DOI: 10.1214/ECP.v1-975

Elect. Comm. in Probab. **1** (1996) 29–47

ELECTRONIC COMMUNICATIONS in PROBABILITY

THE DIMENSION OF THE FRONTIER OF PLANAR BROWNIAN MOTION[1]

G. LAWLER, Department of Mathematics, Box 90320, Duke University,
Durham, NC 27708-0320
e-mail: jose@math.duke.edu

Submitted: September 18, 1995 *Revised:* February 16, 1996.

AMS 1991 Subject classification: 60J65
Keywords and phrases: Planar Brownian Motion, Frontier Dimension, Disconnection Exponents.

Abstract

Let B be a two dimensional Brownian motion and let the frontier of $B[0,1]$ be defined as the set of all points in $B[0,1]$ that are in the closure of the unbounded connected component of its complement. We prove that the Hausdorff dimension of the frontier equals $2(1-\alpha)$ where α is an exponent for Brownian motion called the two-sided disconnection exponent. In particular, using an estimate on α due to Werner, the Hausdorff dimension is greater than 1.015.

1 Introduction

Let $B(t)$ be a Brownian motion taking values in $\mathbb{R}^2$ which we also consider as $\mathbb{C}$. Let $B[0,1]$ be the image of $[0,1]$. For any compact $A \subset \mathbb{C}$ we define the frontier of A, $\mathrm{fr}(A)$ to be the set of points in A connected to infinity. More precisely, $\mathrm{fr}(A)$ is the set of $x \in A$ such that x is in the closure of the unbounded connected component of $\mathbb{C} \setminus A$. Take a typical point $x \in \mathrm{fr}(B[0,1])$. Then locally at x the Brownian motion looks like two independent Brownian motions starting at x with the condition that x is not disconnected from infinity. In this paper, we use this idea to prove that the Hausdorff dimension of $\mathrm{fr}(B[0,1])$ is given in terms of a certain exponent for Brownian motion.

Let B^1, B^2 be independent Brownian motions starting on the unit circle, and let

$$T_n^j = \inf\{t : |B^j(t)| = n\}.$$

Let D_n be the event that 0 is contained in the unbounded connected component of $\mathbb{C} \setminus (B^1[0,T_n^1] \cup B^2[0,T_n^2])$. Let

$$q(n) = \sup \mathbb{P}^{x^1,x^2}\{D_n\},$$

where $\mathbb{P}^{x^1,x^2}$ denotes probabilities assuming $B^1(0) = x^1, B^2(0) = x^2$, and the supremum is over all $|x^1| = |x^2| = 1$. It is easy to see from the strong Markov property and Brownian

[1] Research supported by the National Science Foundation and NSERC.

scaling that for all $n, m \geq 1$,

$$q(nm) \leq q(n)q(m).$$

Standard techniques, using the subadditivity of the function $f(k) = \log q(2^k)$, can be used to show that there exists an $\alpha > 0$ such that as $n \to \infty$,

$$q(n) \approx n^{-2\alpha}, \tag{1}$$

where $\approx$ denotes that the logarithms of boths sides are asymptotic. Moreover, $q(n) \geq n^{-2\alpha}$ for all n. We call α the *two-sided disconnection exponent.* (Sometimes 2α is called the two-sided disconnection exponent.) The value of α is not known. One can give an easy bound of $\alpha \leq 1/2$, by noting that with probability at least $cn^{-1/2}$, a Brownian motion in $\mathbb{C}$ starting at 1 reaches the circle of radius n without hitting the negative real axis (see (7)). If B^1 and B^2 both reach the circle of radius n without hitting the negative real axis, then 0 is connected to infinity in the complement of $B^1[0, T_n^1] \cup B^2[0, T_n^2]$. Werner [9, 10] has recently shown that $2\alpha \leq .985$. A lower bound, $\alpha \geq (2\pi^2+1)/(8\pi^2)$, was given by Burdzy and Lawler [2]. There is an interesting conjecture due to Mandelbrot [7] that $\alpha = 1/3$, and recent simulations of Puckette and Werner [8] are consistent with this conjecture. Mandelbrot's conjecture was really about the Hausdorff dimension of the frontier of Brownian motion, but as we see from the main theorem of this paper the two quantities are related. We let $\dim_h(A)$ denote the Hausdorff dimension of the set A.

Theorem 1.1 *With probability one,*

$$\dim_h(\mathrm{fr}(B[0,1])) = 2(1-\alpha),$$

where α is the two-sided disconnection exponent.

Using Werner's estimate, we get an immediate corollary,

$$\dim_h(\mathrm{fr}(B[0,1])) > 1.015. \tag{2}$$

Bishop, Jones, Pemantle, and Peres [1] have recently given a different argument to show that

$$\dim_h(\mathrm{fr}(B[0,1])) > 1, \tag{3}$$

but their methods are not sufficient to prove (2). In the course of our proof we will rederive the estimate $\alpha < 1/2$, so (3) can be concluded from this paper alone. The main technical tool in proving the theorem is to improve the estimate in (1). We show that

$$q(n) \asymp n^{-2\alpha}, \tag{4}$$

where $\asymp$ denotes that both sides are bounded by a constant times the other side. Using this estimate (and some slight generalizations) we can compute the Hausdorff dimension by covering the set of "frontier times" by approximate "frontier intervals" of size 2^{-n} and then letting $n \to \infty$. The method is similar to that in [6] where the Hausdorff dimension of the set of cut points of a Brownian path is computed. In the next section we give the main proof assuming the key estimate (4). The estimate and generalizations are derived in the last section. This work was done while the author was visiting the University of British Columbia. I would like to thank Chris Burdzy for the argument at the end of Section 2, and I would like to thank the referee for a careful reading and suggestions for improvement of this paper.

2 Main proof

Let $B(t)$ be a Brownian motion taking values in $\mathbb{C}$. Let

$$\Lambda = \mathrm{fr}(B[0,1]).$$

Let L be the set of frontier times for B,

$$L = \{t \in [0,1] : B(t) \in \Lambda\}.$$

It is well known [5] that with probability one Brownian motion doubles the Hausdorff dimension of sets. Hence it suffices to prove that with probability one

$$\dim_h(L) = 1 - \alpha.$$

Throughout this paper we use $\mathcal{B}(x,r)$ to denote the closed disc of radius r about x. For any compact $A \subset \mathbb{C}$ we will write $Q(A)$ for the unbounded connected component of $\mathbb{C} \setminus A$ and $\bar{Q}(A)$ for the closure of $Q(A)$. Note that $A_1 \subset A_2$ implies $Q(A_1) \supset Q(A_2)$. For every n, let $I(j,n) = [(j-1)/2^n, j/2^n]$. We will say that $I(j,n)$ is an approximate frontier interval if

$$B([\frac{j-1}{2^n}, \frac{j}{2^n}]) \cap Q(B[0, \frac{j-2}{2^n}] \cup B[\frac{j+1}{2^n}, 1]) \neq \emptyset.$$

Let $K_{j,n}$ be the indicator of the event "$I(j,n)$ is an approximate frontier interval" and let

$$L_n = \bigcup_{K_{j,n}=1} I(j,n).$$

Note that $L_1 \supset L_2 \supset \cdots$, and $L \subset L_n$.

