DOI: 10.1214/ECP.v13-1362

Elect. Comm. in Probab. **13** (2008), 175–186

ELECTRONIC COMMUNICATIONS in PROBABILITY

ON THE BOUNDEDNESS OF BERNOULLI PROCESSES OVER THIN SETS

RAFAŁ LATAŁA[1]
Institute of Mathematics, University of Warsaw, Banacha 2, 02-097 Warszawa, Poland
email: rlatala@mimuw.edu.pl

Submitted November 8, 2007, *accepted in final form* February 25, 2008

AMS 2000 Subject classification: 60E15, 60G17.
Keywords: Bernoulli process, Bernoulli conjecture, partitioning scheme, majorizing measure.

Abstract
We show that the Bernoulli conjecture holds for sets with small one-dimensional projections, i.e. any bounded Bernoulli process indexed by such set may be represented as a sum of a uniformly bounded process and a process dominated by a bounded Gaussian process.

1 Introduction

Let I be a countable set and $(\varepsilon_i)_{i\in I}$ be a Bernoulli sequence i.e. a sequence of independent symmetric variables taking values ± 1. For $T \subset l^2(I)$ we consider the Bernoulli process $(\sum_{i\in I} \varepsilon_i t_i)_{t\in T}$. The problem we treat in this paper concerns the conditions we need to impose on the set T to guarantee that the Bernoulli process is almost surely bounded. By the concentration property of Bernoulli processes (cf. Theorem 2 below) it is enough to consider the boundedness of the mean.
For a nonempty set $T \subset l^2(I)$ we define

$$b(T) := \mathbf{E} \sup_{t\in T} \sum_{i\in I} \varepsilon_i t_i.$$

(More precisely, to avoid measurability problems one defines $b(T) := \sup_F \mathbf{E} \sup_{t\in F} \sum_{i\in I} \varepsilon_i t_i$, where the supremum is taken over all finite subsets of T.) In a similar way we put

$$g(T) := \mathbf{E} \sup_{t\in T} \sum_{i\in I} g_i t_i,$$

where $(g_i)_{i\in I}$ is a sequence of i.i.d. Gaussian $\mathcal{N}(0,1)$ r.v.'s. The fundamental majorizing measure theorem of Fernique [1] and Talagrand [4] states that $g(T) < \infty$ if and only if $\gamma_2(T) < \infty$ – for precise definition of γ_2 cf. [7, Definition 1.2.5].
It is easy to see that $g(T) \geq \mathbf{E}|g_1| b(T) = \sqrt{\frac{\pi}{2}} b(T)$. Moreover, obviously $b(T) \leq \sup_{t\in T} \sum_{i\in I} |t_i|$ and $b(T_1 + T_2) \leq b(T_1) + b(T_2)$.

[1]RESEARCH SUPPORTED IN PART BY MEIN GRANT 1 PO3A 012 29.

The Bernoulli conjecture (cf. [3, Problem 12] or [7, Conjecture 4.1.3]) states that for any set T with $b(T) < \infty$ we may find a decomposition $T \subset T_1 + T_2$ with $\sup_{t\in T_1} \sum_i |t_i| < \infty$ and $g(T_2) < \infty$.
The aim of this note is to show that the Bernoulli conjecture holds under some additional restrictions on the set T – namely, that all one dimensional projections of T (i.e. the sets $\{t_i : t \in T\}$) are small. In particular, we answer the question posed by M. Talagrand [7, p. 144] – concerning the case when t_i may take only two values 0 and 2^{-k_i} (see Example 1 in section 4).
In the paper we use letter L to denote universal positive constants that may change from line to line, and L_i to denote positive universal constants that are the same at each occurence.

2 Partitioning Scheme

In this section we slightly modify some of Talagrand's results concerning partitioning scheme for a family of distances (gathered in sections 2.6 and 5.1 of [7]) to get the statement expressed in the language that will be suitable for our purposes. The only new point of our approach is Definition 1 below.
Let $r = 2^\nu$ for some integer $\nu \geq 2$. Suppose that $T \subset l^2(I)$ and we have a family of metrics $(d_j)_{j\in\mathbb{Z}}$ on $l^2(I)$ and nonnegative functions F_j defined on all subsets of T such that for all $s, t \in T$, $\emptyset \neq A \subset T$ and $j \in \mathbb{Z}$,

$$d_{j+1}(s,t) \geq r^{-1} d_j(s,t), \tag{1}$$

$$F_{j+1}(A) \leq F_j(A), \tag{2}$$

$$F_j(A) \geq F_j(B) \ \text{ for } \emptyset \neq B \subset A, \tag{3}$$

$$\exists_{j_0\in\mathbb{Z}}\ d_{j_0-1}(s,t) \leq r^{-j_0+1}/2 \ \text{ for all } s,t \in T, \tag{4}$$

$$\exists_{\theta>0}\ d_j^2(s,t) \geq \theta^2 \sum_{i\in I} \min\left\{r^{-2j}, (s_i - t_i)^2\right\} \text{ for all } s,t \in T, j \in \mathbb{Z}. \tag{5}$$

We define for $t \in T$, $a \geq 0$

$$\tilde{B}_j(t,a) := \{s \in T : d_j(s,t) \leq a\}$$

and as in [7] we set $N_n := 2^{2^n}$, $n = 0, 1, \ldots$.

Definition 1. Let $\Gamma > 0$ and $n_0 \in \mathbb{Z}_+$. We say that functionals F_j are (Γ, n_0)- *decomposable* on T if the following holds. Suppose that $C \subset T$, $t \in T$, $j \in \mathbb{Z}$ and $n \geq n_0$ satisfy

$$\emptyset \neq C \subset \tilde{B}_{j-1}(t, 2^{n/2} r^{-j+1}). \tag{6}$$

Then we can split C into m disjoint nonempty sets $C_1, \ldots, C_m$ with $m \leq N_n$ such that for all $i \leq m$ either

$$C_i \subset \tilde{B}_j(t_i, 2^{n/2} r^{-j}) \text{ for some } t_i \in C \tag{7}$$

or

$$\forall_{t\in C_i}\ F_{j+1}\big(C_i \cap \tilde{B}_{j+1}(t, 2^{n/2+2} r^{-j-1/2})\big) \leq F_j(C) - \frac{1}{\Gamma} 2^n r^{-j}. \tag{8}$$

Conditions (1)-(4) are just reformulations of Talagrand's assumptions for a family of distances from [7, Section 5.1]. Condition (5) gives a connection between distances d_j and "cut" l_2-distances induced by the Bernoulli process. A minor change with respect to [7] is present in

Definition 1 – Talagrand's approach yielded the splitting of C with only one set C_i satisfying (8).

