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STRONG LAW OF LARGE NUMBERS UNDER A GENERAL MOMENT CONDITION

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Abstract
We use our maximum inequality for p-th order random variables ($p > 1$) to prove a strong law of large numbers (SLLN) for sequences of p-th order random variables. In particular, in the case $p = 2$ our result shows that $\sum f(k)/k < \infty$ is a sufficient condition for SLLN for f-quasi-stationary sequences to hold. It was known that the above condition, under the additional assumption of monotonicity of f, implies SLLN (Erdös (1949), Gal and Koksma (1950), Gaposhkin (1977), Moricz (1977)). Besides getting rid of the monotonicity condition, the inequality enables us to extend the general result to p-th order random variables, as well as to the case of Banach-space-valued random variables.

Notations

$\mathbf{N}$ stands for the set of positive integers, $\mathbf{N_0} = \mathbf{N} \cup \{0\}$. X denotes a Banach space, real or complex. Let $(\Omega, \mathcal{A}, P)$ be an underlying probability space. By an X-valued random variable we mean a Bochner measurable mapping $\xi : \Omega \to X$.
Given a sequence $(\xi_n), n \in \mathbf{N_0}$ of X-valued random variables denote

$$S_{a,b} = \sum_{k=a}^{a+b-1} \xi_k \ , \quad M_{a,b} = \max_{k \le b} \|S_{a,k}\| \ , \quad a, b \in \mathbf{N_0} \ .$$

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We say that for a sequence $(\xi_n), n \in \mathbf{N_0}$ *the strong law of large numbers* (SLLN) *holds*, if $S_{0,n}/n \to 0$ a.s. as $n \to \infty$.

Main Results

The main objective of this note is to prove the following theorem and some of its consequences.

Theorem 1 *Let* $1 < p < \infty$ *. If for a sequence* $(\xi_n) \subset L_p(X)$

$$\sum_{n=0}^{\infty} \sup_{k \in \mathbf{N_0}} \mathbf{E} \| \frac{S_{k,2^n}}{2^n} \|^p < \infty , \tag{1}$$

then SLLN *holds for* (ξ_n).

We apply Theorem 1 to quasi-stationary sequences.

Corollary 1 *Let* $(\xi_n), n \in \mathbf{N_0}$ *be a sequence of* X*-valued random variables such that for some* $1 < p < \infty$ *and each* $k, n \in \mathbf{N_0}$

$$\mathbf{E}\|S_{k,n}\|^p \le g(n) ,$$

for a numerical function g*. Then*

(i) If

$$\sum_{n=1}^{\infty} \frac{g(2^n)}{2^{np}} < \infty ,$$

then SLLN *holds for* (ξ_n) *.*

(ii) If $g(n)/n^{p+1}$ *is monotone, and*

$$\sum_{n=0}^{\infty} \frac{g(n)}{n^{p+1}} < \infty ,$$

then SLLN *holds for* (ξ_n)*.*

Part (ii) of Corollary 1 has been proved earlier for the case $p = 2$, and 1-dimensional X (see Gal and Koksma, 1950 and Gaposhkin, 1977). Below we also discuss Moricz's, 1977 further contribution.

Let $f(n), n \in \mathbf{N_0}$ be a non-negative function. We say that a real or complex-valued sequence $(\xi_n), n \in \mathbf{N_0}$ is *f-quasi-stationary*, if $\mathbf{E}|\xi_k|^2 < \infty,\ k \in \mathbf{N_0}$, and

$$|\mathbf{E}\xi_l \overline{\xi}_{l+m}| \le f(m) , \quad l, m \in \mathbf{N_0} .$$

The following proposition is a consequence of Theorem 1.

