluster is infinite if it consists of infinitely many vertices. It is proved in [5] that under the explicitly constructed, parameter-dependent Gibbs measure, an infinite homogeneous cluster exists almost surely for some values of parameters, while infinite homogeneous clusters do not exist almost surely for some other values of parameters. The main theorem of this paper is as follows:

Theorem 1.1. *For any translation-invariant Gibbs measure of 1-2 model configurations on the whole-plane hexagonal lattice* $\mathbb{H}$*, almost surely there is at most one infinite homogeneous cluster.*

It is proved by Burton and Keane ([1]) that for any translation-invariant, finite energy measure on $\{0,1\}^{\mathbb{Z}^d}$ configurations, almost surely there is at most one infinite open cluster. However, the proof in [1] does not work for the 1-2 model case, since the Gibbs measure for the 1-2 model does not satisfy the finite-energy condition in [1]. Namely, if F is a finite subset of $V(\mathbb{H})$, the vertex set of the hexagonal lattice $\mathbb{H}$, and $\phi : F \rightarrow \{xyz\}_{x,y,z\in\{0,1\}}$ is a function which associate to each vertex of F a 3-digit binary number. Then we can associate to each $\xi \in \{\{xyz\}_{x,y,z\in\{0,1\}}\}^{V(\mathbb{H})}$ a point $\tilde{\xi} \in \{\{xyz\}_{x,y,z\in\{0,1\}}\}^{V(\mathbb{H})}$ by changing the values of ξ on F to ϕ, i.e.,

$$\tilde{\xi}(z) = \begin{cases} \phi(z) & \text{if } z \in F \\ \xi(z) & \text{otherwise} \end{cases}$$

If $E \subseteq \{\{xyz\}_{x,y,z\in\{0,1\}}\}^{V(\mathbb{H})}$ is an event, we set $\phi(E) = \{\tilde{x} : x \in E\}$. A probability measure μ has **finite energy** if for any $E \subseteq \{\{xyz\}_{x,y,z\in\{0,1\}}\}^{V(\mathbb{H})}$ with $\mu(E) > 0$ and F and ϕ, $\mu(\phi(E)) > 0$. The fact that the probability measure for the 1-2 model does not satisfy the finite energy property can be seen as follows. For instance, an a-configuration can either be an $\{001\}$ configuration or $\{110\}$ configuration. However, in one connected set of vertices with a-configurations (an a-cluster), either all vertices have $\{001\}$-configuration or all vertices have $\{110\}$ configuration because if the $\{001\}$ configuration and the $\{110\}$ configuration coexist in the same a-cluster, then there must be a vertex with 3 incident present edges, which violates the law that each vertex has 1 or 2 incident present edges. If a connected subset of sites has positive probability of having an $\{001\}$-cluster, then the same set by changing the configuration of an interior vertex from $\{001\}$ to $\{110\}$ will be an a-cluster with probability 0, which contradicts the definition of the finite energy property.

We prove the theorem for $\{001\}$ cluster in Section 2, and the result for all the other homogeneous clusters can be proved using exactly the same technique.

2 Infinite Clusters

Lemma 2.1. *Let μ be an ergodic, translation-invariant Gibbs measure for 1-2 model configurations. Let $\mathcal{N}_{\{001\}}$ be the number of infinite $\{001\}$-clusters. For any $1 < k < \infty$,*

$$\mu(\mathcal{N}_{\{001\}} = k) = 0$$

Proof. Recall that in a $\{001\}$ configuration of a vertex, only the horizontal incident edge is present. We prove the lemma by contradiction. Without loss of generality, assume $0 < c \leq b \leq a$. Since μ is ergodic, and $\{\mathcal{N}_{\{001\}} = k\}$ is a translation-invariant event, then either $\mu(\mathcal{N}_{\{001\}} = k) = 0$ or $\mu(\mathcal{N}_{\{001\}} = k) = 1$. Assume there exists $1 < k < \infty$, such that

$$\mu(\mathcal{N}_{\{001\}} = k) = 1. \tag{2.1}$$

Let B_n be an $n \times n$ box of the hexagonal lattice centered at the origin. i.e, a rectangle domain with n vertices incident to vertices outside the domain on each side, as illustrated in Figure 4, in which the subgraph bounded by the outer dashed rhombus is a 5×5 box, and the subgraph bounded by the inner dashed rhombus is a 3×3 box.