Theorem 1. *If $0 \in T$, conditions (1)-(5) hold and functionals F_j are (Γ, n_0) decomposable on T, then we may find decomposition $T \subset T_1 + T_2$ with*

$$\gamma_2(T_1) \leq L\theta^{-1} r(\Gamma F_{j_0+1}(T) + 2^{n_0} r^{-j_0})$$

and

$$\|t\|_1 \leq L\theta^{-2} r(\Gamma F_{j_0+1}(T) + 2^{n_0} r^{-j_0}) + 20\theta^{-1} \sup_{s\in T} \|s\|_2 \text{ for } t \in T_2.$$

Proof. First we will follow the proof of [7, Theorem 5.1.2] with $F_{n,j} := \Gamma F_j$, $\varphi_j := r^{2j} d_j^2$ (notice that $r = 2^{\kappa-4}$ for $\kappa := \nu + 4 \geq 6$). We have $\varphi_{j+1} \geq \varphi_j$ by (1) and the condition (5.7) is obviously implied by (4). Let $B_j(t,c) := \{s \in T \colon \varphi_j(s,t) \leq c\}$ as in [7], then $B_j(t, 2^n) = \tilde{B}_j(t, 2^{n/2} r^{-j})$.
We will not prove the growth conditon in the sense of [7, Definition 5.1.1], but instead we will show that the place, where it was used can be obtained by our assumptions on decomposability. The main point in the proof of Theorem 5.1.2 was the inductive construction of the partitions $\mathcal{A}_n$ and numbers $j(C), q(C), b_0(C), b_1(C)$, $b_2(C)$ for $C \in \mathcal{A}_n$ satisfying (5.11)-(5.17) given on p.147 of [7]. Let us take $C \in \mathcal{A}_n$ and put $j = j(C)$. Then, by (5.11) the condition (6) holds, so we may split C into $m \leq N_n$ disjoint sets $C_1, \ldots, C_m$ satisfying (7) or (8). Let $A = C_i$ for some i. If (8) holds, then for all $t \in A$,

$$\begin{aligned} F_{n+1,j+1}\big(A \cap B_{j+1}(t, 2^{n+\kappa})\big) &= \Gamma F_{j+1}\big(A \cap \tilde{B}_{j+1}(t, 2^{n/2+2} r^{-j-1/2})\big) \leq \Gamma F_j(C) - 2^n r^{-j} \\ &= F_{n,j}(C) - 2^n r^{-j} \end{aligned}$$

(compare with the estimate at the top of page 149 in [7]). So we may put $j(A) = j(C), q(A) = q(C) + 1, b_0(A) = b_0(C), b_1(A) = b_1(C)$ and $b_2(A) = b_0(C) - 2^n r^{-j}$ and check all conditions as on pp.148-149 of [7].
If (7) holds for $C_i = A$ then $A \subset B_j(t_i, 2^n)$ and we can follow the definitions and arguments for the case $A = D_{l-1} \cap B_j(t_i, 2^n)$ given on pp. 149-150 of [7].
Hence following the proof of Theorem 5.1.2 we construct an increasing sequence $(\mathcal{A}_n)_{n\geq 0}$ of partitions of T with $\mathcal{A}_0 = \{T\}$, $\#\mathcal{A}_n \leq N_n$ and for each $A \in \mathcal{A}_n$ an integer $j(A)$ satisfying the following conditions (for the sake of convienience from now on our $j(A)$ is $j(A) - 1$ from [7], we also put $\mathcal{A}_n := \{T\}$ for $n < n_0$): $j(T) = j_0 - 1$,

$$A \in \mathcal{A}_n,\ B \in \mathcal{A}_{n-1},\ A \subset B \ \Rightarrow\ j(B) \leq j(A) \leq j(B) + 1, \tag{9}$$

$$\forall_{t\in T} \sum_{n\geq 0} 2^n r^{-j(A_n(t))} \leq Lr(\Gamma F_{j_0+1}(T) + 2^{n_0} r^{-j_0}) \tag{10}$$

and

$$\forall_{A\in\mathcal{A}_n} \exists_{t(A)\in T}\ A \subset \tilde{B}_{j(A)}(t(A), r^{-j(A)} 2^{n/2}), \tag{11}$$

where $A_n(t)$ denotes the unique set in $\mathcal{A}_n$ such that $t \in A_n(t)$.
Now we apply Theorem 2.6.3 of [7] with the constructed partition and numbers $j(A)$. Let $V := r$, $\delta(A) := \theta^{-1} 2^{n/2+1} r^{-j(A)}$ and μ be a counting measure on $\Omega = I$. Conditions (2.98) and (2.99) are implied by (10) and (9) respectively. If $A \subset B$, $A \in \mathcal{A}_n, B \in \mathcal{A}_{n'}$, $n' \leq n$ and if

additionally $j(A)=j(B)$ then $\delta(B)\le\delta(A)$ so (2.100) holds. To verify the assumption (2.101) take $s,t\in A$, then by (5) and (11),

$$\Big(\sum_{i\in I}\min\big\{(s_i-t_i)^2,r^{-2j(A)}\big\}\Big)^{1/2}$$
$$\le\Big(\sum_{i\in I}\min\big\{(s_i-t(A)_i)^2,r^{-2j(A)}\big\}\Big)^{1/2}+\Big(\sum_{i\in I}\min\big\{(t_i-t(A)_i)^2,r^{-2j(A)}\big\}\Big)^{1/2}$$
$$\le\theta^{-1}(d_{j(A)}(s,t(A))+d_{j(A)}(t,t(A)))\le\delta(A).$$

