RECURSIONS AND TIGHTNESS FOR THE MAXIMUM OF THE DISCRETE, TWO DIMENSIONAL GAUSSIAN FREE FIELD

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Abstract

We consider the maximum of the discrete two dimensional Gaussian free field in a box, and prove the existence of a (dense) deterministic subsequence along which the maximum, centered at its mean, is tight. The method of proof relies on an argument developed by Dekking and Host for branching random walks with bounded increments and on comparison results specific to Gaussian fields.

1   Introduction and main result

We consider the discrete Gaussian Free Field (GFF) in a two-dimensional box of side $N + 1$, with Dirichlet boundary conditions. That is, let $V_N = ([0,N] \cap \mathbb{Z})^2$, $V_N^o = ((0,N) \cap \mathbb{Z})^2$, and let $\{w_m\}_{m \geq 0}$ denote a simple random walk started in V_N and killed at $\tau = \min\{m : w_m \in \partial V_N\}$ (that is, killed upon hitting the boundary $\partial V_N = V_N \setminus V_N^o$). For $x, y \in V_N$, define $G_N(x,y) = E^x(\sum_{m=0}^{\tau} \mathbf{1}_{w_m = y})$, where E^x denotes expectation with respect to the random walk started at x. The GFF is the zero-mean Gaussian field $\{X_z^N\}_z$ indexed by $z \in V_N$ with covariance G_N.

Throughout, for any finite set T and random field $\{W_z\}_{z \in T}$, we write $W^* = \max_{z \in T} W_z$. In particular, we write $X_N^* = \max_{z \in V_N} X_z^N$. It was proved in [2] that $X_N^*/(\log N) \to c$ with $c = 2\sqrt{2/\pi}$, and the proof is closely related to the proof of the law of large numbers for the maximal displacement

of a branching random walk (in $\mathbb{R}$). Based on the analogy with the maximum of independent Gaussian variables and the case of branching random walks, the following is a natural conjecture.

Conjecture 1. *The sequence of random variables* $Y_N := X_N^* - EX_N^*$ *is tight.*

To the best of our knowledge, the sharpest result in this direction (prior to this paper) is due to [8], who shows that the variance of Y_N is $o(\log N)$; in the same paper, Chatterjee also analyzes related Gaussian fields, but in all these examples, does not prove tightness. We defer to Section 3 for some pointers to the relevant literature concerning the Gaussian free field and the origin of Conjecture 1.

The goal of this note is to prove a weak form of the conjecture. Namely, we will prove the following.

Theorem 1. *There is a deterministic sequence* $\{N_k\}_{k \geq 1}$ *such that the sequence of random variables* $\{Y_{N_k}\}_{k \geq 1}$ *is tight.*

More information on the sequence $\{N_k\}_{k \geq 1}$ is provided below in Remark 1.

It is of course natural to try to improve the tightness from subsequences to the full sequence. As will be clear from the proof, for that it is enough to prove the existence of a constant C such that $EX_{2N}^* \leq EX_N^* + C$. This is weaker than, and implied by, the conjectured behavior of EX_N^*, which is

$$EX_N^* = c \log N - c_2 \log \log N + O(1), \tag{1}$$

for $c = 2\sqrt{2/\pi}$ and an appropriate c_2, see e.g. [7] and Remark 3. (After this work was completed, the conjecture (1) was proved by Bramson and Zeitouni [6], consequently establishing Theorem 1 without the need for taking subsequences.)

Finally, although we deal here exclusively with the GFF, it should be clear from the proof that the analysis applies to a much wider class of models.

2   Preliminary considerations

Our approach is motivated by the Dekking-Host [10] proof of tightness of branching random walks (BRW) with independent bounded increments (see also the argument in [4]). Although the GFF cannot be represented as a branching random walk, it does possess a branching-like structure that we describe in this section; this structure plays a crucial role in the proof of Theorem 1.