Lemma 2.1 *With probability one,*

$$L = \bigcap_{n=1}^{\infty} L_n.$$

Proof: It is easy to see that with probability one there exists an n such that $0, 1 \notin L_n$. Let $0 < t < 1$. Suppose $t \notin L$, i.e., $B(t) \notin \bar{Q}(B[0,1])$. Since $\bar{Q}(B[0,1])$ is a closed set, there is a $\delta > 0$ with

$$\mathcal{B}(B(t), 2\delta) \cap \bar{Q}(B[0,1]) = \emptyset.$$

It is easy to see that this implies that

$$Q(B[0,1] \cap \{z : |z - B(t)| \geq \delta\}) = Q(B[0,1]),$$

and

$$\mathcal{B}(B(t), \delta) \cap Q(B[0,1]) = \emptyset.$$

But by continuity of B, for all n sufficiently large,

$$B([\frac{j-2}{2^n}, \frac{j+2}{2^n}]) \subset \mathcal{B}(B(t), \delta),$$

where $j = j_t$ is chosen with $j - 1 < t2^n \leq j$. Hence $I(j,n)$ and $I(j+1,n)$ are not frontier intervals and hence $t \notin L_n$. ∎

The next lemma will be proved in the next section (see Lemma 3.15). In this paper we use c, c_1, c_2 for positive constants; the exact values of c, c_1, c_2 may vary from line to line.

Lemma 2.2 *There exist c_1, c_2 such that for $2^{n-2} \leq j \leq k \leq 3 \cdot 2^{n-2}$,*

$$c_1 2^{-n\alpha} \leq \mathbb{E}\{K_{j,n}\} \leq c_2 2^{-n\alpha},$$

and

$$\mathbb{E}\{K_{j,n} K_{k,n}\} \leq c_2 2^{-n\alpha}(k-j+1)^{-\alpha}.$$

Let μ_n be the (random) measure on $[0,1]$ whose density with respect to Lebesgue measure is $2^{\alpha n}$ on every interval $I(j,n)$ with $2^{n-2} < j \leq 3 \cdot 2^{n-2}$ and $K_{j,n} = 1$. On all other intervals the density is zero. Note that μ_n is supported on $L_n \cap [1/4, 3/4]$ and hence any weak limit of the μ_n is supported on $L \cap [1/4, 3/4]$. Let

$$J_n = \sum_{2^{n-2} < j \leq 3 \cdot 2^{n-2}} K_{j,n}.$$

Note that $\mu_n[0,1] = \mu_n[1/4, 3/4] = 2^{n(\alpha-1)} J_n$.
The upper bound on $\dim_h(L \cap [1/4, 3/4])$ is easy. We can cover $L \cap [1/4, 3/4]$ by J_n intervals of length 2^{-n}. It follows from Lemma 2.2, that

$$\mathbb{E}\{J_n\} \leq c_2 2^{(1-\alpha)n}.$$

If $\beta > 1 - \alpha$, Markov's inequality gives

$$\mathbb{P}\{J_n \geq 2^{\beta n}\} \leq c_2 2^{(1-\alpha-\beta)n},$$

and hence by the Borel-Cantelli Lemma,

$$\mathbb{P}\{J_n \geq 2^{\beta n} \text{ i.o.}\} = 0.$$

By standard arguments this implies that $\dim_h(L \cap [1/4, 3/4]) \leq \beta$ with probability one and since this holds for all $\beta > 1 - \alpha$,

$$\mathbb{P}\{\dim_h(L \cap [\frac{1}{4}, \frac{3}{4}]) \leq 1 - \alpha\} = 1.$$

To get the lower bound we first note that Lemma 2.2 implies

$$\mathbb{E}\{J_n\} \geq c_1 2^{(1-\alpha)n}.$$

$$\mathbb{E}\{(J_n)^2\} \leq c_2 2^{2(1-\alpha)n}.$$

Hence by standard arguments, there exists a $c_3 > 0$ such that for all n,

$$\mathbb{P}\{J_n \geq c_3 2^{(1-\alpha)n}\} \geq c_3,$$

and hence

$$\mathbb{P}\{\mu_n[\frac{1}{4}, \frac{3}{4}] \geq c_3 \text{ i.o.}\} \geq c_3. \tag{5}$$

Let $I_\beta(\mu)$ denote the β-energy of a measure μ on $[0,1]$, i.e.,

$$I_\beta(\mu) = \int_0^1 \int_0^1 |s-t|^{-\beta} \, d\mu(s) \, d\mu(t).$$

Suppose $\beta = 1 - \alpha - \epsilon$ with $\epsilon > 0$. Then (using Lemma 2.2),

$$\begin{aligned}
\mathbb{E}\{I_\beta(\mu_n)\} &= \sum 2^{2\alpha n}\left[\int_{(j-1)2^{-n}}^{j2^{-n}}\int_{(k-1)2^{-n}}^{k2^{-n}}(s-t)^{-\beta}\,ds\,dt\right]\mathbb{E}\{K_{j,n}K_{k,n}\} \\
&\leq u_\beta \sum 2^{2\alpha n}[2^{(\beta-2)n}(|k-j|+1)^{-\beta}]2^{-n\alpha}(|k-j|+1)^{-\alpha} \\
&\leq u_\beta \sum 2^{-n}2^{-\epsilon n}(|k-j|+1)^{\epsilon-1} \\
&\leq u_\beta.
\end{aligned}$$

Here the sums are over all $2^{n-2} < j,k \leq 3\cdot 2^{n-2}$ and u_β is a positive constant, depending on β, whose value may change from line to line. In particular,

$$\mathbb{P}\{I_\beta(\mu_n) \geq 2u_\beta/c_3\} \leq \frac{1}{2}c_3.$$

Therefore, using (5),

$$\mathbb{P}\{\mu_n([\frac{1}{4},\frac{3}{4}]) \geq c_3;\ I_\beta(\mu_n) \leq 2u_\beta/c_3 \text{ i.o.}\} \geq \frac{1}{2}c_3.$$

On the event above, let μ be any weak limit of the μ_n. Then it is easy to verify that μ is supported on $L\cap[1/4,3/4]$; $\mu(L) \geq c_3$; and $I_\beta(\mu) \leq 2u_\beta/c_3$. By standard arguments (see [4, Theorem 4.13]) this implies that with probability at least $c_3/2$, $\dim_h(L\cap[1/4,3/4]) \geq \beta$. Since this is true for every $\beta < 1-\alpha$, we can conclude

$$\mathbb{P}\{\dim_h(L\cap[\frac{1}{4},\frac{3}{4}]) = 1-\alpha\} \geq \frac{c_3}{2}.$$

There is nothing special about the interval $[1/4,3/4]$. By a similar argument we can show that for every $0 \leq a < b \leq 1$, there exists a $u = u(a,b) > 0$ such that

$$\mathbb{P}\{\dim_h(L\cap[a,b]) = 1-\alpha\} = u,$$

$$\mathbb{P}\{\dim_h(L\cap[a,b]) \leq 1-\alpha\} = 1.$$

We can adapt the proof above to prove a slightly different result. Let $\tilde{K}_{j,n}$ be the indicator function of the event "$I(j,n)$ is an approximate frontier interval; $B(I(j,n)) \subset \{1 \leq \Im(z) \leq 2\}$; $\Im(B(1)) \geq 3$". Similarly to the proof of Lemma 2.2 we can show that for $2^{n-2} \leq j \leq 3\cdot 2^{n-2}$,

$$\mathbb{E}\{\tilde{K}_{j,k}\} \geq c2^{-n\alpha}.$$

Hence if

$$\tilde{J}_n = \sum_{2^{n-2}<j\leq 3\cdot 2^{n-2}} \tilde{K}_{j,n},$$

we can show that

$$\mathbb{E}\{\tilde{J}_n\} \geq c_1 2^{(1-\alpha)n}, \quad \mathbb{E}\{[\tilde{J}_n]^2\} \leq c_2 2^{2(1-\alpha)n}.$$