Corollary 2 *Let* $(\xi_n), n \in \mathbf{N_0}$ *be an* f*-qusi-stationary sequence. If*

$$f(0) + \sum_{m=1}^{\infty} \frac{f(m)}{m} < \infty , \tag{2}$$

then SLLN *holds for* (ξ_n) *.*

Corollary 2 was known earlier under the additional condition of monotonicity of f . It has been established first by Erdös, 1949 for monotone $f(m) = O(\log^{-\alpha} m), \alpha > 1$. In Gal and Koksma, 1950 it was extended to monotone sequences $f(m)$ satisfying (2). Gaposhkin, 1975 has shown that condition (2) for monotone f is in a sense necessary: If

$$\sum_{m=1}^{\infty} \frac{f(m)}{m} = \infty,$$

then there is an f-quasi-stationary sequence $(\xi_n), n \in \mathbf{N_0}$ for which SLLN fails.
Regarding a general norming in SLLN for an f-quasi-stationary sequence, the reader is referred to the papers by Moricz,1977 and Serfling, 1978. In the case of classical norming $(\lambda_n = 1/n)$ Moricz has proved Theorem 1 above for real valued random variables in the case $p = 2$, and our Corollary 2 (see Moricz, 1977, Theorem 2′, p.228 and Theorem 2, p.227 respectively), both under some additional conditions (see (1.16) and (1.17), respectively, p.227). His main condition (1.16) is in fact equivalent to

$$\sum_m \overline{\varphi}(2^m) < \infty \quad \text{and} \quad \sum_m \frac{\overline{f}(m)}{m} < \infty,$$

where

$$\varphi(m) = \sup_{k \in \mathbf{N_0}} [\mathbf{E}|\frac{S_{k,m}}{m}|^2] \text{ , and } \quad \overline{a}_m = \max_{n \geq m}\{a_n\}.$$

Moricz's second condition (1.17) is not relevant for the purpose of comparison with our paper so we do not discuss it.

Example. Let us show that $\sum_m f(m)/m$ might be finite, whereas $\sum_m \overline{f}(m)/m$ is infinite. This would show that Moricz's condition (1.16) is restrictive. Notice first that for every $f, \quad 0 \leq f(m) \leq 1,\ m \in \mathbf{N_0}$ there is a sequence (ξ_k) of real random variables so that

$$\mathbf{E}\xi_k^2 = 1,\ \mathbf{E}\xi_k) = 0 \quad \text{and} \quad f(m) = \sup_k |\mathbf{E}\xi_k \xi_{k+m}|.$$

Then we put $f(m) = 1/\log m$, if $m = n^2,\ n \in \mathbf{N}$, and $f(m) = 0$ otherwise. It is worthy to note that for weakly stationary sequences condition (2) can be replaced by a weaker condition of convergence (conditional) of the series

$$\sum_{m=1}^{\infty} \frac{R(m)}{m \log m} \log\log m \ ,$$

where R is the correlation function of the sequence (Gaposhkin, 1977).

Proofs

The proof of Theorem 1 is based on the following proposition proved in Chobanyan, Levental and Salehi, 2004.

Theorem 2 *Let $1 < p < \infty$. For any sequence $(\xi_n) \subset L_p(X)$ we have*

$$\sum_{n=0}^{\infty} \mathbf{E}\frac{M^p_{2^n,2^n} - \|S_{2^n,2^n}\|^p}{2^{np}} \leq \frac{2^{p+1}}{2^p - 2} \sum_{n=0}^{\infty} G_n \ ,$$

where

$$G_n = \sup_{k\in\mathbf{N_0}} (\frac{1}{2}\mathbf{E}\|\frac{S_{k,2^n}}{2^n}\|^p + \frac{1}{2}\mathbf{E}\|\frac{S_{k+2^n,2^n}}{2^n}\|^p - \mathbf{E}\|\frac{S_{k,2^{n+1}}}{2^{n+1}}\|^p).$$

For the sake of completeness we outline the proof of Theorem 2. We have for any $k \in \mathbf{N}_0$ $n \in \mathbf{N}_0$

$$M_{k,2^{n+1}} \ \leq \ \max \{M_{k,2^n}, \ \|S_{k,2^n}\| + M_{k+2^n,2^n}\} \quad .$$

Making use of the following elementary inequality $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$, we get

$$\begin{aligned} M^p_{k,2^{n+1}} \ &\leq \ \max \{M^p_{k,2^n}, \ 2^{p-1}(\|S_{k,2^n}\|^p \ + \ M^p_{k+2^n,2^n})\} \ \leq \\ &(2^{p-1}-1)\|S_{k,2^n}\|^p \ + \ M^p_{k,2^n} \ + \ 2^{p-1}M^p_{k+2^n,2^n} \quad . \end{aligned} \tag{3}$$