2.1   The basic branching structure

We consider $N = 2^n$ in what follows, write $Z_n = X_N^*$ and identify an integer $m = \sum_{i=0}^{n-1} m_{i+1} 2^i \leq N$ with its binary expansion $(m_n, m_{n-1}, \ldots, m_1)$, where $m_i \in \{0, 1\}$. For $k \geq 1$, introduce the sets of k-diadic integers

$$A_k = \{m \in \{1, \ldots, N\} : m = (2l + 1)N/2^k \quad \text{for some integer } l\}.$$

Note that if $m \in A_k$ then $m_i = 0$ for $i \leq n - k$ and $m_{n-k} = 1$. Set

$$B_k = \{z = (x, y) \in V_N, x \text{ or } y \in A_k\}.$$

Then, define the σ-algebras

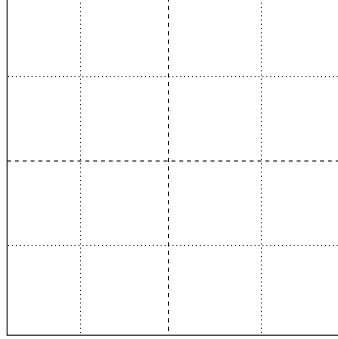$$\mathscr{A}_k = \sigma(X_z^N : z \in \cup_{i \leq k} B_i).$$

Figure 1: Indices in B_1 (dashed lines) and B_2 (dotted lines). $\mathscr{A}_2$ is the σ-algebra generated by variables with indices on either dashed or dotted lines.

Finally, for every $z = (x, y) \in V_N^o$, write $z_i = (x_i, y_i)$ with x_i, y_i denoting as above the ith digit in the binary expansion of x, y. We introduce the random variables

$$\xi_{z_{k+1},\ldots,z_n}^{z_1,\ldots,z_k} = E[X_z^N | \mathscr{A}_k]. \tag{2}$$

Define

$$X_{z_2,\ldots,z_n}^{z_1} = X_z^N - \xi_{z_2,\ldots,z_n}^{z_1}. \tag{3}$$

Note that the random variables $\{X_{z_2,\ldots,z_n}^{z_1}\}$ are independent of $\mathscr{A}_1$. Further, by the Markov property of the GFF, the collections $\{X_{\cdot}^{z_1}\}_{z_1 \in V_1}$ are i.i.d. copies of the GFF in the box $V_{N/2}$. Iterating, we have the representation

$$X_z^N = \xi_{z_2,\ldots,z_n}^{z_1} + \xi_{z_3,\ldots,z_n}^{z_1,z_2} + \ldots + \xi_{z_n}^{z_1,z_2,\ldots,z_{n-1}}, \tag{4}$$

where all the summands in the right side of (4) are independent.

For motivation purpose, we explain the relation with branching random walks; this is not used in the sequel. Let $\{g^{z_1,\ldots,z_k}\}_{k \geq 1, z_i \in V_i}$ be independent random variables and define

$$B_z^N = g^{z_1} + g^{z_1,z_2} + \ldots + g^{z_1,\ldots,z_n}.$$

Then B_z^N determines a branching random walk (on a quaternary tree), possibly with time-dependent increments. Thus, should the random variables in the right side of (4) not depend on the subscript, (4) would correspond to a branching random walk. For such a BRW, a functional recursion for the law of X_N^* can be written down, and used to prove tightness (see [5] and [1]). Unfortunately, no such simple functional recursions are available in the case (4). For this reason, we only use (3), and then adapt an argument of [10], originally presented in the context of BRW.

Returning to the setup of the GFF, note that with $w \in V_{N/2}$ identified with the sequence $(z_2, \ldots, z_n)$ and $D_w^{z,N} = \{\xi_{z_2,\ldots,z_n}^z\}$, (3) implies that

$$Z_n = X_N^* = \max_{z \in V_1}((X^{N/2,z} + D^{z,N})^*), \tag{5}$$

where the random fields $\{(X^{N/2,z})_w\}_{w \in V_{N/2}}$, $z \in V_1$, are four i.i.d. copies of $X^{N/2}$, and the $\{(D^{z,N})_w\}_{w \in V_{N/2}}$, $z \in V_1$, are complicated zero-mean Gaussian fields, independent of the fields $\{X^{N/2,z}\}$. Unfortunately, the $D^{z,N}$ random fields are far from being uniformly bounded, and this fact prevents the application of the argument from [10].

2.2 Two basic lemmas

In this subsection we present two preliminary lemmas that will allow for a comparison of the GFF between different scales. The first shows that the maximum of the sum of two zero mean fields tends to be larger than each of the fields.