By using the same argument as above, we can then show that

$$\mathbb{P}\{\dim_h[B[0,1]\cap\{1\leq\Im(z)\leq 2\}] = 2(1-\alpha); \Im(B(1)) \geq 3\} > 0. \tag{6}$$

To prove the main theorem, we have to show that $u(0,1) = 1$. When searching for cutpoints for Brownian motion [6], a similar problem arose. An argument similar to that above showed

that for every interval, with positive probability the Hausdorff dimension of the set of "cut times" in that interval was given by a certain value, $1-\zeta$. In order to show that this result holds with probability one, it was shown that there were cut times arbitrarily close to $t=0$ and then a zero-one law could be used. This idea will not work in the case of the frontier of Brownian motion because $B(0)$ is not a point on the frontier. However, by concentrating on a different point, the point at which the imaginary part of B_t is a minimum, we can show that the dimension of L is $1-\alpha$ with probability one. We will only sketch the argument which is due to K. Burdzy. (It is not too difficult to give the complete details, but this is such an "obvious" fact that it should not take up too much of this paper.)
Write $B_t = X_t + iY_t$ where X_t, Y_t are independent one-dimensional Brownian motions. Let

$$R = \inf\{Y_t : 0 \le t \le 1\}.$$

It is easy to check that with probability one there is a unique (random) time $\sigma \in (0,1)$ with $Y_\sigma = R$. Let V_n be the event that

$$\dim_h[\mathrm{fr}(B[0,1]) \cap \mathcal{B}(B_\sigma, 2^{-n})] = 2(1-\alpha).$$

It suffices to prove that

$$\mathbb{P}\left\{\bigcap_{n=1}^{\infty} V_n\right\} = 1.$$

Let W, Z be independent h-processes in the upper half plane starting at 0, i.e., W, Z are independent Brownian motions "conditioned to have positive imaginary part for all positive times." It is intuitively clear that near B_σ, the Brownian motion B looks like two independent h-processes conditioned to have imaginary part greater than R. Burdzy and San Martin made this notion precise (see [3, Lemma 2.1]) and showed that any "local" event at B_σ has the same probability as the corresponding local event for W, Z. Let E_n be the event that

$$\dim_h[\mathrm{fr}(Z[0,1] \cup W[0,1]) \cap \mathcal{B}(0, 2^{-n})] = 2(1-\alpha).$$

Then by the result of Burdzy and San Martin,

$$\mathbb{P}\left\{\bigcap_{n=1}^{\infty} V_n\right\} = 1 \leftrightarrow \mathbb{P}\left\{\bigcap_{n=1}^{\infty} E_n\right\} = 1.$$

It is not hard to show using 0-1 Laws that

$$\mathbb{P}\left\{\bigcap_{n=1}^{\infty} E_n\right\} = 0 \text{ or } 1.$$

Let τ_n^W be the first time that the imaginary part of W is at least 2^{-n} and similarly define τ_n^Z. Using (6), one can show that there is a $c > 0$ such that for each n

$$\mathbb{P}\{\dim_h[\{2^{-n} < \Im(z) < 2^{-n+1}\} \cap \mathrm{fr}(W[0,\tau_{n-3}^W] \cup Z[0,\tau_{n-3}^Z] \cup \{\Im(z) \ge 2^{-n+3}\})] = 2(1-\alpha)\} \ge c.$$

By the strong Markov property and the transience of the h-processes this implies that there is a $c > 0$ such that

$$\mathbb{P}\{\dim_h[\{2^{-n} < \Im(z) < 2^{-n+1}\} \cap \mathrm{fr}(W[0,\infty) \cup Z[0,\infty))] = 2(1-\alpha)\} \ge c.$$

(We have extended the definition of frontier in a natural way to include some noncompact A.) In particular,

$$\mathbb{P}\{E_n\} \geq c > 0.$$

Hence

$$\mathbb{P}\left\{\bigcap_{n=1}^{\infty} V_n\right\} = \mathbb{P}\left\{\bigcap_{n=1}^{\infty} E_n\right\} = 1.$$

3 Brownian motion estimates

Let B^1, B^2 be independent Brownian motions taking values in $\mathbb{R}^2 = \mathbb{C}$ starting at $|x^1| = |x^2| = 1$. For the first part of this section, we will also assume an initial configuration is given. An initial configuration will be an ordered pair $\tilde{\Gamma} = (\Gamma^1, \Gamma^2)$ of closed subsets of $\{z : 0 < |z| \leq 1\}$ such that

$$\Gamma^j \cap \{z : |z| = 1\} = \{x^j\}, \;\; j = 1, 2.$$

We set $T_1^j = 0$; for $n > 1$ we set, as before,

$$T_n^j = \inf\{t : |B^j(t)| = n\};$$

and we let

$$\Gamma_n^j = \Gamma^j \cup B^j[0, T_n^j],$$

$$\Gamma_n = \Gamma_n^1 \cup \Gamma_n^2.$$

We let $D_n = D_n(\tilde{\Gamma})$ be the event that 0 is connected to infinity in $\mathbb{C} \setminus \Gamma_n$,

$$D_n = \{0 \in Q(\Gamma_n)\}.$$

For each $n \geq 1$, let $\Theta^j(n)$ be the argument of $B^j(T_n^j)$. We will consider arguments modulo 2π, i.e., 0 and 2π are the same argument.
Let

$$q(n) = \sup \mathbb{P}^{x^1, x^2}\{D_n\},$$

where the supremum is over all $|x^1| = |x^2| = 1$ and the initial configuration is given by $\Gamma^j = \{x^j\}$. By Brownian scaling and the strong Markov property,

$$q(nm) \leq q(n)q(m),$$

and hence by standard subadditivity arguments there exists an α such that as $n \to \infty$,

$$q(n) \approx n^{-2\alpha}.$$

Here $\approx$ denotes that the logarithms of both sides are asymptotic. Moreover, $q(n) \geq n^{-2\alpha}$. We know (see Section 1) that $\alpha \in (0, 1)$.
Let $Y_n = Y_n(\tilde{\Gamma})$ be the supremum of all $\epsilon > 0$ such that

$$0 \in Q[\Gamma_n \cup \mathcal{B}(B^1(T_n^1), n\epsilon) \cup \mathcal{B}(B^2(T_n^2), n\epsilon)].$$

Note that $\{Y_n > 0\} = D_n$. In the proofs we will use a standard estimate about Brownian motion. Let

$$A_\theta = \{z \in \mathbb{C} : \frac{-\theta}{2} < \operatorname{Arg}(z) < \frac{\theta}{2}\}.$$

Then if $x = 1, 0 < \theta \le \pi$,

$$\mathbb{P}^x\{B^1[0, T^1_n] \subset A_\theta\} \asymp n^{-\pi/2\theta}. \tag{7}$$

The $\theta = \pi/2$ result is the standard "gambler's ruin" estimate for Brownian motion, and the estimate for other θ can be derived by conformal mapping.