Thus all assumptions of Theorem 2.6.3 are satisfied and hence we may find a decomposition $T\subset T_1+T_2+T_3$ satisfying (2.102)-(2.105). By (2.102) and (10) we have

$$\gamma_2(T_1)\le L\sup_{t\in T}\sum_{n\ge0}2^{n/2}\delta(A_n(t))\le L\theta^{-1}\sup_{t\in T}\sum_{n\ge0}2^nr^{-j(A_n(t))}\le L\theta^{-1}r(\Gamma F_{j_0+1}(T)+2^{n_0}r^{-j_0}).$$

Using (2.104) with $p=1$ and the definition of δ we get by (10) for any $t\in T_2$,

$$\|t\|_1\le L\theta^{-2}\sup_{t\in T}\sum_{n\ge0}2^nr^{-j(A_n(t))}\le L\theta^{-2}r(\Gamma F_{j_0+1}(T)+2^{n_0}r^{-j_0}).$$

Finally since $0\in T$ and $T\in\mathcal{A}_0$ we get by (11) for any $s\in T$,

$$\Big(\sum_{i\in I}\min\big\{s_i^2,r^{-2j(T)}\big\}\Big)^{1/2}$$
$$\le\Big(\sum_{i\in I}\min\big\{(s_i-t(T)_i)^2,r^{-2j(T)}\big\}\Big)^{1/2}+\Big(\sum_{i\in I}\min\big\{t(T)_i^2,r^{-2j(T)}\big\}\Big)^{1/2}$$
$$\le\theta^{-1}(d_{j(T)}(s,t(T))+d_{j(T)}(0,t(T)))\le2\theta^{-1}r^{-j(T)}.$$

In particular $\#\{i\colon|s_i|\ge r^{-j(T)}/2\}\le16\theta^{-2}$ and by (2.105) for any $t\in T_3$ we can find $s\in T$ such that

$$\|t\|_1\le5\sum_{i=1}^N|s_i|I_{\{2|s_i|\ge r^{-j(T)}\}}\le20\theta^{-1}\|s\|_2.$$

Thus we may take T_2+T_3 from [7, Theorem 2.6.3] for T_2 in the statement of our theorem. □

3 Estimates for Bernoulli processes

We begin this section with recalling several well known estimates for suprema of Bernoulli processes and deriving their simple consequences. First result is the concentration property of Bernoulli processes (cf. [5] or [2, Corollary 4.10]).

Theorem 2. *Let $(a_t)_t\in T$ be a sequence of real numbers indexed by a set T and $S:=\sup_{t\in T}(a_t+\sum_{i\in I}t_i\varepsilon_i)$ be such that $|S|<\infty$ a.s. Then*

$$\mathbf{P}(|S-\mathrm{Med}(S)|\ge u)\le4\exp\Big(-\frac{u^2}{16\sigma^2}\Big)\ \textit{for } u>0,$$

where $\sigma:=\sup_{t\in T}\|t\|_2$. In particular $\mathbf{E}|S|<\infty$, $|\mathbf{E}S-\mathrm{Med}(S)|\le L\sigma$ and

$$\mathbf{P}(|S-\mathbf{E}(S)|\ge u)\le L\exp\Big(-\frac{u^2}{L\sigma^2}\Big)\ \textit{for } u>0.\tag{12}$$

Corollary 1. *Let $(Y_t^k)_{t\in T}$, $1\le k\le m$ be i.i.d. Bernoulli processes and $\sigma := \sup_{t\in T}\|Y_t^1\|_2$. Then for any process $(Z_t)_{t\in T}$ independent of $(Y_t^k \colon t\in T, k\le m)$ we have*

$$\mathbf{E}\max_{1\le k\le m}\sup_{t\in T}(Y_t^k+Z_t)\le \mathbf{E}\sup_{t\in T}(Y_t^1+Z_t)+L_1\sigma\sqrt{\log m}. \tag{13}$$

Proof. By the Fubini Theorem it is enough to consider the case when $\mathbf{P}(\forall_t\ Z_t=z_t)=1$ for some deterministic sequence $(z_t)_{t\in T}$. By (12) we have for all $u>0$, $k\le N$,

$$\mathbf{P}\Big(\sup_{t\in T}(Y_t^k+z_t)\ge \mathbf{E}\sup_{t\in T}(Y_t^k+z_t)+u\Big)\le L\exp\Big(-\frac{u^2}{L\sigma^2}\Big).$$

Thus

$$\mathbf{P}\Big(\max_{k\le m}\sup_{t\in T}(Y_t^k+z_t)\ge \mathbf{E}\sup_{t\in T}(Y_t^1+z_t)+u\Big)\le \min\Big\{1, mL\exp\Big(-\frac{u^2}{L\sigma^2}\Big)\Big\}$$

and (13) follows by integration by parts. □

In the same way (using $b(T-s)=b(T)$) we show

Corollary 2. *If $t_0\in l^2(I)$ and $T=\bigcup_{k=1}^m T_k\subset l^2(I)$, then*

$$b(T)\le \max_k b(T_k)+L_1\sigma\sqrt{\log m},$$

where $\sigma := \sup_{t\in T}\|t-t_0\|_2$.

Theorem 3 ([7, Theorem 4.2.4])**.** *Suppose that vectors $t_1,\ldots,t_m\in l^2(I)$ and numbers $a,b>0$ satisfy*

$$\forall_{l\neq l'}\ \|t_l-t_{l'}\|_2\ge a \ \text{and}\ \ \forall_l\ \|t_l\|_\infty\le b. \tag{14}$$

Then

$$\mathbf{E}\sup_{l\le m}\sum_{i\in I}t_{l,i}\varepsilon_i\ge \frac{1}{L_2}\min\Big\{a\sqrt{\log m},\frac{a^2}{b}\Big\}.$$

Corollary 3 ([7, Proposition 4.2.2])**.** *Consider vectors $t_1,\ldots,t_m\in l^2(I)$ and numbers $a,b>0$ such that (14) holds. Then for any $\sigma>0$ and any sets $H_l\subset B_{l^2(I)}(t_l,\sigma)$,*

$$b\Big(\bigcup_{l\le m}H_l\Big)\ge\frac{1}{L_2}\min\Big\{a\sqrt{\log m},\frac{a^2}{b}\Big\}-L_3\sigma\sqrt{\log m}+\min_{l\le m}b(H_l).$$