(3) can be rewritten as

$$\begin{aligned} M^p_{k,2^{n+1}} - \|S_{k,2^{n+1}}\|^p \ &\leq M^p_{k,2^n} \ - \|S_{k,2^n}\|^p + 2^{p-1}(M^p_{k+2^n,2^n} - \|S_{k+2^n,2^n}\|^p) \\ &-\|S_{k,2^{n+1}}\|^p + 2^{p-1}\|S_{k,2^n}\|^p + 2^{p-1}\|S_{k+2^n,2^n}\|^p \quad . \end{aligned}$$

Dividing both sides by $2^{(n+1)p}$, taking expectations, and then maximums over all k's, we get

$$F_{n+1} \leq \frac{1}{2^p}F_n \ + \frac{1}{2}F_n + G_n \ , \quad n \in \mathbf{N}_0, \tag{4}$$

where

$$F_n = \sup_{k\in\mathbf{N}_0} \mathbf{E}\big(\frac{M^p_{k,2^n} - \|S_{k,2^n}\|^p}{2^{np}}\big) \quad ;$$

$$G_n = \sup_{k\in\mathbf{N}_0} \ \big(\ \frac{1}{2} \ \mathbf{E} \ \|\frac{S_{k,2^n}}{2^n}\|^p \ + \frac{1}{2} \ \mathbf{E} \ \|\frac{S_{k+2^n,2^n}}{2^n}\|^p \ - \ \mathbf{E} \ \|\frac{S_{k,2^{n+1}}}{2^{n+1}}\|^p \ \big) \ .$$

It is easy to make sure by induction in n that

$$F_{n+1} \leq \sum_{k=0}^{n} c^{n-k}G_k \ , \quad n \in \mathbf{N}_0 \ ,$$

where $c = \frac{1}{2} + \frac{1}{2^p}$. Summing up (4) from $n=0$ to $n=N$, we come to Theorem 2.

Proof of Theorem 1. Assuming (1) holds we get

$$\sum_{n=0}^{\infty} G_n \leq \sum_{n=0}^{\infty} \sup_{k\in\mathbf{N_0}} \mathbf{E}\|\frac{S_{k,2^n}}{2^n}\|^p < \infty \ .$$

Therefore, by Theorem 2,

$$\frac{M^p_{2^n,2^n} - \|S_{2^n,2^n}\|^p}{2^{np}} \to 0 \quad \text{a.s.} \tag{5}$$

But (1) also implies that

$$\frac{\|S_{2^n,2^n}\|^p}{2^{np}} \to 0 \quad \text{a.s.}$$

This convergence along with (5) implies

$$\frac{\|M_{2^n,2^n}\|}{2^n} \to 0 \quad \text{a.s.,}$$

which is equivalent to SLLN (Chobanyan, Levental and Mandrekar, 2004). □

Proof of Corollary 2. Assume that $(\xi_n), n \in \mathbf{N_0}$ is an f-quasi-stationary sequence. Then we have for any $k \in \mathbf{N_0}$ and any $n \in \mathbf{N_0}$

$$\mathbf{E}|\frac{S_{k,2^n}}{2^n}|^2 \leq \sum_{m=0}^{2^n-1} \frac{f(m)(2^n-m)}{2^{2n}} \leq \frac{1}{2^n}\sum_{m=0}^{2^n-1} f(m)\ .$$

This implies

$$\sum_{n=0}^{\infty} \sup_k \mathbf{E}|\frac{S_{k,2^n}}{2^n}|^2 \leq \sum_{n=0}^{\infty}\sum_{m=0}^{2^n} \frac{f(m)}{2^n} \leq 2f(0) + \sum_{m=1}^{\infty} f(m) \sum_{n=[\log_2 m]}^{\infty} \frac{1}{2^n} \leq 2f(0) + 2\sum_{m=1}^{\infty}\frac{f(m)}{m}.$$

Corollary 2 is proved. □

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