Lemma 3.1 *There exists a constant $c_1 > 0$ such that if $\tilde{\Gamma}$ is any initial configuration with $Y_1 > 0$, then*

$$\mathbb{P}\{Y_2 \ge \frac{1}{4}\} \ge c_1 (Y_1)^4.$$

Proof: If $Y_1 > 1/10$ then it is easy to prove the lemma by direct construction of an event so we assume $Y_1 = r \le 1/10$. Without loss of generality we will assume that $0 = \Theta^1(1) \le \Theta^2(1) = \theta \le \pi$. Consider the events

$$U^1 = \{B^1[0, T^1_2] \subset \mathcal{B}(x^1, r/4) \cup \{z : |z| > 1, \mathrm{Arg}(z) \in (-\pi/4, -r/16)\};$$
$$\mathrm{Arg}(B^1(t)) \in (-\pi/4, -\pi/8),\ T^1_{3/2} \le t \le T^1_2\},$$
$$U^2 = \{B^2[0, T^2_2] \subset \mathcal{B}(x^2, r/4) \cup \{z : |z| > 1, \mathrm{Arg}(z) \in (\theta + (r/16), \theta + (\pi/4))\};$$
$$\mathrm{Arg}(B^2(t)) \in (\theta + (\pi/8), \theta + (\pi/4)),\ T^2_{3/2} \le t \le T^1_2\}.$$

By standard estimates (using conformal mapping) it can be shown that

$$\mathbb{P}\{U^j\} \ge cr^2.$$

It is easy to see that $U^1 \cap U^2 \subset \{Y_2 \ge 1/4\}$, and hence we get the result. ∎

The next lemma is a very important technical lemma. It states in a uniform way that two Brownian motions conditioned not to disconnect the origin from infinity have a reasonable chance of being "not too close to disconnecting." Let $\mathcal{F}_s$ be the σ-algebra generated by

$$\{B^j(t) : 0 \le t \le T^j_s, j = 1, 2\}.$$

Lemma 3.2 *There exists a constant $c_1 > 0$ such that if $\tilde{\Gamma}$ is any initial configuration with $Y_1 > 0$,*

$$\mathbb{P}\{Y_2 \ge \frac{1}{4}\} \ge c_1 \mathbb{P}\{D_2\}.$$

Proof: For every $3/2 \le \rho \le 2$, let $V(\rho)$ be the event

$$V(\rho) = \{Y_r \ge \frac{1}{4}, \rho \le r \le 2\}.$$

For any $\epsilon > 0$, it is easy to see by direct construction of an event (see, e.g., the proof of Lemma 3.1) that there is a $u_\epsilon > 0$ such that for any initial configuration with $Y_1 \ge \epsilon$,

$$\mathbb{P}\{V(\frac{3}{2})\} \ge u_\epsilon.$$

Choose integer N sufficiently large so that

$$\exp[\sum_{n=N}^{\infty} n^2 2^{-n}] \le \frac{5}{4}.$$

Let $h_N = 3/2$ and for $n > N$, let

$$h_n = h_{n-1} \exp\{n^2 2^{-n}\},$$

so that $h_n \leq 15/8$ for all n. Let

$$r(n) = \inf \frac{\mathbb{P}\{V(h_n)\}}{\mathbb{P}\{D(h_n)\}},$$

where the infimum is over all initial configurations with $Y_1 \geq 2^{-n}$. By the comment above, $r(n) > 0$ for each fixed n. We will show below that there is a $c_2 > 0$ such that for all $n > N$,

$$r(n) \geq (1 - \frac{c_2}{n^2}) r(n-1), \tag{8}$$

and hence there exists a $c_1 > 0$ such that $r(n) \geq c_1$ for all n. This clearly gives the lemma.
By the strong Markov property, Brownian scaling, and the definition of $r(n)$, we can see that if $n > N$, $1 \leq s \leq \exp\{n^2 2^{-n}\}$ and $A \in \mathcal{F}_s$ with

$$A \subset \{Y_s \geq 2^{-(n-1)}\},$$

then

$$\mathbb{P}\{A \cap V(sh_{n-1})\} \geq r(n-1) \mathbb{P}\{A \cap D(sh_{n-1})\}.$$

Hence

$$\mathbb{P}\{A \cap V(h_n)\} \geq r(n-1) \mathbb{P}\{A \cap D(h_n)\}. \tag{9}$$

Choose $n > N$, and let $\tilde{\Gamma}$ be an initial configuration with $Y_1 \geq 2^{-n}$. By Lemma 3.1,

$$\mathbb{P}\{D_2\} \geq c 2^{-4n}. \tag{10}$$

Let $s_j = s_{j,n} = \exp\{j 2^{-n}\}$ and let

$$\sigma = \sigma_n = \inf\{j : Y_{s_j} \geq 2^{-(n-1)}\}.$$

By (9), if $j < n^2$,

$$\mathbb{P}\{V(h_n) \cap \{\sigma = j\}\} \geq r(n-1) \mathbb{P}\{D(h_n) \cap \{\sigma = j\}\},$$

and hence

$$\mathbb{P}\{V(h_n) \cap \{\sigma < n^2\}\} \geq r(n-1) \mathbb{P}\{D(h_n) \cap \{\sigma < n^2\}\}.$$

If we can show that

$$\mathbb{P}\{D(h_n) \cap \{\sigma < n^2\}\} \geq (1 - \frac{c_2}{n^2}) \mathbb{P}\{D(h_n)\}, \tag{11}$$

we will have derived (8).
Let $\beta > 0$ be the probability that a Brownian motion starting at the origin, stopped upon reaching the sphere of radius 2, disconnects the ball of radius 1 from the sphere of radius 2. Then, for n sufficiently large, we can see that if $\tilde{\Gamma}$ is any initial configuration with $Y_1 \leq 2^{-(n-1)}$,

$$\mathbb{P}\{Y_{s_5} = 0\} \geq \beta^2.$$

Likewise, by scaling we can see that for all $j < n^2$,

$$\mathbb{P}\{Y_{s_{j+5}} = 0 \mid Y_{s_j} \leq 2^{-(n-1)}\} \geq \beta^2.$$

Iterating this, we can conclude that

$$\mathbb{P}\{\sigma \geq n^2; Y_{s_{n^2}} \neq 0\} \leq c_1 e^{-c_2 n^2}.$$

But,

$$\mathbb{P}\{D(h_n) \cap \{\sigma \geq n^2\}\} \leq \mathbb{P}\{\sigma \geq n^2; Y_{s_{n^2}} \neq 0\}.$$

Combining this with (10), we can conclude (11). ∎

By applying this lemma to the configuration Γ_{n-1}, we see (using Brownian scaling) that for $n \geq 2$, and any initial configuration,

$$\mathbb{P}\{Y_n \geq \frac{1}{4} \mid \mathcal{F}_{n-1}\} \geq c\mathbb{P}\{D_n \mid \mathcal{F}_{n-1}\}, \tag{12}$$

and hence

$$\mathbb{P}\{Y_n \geq \frac{1}{4} \mid D_n\} \geq c. \tag{13}$$

Since paths with $Y_n \geq 1/4$ can be extended with positive probabilty to distance $2n$ without producing a disconnection, we get the following corollary.

Corollary 3.3 *There exists a $c > 0$ such that*

$$q(2n) \geq cq(n).$$

For the remainder of this paper, we will not need to consider any initial configurations. Let B^1, B^2 be independent Brownian motions starting on the unit circle. For any t_1, t_2 (perhaps random), let $J(t_1, t_2)$ denote the set of θ such that there exists a continuous curve $\gamma : [0, \infty) \to \mathbb{C}$ satisfying:

$$\gamma(0) = 0, \quad \gamma(1) = e^{i\theta}, \quad |\gamma(t)| \to \infty, \ t \to \infty$$

$$|\gamma(t)| > 1, \quad 1 < t < \infty,$$

$$\gamma[0, \infty) \cap (B^1[0, t_1] \cup B^2[0, t_2]) = \emptyset.$$

We consider $J(t_1, t_2)$ as a subset of the unit circle. It is easy to verify that $J(t_1, t_2)$ is one of: empty, an open interval of the unit circle, or the union of two disjoint open invervals. Moreover, if $J(t_1, t_2) = \emptyset$, then $J(s_1, s_2) = \emptyset$ for all $s_1 \geq t_1, s_2 \geq t_2$. Let $X_n = l[J(T_n^1, T_n^2)]$ where l denotes Lebesgue measure on the unit circle. Note that $D_n = \{X_n > 0\}$. It seems intuitively clear that if D_n holds then there should be a reasonable chance that $J(T_n^1, T_n^2)$ is not very small. The next lemma gives a rigorous statement of this.

