Probability Surveys
Vol. 21 (2024) 1–27
ISSN: 1549-5787
https://doi.org/10.1214/24-PS25

Universality conjectures for activated random walk

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Abstract: Activated Random Walk is a particle system displaying Self-Organized Criticality, in that the dynamics spontaneously drive the system to a critical state. How universal is this critical state? We state many interlocking conjectures aimed at different aspects of this question: scaling limits, microscopic limits, temporal and spatial mixing, incompressibility, and hyperuniformity.

MSC2020 subject classifications: 60K35, 82C22, 82C26, 82B26, 82B27.
Keywords and phrases: Abelian network, activated random walk, hyperuniformity, incompressibility, microscopic limit, quadrature inequality, scaling limit, spatial mixing, stationary distribution, temporal mixing.

Received June 2023.

Contents

arXiv: 2306.01698

1. Introduction

Many complex systems in nature are driven by a steady source of energy which builds up slowly and is released in intermittent bursts. An example is the steady accumulation of pressure between continental plates, which is released in sudden bursts in the form of earthquakes. Wildfires, landslides, avalanches, and financial crises all have this character.

In a famous 1987 paper [4], Bak, Tang and Wiesenfeld proposed both a general mechanism for how such systems arise, and a prototypical example. Their term for the mechanism was **self-organized criticality (SOC)**, and their example was a pile of sand resting on the surface of a table. Given the current slope, ζ, of the pile, we can sprinkle more sand on top and measure two things:

- How much sand falls off the table?
- Does ζ tend to increase or decrease?

In experiments, one finds that the system drives itself to a **critical slope** ζ_c: If the pile is too flat ($\zeta < \zeta_c$), then very little sand falls off the table and ζ increases as sand is added. If the pile is too steep ($\zeta > \zeta_c$), then the sprinkling causes a lot of sand to fall off the table, so that ζ *decreases* as sand is added.

No matter the initial sand profile, the effect of adding more sand is therefore to drive the pile toward its critical slope ζ_c.

This new idea of self-organization to criticality led initially to a lot of excitement (exemplified by Per Bak's ambitiously titled book *How Nature Works*), but its success in making specific testable predictions has been modest so far. We do not know of a "universal model of SOC" in the way that, for instance, Brownian motion is a "universal model of diffusion."

1.1. In search of a universal model of SOC

The most intensively studied model of SOC is called the Abelian Sandpile. In this model, the pile of sand is a collection of indistinguishable particles on the vertices of a fixed graph, for example the d-dimensional cubic lattice $\mathbb{Z}^d$. When a vertex has at least as many particles as the number of neighbors in the graph ($2d$, in the case of $\mathbb{Z}^d$), it *topples* by sending one particle to each neighboring vertex. As a result, some of those neighboring vertices may now have enough particles to topple, enabling some of their neighbors in turn to topple, and so on, thus creating an avalanche. Dhar [12] discovered a beautiful algebraic structure underlying this model.

Abelian networks [3] form a larger class of SOC models. Among these, the Stochastic Sandpile [13], the Oslo model [21], and Activated Random Walk [23, 44] (but not the original Abelian Sandpile!) seem to have some "universality" in the sense that when the system is large, its behavior does not depend much on details like the initial condition, the boundary conditions, or the underlying graph. However, the meaning of "universality" is rarely spelled out. The purpose of this survey is to state precisely several senses in which one of these models, Activated Random Walk, seems to be "universal".

Activated Random Walk (ARW) is a particle system with two species, active particles ($\mathtt{a}$) and sleeping particles ($\mathtt{s}$) that become active if an active particle encounters them ($\mathtt{a} + \mathtt{s} \to 2\mathtt{a}$). Active particles perform random walk at rate 1. When an active particle is alone, it falls asleep ($\mathtt{a} \to \mathtt{s}$) at rate λ. A sleeping particle stays asleep until an active particle steps to its location. The parameter $\lambda > 0$ is called the *sleep rate*. We denote this dynamics by $\mathrm{ARW}(\mathbb{Z}^d, \lambda)$.