Before stating the last result, which is the main new observation of this section, let us introduce some additional notation. For $\emptyset\neq J\subset I$, $t\in l^2(I)$, $T\subset l^2(I)$ we define $t_J := (t_i)_{i\in J}\in l^2(J)$ and

$$b_J(T) := \mathbf{E}\sup_{t\in T}\sum_{i\in J}\varepsilon_i t_i.$$

We also set

$$d_J(t,s) := \|t_J-s_J\|_2,\quad t,s\in l^2(I)$$

and

$$B_J(t,a) := \{s\in l^2(I)\colon d_J(s,t)\le a\},\quad t\in l^2(I), a\ge 0.$$

Proposition 1. *Suppose that m is a positive integer, numbers $b, \sigma > 0$ satisfy $b\sqrt{\log m} \leq \sigma$ and $T \subset l^2(I)$ is such that for constants $c, \tilde{c} > 0$,*

$$\forall_{t,s\in T}\ d_I(t,s) \leq c, \quad d_J(t,s) \leq \tilde{c}, \quad \|t-s\|_\infty \leq b. \tag{15}$$

Then there exist $t_1, \ldots, t_m \in T$ such that either $T \subset \bigcup_{l\leq m} B_I(t_l, \sigma)$ or

$$b_J\Big(T \setminus \bigcup_{l\leq m} B_I(t_l,\sigma)\Big) \leq b_I(T) - \Big(\frac{1}{L_4}\sigma - 2L_1\tilde{c}\Big)\sqrt{\log m} + L_5 c. \tag{16}$$

Proof. Since $b_J(T) = b_J(T-t)$ for any $t \in l^2(I)$, we may and will assume that $0 \in T$. Moreover to show (16) it is enough to consider the case $m \geq 2$, $\tilde{c} \leq \min(c, \sigma/4)$ and $N(T, d_I, \sigma) > m$ (where $N(T, d, a)$ denotes the minimal number of balls in metric d with radius a that cover T).

We set

$$\alpha := \inf_{t_1,\ldots,t_m \in T} b_J\Big(T \setminus \bigcup_{l\leq m} B_I(t_l,\sigma)\Big).$$

Let $\varepsilon_i^{(k)}$, $i \in J$, $k = 1, \ldots, 4m$ be independent Bernoulli r.v.'s, independent of $(\varepsilon_i)_{i\in I}$. Let

$$Y_t^{(k)} := \sum_{i\in J} t_i \varepsilon_i^{(k)}, \quad Z_t := \sum_{i\in I\setminus J} t_i\varepsilon_i$$

and

$$S_k := \{t \in T \colon Y_t^{(k)} > \alpha - L\tilde{c}\}.$$

First we will show that if L is sufficiently large then

$$p := \mathbf{P}\Big(N\Big(\bigcup_{l\leq 4m} S_l, d_I, \frac{\sigma}{2}\Big) \geq m\Big) \geq \frac{1}{4}. \tag{17}$$

Suppose that $p \leq 1/4$ and put

$$\tilde{S} := \Big\{t \in T \colon d_I(t,s) \leq \frac{\sigma}{2} \text{ for some } s \in \bigcup_{l\leq 4m-1} S_l\Big\},$$

then

$$\mathbf{P}(N(\tilde{S}, d_I, \sigma) > m) \leq \mathbf{P}\Big(N\Big(\bigcup_{l\leq 4m-1} S_l, d_I, \frac{\sigma}{2}\Big) > m\Big) \leq p \leq \frac{1}{4}.$$

Let us fix $(\varepsilon_i^{(k)})_{k\leq 4m-1}$ such that $N(\tilde{S}, d_I, \sigma) \leq m$, then $b_J(T \setminus \tilde{S}) \geq \alpha$. Denote by $\mathbf{P}_{4m}$ the probability with respect to variables $(\varepsilon_i^{(4m)})$. We have

$$\begin{aligned}\mathbf{P}_{4m}(S_{4m} \subset \tilde{S}) &= \mathbf{P}_{4m}\Big(\sup_{t\in T\setminus\tilde{S}} \sum_{i\in J} t_i\varepsilon_i^{(4m)} \leq \alpha - L\tilde{c}\Big) \\ &\leq \mathbf{P}_{4m}\Big(\sup_{t\in T\setminus\tilde{S}} \sum_{i\in J} t_i\varepsilon_i^{(4m)} \leq b_J(T\setminus\tilde{S}) - L\tilde{c}\Big) \leq \frac{1}{4}\end{aligned}$$

for sufficiently large L by Theorem 2. Hence

$$\mathbf{P}(S_{4m} \subset \tilde{S}) \leq \mathbf{P}(N(\tilde{S}, d_I, \sigma) > m) + \frac{1}{4}\mathbf{P}(N(\tilde{S}, d_I, \sigma) \leq m) \leq p + \frac{1-p}{4} \leq \frac{1}{2},$$

i.e.

$$\mathbf{P}\Big(\exists_{t\in S_{4m}}\ d_I\Big(t, \bigcup_{l\leq 4m-1} S_l\Big) \geq \frac{\sigma}{2}\Big) \geq 1/2.$$

By the symmetry we have for any $1 \leq k \leq 4m$,

$$\mathbf{P}\Big(\exists_{t\in S_k}\ d_I\Big(t, \bigcup_{l\leq 4m, l\neq k} S_l\Big) \geq \frac{\sigma}{2}\Big) \geq 1/2.$$

Define

$$A := \mathrm{card}\Big\{k \leq 4m\colon \exists_{t\in S_k}\ d_I\Big(t, \bigcup_{l\leq 4m, l\neq k} S_l\Big) \geq \frac{\sigma}{2}\Big\},$$

then $\mathbf{E}A \geq 2m$ and thus (since $0 \leq A \leq 4m$) $\mathbf{P}(A \geq m) \geq 1/4$. However if $A \geq m$, then $N(\bigcup_{l\leq 4m} S_l, d_I, \sigma/2) \geq m$, so (17) holds.