Lemma 3.4 *There exists a $\delta > 0$ such that for all $n \geq 1$,*

$$\sup \mathbb{P}^{x^1, x^2}\{X_n > \delta\} \geq \frac{1}{2} q(n),$$

where the supremum is over all $|x^1| = |x^2| = 1$.

Proof: It suffices to prove the result for $n > 2$. For any $\delta > 0$, let $V_\delta(t_1, t_2)$ be the event that $J(t_1, t_2)$ contains an open interval (a, b) with $a, b \notin J(t_1, t_2)$ and $0 < |b - a| \leq \delta$. We will show that for δ sufficiently small and $|x^1| = |x^2| = 1$,

$$\mathbb{P}^{x^1,x^2}\{V_\delta(T_n^1, T_n^2)\} \leq \frac{1}{2}q(n),$$

and hence for some $|x^1| = |x^2| = 1$,

$$\mathbb{P}^{x^1,x^2}\{X_n > \delta\} \geq \frac{1}{2}q(n).$$

Let $\tau_0^1 = \tau_0^2 = \sigma_1^1 = \sigma_1^2 = 0$ and define inductively

$$\tau_k^j = \inf\{t > \sigma_k^j : |B_t^j| = 2\},$$

$$\sigma_{k+1}^j = \inf\{t > \tau_k^j : |B_t^j| = 1\}.$$

First consider the event $V_\delta(\tau_1^1, 0)$. Let ρ be the first time such that $V_\delta(\rho, 0)$ holds. Then it is easy to check that $B^1(\rho)$ is on the unit circle and is an endpoint of an interval of length at most δ contained in $J(\rho, 0)$. Let u_ϵ denote the probability that a Brownian motion starting at the origin does not make a closed loop about $\mathcal{B}(0, \epsilon)$ before reaching distance 1. Note that $u_\epsilon \to 0$ as $\epsilon \to 0$. If $\rho < \tau_1^1$, and we start the Brownian motion at $B^1(\rho)$, the probability that the interval of size δ is not surrounded by time τ_1^1 is bounded above by u_δ. If it is surrounded, there is at most one interval in $J(\tau_1^1, 0)$. We can do the same argument on this interval if its length ever becomes smaller than δ. We conclude

$$\mathbb{P}^{x^1,x^2}\{V_\delta(\tau_1^1, 0)\} \leq 2u_\delta.$$

Similarly,

$$\mathbb{P}^{x^1,x^2}\{V_\delta(0, \tau_1^2)\} \leq 2u_\delta.$$

Let

$$W(k, m) = V_\delta(\tau_k^1, \tau_m^2) \setminus [V_\delta(\tau_{k-1}^1, \tau_m^2) \cup V_\delta(\tau_k^1, \tau_{m-1}^2)],$$

where for convenience we define $V_\delta(\tau_k^1, \tau_m^2)$ to be the empty set if $k = m = 0$ or if either k or m is negative. Then if $k > 0$, $m \geq 0$,

$$\mathbb{P}^{x^1,x^2}\{W(k, m)\} \leq \mathbb{P}^{x^1,x^2}\{J(\tau_{k-1}^1, \tau_m^2) \neq \emptyset\}\mathbb{P}^{x^1,x^2}\{V_\delta(\tau_k^1, \tau_m^2) \mid V_\delta(\tau_{k-1}^1, \tau_m^2)^c; J(\tau_{k-1}^1, \tau_m^2) \neq \emptyset\},$$

with a similar expression if $k \geq 0, m > 0$. The argument in the previous paragraph can be repeated to show that

$$\mathbb{P}^{x^1,x^2}\{V_\delta(\tau_k^1, \tau_m^2) \mid V_\delta(\tau_{k-1}^1, \tau_m^2)^c; J(\tau_{k-1}^1, \tau_m^2) \neq \emptyset\} \leq 2u_\delta.$$

Let v be the probability that a Brownian motion starting distance 1 from the origin makes a closed loop about $\mathcal{B}(0, 1)$ before reaching the circle of radius 2. It is easy to see that $v > 0$, and by the strong Markov property we can easily see that

$$\mathbb{P}^{x^1,x^2}\{J(\tau_{k-1}^1, \tau_m^2) \neq \emptyset\} \leq (1 - v)^{k+m-1}.$$

Hence

$$\mathbb{P}^{x^1,x^2}\{W(k, m)\} \leq 2(1 - v)^{k+m-1}u_\delta.$$

By the strong Markov property, we get (for $n > 2$),

$$\begin{aligned}
&\mathbb{P}^{x^1,x^2}\{W(k,m); T_n^1 > \tau_k^1; T_n^2 > \tau_m^2; V_\delta(T_n^1,T_n^2)\} \\
&\qquad \leq \mathbb{P}^{x^1,x^2}\{W(k,m)\}\mathbb{P}^{x^1,x^2}\{D_n \mid W(k,m); T_n^1 > \tau_k^1; T_n^2 > \tau_m^2\} \\
&\qquad \leq cu_\delta(1-v)^{k+m-1}q(n/2) \\
&\qquad \leq cu_\delta(1-v)^{k+m}q(n).
\end{aligned}$$

Hence

$$\mathbb{P}^{x^1,x^2}\left\{\bigcup_{k=0}^{\infty}\bigcup_{m=0}^{\infty}[V_\delta(T_n^1,T_n^2)\cap W(k,m)]\right\} \leq cu_\delta q(n) \leq \frac{1}{2}q(n),$$

if δ is chosen sufficiently small. But it easy to verify that

$$V_\delta(T_n^1,T_n^2) = \bigcup_{k=0}^{\infty}\bigcup_{m=0}^{\infty}[V_\delta(T_n^1,T_n^2)\cap W(k,m)]. \quad \blacksquare$$

Lemma 3.5 *Let δ be as in Lemma 3.4. Let $V_n^j = V_n^j(\epsilon)$ be the event*

$$V_n^j(\epsilon) = \{B^j[0,T_n^j]\cap\mathcal{B}(0,1)\subset\mathcal{B}(B^j(0),\epsilon)\}.$$

Then there exists a $c > 0$ such that for each $n > 1, \epsilon > 0$

$$\sup\mathbb{P}^{x^1,x^2}\{V_n^1\cap V_n^2; X_n > \delta\} \geq c\epsilon^2 q(n),$$

where the supremum is over all $|x^1| = |x^2| = 1$.