To draw out the analogy between ARW and sandpiles: The sleeping particles play the role of sand grains, the movement of the active particles plays the role of toppling, and the awakening of sleeping particles by active particles can trigger an avalanche in which many particles wake up. The density of particles in the system plays the role of the slope of the sandpile. Just as a pile of high slope can easily be destabilized by adding a single sand grain, an ARW configuration with a high density of sleeping particles can easily be destabilized by adding a single active particle.

1.2. Our perspective on "universality"

How can we make accurate predictions about an interacting particle system? "Universality" is an attempt to quantify commonalities among particle systems, so that observations of one system can be transferred to make predictions about other systems. In particular, there are families of particle systems that acquire commonalities as one or more of

$$\{\text{number of particles, spatial extent, length of time evolution}\}$$

becomes large. To state these commonalities precisely, it is often useful to take a limit as the number of particles, or the spatial or temporal system size, tends to infinity. *Existence* of this limit can be a key open question.

Now there are two views on universality:

- From the perspective of the limit: An infinite system is "universal" to the extent that many different finite systems have it as a limit.

But often the limit is only conjectural, so we would like to express universality as a statement about the finite systems themselves!

- From the perspective of the finite system: A finite system is "universal" to the extent that small-scale details have a negligible effect on large-scale observables.

Examples of small-scale details could be the underlying lattice, the boundary conditions, the initial condition at time zero, or the exact rules governing how the particles interact. Examples of large-scale observables are spatial or temporal averages, such as the density of particles in a given macroscopic region. From this perspective, the irrelevance of small details is a *symptom* of the existence of the universal limit, even if we cannot yet prove that the limit exists.

In the specific case of Activated Random Walk, our universality conjectures address the following themes.

- Existence of scaling limits (Conjectures 1, 2, 6, 11, 17, 21) and microscopic limits (Conjectures 3, 7, 13)
- Extra symmetry acquired in the limit (Conjectures 1, 2, 7, Question 14)
- Statistical properties of the limit: incompressibility (Conjecture 20) and hyperuniformity (Conjectures 26, 27, Question 28)
- Temporal mixing: The system quickly forgets its initial condition (Conjectures 9, 19, 24)
- Spatial mixing: Boundary conditions do not affect observables in the bulk (Question 14, Conjecture 15)
- Slow-to-fast phase transition (Question 22, Conjecture 25)
- Commonalities of different experimental conditions (Conjecture 12, Question 14, Conjecture 23)

How did we arrive at all these conjectures, and how are they related? The answer is that some of them are natural extensions of known results, and most of the rest arose from our attempts to pinpoint the main difficulties of proving those. For example, Conjecture 1, on the limiting shape of ARW with a single point source, looks like a minor variant of known results for internal DLA, so it surprised us that it seems hard to prove! In our attempt [39] to prove even a very soft version of this conjecture it became apparent that the main difficulty is ruling out the existence of “dense clumps” in the particle aggregate. This led us to test numerically for dense clumps, and find that not only are macroscopic dense clumps absent, but that particles have a repulsion that spreads them out very evenly at the microscopic scale (cf. Section 9.3). The realization that macroscopic shape depends on microscopic structure (most obviously through density, but also through uniformity, the mechanism of which seems to be microscopic repulsion) led us to conjecture existence of the microscopic limit (Conjecture 3).

A second route for arriving at conjectures was to identify root causes of the failure of universality in the abelian sandpile model, and investigate whether the same causes also operate in ARW. For example, Conjecture 9 fails for the abelian sandpile [17] due to slow mixing [24]. The fact that ARW mixing is known to be somewhat faster [34] and believed to be much faster (Conjecture 19) gives us some confidence that Conjecture 9 holds for ARW.

1.3. Plan of the paper

This survey complements the extensive review by L. Rolla [46], to which we refer the reader for a detailed introduction to ARW.

We discuss ARW dynamics in six settings, differing in the initial condition, underlying graph, or boundary conditions. Our conjectures focus on the approach to criticality, and on shared properties of the corresponding critical states. Sections 2 and 3 consider configurations of finitely many particles on the d-dimensional lattice $\mathbb{Z}^d$. Our conjectures touch on the existence of scaling limits (Conjectures 1, 2, 6), microscopic limits (Conjectures 3, 5, 7), and extra symmetry acquired in the limit.