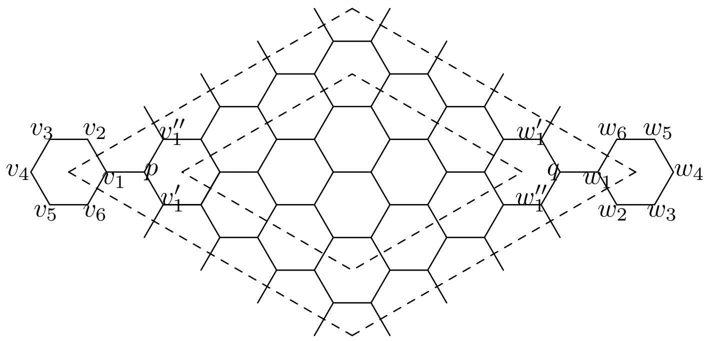

Figure 4: $n \times n$ box

Let $\mathcal{S}_n$ be the event that B_n intersects all of the k infinite $\{001\}$ clusters. Then

$$1 = \mu(\cup_n \mathcal{S}_n) = \lim_{n \to \infty} \mu(\mathcal{S}_n).$$

Hence there exists N, such that $\mu(\mathcal{S}_N) > \frac{1}{2}$. Let ω be a configuration in $\mathcal{S}_N$. Consider the box B_{N+2}. There are two types of edges in B_{N+2}: either an interior edge whose both endpoints are in B_{N+2}; or a boundary edge connecting one vertex in B_{N+2} and one vertex outside B_{N+2}. The boundary edges are those intersecting the outer dashed rhombus in Figure 4. We are going to change the configurations in B_{N+2} to derive a contradiction.

For the time being, we keep the configuration for all boundary edges; while for each interior edge of B_{N+2}, it is present if and only if it is horizontal (an a-type edge). There are three types of vertices in B_{N+2}: type I: a vertex whose all three neighbors are still in B_{N+2}; type II: a vertex with two neighbors in B_{N+2}, but one neighbor outside B_{N+2}; type III: a vertex with one neighbor in B_{N+2}, but two neighbors outside B_{N+2}. After the first step of changing configurations as described above, all type I vertices of B_{N+2} have a configuration $\{001\}$. In particular, all the vertices of B_N are type I vertices of B_{N+2}, hence all vertices of B_N has a configuration $\{001\}$. All the type II vertices has at least one present edge and one unpresent edge, hence the configurations at type II vertices do not violate the rule that each vertex has degree 1 or 2. Now consider the type III vertices of B_{N+2}; there are only 2 type III vertices, lying in the two corners of B_{N+2}, labeled by v_1 and w_1 as in Figure 4. They have at least one incident present edge, since the horizontal incident edge is present. We will describe the change of configurations around v_1 here; the change of configurations around w_1 are very similar. If v_1 has degree 1 or 2, then we are done and do not need to change configurations any more. If v_1 has degree 3, we consider the hexagon, denoted by h_1, which includes v_1 but does not include any other vertices of B_{N+2}. Let $v_1, v_2, v_3, v_4, v_4, v_5, v_6$ be all the vertices of h_1 in cyclic order, see Figure 2. If v_3 has configuration $\{001\}$, remove the edge v_1v_2, and we get a 1-2 model configuration. Similarly, if v_5 has configuration $\{001\}$, remove the edge v_1v_6, and we get a 1-2 model configuration. If neither v_3 nor v_5 have configuration $\{001\}$, we change the configuration as follows. Remove v_1v_2. After removing v_1v_2, if v_2 has degree 1, we are done. if v_2 has degree 0, add v_2v_3. After adding v_2v_3, if v_3 has degree 2, we are done. If v_3 has degree 3, remove v_3v_4. After removing v_3v_4, if v_4 has degree 1, we are done. If v_4 has degree 0, add v_4v_5. After adding v_4v_5, if v_5 has degree 2, we are done. If v_5 has degree 3, remove v_5v_6. Then v_6 has at least one present incident edge v_1v_6, and 1 unpresent incident edge v_5v_6. Hence v_6 has degree 1 or 2, we are done. Similar process applies for w_1 on the other corner.