Let us fix $(\varepsilon_i^{(k)})_{k\leq 4m}$ such that $N(\bigcup_{l\leq 4m} S_l, d_I, \sigma/2) \geq m$. Then there exist $t_l \in T$ and $1 \leq k_l \leq 4m$, $l = 1,\ldots,m$ such that $t_l \in S_{k_l}$ and $d_I(t_i,t_j) \geq \sigma/2$ for $i \neq j$. We have $d_{I\setminus J}(t_i,t_j) \geq d_I(t_i,t_j) - d_J(t_i,t_j) \geq \sigma/2 - \tilde{c} \geq \sigma/4$ for $i \neq j$. Let $\mathbf{E}_{I\setminus J}$ ($\mathbf{P}_{I\setminus J}$) denote the integration (resp. probability) with respect to $(\varepsilon_i)_{i\in I\setminus J}$, then by Theorem 3,

$$\begin{aligned}\mathbf{E}_{I\setminus J} \max_{1\leq k\leq 4m} \sup_{t\in T}(Y_t^{(k)} + Z_t) &\geq \mathbf{E}_{I\setminus J} \max_{1\leq l\leq m}(Y_{t_l}^{(k_l)} + Z_{t_l}) \geq \alpha - L\tilde{c} + \mathbf{E}\max_{1\leq l\leq m} Z_{t_l}\\ &\geq \alpha - L\tilde{c} + \frac{1}{L_2}\min\Big\{\frac{\sigma}{4}\sqrt{\log m}, \frac{\sigma^2}{16b}\Big\} \geq \alpha - L\tilde{c} + \frac{1}{16L_2}\sigma\sqrt{\log m}.\end{aligned}$$

Since $0 \in T$, we have $\sup_{t\in T}\|t_{I\setminus J}\|_2 \leq c$ by (15), hence by (12) (recall that $\tilde{c} \leq c$ and according to our convention L may differ at each occurence),

$$\mathbf{P}_{I\setminus J}\Big(\max_{1\leq k\leq 4m}\sup_{t\in T}(Y_t^{(k)} + Z_t) \geq \alpha + \frac{1}{16L_2}\sigma\sqrt{\log m} - Lc\Big) \geq \frac{1}{2},$$

therefore

$$\mathbf{P}\Big(\max_{1\leq k\leq 4m}\sup_{t\in T}(Y_t^{(k)} + Z_t) \geq \alpha + \frac{1}{16L_2}\sigma\sqrt{\log m} - Lc\Big) \geq \frac{1}{2}p \geq \frac{1}{8}.$$

This implies (using again (12))

$$\mathbf{E}\max_{1\leq k\leq 4m}\sup_{t\in T}(Y_t^{(k)} + Z_t) \geq \alpha + \frac{1}{16L_2}\sigma\sqrt{\log m} - L_5c.$$

Corollary 1 yields

$$\mathbf{E}\max_{1\leq k\leq 4m}\sup_{t\in T}(Y_t^{(k)} + Z_t) \leq b_I(T) + L_1\tilde{c}\sqrt{\log 4m} \leq b_I(T) + \sqrt{3}L_1\tilde{c}\sqrt{\log m},$$

hence

$$\alpha \leq b_I(T) - \Big(\frac{1}{16L_2}\sigma - \sqrt{3}L_1\tilde{c}\Big)\sqrt{\log m} + L_5c,$$

which yields (16) provided $L_4 \geq 16L_2$. □

4 Thin sets

Definition 2. We say that a set $A \subset \mathbb{R}$ is *θ-thin* for some $\theta > 0$ if there exists a sequence of functions $(f_{k,l})_{k\in\mathbb{Z},l\in I_k}$ satisfying the following conditions
i) $f_{k,l} \colon \mathbb{R} \to [-2^{-k}, 2^{-k}]$, $f_{k,l}(0) = 0$,
ii) $\sum_{k,l} |f_{k,l}(x) - f_{k,l}(y)| \le |x - y|$ for all $x, y \in \mathbb{R}$,
iii) $\sum_{k\ge j,l\in I_k} |f_{k,l}(x) - f_{k,l}(y)|^2 \ge \theta^2 \min\{2^{-2j}, |x-y|^2\}$ for all $j \in \mathbb{Z}$, $x, y \in A$.

Example 1. $T = \{0, a\}$ is 1/2-thin.
Indeed let $I_k = \{1\}$ and $f_{k,1}(x) = f_k(x) := \min\{(|x| - 2^{-k})_+, 2^{-k}\}$. Suppose that $2^{-i} \le |a| < 2^{-i+1}$, then for $j > i$, $\sum_{k\ge j} |f_k(a) - f_k(0)|^2 = \sum_{k\ge j} 2^{-2k} = 2^{2-2j}/3$ and for $j \le i$, $\sum_{k\ge j} |f_k(a) - f_k(0)|^2 = \sum_{k\ge i+1} 2^{-2k} + (|a| - 2^{-i})^2 = 2^{-2i}/3 + (|a| - 2^{-i})^2 \ge a^2/4$.

Example 2. $T = \{2^k \colon k \in \mathbb{Z}\} \cup \{0\}$ is 1/4-thin.
Let $I_k = \mathbb{Z}$ and $f_{k,l}(x) := \min\{(x - 2^{l-1} - 2^{-k})_+, (2^{l-1} - 2^{-k})_+, 2^{-k}\}$. Then if $x = 2^{i_1} > y = 2^{i_2}$,

$$\sum_{k\ge j,l} |f_{k,l}(x) - f_{k,l}(y)|^2 \ge \sum_{k\ge j} |f_{k,i_1}(2^{i_1}) - f_{k,i_1}(2^{i_1-1})|^2 \ge \frac{1}{4} \min\left\{2^{-2j}, |2^{i_1-1}|^2\right\}$$
$$\ge \frac{1}{16} \min\left\{2^{-2j}, |x-y|^2\right\},$$

where the first inequality follows by the monotonicity of f_{k,i_1} and the second one by the same calculation as in Example 1.