In Section 4 we discuss infinite (stationary ergodic) particle configurations on $\mathbb{Z}^d$. We conjecture the existence of a microscopic limit as the threshold density is approached from below (Conjecture 9).

In Sections 5, 6, 7 we consider three different Markov chains on ARW configurations on a finite graph. The main themes here are temporal mixing (the system quickly forgets its initial condition: Conjectures 19, 24), spatial mixing (the boundary condition does not affect observables in the bulk: Question 14, Conjecture 15), and a slow-to-fast phase transition (Question 22, Conjecture 25).

Sections 8 and 9 discuss statistical properties of these ARW systems: hyperuniformity (Conjectures 26, 27, Question 28) and site correlations (Tables 1, 2, 3).

We conclude in Section 10 by contrasting the conjectured behavior of ARW with what is known about the Abelian Sandpile model.

2. Point source

2.1. Spherical limit shape

Consider ARW$(\mathbb{Z}^d, \lambda)$ with initial configuration $n\delta_0$, consisting of n active particles at the origin and all other sites empty. After a dynamical phase in which

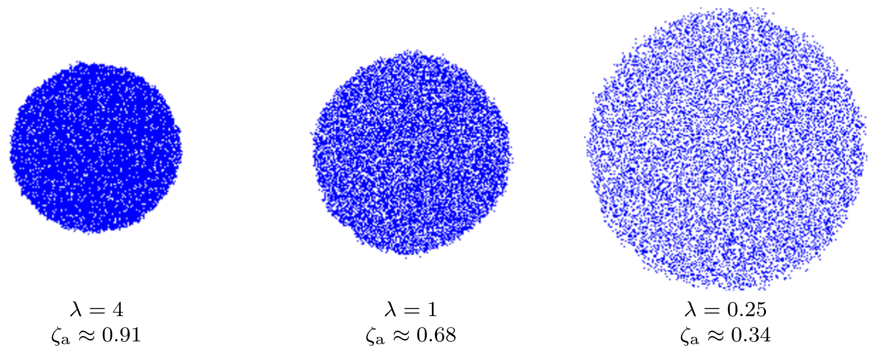


Fig 1. *ARW aggregates formed by stabilizing a point source of* 10000 *active particles at the origin in the square lattice* $\mathbb{Z}^2$*, at three different sleep rates* λ*. Each pixel represents a site of* $\mathbb{Z}^2$*: sites with a sleeping particle are colored blue, and empty sites are colored white. Particles spread farther when the sleep rate is lower, so the aggregate density* ζ_{a} *is an increasing function of* λ*.*

each particle performs random walk, and may fall asleep and be awakened many times, activity will die out when all particles fall asleep at distinct sites. We refer to the final configuration of n sleeping particles as the *ARW aggregate* (Figure 1).

Conjecture 1. (Aggregate density ζ_{a}) *Let A_n denote the random set of sites visited by at least one walker during the dynamical phase of ARW started from n particles at the origin in $\mathbb{Z}^d$.*

There exists a positive constant $\zeta_{\mathrm{a}} = \zeta_{\mathrm{a}}(\mathbb{Z}^d, \lambda)$ such that for any $\epsilon > 0$, with probability tending to 1 as $n \to \infty$, the random set A_n contains all sites of $\mathbb{Z}^d$ that belong to the origin-centered Euclidean ball of volume $(1-\epsilon)n/\zeta_{\mathrm{a}}$; and A_n is contained in the origin-centered Euclidean ball of volume $(1+\epsilon)n/\zeta_{\mathrm{a}}$.

A weak form of Conjecture 1 in dimension 1 is proved in [39]. As the sleep rate λ increases to $+\infty$, Activated Random Walk degenerates to Internal DLA, whose limit shape is proved to be a Euclidean ball [31]. The main barrier to applying Internal DLA methods is proving that sleeping particles are spread uniformly, which is the topic of our next conjecture.