Let ω' be the new configuration obtained from ω by the configuration-changing process described above. We will prove that ω' has exactly one infinite $\{001\}$-cluster. Note that if a vertex has $\{001\}$ configuration in ω, then it has $\{001\}$ configuration in ω'. Hence each infinite $\{001\}$ cluster of ω must be a subset of an infinite $\{001\}$ cluster of ω'. Moreover, the configuration-changing process described above only changes configurations at a finite number of vertices, ω' cannot have infinite $\{001\}$ clusters which do not include an infinite cluster $\{001\}$ cluster of ω. Since all the infinite $\{001\}$ clusters of ω intersect B_N, all the infinite $\{001\}$ clusters of ω' intersect B_N. But all the vertices in B_N have the configuration $\{001\}$. As a result, there is exactly one infinite $\{001\}$ cluster in ω'. We consider the probability that such an configuration ω' occurs. This probability is bounded below by the probability of the event $\mathcal{S}_N$, multiplying a factor due to the change of configurations at finitely many vertices. Namely,

$$\mu(\mathcal{N}_{\{001\}} = 1) > \frac{1}{2}\left(\frac{c}{6a}\right)^{2(N+2)^2+10} > 0,$$

which is a contradiction to (2.1), and the lemma follows. □

Before proving the theorem, we shall introduce the following notation. Let Y be a finite set with at least three elements. A partition of Y is a collection $P = \{P_1, P_2, P_3\}$ of the three non-empty disjoint subset of Y whose union is Y. Partitions P and Q are compatible if there is an ordering of each such that $Q_2 \cup Q_3 \subset P_1$. A collection $\mathcal{P}$ of partitions is compatible if each pair P, Q is compatible.

Lemma 2.2. *([1])If $\mathcal{P}$ is a collection of compatible partitions of Y, then*

$$|\mathcal{P}| \leq |Y| - 2$$

where $|\cdot|$ denote the cardinality of a set.

Theorem 2.3. *Let μ be a translation-invariant Gibbs measure on 1-2 model configurations Ω. Then μ-almost surely every $\omega \in \Omega$ has at most one infinite $\{001\}$-cluster.*

Proof. By ergodic decomposition, we may assume without loss of generality that μ is ergodic, so that $\mathcal{N}_{\{001\}}$, the total number of infinite $\{001\}$ clusters is constant μ-almost surely. Then by Lemma 2.1, $\mathcal{N}_{\{001\}}$ is either zero, one, or infinity. If $\mathcal{N}_{\{001\}}$ is zero or one, we are finished, so assume $\mathcal{N}_{\{001\}}$ is infinity.

A box B_n is an encounter box if the following two conditions hold

- There exists an infinite $\{001\}$ cluster $\mathcal{C}$, such that $B_n \subset \mathcal{C}$;
- the set $\mathcal{C} \setminus B_{n+2}$ has no finite components and exactly three infinite components.

We claim that if $\mu(\mathcal{N}_{\{001\}} = \infty) = 1$, there exists N, such that the probability that the box B_N centered at $(0,0)$ is an encounter box is strictly positive. To see that, let B_n be an $n \times n$ box centered at the origin, as defined in the proof of Lemma 2.1. Let $\mathcal{N}_{B_n,\{001\}}$ be the number of infinite $\{001\}$ clusters intersecting B_n. Then

$$\lim_{n\rightarrow\infty} \mu(\mathcal{N}_{B_n,\{001\}} \geq 31) = 1$$

As a result, there exists N, such that

$$\mu(\mathcal{N}_{B_{N+2},\{001\}} \geq 31) > \frac{1}{2}$$

Let ω be a 1-2 model configuration satisfying $\mathcal{N}_{B_{N+2},\{001\}} \geq 31$, then we can find three boundary vertices u_1, u_2, u_3 (vertices in B_{N+2} with at least one neighbor outside B_{N+2}) of B_{N+2}, such that