Example 3. Suppose that T is a "Cantor-like set" such that $0 \in T$ and for some $\alpha > 0$,

$$\forall_{s,t\in T, s<t}\ \exists_{\tilde{s},\tilde{t}\in[s,t],\tilde{s}<\tilde{t}}\ (\tilde{s}, \tilde{t}) \cap T = \emptyset \text{ and } \tilde{t} - \tilde{s} \ge \alpha(t - s).$$

Then T is $\alpha/2$-thin.
Let $\mathbb{R} \setminus \overline{T} = \bigcup_{n=1}^{N} (a_n, b_n)$, $N \le \infty$. We put $I_k := \{1, \ldots, N\}$ and

$$f_{k,l}(x) := \min\left\{(x - a_l - 2^{-k})_+, (b_l - a_l - 2^{-k})_+, 2^{-k}\right\} \text{ if } b_l > a_l \ge 0,$$
$$f_{k,l}(x) := \min\left\{(b_l - x - 2^{-k})_+, (b_l - a_l - 2^{-k})_+, 2^{-k}\right\} \text{ if } a_l < b_l \le 0.$$

If $x, y \in T$, $x < y$, then $x < a_n < b_n < y$ for some $n \le N$ with $(b_n - a_n) \ge \alpha(y - x)$ and

$$\sum_{k\ge j,l} |f_{k,l}(x) - f_{k,l}(y)|^2 \ge \sum_{k\ge j} |f_{k,n}(b_n) - f_{k,n}(a_n)|^2 \ge 4^{-1} \min\left\{2^{-2j}, |b_n - a_n|^2\right\}$$
$$\ge \left(\frac{\alpha}{2}\right)^2 \min\left\{2^{-2j}, |x-y|^2\right\},$$

where the first inequality follows by the monotonicity of $f_{k,n}$ and the second one by the same calculation as in Example 1.

Example 4. If T contains some nonempty open interval (a, b) then T is not θ-thin for any $\theta > 0$.

Suppose on the contrary that T is θ-thin and functions $(f_{k,l})_{k\in\mathbb{Z},l\in I_k}$ satisfy conditions i)-iii) of Definition 2. By condition ii) the functions $f_{k,l}$ are a.e. differentiable and $\sum_{k,l}|f'_{k,l}(z)|\leq 1$ for a.e. $z\in\mathbb{R}$. Hence there exists j_0 such that

$$\int_a^b \sum_{k\geq j_0,l}|f'_{k,l}(z)|dz<\theta(b-a).$$

Thus for any $n\in\mathbb{Z}_+$ we can find $x,y\in(a,b)$ with $y-x=(b-a)/n$ such that

$$\sum_{k\geq j_0,l}|f_{k,l}(y)-f_{k,l}(x)|\leq\int_x^y\sum_{k\geq j_0,l}|f'_{k,l}(z)|dz<\theta(y-x).$$

Hence

$$\sum_{k\geq j_0,l}|f_{k,l}(y)-f_{k,l}(x)|^2<\theta^2|y-x|^2=\theta^2\min\left\{2^{-2j_0},|y-x|^2\right\}$$

if n is sufficiently large, and this contradicts condition iii).

In a similar way one can show that a θ-thin set cannot have positive Lebesgue measure.

Lemma 1. *Suppose that A is a θ-thin subset of $\mathbb{R}$ and $r=2^\nu$ for some positive integer ν. Then there exist functions $(\tilde{f}_{k,l})_{k\in\mathbb{Z},l\in\tilde{I}_k}$ such that*
i) $\tilde{f}_{k,l}\colon\mathbb{R}\to[-r^{-k},r^{-k}]$, $\tilde{f}_{k,l}(0)=0$,
ii) $\sum_{k,l}|\tilde{f}_{k,l}(x)-\tilde{f}_{k,l}(y)|\leq 2|x-y|$ for all $x,y\in\mathbb{R}$,
iii) $\rho_j^2(x,y):=\sum_{k\geq j,l\in\tilde{I}_k}|\tilde{f}_{k,l}(x)-\tilde{f}_{k,l}(y)|^2\geq\theta^2\min\{r^{-2j},|x-y|^2\}$ for all $j\in\mathbb{Z}$, $x,y\in A$.
iv) $\rho_{j+1}(x,y)\geq r^{-1}\rho_j(x,y)$ for all $x,y\in A$.

Proof. Let $(f_{k,l})_{k\in\mathbb{Z},l\in I_k}$ be as in Definition 2. Let us put

$$\tilde{I}_k:=\{(l_1,l_2,l_3)\colon 0\leq l_1\leq\nu-1,l_2\geq 0,l_3\in I_{\nu(k-l_2)+l_1}\},$$
$$\tilde{f}_{k,(l_1,l_2,l_3)}:=r^{-l_2}f_{\nu(k-l_2)+l_1,l_3}\text{ for }(l_1,l_2,l_3)\in\tilde{I}_k.$$

Notice that

$$\|\tilde{f}_{k,(l_1,l_2,l_3)}\|_\infty\leq r^{-l_2}2^{-\nu(k-l_2)-l_1}=r^{-k}2^{-l_1}\leq r^{-k},$$

$$\sum_{k,(l_1,l_2,l_3)\in\tilde{I}_k}|\tilde{f}_{k,(l_1,l_2,l_3)}(x)-\tilde{f}_{k,(l_1,l_2,l_3)}(y)|\leq\sum_{l_2\geq 0}r^{-l_2}\sum_{k,l\in I_k}|f_{k,l}(x)-f_{k,l}(y)|\leq\frac{r}{r-1}|x-y|$$
$$\leq 2|x-y|.$$

We also have for $x,y\in A$,

$$\sum_{k\geq j,(l_1,l_2,l_3)\in\tilde{I}_k}|\tilde{f}_{k,(l_1,l_2,l_3)}(x)-\tilde{f}_{k,(l_1,l_2,l_3)}(y)|^2\geq\sum_{k\geq\nu j,l\in I_k}|f_{k,l}(x)-f_{k,l}(y)|^2$$
$$\geq\theta^2\min\left\{2^{-2\nu j},|x-y|^2\right\}=\theta^2\min\left\{r^{-2j},|x-y|^2\right\}.$$

Moreover,

$$\sum_{k\geq j+1,(l_1,l_2,l_3)\in\tilde{I}_k}|\tilde{f}_{k,(l_1,l_2,l_3)}(x)-\tilde{f}_{k,(l_1,l_2,l_3)}(y)|^2$$
$$\geq r^{-2}\sum_{k\geq j,(l_1,l_2,l_3)\in\tilde{I}_k}|\tilde{f}_{k,(l_1,l_2,l_3)}(x)-\tilde{f}_{k,(l_1,l_2,l_3)}(y)|^2,$$

since $\tilde{f}_{k+1,(l_1,l_2+1,l_3)}=r^{-1}\tilde{f}_{k,(l_1,l_2,l_3)}$ for $k\in\mathbb{Z}$, $(l_1,l_2,l_3)\in\tilde{I}_k$. □

5 Main Result

In this section we prove the main result of this note, which is the following theorem.