2.2. Macroscopic structure of the aggregate

An *ARW configuration* in $\mathbb{Z}^d$ is a map

$$\eta : \mathbb{Z}^d \to \mathbb{N} \cup \{\mathtt{s}\}$$

where $\eta(x) = \mathtt{s}$ indicates that there is a sleeping particle at $x \in \mathbb{Z}^d$, and $\eta(x) = k$ indicates that there are k active particles at x. We write

$$\mathcal{S} : (\mathbb{N} \cup \{\mathtt{s}\})^{\mathbb{Z}^d} \to \{0, \mathtt{s}\}^{\mathbb{Z}^d}$$

for the operation of *stabilizing* an ARW configuration η: running ARW dynamics until all particles fall asleep[1].

Consider $\mathcal{S}(n\delta_0)$, the ARW aggregate formed by stabilizing n particles at the origin in $\mathbb{Z}^d$. It is a consequence of the Abelian property (cf. [46], Lemma 2.4) that the aggregates $\mathcal{S}(n\delta_0)$ and $\mathcal{S}((n+1)\delta_0)$ can be coupled so that

$$\mathcal{S}((n+1)\delta_0) = \mathcal{S}(\mathcal{S}(n\delta_0) + \delta_0)$$

for any $n \geq 1$. Here $\mathcal{S}(\mathcal{S}(n\delta_0) + \delta_0)$ is the particle configuration obtained by stabilising n particles at the origin, then adding a new particle at the origin and stabilising again. We will rescale the aggregate and take a limit as $n \to \infty$. For $x \in \mathbb{R}^d$ let

$$a_n(x) = 1_{\left\{\mathcal{S}(n\delta_0)(\lfloor n^{1/d}x \rfloor) = \mathtt{s}\right\}}.$$

Write $f_n \xrightarrow{*} f$ for weak-$*$ convergence: $\int f_n \phi \, \mathrm{d}x \to \int f \phi \, \mathrm{d}x$ for all bounded continuous test functions ϕ on $\mathbb{R}^d$, where $\mathrm{d}x$ is Lebesgue measure on $\mathbb{R}^d$.

Conjecture 2. (Uniformity of the aggregate) *The rescaled ARW aggregates a_n satisfy*

$$a_n \xrightarrow{*} \zeta_a \mathbf{1}_B$$

with probability one, where B is the origin-centered ball of volume $1/\zeta_a$ in $\mathbb{R}^d$.

In other words, in the weak-$*$ scaling limit, the random locations of the sleeping particles in the aggregate blur out to a constant density ζ_a everywhere in the ball.

2.3. Microscopic structure of the aggregate

The next conjecture zooms in to the fine scale random structure of the aggregate near the origin (Figure 2).

Write α_n for the law of the aggregate $\mathcal{S}(n\delta_0)$. This is a probability measure on $\{0, \mathtt{s}\}^{\mathbb{Z}^d}$. We examine its marginals [2] on a finite subset of $\mathbb{Z}^d$, as $n \to \infty$.

Conjecture 3. (Microscopic limit of the aggregate) *For all finite $V \subset \mathbb{Z}^d$ and all $\xi \in \{0, \mathtt{s}\}^V$, the sequence $\alpha_n|_V(\xi)$ converges as $n \to \infty$.*

This conjecture would imply, by Kolmogorov's extension theorem, the existence of the infinite-volume limit

$$\alpha := \lim_{V \uparrow \mathbb{Z}^d} \lim_{n \to \infty} \alpha_n|_V,$$

[1] It can be shown using the Abelian property (cf. [46], Lemma 2.4) that $\mathcal{S}(\eta)$ is always defined if η has finitely many particles, but it may be undefined in general. The situation of infinite η is discussed in Section 4.

[2] For a probability measure μ on $\{0, \mathtt{s}\}^{\mathbb{Z}^d}$ and a finite set $V \subseteq \mathbb{Z}^d$, we write $\mu|_V$ for the marginal distribution on $\{0, \mathtt{s}\}^V$, that is $\mu|_V(\xi) := \mu(\{\eta \in \{0, \mathtt{s}\}^{\mathbb{Z}^d} : \eta(v) = \xi(v) \ \forall v \in V\})$, for $\xi \in \{0, \mathtt{s}\}^V$.