- u_1, u_2, u_3 are in three different infinite $\{001\}$ clusters of ω;
- let $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ be the three different infinite clusters including u_1, u_2, u_3, then $(\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3) \cap (\bar{h}_1 \cup \bar{h}_2 \cup \{v_1', v_1'', w_1', w_1''\}) = \emptyset$, where $\bar{h}_1$ (resp. $\bar{h}_2$) consists of all the vertices in the hexagon h_1 (resp. h_2), as well as vertices incident to h_1 (resp. h_2), and h_1, h_2 are the two hexagons outside the two corners of B_{N+2}, as shown in Figure 4.

Since $|\bar{h}_1 \cup \bar{h}_2 \cup \{v_1', v_1'', w_1', w_1''\}| = 28$, the number of infinite $\{001\}$ clusters intersecting $\bar{h}_1 \cup \bar{h}_2 \cup \{v_1', v_1'', w_1', w_1''\}$ is at most 24. Moreover, since in ω, the number of infinite $\{001\}$ clusters intersecting B_{N+2} is at least 31, we can always find u_1, u_2, u_3, satisfying the conditions listed above.

To make B_N an encounter box, we change configurations in B_{N+2} as follows. First of all, we define the outer contour of B_{N+2} to be the closed contour consisting of all the interior edges (edges connecting two vertices in B_{N+2}) sharing a vertex with a boundary edge of B_{N+2}. Let all the horizontal edges (a-type edges) on the outer contour of B_{N+2} be present. Let u be a boundary vertex of B_{N+2} (vertex with at least one neighbor outside B_{N+2}), other than u_1, u_2, u_3, w_1, v_1. u is incident to three edges: e_h, the horizontal edge; e_b, the boundary edge; and e_i, the edge other than e_h and e_b. e_h is present for any such u after the first step of changing configurations described above. We change the configurations of e_i, if necessary, such that if e_b is present, then e_i is not present; and if e_b is not present, then e_i is present. This way, all the boundary vertices of B_{N+2} except u_1, u_2, u_3, w_1, w_4 have degree 2, and do not have a $\{001\}$ configuration. Let p (resp. q) be the vertex adjacent to v_1 (resp. w_1) through a horizontal edge, see Figure 4. The configurations of edges $v_1 p$ and $w_1 q$ are rearranged according to the configurations of on the four other incident edges of p and q, such that at vertices p and q, the rule that one or two incident edges are present (1-2 law) is not violated. To make sure the configurations at v_1 and w_1 satisfy the 1-2 law, we will change the configurations on the edges of the hexagons h_1 and h_2 (see Figure 4). We discuss the case of h_1 here, the case of h_2 is exactly the same. We give h_1 a configuration such that each alternating side of h_1 is present. This way no matter what configurations are outside h_1, we always get a configuration on vertices of h_1 which does not violate the 1-2 law.

Obviously, after such a change of configurations, the only possible ways to connect $\{001\}$ clusters outside B_{N+2} to $\{001\}$ clusters in B_N is through vertices u_1, u_2, u_3. That is because any boundary vertices of B_{N+2} except u_1, u_2, u_3, v_1, w_1 do not have a $\{001\}$ configuration. Moreover, the method to arrange configurations on h_1 (resp. h_2) implies that at least one non-horizontal edge incident to v_1 (resp. w_1) is present, hence v_1 (resp. w_1) does not have the $\{001\}$ configuration. Now consider the box B_N centered at the origin. Remove the boundary edges of B_N from the configuration. For each interior edge of B_N, it is present if and only if it is horizontal. This way all the vertices in B_N have a $\{001\}$-configuration. Moreover, all the vertices in B_{N+2} do not violate the 1-2 law. To check this claim, we only need to check the vertices of B_{N+2} which are neighbors of vertices of B_N. Any such vertex has at least one present incident edges, namely the horizontal edge, and at least one non-present incident edges, namely the boundary edge of B_N, hence the 1-2 law is satisfied.