Proof: We will first show that there exists a c such that for each n, there exist $|x^1| = |x^2| = 1$ with

$$\mathbb{P}^{x^1,x^2}\{V_n^1; X_n \geq \delta\} \geq c\epsilon q(n).$$

Choose x^1, x^2 (depending on n) that maximize

$$p(n) = \mathbb{P}^{x^1,x^2}\{X_n \geq \delta\}.$$

By Lemma 3.4, we know that $p(n) \geq q(n)/2$. Let

$$\rho = \rho(\epsilon) = \inf\{t : B^1(t)\in\mathcal{B}(0,1)\setminus\mathcal{B}(B^1(0),\epsilon)\},$$

$$\sigma = \sigma(\epsilon) = \inf\{t : |B^1(t)| = 1-\epsilon\},$$

$$\tau = \tau(\epsilon) = \inf\{t \geq \rho : |B^1(t)| = 1\},$$

It is easy to check that there is a $u > 0$ (independent of ϵ) such that

$$\mathbb{P}\{\sigma < \tau \mid \rho < T_n^1\} \geq u.$$

If a Brownian motion is at distance $1-\epsilon$, there is a probability of at least $c\epsilon$ that it will hit the circle of radius $1/2$ before hitting the circle of radius 1. Starting on the ball of radius $1/2$ there is a positive probability that the Brownian motion will make a closed loop about the disc of radius $1/2$ before leaving the circle of radius 1. From all this we see that the probability that

$B^1[0,\tau]$ disconnects 0 from infinity given that $\rho < T_n^1$ is greater than $c\epsilon$. If $B^1[0,\tau]$ does not disconnect 0 from infinity, the strong Markov property says that the probability that $X_n \geq \delta$ is at most $p(n)$. Therefore

$$\mathbb{P}^{x^1,x^2}\{(V_n^1)^c; X_n \geq \delta\} \leq (1-c\epsilon)p(n),$$

and hence

$$\mathbb{P}^{x^1,x^2}\{V_n^1; X_n \geq \delta\} \geq c\epsilon p(n) \geq c\epsilon q(n).$$

Now choose x^1, x^2 that maximize $\mathbb{P}^{x^1,x^2}\{V_n^1; X_n \geq \delta\}$ and do the same argument on B^2. ∎

Let δ be as in Lemma 3.4 and fix $\epsilon = \delta/20$. For any $\lambda > 0, |x| = 1$, let $W^j(\lambda, x)$ be the event that $B^j[0, T_2^j]$ disconnects $\mathcal{B}(x,\lambda)$ from the sphere of radius 2. Let

$$p(\lambda) = \mathbb{P}^{x^1,x^2}\{W^1(\lambda, x^1) \cap W^2(\lambda, x^2)\},$$

where $|x^1| = |x^2| = 1$. Note that the probability is indepedent of the x^1, x^2 chosen and $p(\lambda) \to 1$ as $\lambda \to 0$. By the strong Markov property and Corollary 3.3,

$$\mathbb{P}^{x^1,x^2}\{D_n \cap [W^1(\lambda, x^1) \cap W^2(\lambda, x^2)]^c\} \leq c(1-p(\lambda))q(n).$$

Since

$$V_n^1 \cap V_n^2 \cap \{X_n > \delta\} \subset D_n,$$

by choosing λ sufficiently small we can conclude the following from Lemma 3.5.

Corollary 3.6 *Let δ, ϵ be as above and $V_n^j = V_n^j(\epsilon)$ as in Lemma 3.5. Let $W^j(x) = W^j(\lambda, x)$ be as above. There exist $\lambda > 0$ and $c > 0$ such that*

$$\sup \mathbb{P}^{x^1,x^2}\{V_n^1 \cap V_n^2 \cap W^1(x^1) \cap W^2(x^2); X_n \geq \delta\} \geq cq(n),$$

where the supremum is over all $|x^1| = |x^2| = 1$.

We now fix a λ that satisfies Corollary 3.6. Suppose we start the Brownian motions at y^j with $|y^j - x^j| \leq \lambda/2$. To determine whether or not the event

$$V_n^1 \cap V_n^2 \cap W^1(x^1) \cap W^2(x^2) \cap \{X_n \geq \delta\}$$

occurs, we need only view the paths after the first time they hit the sphere of radius λ about x^j. Hence, using either the exact form of the Poisson kernel or the Harnack inequality for harmonic functions, we can see for any $|y^1| = |y^2| = 1$ with $|y^j - x^j| \leq \lambda/2$,

$$\mathbb{P}^{y^1,y^2}\{V_n^1 \cap V_n^2 \cap W^1(x^1) \cap W^2(x^2); X_n > \delta\} \geq c\mathbb{P}^{x^1,x^2}\{V_n^1 \cap V_n^2 \cap W^1(x^1) \cap W^2(x^2); X_n \geq \delta\}.$$

Hence we can conclude the following.

Corollary 3.7 *Let δ, ϵ be as above and $V_n^j = V_n^j(\epsilon)$ as in Lemma 3.5. Let $W^j(x^j)$ be as above. There exist $\lambda > 0$ and $c > 0$ such that*

$$\sup_{x^1,x^2} \inf_{y^1,y^2} \mathbb{P}^{y^1,y^2}\{V_n^1 \cap V_n^2 \cap W^1(x^1) \cap W^2(x^2); X_n \geq \delta\} \geq cq(n),$$

where the supremum is over all $|x^1| = |x^2| = 1$ and the infimum is over all $|y^1| = |y^2| = 1$ with $|x^i - y^i| \leq \lambda/2$.

Proposition 3.8 *There exists a constant $c > 0$ such that for all $n, m \geq 1$,*

$$q(nm) \geq cq(n)q(m).$$

Proof: Let $\epsilon, \delta, \lambda$ be as in Corollary 3.7. Let $A = \{e^{i\theta} : -\delta/4 < \theta < \delta/4\}$. It follows from Corollary 3.7 and the rotational invariance of Brownian motion that for every m, there exist $|x^1| = |x^2| = 1$ and $c > 0$ such that for every $|y^1| = |y^2| = 1$ with $|y^j - x^j| \leq \lambda/2$,

$$\mathbb{P}^{y^1,y^2}\{V_m^1 \cap V_m^2 \cap W^1(x^1) \cap W^2(x^2); A \subset J(T_m^1, T_m^2)\} \geq cq(m).$$

Let $x^1 = e^{i\theta_1}, x^2 = e^{i\theta_2}$. Clearly, $x^1, x^2 \notin A$.
Let $W_n = W_n(x^1, x^2, \epsilon, \delta, \lambda)$ be the event

$$W_n = \{|B^j(T_n^j) - nx^j| \leq \frac{\lambda n}{2}, j = 1, 2;$$

$$0 \text{ is } \epsilon - \text{connected to } nA \text{ in } \mathcal{B}(0,n) \setminus (B^1[0,T_n^1] \cup B^2[0,T_n^2])\}.$$

To say that 0 is ϵ-connected to nA in $\mathcal{B}(0,n) \setminus (B^1[0,T_n^1] \cup B^2[0,T_n^2])$ means that there is a continuous function $\gamma : [0,1] \to \mathbb{C}$ satisfying $\gamma(0) = 0; \gamma(1) \in nA; |\gamma(t)| < n, t < 1$; and

$$\gamma(0,1) \cap [B^1[0,T_n^1] \cup B^2[0,T_n^2] \cup \mathcal{B}(B^1(T_n^1), n\epsilon) \cup \mathcal{B}(B^2(T_n^2), n\epsilon)] = \emptyset.$$

If we can show that there is a c such that for all n and all $x^1, x^2 \notin A$,

$$\sup_{|z^1|=|z^2|=1} \mathbb{P}^{z^1,z^2}\{D_n \cap W_n\} \geq cq(n), \tag{14}$$

then by the strong Markov property and Corollary 3.7,

$$q(mn) \geq cq(n)q(m).$$

But (14) can be derived easily from (12) by attaching an appropriate event of positive probability. We omit the details. ∎

By considering the superadditive function

$$f(k) = \log q(2^k) + \log c,$$

it follows from standard arguments that there exist c_1, c_2 such that for all n

$$n^{-2\alpha} \leq q(n) \leq c_2 n^{-2\alpha}.$$

In fact, we can make an improvement on this. If we start the Brownian motions at any points x^1, x^2 on the unit circle, then with at least some positive probability (independent of the starting points), $Y_2 \geq 1/4$. Hence we can then connect as in the proof of Proposition 3.8 to a configuration that does not produce a disconnection to conclude that

$$\mathbb{P}^{x^1,x^2}\{Y_n > 0\} \geq cn^{-2\alpha}.$$

By using (12), we can see that

$$\mathbb{P}^{x^1,x^2}\{Y_n > \frac{1}{4} \mid Y_n > 0\} \geq c.$$

We therefore get the following corollary.