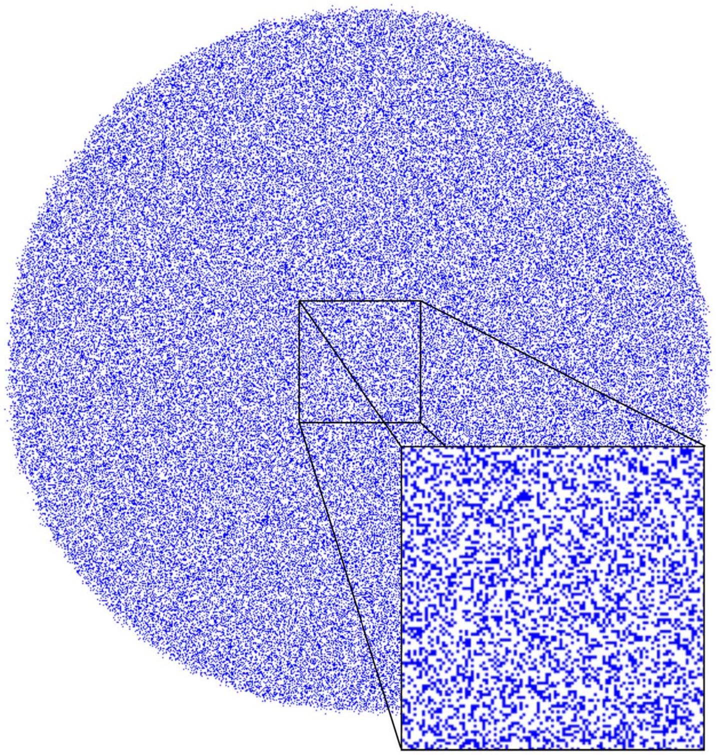


Fig 2. *An ARW aggregate of 10^5 particles in $\mathbb{Z}^2$ at sleep rate $\lambda = 0.25$, with a zoom-in of the microscopic structure deep inside.*

which is a probability measure on the set of infinite stable configurations $\{0, \mathtt{s}\}^{\mathbb{Z}^d}$. The outer limit is over an exhaustion of $\mathbb{Z}^d$, that is, a sequence of finite sets $V_1 \subset V_2 \subset \cdots$ such that $\bigcup_{n \geq 1} V_n = \mathbb{Z}^d$. To spell the limit out: For any finite $V \subset \mathbb{Z}^d$ and any configuration $\xi \in \{0, \mathtt{s}\}^V$,

$$\alpha_n|_V(\xi) \to \alpha|_V(\xi).$$

Note the order of limits: we are restricted to a fixed window V as the size of the aggregate $n \to \infty$. Even though α is supported on configurations with an infinite number of particles, a sample from α is best imagined as a tiny piece of an even larger aggregate!

Conjecture 4. *The limit α is invariant with respect to translations of $\mathbb{Z}^d$.*

To explain why we believe that the origin plays no special role in the conjectural limiting measure α, notice that, by the Abelian property, one could start-off by partially stabilising the $n\delta_0$ configuration to density 1. This fills up a large ball with high probability [31], so locally the origin looks like any other nearby point.