Again let ω' be the new configuration obtained from ω by the configuration-changing process described above. It is trivial to check by definition that B_N is an encounter box for the configuration ω'. Consider the probability of those configurations in which B_N is an encounter box, this probability is bounded below by the probability $\mu(\mathcal{N}_{B_{N+2},\{001\}} \geq 31)$, multiplying a factor caused by changing configurations at finitely many vertices. Namely,

$$\mu(B_N \text{ is an encounter box}) \geq \frac{1}{2}\left(\frac{c}{6a}\right)^{2(N+2)^2+10} > 0$$

Let $B_{s(N+2)}$ be an $s(N+2) \times s(N+2)$ box centered at the origin, consisting of s^2 non-overlapping $(N+2) \times (N+2)$ boxes (each one is a translation of the other). Let $B_N(i,j)$ be the $N \times N$ box centered at (i,j). Let $\mathcal{B}_N^s$ be the collection of all the $N \times N$ box included in one of the s^2 non-overlapping $(N+2) \times (N+2)$ boxes in $B_{s(N+2)}$, such that each $N \times N$ box and the corresponding $(N+2) \times (N+2)$ box have the same center.

By translation invariance, the probability that each $B_N(i,j)$ is an encounter box is at least $\frac{1}{2}\left(\frac{c}{6a}\right)^{2(N+2)^2+10}$, so the expected number of encounter boxes in $\mathcal{B}_N^s$ is at least

$$\frac{1}{2}\left(\frac{c}{6a}\right)^{2(N+2)^2+10} s^2 \tag{2.2}$$

Let $\mathcal{C}$ be a fixed infinite $\{001\}$ cluster of ω. Define

$$Y = \mathcal{C} \cap \{\text{outer boundary of } B_{s(N+2)}\},$$

where the outer boundary of $B_{s(N+2)}$ are the set of those vertices outside $B_{s(N+2)}$ and incident to vertices in $B_{s(N+2)}$.

If $B_N(i,j) \in \mathcal{B}_N^s$ is an encounter box for ω with respect to $\mathcal{C}$, then the removal of $B_{N+2}(i,j)$ from $\mathcal{C}$ defines a partition

$$P = \{P_1, P_2, P_3\}$$

of the set Y, such that $P_i \neq \phi$ for $1 \leq i \leq 3$. Namely, if $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$ be the three components of $\mathcal{C} \setminus B_{N+2}(i,j)$, then let $P_i = \mathcal{D}_i \cap Y$.

Moreover, if $B_N(i',j') \in \mathcal{B}_N^s$ is another encounter box with respect to the same infinite $\{001\}$ cluster $\mathcal{C}$ such that $(i',j') \neq (i,j)$ then $B_N(i',j')$ gives another partition $Q = \{Q_1, Q_2, Q_3\}$ of Y, and the indices of P and Q can be chosen in such a way that $Q_2 \cup Q_3 \subset P_1$; simply choose Q_1 to correspond to the component of $\mathcal{C} \setminus B_{N+2}(i',j')$ containing $B_N(i,j)$. Hence the set of partitions corresponding to encounter boxes of $\mathcal{C}$ in $\mathcal{B}_N^s$ forms a compatible partition of Y. By Lemma 2.2, the number of compatible partitions is at most $|Y|-2$, where $|Y|$ is the number of vertices in Y. Summing over all different infinite clusters, we have the total number of encounter boxes in $\mathcal{B}_N^s$ is bounded above by $4s(N+4)$, which is less than (2.2) when s is large. The contradiction shows that it is μ-a.s. impossible that there are infinite many infinite $\{001\}$ clusters. By Lemma 2.1 almost surely there is at most one infinite $\{001\}$ cluster. □

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