Corollary 3.9 *There exist c_1, c_2 such that for all $|x^1| = |x^2| = 1$,*

$$c_1 n^{-2\alpha} \leq \mathbb{P}^{x^1,x^2}\{Y_n \geq \frac{1}{4}\} \leq \mathbb{P}^{x^1,x^2}\{Y_n > 0\} \leq c_2 n^{-2\alpha}.$$

Let $D^n = D_{2^n}$ and let W^n be the event that the Brownian paths reach distance 2^n without hitting the negative real axis,

$$W^n = \{(B^1[0, T^1_{2^n}] \cup B^2[0, T^2_{2^n}]) \cap \{z \in \mathbb{C} : z < 0\} = \emptyset\}.$$

A standard estimate gives that for $x^1 = x^2 = 1$,

$$\mathbb{P}^{x^1,x^2}\{W^n\} \geq c(2^n)^{-1}.$$

However ideas similar to the one above (i.e., using Lemma 3.2 and attaching appropriate events of positive probability) can be used to show that for $j < n - 2$,

$$\mathbb{P}^{x^1,x^2}\{D^n \cap (W^{j+2})^c \mid W^j\} \geq c_1(2^{n-j})^{-2\alpha},$$

$$\mathbb{P}^{x^1,x^2}\{D^n \mid W^j\} \leq c_2(2^{n-j})^{-2\alpha}.$$

Let λ be the first j such that W^j does not hold. Then the above inequalities imply that there is a $u > 0$ such that

$$\mathbb{P}^{x^1,x^2}\{\lambda \leq j + 2 \mid D^n, \lambda > j\} \geq c.$$

By iterating, we see that there is a $\beta > 0$ such that

$$\mathbb{P}^{x^1,x^2}\{W^n \mid D^n\} = \mathbb{P}^{x^1,x^2}\{\lambda > n \mid D^n\} \leq (2^n)^{-\beta}.$$

Since $W^n \subset D^n$ and $\mathbb{P}\{W_n\} \geq c2^{-n}$,

$$\mathbb{P}^{x^1,x^2}\{D_n\} \geq (2^n)^{-1+\beta}.$$

We have derived the following. As remarked previously, Werner [9, 10] has given a stronger bound.

Proposition 3.10 *If α is the two-sided disconnection exponent, then $\alpha < 1/2$.*

We need to consider a slightly different event. Let $\hat{D}_n$ be the event

$$\hat{D}_n = \{\bar{Q}(B^1[0, T^1_n] \cup B^2[0, T^2_n]) \cap \mathcal{B}(0, 1) \neq \emptyset\},$$

where $\bar{Q}$ is as defined in the previous section. Note that $D_n \subset \hat{D}_n$.

Lemma 3.11 *There exists a c such that for any $|x^1| = |x^2| = 1$,*

$$\mathbb{P}^{x^1,x^2}\{\hat{D}_n\} \leq cn^{-2\alpha}.$$

Proof: Let $\tilde{D}_n$ be the event

$$\tilde{D}_n = \{\bar{Q}(B^1[0, T^1_n] \cup B^2[0, T^2_n]) \cap \mathcal{B}(0, \frac{1}{2}) \neq \emptyset\}.$$

We will show that $\mathbb{P}^{x^1,x^2}\{\tilde{D}_n\} \leq cn^{-2\alpha}$. To get the lemma from this result, note that

$$\begin{aligned}\mathbb{P}^{x^1,x^2}\{\hat{D}_n\} &\leq \mathbb{P}^{x^1,x^2}\{\bar{Q}(B^1[T_2^1,T_n^1] \cup B^2[T_2^2,T_n^2]) \cap \mathcal{B}(0,1) \neq \emptyset\} \\ &\leq \sup_{|y^1|=|y^2|=1} \mathbb{P}^{y^1,y^2}\{\tilde{D}_{n/2}\}.\end{aligned}$$

Define sequences of random times as follows. Let $\sigma_0^j = 0$ and for $k > 0$,

$$\tau_k^j = \inf\{t > \sigma_{k-1}^j : |B^j(t)| = \frac{1}{2}\},$$

$$\sigma_k^j = \inf\{t > \tau_k^j : |B^j(t)| = 1\}.$$

Let

$$\rho^j = \rho^j(n) = \sup\{k : \sigma_k^j < T_n^j\}.$$

Let $\tilde{D}_n(k,m) = \tilde{D}_n \cap \{\rho^1 = k, \rho^2 = m\}$. Every time a Brownian motion reaches the circle of radius $1/2$ there is a positive probability, say p, that the Brownian motion will make a closed loop around $\mathcal{B}(0,1/2)$ before reaching distance 1. Hence the probability that $B^1[0,\sigma_k^1] \cup B^2[0,\sigma_m^2]$ does not disconnect $\mathcal{B}(0,1/2)$ from infinity is bounded by $(1-p)^{k+m}$. The probability that

$$B^j[0,T_n^j] \cap \mathcal{B}(0,\frac{1}{2}) = \emptyset, \quad j = 1,2,$$

and

$$\mathcal{B}(0,\frac{1}{2}) \cap \bar{Q}(B^1[0,T_n^1] \cap B^2[0,T_n^1]) \neq \emptyset,$$

is the same as the probability that

$$B^j[0,T_n^j] \cap \mathcal{B}(0,\frac{1}{2}) = \emptyset, \ \ j = 1,2,$$

and

$$0 \in \bar{Q}(B^1[0,T_n^1] \cap B^2[0,T_n^2]).$$

The latter probability is clearly bounded by $q(n) \leq cn^{-2\alpha}$. Hence

$$\mathbb{P}^{x^1,x^2}\{\tilde{D}_n(k,m)\} \leq c(1-p)^{k+m}n^{-2\alpha}.$$

If we sum over all values of k, m, we get the result. ∎

We will need the results for fixed times. Let F_n be the event

$$F_n = \{0 \in Q(B^1[0,n] \cup B^2[0,n])\},$$

and let G_n be the event

$$G_n = \{0 \in Q(B^1[0,T_n^1 \wedge n^2] \cup B^2[0,T_n^2 \wedge n^2])\}.$$

Let $\hat{F}_n$ and $\hat{G}_n$ be the corresponding events,

$$\hat{F}_n = \{\mathcal{B}(0,1) \cap \bar{Q}(B^1[0,n] \cup B^2[0,n]) \neq \emptyset\},$$

$$\hat{G}_n = \{\mathcal{B}(0,1) \cap \bar{Q}(B^1[0,T_n^1 \wedge n^2] \cup B^2[0,T_n^2 \wedge n^2]) \neq \emptyset\}.$$

The proofs of the following lemmas are identical to proofs in [6] and hence we omit them, only referring to the appropriate lemma in that paper.