Lemma 1. *Let $\{X_i\}_{i \in V_N}$ and $\{Y_i\}_{i \in V_N}$ be two independent, random fields, indexed by V_N, with $E|X_i| < \infty$ and $EY_i = 0$ for all i. Then,*

$$E \max_{i \in V_N}(X_i + Y_i) \geq E \max_{i \in V_N} X_i \,. \tag{6}$$

Proof Let $\alpha \in V_N$ be such that $\max_{i \in V_N} X_i = X_\alpha$ (in case several αs satisfy the above equality, choose the first according to lexicographic order). We then have

$$E \max_{i \in V_N}(X_i + Y_i) \geq E(X_\alpha + Y_\alpha) = EX_\alpha + EY_\alpha = EX_\alpha = E \max_{i \in V_N} X_i \,,$$

where the second equality is due to the independence of the fields and the fact that $EY_i = 0$ for all i. □

By (5) and Lemma 1, we have that

$$EZ_{n+1} \geq EZ_n \,. \tag{7}$$

The following lemma gives a control in the opposite direction.

Lemma 2. *There exists a sequence $n_k \to \infty$ and a constant C such that*

$$EZ_{n_k+1} \leq EZ_{n_k} + C \,. \tag{8}$$

Proof From [2] there exists a constant $c > 0$ so that $EZ_n/n \to c$. Fixing arbitrary K and defining $I_{n,K} = \{i \in [n, 2n] : EZ_{i+1} > EZ_i + K\}$, one has from (7) and the existence of the limit $EZ_n/n \to c$ that

$$\limsup_{n \to \infty} \frac{|I_{n,K}|}{2n} \leq \frac{c}{K} \,.$$

In particular, choosing $K = 3c$ it follows that for all n large, there exists an $n' \in [n, 2n]$ so that

$$EZ_{n'} \leq EZ_{n'-1} + K \,,$$

as claimed. □

2.3 Proof of Theorem 1

By (5) and Lemma 1, we get that

$$EX_{2N}^* \geq E \max_{z \in V_1}(X_N^*)^z \geq E \max(X_N^*, \tilde{X}_N^*),$$

where $\tilde{X}_N^*$ is an independent copy of X_N^*. Using the equality $\max(a, b) = (a + b + |a - b|)/2$, we get that

$$EZ_{n+1} - EZ_n \geq E|X_N^* - \tilde{X}_N^*|/2 \,.$$

For the sequence n_k and the constant C from Lemma 2, we thus get that

$$2C \geq E|X_{2^{n_k}}^* - \tilde{X}_{2^{n_k}}^*| \geq E|X_{2^{n_k}}^* - E\tilde{X}_{2^{n_k}}^*| = E|X_{2^{n_k}}^* - EX_{2^{n_k}}^*| \,,$$

where the second inequality follows from Jensen's inequality and the independence of X_N^* and $\tilde{X}_N^*$. This shows that the sequence $\{X_{2^{n_k}}^* - EX_{2^{n_k}}^*\}_{k \geq 1}$ is tight and completes the proof of Theorem 1. □

Remark 1. *The subsequence n_k provided in Lemma 2 can be taken with density arbitrary close to* 1*, as can be seen from the following modification of the proof. Fixing arbitrary K and ϵ and defining $I_{n,\epsilon,K} = \{i \in [n, n(1+\epsilon)] : EZ_{n+1} > EZ_n + K\}$, one has from* (7) *and the existence of the limit $EZ_n/n \to c$ (with $c = 2\sqrt{2/\pi}$) that*

$$\limsup_{n\to\infty} \frac{|I_{n,\epsilon,K}|}{n\epsilon} \leq \frac{c}{K}.$$

It is of course of interest to see whether one can take $n_k = k$. Minor modifications of the proof of Theorem 1 would then yield Conjecture 1.

Remark 2. *Minor modifications of the proof of Theorem 1 also show that if there exists a constant C so that $EX_{2N}^* \leq EX_N^* + C$ for all integer N, then Conjecture 1 holds.*

Remark 3. *For Branching Random Walks, under suitable assumptions it was established in [1] that* (1) *holds. Running the argument above then immediately implies the tightness of the minimal (maximal) displacement, centered around its mean.*

3   Some bibliographical remarks

The Gaussian free field has been extensively studied in recent years, in both its continuous and discrete forms. For an accessible review, we refer to [14]. The fact that the GFF has a logarithmic decay of correlation invites a comparison with branching random walks, and through this analogy a form of Conjecture 1 is implicit in [7]. This conjecture is certainly "folklore", see e.g. open problem #4 in [8]. For some one-dimensional models (with logarithmic decay of correlation) where the structure of the maxima can be analyzed, we refer to [11, 12]. The analogy with branching random walks has been reinforced by the study of the so called *thick points* of the GFF, both in the discrete form [9] and in the continuous form [13].

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