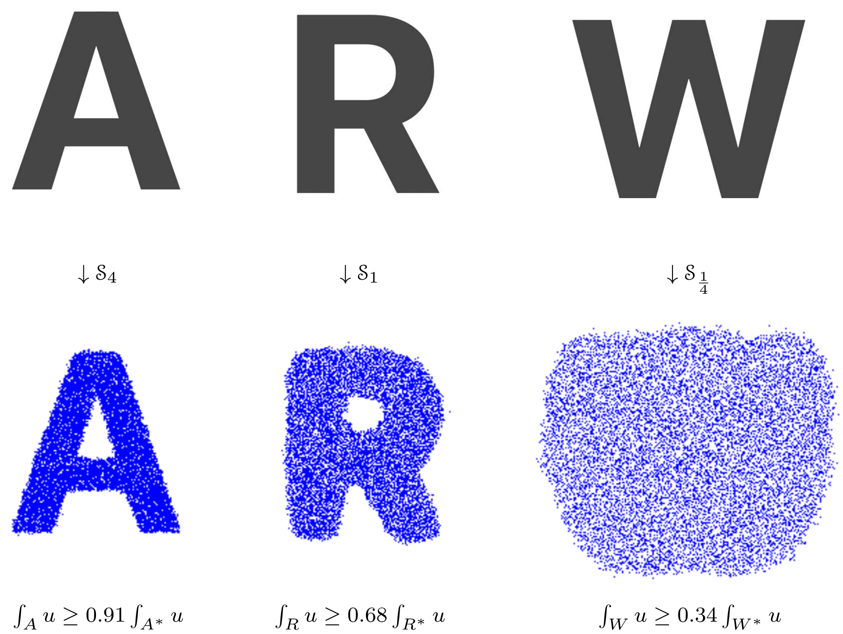


FIG 3. *Top: The capital letters A, R, W viewed as subsets of $\mathbb{R}^2$. Middle: The stabilization of $1_{A\cap\epsilon\mathbb{Z}^2}$ at sleep rate 4, and $1_{R\cap\epsilon\mathbb{Z}^2}$ at sleep rate 1, and $1_{W\cap\epsilon\mathbb{Z}^2}$ at sleep rate $\frac{1}{4}$, with $\epsilon = \frac{1}{500}$. Bottom: According to Conjecture 6, the resulting supports A^*, R^*, W^* are characterized by quadrature inequalities for superharmonic functions. The constant appearing in the inequality is the aggregate density ζ_a ($\approx 0.91, 0.68, 0.34$ for sleep rates $4, 1, \frac{1}{4}$).*

Conjecture 5. *The limit α is supported on configurations of density ζ_a.*

This follows from a much stronger property of the system, namely a form of "repulsion" between close particles. See Section 9.3 for a detailed discussion.

3. Multiple sources

Let $A \subset \mathbb{R}^d$ be a bounded open set satisfying $\int_{\bar{A}\setminus A} \mathrm{d}x = 0$ where $\mathrm{d}x$ denotes d-dimensional Lebesgue measure. For $\epsilon > 0$, let $\mathcal{S}(1_A \cap \epsilon\mathbb{Z}^d)$ be the configuration of sleepers that results from starting one active particle at each point of $A \cap \epsilon\mathbb{Z}^d$ and running activated random walk on $\epsilon\mathbb{Z}^d$ with sleep rate λ. The following conjectured scaling limit for ARW is inspired by Theorem 1.2 of [37], which describes the scaling limit of internal DLA in $\mathbb{Z}^d$.

Conjecture 6. (Quadrature inequality) *As $\epsilon \to 0$,*

$$\mathcal{S}(1_A \cap \epsilon\mathbb{Z}^d) \xrightarrow{*} \zeta_a 1_{A^*}$$

where A^* is the unique (up to measure zero) open subset of $\mathbb{R}^d$ satisfying

$$\int_A u \,\mathrm{d}x \geq \zeta_a \int_{A^*} u \,\mathrm{d}x \tag{3.1}$$

for all integrable superharmonic functions u on A^*.

The intuition behind this conjecture is that the uniform density 1 on A spreads out to uniform density $\zeta_{\mathrm{a}} < 1$ on the larger set A^*. If u is a superharmonic function on A^*, then the sum of the values of u at all particle locations is approximately a supermartingale, leading to (3.1) by optional stopping. In the case of multiple point sources, A^* is a smash sum of Euclidean balls [37, Theorem 1.4].

In general, the existence and uniqueness of A^* is a nontrivial theorem. One proof of existence ([49], see also [9]) is via an obstacle problem: one constructs a suitable function f (the "obstacle") and the pointwise smallest superharmonic function $u \geq f$, then takes $A^* = \{u > f\}$. The set A^* has the following physical interpretation [14]: A viscous fluid is surrounded by an inviscid fluid in the gap between two parallel plates. If A is the region initially occupied by the viscous fluid, and the plates are squeezed together so that the gap between them shrinks by a factor of ζ_a, then A^* is the region occupied by the viscous fluid after squeezing.