Theorem 4. *Suppose that $T \subset l^2(I)$ is such that $0 \in T$, $b(T) < \infty$ and all one dimensional projections of T, $(\{t_i : t \in T\})_{i \in I}$ are θ-thin. Then $T \subset T_1 + T_2$ with $\sup_{t\in T_2} \|t\|_1 \le L\theta^{-2} b(T)$ and $g(T_1) \le L\theta^{-1} b(T)$.*

To prove the theorem we will first construct distances d_j and functionals F_j satisfying (1)-(5). Let $r = 2^\nu$ with $\nu \ge 2$ to be chosen later.
By Lemma 1 there exist functions $f_{i,k,l}$ such that

$$f_{i,k,l} : \mathbb{R} \to [-r^{-k}, r^{-k}],\ f_{i,k,l}(0) = 0, \tag{18}$$

$$\forall_{i\in I} \forall_{x,y \in \mathbb{R}} \sum_{k,l} |f_{i,k,l}(x) - f_{i,k,l}(y)| \le 2|x - y| \tag{19}$$

and a decreasing family of metrics $(d_j)_{j\in\mathbb{Z}}$ on $l^2(I)$ defined by

$$d_j(s,t) := \Big(\sum_{i,k\ge j,l} |f_{i,k,l}(t_i) - f_{i,k,l}(s_i)|^2 \Big)^{1/2}$$

satisfies (1) and (5).
For $\emptyset \ne A \subset T$ let

$$F_j(A) := \mathbf{E} \sup_{t\in T} \sum_{i,k\ge j,l} f_{i,k,l}(t_i) \varepsilon_{i,k,l},$$

where $(\varepsilon_{i,k,l})$ is a multiindexed Bernoulli sequence. Obviously F_j satisfies (2) and (3). Moreover (19) and the comparison theorem for Bernoulli processes [6, Theorem 2.1] (cf. the proof of [7, Proposition 4.3.7]) implies

$$\forall_{j\in\mathbb{Z}}\ F_j(T) \le 2b(T). \tag{20}$$

Notice that by (19),

$$\sup_{s,t\in T} d_j(t,s) \le 2 \sup_{t,s\in T} \|t - s\|_2 \le 8b(T),$$

hence the condition (4) holds if $r^{1-j_0} \ge 16b(T)$.
In the next few lemmas we are going to show that functional F_j are (Γ, n_0)-decomposable for large r and sufficiently chosen Γ and n_0.

Lemma 2. *If C is a nonempty subset of T, then there exist vectors $t_1, \dots, t_{m-1} \in C$ such that the set $D := C \setminus \bigcup_{i=1}^{m-1} \tilde{B}_{j+1}(t_i, a)$ is empty or for all $t \in D$,*

$$F_{j+1}(D \cap \tilde{B}_{j+1}(t,\sigma)) \le F_{j+1}(C) - \frac{1}{L_2} \min\left\{a\sqrt{\log m}, a^2 r^{j+1}\right\} + 2L_3\sigma\sqrt{\log m}.$$

Proof. We follow the standard greedy algorithm based on Corollary 3. We may obviously assume that $m \ge 2$ and $N(C, d_{j+1}, a) \ge m$. Let us take any $0 < \delta < L_3\sigma\sqrt{\log m}$, we will inductively choose vectors t_i. Let $D_1 = C$ and $t_1 \in C$ be such that

$$F_{j+1}(C \cap \tilde{B}_{j+1}(t_1,\sigma)) \ge \sup_{t\in C} F_{j+1}(C \cap \tilde{B}_{j+1}(t,\sigma)) - \delta.$$

If $t_1, \ldots, t_k$, $1 \le k < m-1$ are already chosen, we set $D_{k+1} := C \setminus \bigcup_{i=1}^{k} \tilde{B}_{j+1}(t_i, a)$ and take $t_{k+1} \in D_{k+1}$ such that

$$F_{j+1}(D_{k+1} \cap \tilde{B}_{j+1}(t_{k+1}, \sigma)) \ge \sup_{t \in D_{k+1}} F_{j+1}(D_{k+1} \cap \tilde{B}_{j+1}(t, \sigma)) - \delta.$$

Let $t = t_m$ be an arbitrary point in $D = D_m = C \setminus \bigcup_{i=1}^{m-1} \tilde{B}_{j+1}(t_i, a)$ and let $H_l := D_l \cap \tilde{B}_{j+1}(t_l, \sigma)$, $1 \le l \le m$. Then $d_{j+1}(t_k, t_l) \ge a$ for all $1 \le k \ne l \le m$ and $H_l \subset \tilde{B}_{j+1}(t_l, \sigma)$, so we may apply Corollary 3 with $b = r^{-j-1}$ and get

$$F_{j+1}(C) \ge F_{j+1}\Big(\bigcup_{i=1}^{m} H_i\Big) \ge \frac{1}{L_2} \min\left\{a\sqrt{\log m}, a^2 r^{j+1}\right\} - L_3 \sigma \sqrt{\log m} + \min_i F_{j+1}(H_i).$$

But the construction of t_i yields

$$\min_i F_{j+1}(H_i) \ge F_{j+1}(D \cap \tilde{B}_{j+1}(t, \sigma)) - \delta \ge F_{j+1}(D \cap \tilde{B}_{j+1}(t, \sigma)) - L_3 \sigma \sqrt{\log m}.$$