Corollary 3.12 *[6, Lemma 3.14] There exist $u > 0$ and $c < \infty$ such that for all n, and all $|x^1| = |x^2| = 1$, $a > 0$,*

$$\mathbb{P}^{x^1,x^2}\{\hat{D}_n; \min(T_n^1, T_n^2) \le an^2\} \le cn^{-2\alpha} e^{-u/a},$$

$$\mathbb{P}^{x^1,x^2}\{\hat{D}_n; \max(T_n^1, T_n^2) \ge an^2\} \le cn^{-2\alpha} e^{-ua}.$$

Corollary 3.13 *[6, Lemma 3.15] There exist $c_1, c_2 > 0$ such that for all $|x^1| = |x^2| = 1$,*

$$c_1 n^{-2\alpha} \le \mathbb{P}^{x^1,x^2}\{F_{n^2}\} \le \mathbb{P}^{x^1,x^2}\{G_n\} \le c_2 n^{-2\alpha},$$

$$c_1 n^{-2\alpha} \le \mathbb{P}^{x^1,x^2}\{\hat{F}_{n^2}\} \le \mathbb{P}^{x^1,x^2}\{\hat{G}_n\} \le c_2 n^{-2\alpha}.$$

The following can easily be derived from the corollaries. Assume that $B^1(0) = B^2(0) = 0$ and let

$$H_n = \{0 \in Q(B^1[1,n] \cup B^2[1,n])\},$$
$$U_n = \{0 \in Q(B^1[1, T_n^1 \wedge n^2] \cup B^2[1, T_n^2 \wedge n^2])\},$$
$$\hat{H}_n = \{(B^1[0,1] \cup B^2[0,1]) \cap \bar{Q}(B^1[1,n] \cup B^2[1,n]) \ne \emptyset\},$$
$$\hat{U}_n = \{(B^1[0,1] \cup B^2[0,1]) \cap \bar{Q}(B^1[1, T_n^1 \wedge n^2] \cup B^2[1, T_n^2 \wedge n^2]) \ne \emptyset\}.$$

Corollary 3.14 *There exist $c_1, c_2 > 0$ such that for all $|x^1| = |x^2| = 1$,*

$$c_1 n^{-2\alpha} \le \mathbb{P}^{x^1,x^2}\{H_{n^2}\} \le \mathbb{P}^{x^1,x^2}\{U_n\} \le c_2 n^{-2\alpha},$$

$$c_1 n^{-2\alpha} \le \mathbb{P}^{x^1,x^2}\{\hat{H}_{n^2}\} \le \mathbb{P}^{x^1,x^2}\{\hat{U}_n\} \le c_2 n^{-2\alpha}.$$

We are now is a position to prove Lemma 2.2. By scaling we can restate the lemma as follows. For $0 < j \le n$, let $S_{j,n}$ be the indicator function of the event

$$B([j-1, j]) \cap Q(B[0, j-2] \cup B[j+1, n]) \ne \emptyset.$$

Then the following is equivalent to Lemma 2.2.

Lemma 3.15 *There exists c_1, c_2 such that for $n/4 \le j \le k \le 3n/4$,*

$$c_1 n^{-\alpha} \le \mathbb{E}\{S_{j,n}\} \le c_2 n^{-\alpha}, \tag{15}$$

$$\mathbb{E}\{S_{j,n} S_{k,n}\} \le c_2 n^{-\alpha}(k - j + 1)^{-\alpha}. \tag{16}$$

Proof: The relation (15) follows immediately from Corollary 3.14. To get (16), let

$$m = \min\{\frac{k-j}{3}, \frac{n}{10}\},$$

and let

$$\rho_1 = \inf\{t : |B_j - B_s| \ge \sqrt{m} \text{ for some } s \in [j-t, j+t]\},$$
$$\rho_2 = \inf\{t : |B_k - B_s| \ge \sqrt{m} \text{ for some } s \in [k-t, k+t]\},$$
$$\gamma_i = \min\{\rho_i, m\}.$$

Let

$$U_1 = \{B[j-1,j] \cap Q(B[j-\gamma_1, j-2] \cup B[j+1, j+\gamma_1]) \neq \emptyset\},$$
$$U_2 = \{B[k-1,k] \cap Q(B[k-\gamma_2, j-2] \cup B[k+1, k+\gamma_2]) \neq \emptyset\},$$
$$U_3 = \{B[j-m, k+m] \cap Q(B[0, j-m] \cup B[k+m, n]) \neq \emptyset\}.$$

Note that

$$\mathbb{E}\{S_{j,n} S_{k,n}\} \leq \mathbb{P}\{U_1 \cap U_2 \cap U_3\} = \mathbb{P}\{U_1\}\mathbb{P}\{U_2\}\mathbb{P}\{U_3 \mid U_1 \cap U_2\}.$$

If $(k-j) \geq n/20$, then by Corollary 3.14,

$$\mathbb{P}\{U_1\}\mathbb{P}\{U_2\} \leq cn^{-2\alpha} \leq cn^{-\alpha}(k-j+1)^{-\alpha}.$$

If $(k-j) \leq n/20$, then by Corollary 3.14,

$$\mathbb{P}\{U_1\}\mathbb{P}\{U_2\} \leq c(k-j+1)^{-2\alpha}.$$

In the second case, let

$$Y = \sup\{|B_s - B_t| : j-m \leq s, t \leq k+m\}.$$

Then

$$\mathbb{P}\{U_3 \mid U_1 \cap U_2\} =$$
$$\sum_{a=1}^{\infty} \mathbb{P}\{(a-1)\sqrt{m} \leq Y \leq a\sqrt{m} \mid U_1 \cap U_2\} \mathbb{P}\{U_3 \mid U_1 \cap U_2; (a-1)\sqrt{m} \leq Y \leq a\sqrt{m}\}.$$

The strong Markov property and a standard estimate for normal random variables can be used to find c, u such that

$$\mathbb{P}\{(a-1)\sqrt{m} \leq Y \leq a\sqrt{m} \mid U_1 \cap U_2\} \leq ce^{-au}.$$

Corollary 3.13 and Brownian scaling can be used to conclude

$$\mathbb{P}\{U_3 \mid U_1 \cap U_2; (a-1)\sqrt{m} \leq Y \leq a\sqrt{m}\} \leq c(\frac{n}{a^2 m})^{-\alpha}.$$

Hence by summing over a we can conclude that

$$\mathbb{P}\{U_3 \mid U_1 \cap U_2\} \leq c(\frac{n}{m})^{-\alpha} \leq n^{-\alpha}(k-j+1)^{\alpha}. \quad \blacksquare$$

References

[1] C. Bishop, P. Jones, R. Pemantle and Y. Peres (1995): The dimension of the Brownian frontier is greater than 1, preprint.

[2] K. Burdzy and G. Lawler (1990): Non-intersection exponents for random walk and Brownian motion. Part II; Estimates and applications to a random fractal. *Ann. Probab.* **18**, 981-1009.

[3] K. Burdzy and J. San Martin (1989): Curvature of the convex hull of planar Brownian motion near its minimum point. *Stoch. Proc. Their Appl.* **33**, 89-103.

[4] K. Falconer (1990): *Fractal Geometry: Mathematical Foundations and Applications.* Wiley.

[5] R. Kaufman (1969): Une propriété métrique du mouvement brownien. *C. R. Acad. Sci. Paris ser. A* **268**, 727-728.

[6] G. Lawler (1996): Hausdorff dimension of cut points for Brownian motion. *Electronic J. Probab.* **1**, paper no. 2 .

[7] B. Mandelbrot (1983): *The Fractal Geometry of Nature.* W. H. Freeman.

[8] E. Puckette and W. Werner (1995): Simulations and conjectures for disconnection exponents, (preprint).

[9] W. Werner (1995): An upper bound to the disconnection exponent for two-dimensional Brownian motion. *Bernoulli* **1**, 371-380.

[10] W. Werner (1996): Bounds for Disconnection Exponents. *Electronic Comm. Probab.* **1**, 19-28.