The next conjecture examines the microstructure of the aggregate near $\mathbf{0}$.

Conjecture 7. (Microstructure looks the same everywhere) *Assume $\mathbf{0} \in A^*$. For any finite $V \subset \mathbb{Z}^d$, the law of $\mathcal{S}(1_A \cap \epsilon\mathbb{Z}^d)|_{\epsilon V}$ has a limit as $\epsilon \to 0$, in the sense that for any configuration $\eta \in \{0, \mathtt{s}\}^V$*

$$\mathbb{P}\big(\mathcal{S}(1_A \cap \epsilon\mathbb{Z}^d)(\epsilon v) = \eta(v) \text{ for all } v \in V\big) \to \alpha|_V(\eta).$$

The limiting probability measure α on $\{0, \mathtt{s}\}^{\mathbb{Z}^d}$ is the same as in Conjecture 3. In particular, α does not depend on A.

So far we have examined initial conditions with a finite number of particles only. The next section examines infinite configurations.

4. Stationary ergodic

For ARW$(\mathbb{Z}^d, \lambda)$, start with a stationary ergodic configuration $\eta : \mathbb{Z}^d \to \mathbb{N}$, where all particles are initially active. Running ARW dynamics, will all particles fall asleep? If each site of $\mathbb{Z}^d$ is visited only finitely often, then we say that η *stabilizes*. Rolla, Sidoravicius, and Zindy proved the remarkable fact that stabilizing depends only on the mean number of particles per site

$$\zeta := \mathbb{E}(\eta(\mathbf{0})).$$

Theorem 8. (Universality of threshold density ζ_{c}, [45]) *There exists a constant $\zeta_{\mathrm{c}} = \zeta_{\mathrm{c}}(\mathbb{Z}^d, \lambda)$ such that if $\zeta < \zeta_{\mathrm{c}}$ then η stabilizes with probability 1, and if $\zeta > \zeta_{\mathrm{c}}$ then with probability 1, η does not stabilize.*

4.1. Approaching the threshold from below

Theorem 8 ensures that the stabilization $\mathcal{S}(\eta)$ is always defined if $\zeta < \zeta_{\mathrm{c}}$. What happens to the microstructure of $\mathcal{S}(\eta)$ as $\zeta \uparrow \zeta_{\mathrm{c}}$? Start with a stationary ergodic configuration $\eta_0 : \mathbb{Z}^d \to \mathbb{N} \cup \{\mathbf{s}\}$ with mean $\zeta_0 < \zeta_c$, and sprinkle some extra active particles: Letting $(\xi_t(x))_{x\in\mathbb{Z}^d}$ be independent Poisson random variables with mean $t < \zeta_c - \zeta_0$, the configuration $\eta_0 + \xi_t$ stabilizes with probability 1.

Conjecture 9. (Universal limit of subcritical measures) *Fix $\lambda > 0$ and let μ_t be the law of the ARW stabilization of $\eta_0 + \xi_t$ with sleep rate λ. There exists a limiting measure*

$$\mu := \lim_{t\uparrow\zeta_c-\zeta_0} \mu_t$$

supported on configurations $\eta \in \{0,\mathbf{s}\}^{\mathbb{Z}^d}$ of density ζ_c. Moreover, μ depends only on λ and not on the initial configuration η_0.

5. The wired Markov chain

Fix a finite set $V \subset \mathbb{Z}^d$, and consider the particle system $\mathrm{ARW}(V,\lambda)$ in which particles evolve as in ARW with sleep rate λ, with the additional rule that when a particle exits V it is killed (i.e. removed from the system). Fix $v \in V$. The ARW *wired Markov chain* $(w_k)_{k\geq 0}$ on the state space $\{0,\mathbf{s}\}^V$ has the update rule: add one active particle at v and stabilize, i.e.

$$w_{k+1} = \mathcal{S}_V(w_k + \delta_v),$$

where $\mathcal{S}_V$ denotes ARW stabilization with killing of any particles that exit V.