□

Lemma 3. *If $\emptyset \ne C \subset T$ then we may decompose $C = \bigcup_{i=1}^{m} D_i$ into $m \le N_{n-1}$ disjoint sets such that for $i \le m-1$, $D_i \subset \tilde{B}_{j+1}(t_i, L_6 2^{n/2} r^{-j-1/2})$ for some $t_i \in C$ and*

$$\forall_{t \in D_m}\ F_{j+1}(D_m \cap \tilde{B}_{j+1}(t, 2^{n/2+2} r^{-j-1/2})) \le F_{j+1}(C) - 2^n r^{-j-1/2}.$$

Proof. We use Lemma 2 with $m := N_{n-1}$, $\sigma := 2^{n/2+2} r^{-j-1/2}$ and $a := L_6 2^{n/2} r^{-j-1/2}$. Then

$$\frac{1}{L_2} \min\left\{a\sqrt{\log m}, a^2 r^{j+1}\right\} - 2L_3 \sigma \sqrt{\log m} = \Big(\frac{L_6}{L_2} - 8L_3\Big) 2^{n-1/2} r^{-j-1/2} \ge 2^n r^{-j-1/2}$$

if $L_6 \ge L_2(2^{1/2} + 8L_3)$. □

Lemma 4. *If $r \ge L_7$ and $2^{n_0/2} \ge L_8 r$, then functionals F_j are $(r^{1/2}, n_0)$-decomposable.*

Proof. Let us take $C \subset \tilde{B}_{j-1}(t_0, 2^{n/2} r^{-j+1}) \subset \tilde{B}_j(t_0, 2^{n/2} r^{-j+1})$ for some $t_0 \in T$ and $n \ge n_0$. We apply Lemma 3 to C and get a decomposition $C = \bigcup_{i \le m} D_i$, $m \le N_{n-1}$. The set D_m satisfies the condition (8) (with $C_i = D_m$ and $\Gamma = r^{1/2}$) and $(N_{n-1} - 1)(N_{n-2} + 1) + 1 \le N_n$, so it is enough to show that each of the sets D_l, $l \le m-1$, may be decomposed into at most $N_{n-2} + 1$ sets C_i satisfying (7) or (8). Let us fix $l \le m-1$, then

$$D_l \subset \tilde{B}_j(t_0, 2^{n/2} r^{-j+1}) \cap \tilde{B}_{j+1}(t_l, L_6 2^{n/2} r^{-j-1/2})$$

for some $t_0, t_l \in T$. Thus

$$d_j(t, s) \le 2^{n/2+1} r^{-j+1},\ d_{j+1}(t, s) \le L_6 2^{n/2+1} r^{-j-1/2} \text{ for all } t, s \in D_l.$$

Hence we may apply Proposition 1 with $c = 2^{n/2+1} r^{-j+1}$, $\tilde{c} = L_6 2^{n/2+1} r^{-j-1/2}$, $b = 2r^{-j}$, $m = N_{n-2}$ and $\sigma = 2^{n/2} r^{-j}$ and get that $D_l = \bigcup_{i=1}^{m+1} C_i$ with C_i satisfying (7) for $i \le m$ and

$$F_{j+1}(C_{m+1}) \le F_j(D_l) - \Big(\frac{1}{L_4}\sigma - 2L_1 \tilde{c}\Big) \sqrt{\log m} + L_5 c.$$

Notice that

$$\Big(\frac{1}{L_4}\sigma - 2L_1\tilde{c}\Big)\sqrt{\log m} - L_5 c = 2^n r^{-j-1/2}\Big(\frac{1}{2L_4} r^{1/2} - 2L_1L_6 - 2L_5 r^{3/2}2^{-n/2}\Big) \geq 2^n r^{-j-1/2}$$

if L_7 and L_8 are large enough, so C_{m+1} satisfies (8). □

Proof of Theorem 4. Let us choose $r = 2^\nu \in [L_7, 2L_7)$ and $n_0 \geq 1$ such that $2^{n_0/2} \in [L_8 r, 2^{1/2}L_8 r)$, then by Lemma 4 functionals F_j are (Γ, n_0) decomposable with $\Gamma = r^{1/2}$. Let $j_0 \in \mathbb{Z}$ be such that $r^{-j_0} \leq 16b(T) \leq r^{1-j_0}$. Then all assumptions of Theorem 1 are satisfied. Notice that $\theta \leq 1$, $\sup_{s\in T}\|s\|_2 \leq 4b(T)$,

$$\Gamma F_{j_0+1}(T) + 2^{n_0} r^{-j_0} \leq 2r^{1/2}b(T) + 32L_8^2 r^2 b(T) \leq Lb(T).$$

Hence by Theorem 1 we get $T \subset T_1 + T_2$ with

$$g(T_1) \leq L\gamma_2(T_1) \leq L\theta^{-1}b(T)$$

and

$$\sup_{t\in T_2}\|t\|_1 \leq L\theta^{-2}rb(T) + L\theta^{-1}b(T) \leq L\theta^{-2}b(T).$$

□

Acknowledgments. The author would like to thank the anonymous referee for valuable suggestions, which helped to improve the presentation of the results.

References

[1] X. Fernique, *Regularité des trajectoires des fonctions aléatoires gaussiennes*, École d'Été de Probabilités de Saint-Flour, IV-1974, Lecture Notes in Mathematics 480, 1–96, Springer, Berlin, 1975. MR0413238

[2] M. Ledoux, *The concentration of measure phenomenom*, AMS, Providence, RI, 2001.

[3] M. Ledoux, M. Talagrand, *Probability in Banach spaces. Isoperimetry and processes*, Springer-Verlag, Berlin, 1991. MR1102015

[4] M. Talagrand, *Regularity of Gaussian processes*, Acta Math. 159 (1987), 99–149. MR0906527

[5] M. Talagrand, *An isoperimetric theorem on the cube and the Kintchine-Kahane inequalities*, Proc. Amer. Math. Soc. 104 (1988), 905–909. MR0964871

[6] M. Talagrand, *Regularity of infinitely divisible processes*, Ann. Probab. 21 (1993), 362–432. MR1207231

[7] M. Talagrand, *The generic chaining. Upper and lower bounds of stochastic processes*, Springer-Verlag, Berlin, 2005. MR2133757