5.1. Stationary distribution

The stationary distribution of the Markov chain $(w_k)_{k\geq 0}$ does not depend on the choice of the site v where particles are added, as by the Abelian property the Markov transition operators for different v commute! The next result gives an efficient way to sample exactly from the stationary distribution of this chain.

Start with the configuration $\mathbf{1}_V$, consisting of one active particle on each site of V, and let the particles perform $\mathrm{ARW}(V,\lambda)$ until no active particles remain. Some particles exit the system, and the remaining particles fall asleep in V. Denote by $\mathcal{S}_V(\mathbf{1}_V)$ the resulting random configuration of sleepers.

Proposition 10 (Exact sampling, [34])**.** *The law of $\mathcal{S}_V(\mathbf{1}_V)$ is the unique stationary distribution of the ARW wired Markov chain on V with sleep rate λ.*

For any given set $V \subset \mathbb{Z}^d$ let ∂V denote its boundary and $\#V$ denote its cardinality. Write $|\mathcal{S}_V(\mathbf{1}_V)|$ for the total number of (sleeping) particles in $\mathcal{S}_V(\mathbf{1}_V)$.

Conjecture 11. (Stationary density ζ_s) *There exists a constant $\zeta_s = \zeta_s(\mathbb{Z}^d,\lambda)$ such that for any exhaustion $V_1 \subset V_2 \subset \cdots \subset \mathbb{Z}^d$ satisfying $\#(\partial V_n)/\#V_n \to 0$ as $n \to \infty$,*

$$\lim_{n\to\infty} \frac{|\mathcal{S}_{V_n}(\mathbf{1}_{V_n})|}{\#V_n} = \zeta_s$$

in probability.

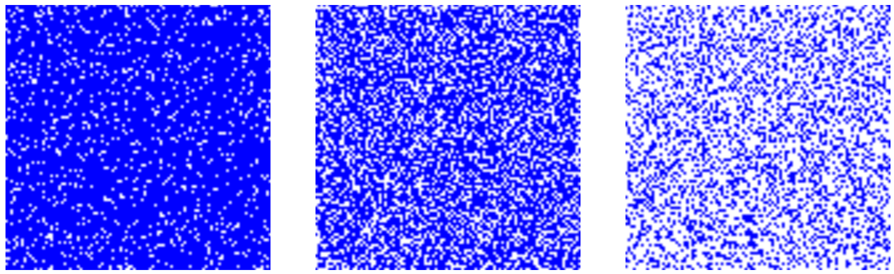


FIG 4. *Stationary configurations for the ARW wired chain on a* 100×100 *box, at three different sleep rates* λ. *The stationary density* ζ_s *is an increasing function of* λ.

Conjecture 12. *The critical densities from Sections 2, 4 and 5 coincide:*

$$\zeta_a = \zeta_c = \zeta_s.$$

5.2. Infinite volume limit

Let π_V denote the stationary distribution of the ARW wired Markov chain, as defined above. For a subset $W \subset V$, write $\pi_V|_W$ for the restriction of π_V to W.

Conjecture 13. *For any fixed finite set* $W \subset \mathbb{Z}^d$, *the measures* $\pi_V|_W$ *have a limit as* $V \uparrow \mathbb{Z}^d$, *and this limit does not depend on the exhaustion of* $\mathbb{Z}^d$.

This conjecture would imply, by Kolmogorov's extension theorem, the existence of a limiting probability measure

$$\pi = \lim_{W \uparrow \mathbb{Z}^d} \lim_{V \uparrow \mathbb{Z}^d} \pi_V|_W. \tag{5.1}$$

on the space of infinite stable configurations $w : \mathbb{Z}^d \to \{0, \mathtt{s}\}$. We can then ask how this limit relates to the measures α and μ from Sections 2 and 4 above.

Question 14. *Is* $\pi = \mu = \alpha$*?*

Can the wired boundary condition be felt deep inside V? We conjecture that as $V \uparrow \mathbb{Z}^d$, the particle density deep inside V coincides with the overall density ζ_s.

Conjecture 15. *For* $w \sim \pi$ *we have* $\pi\{w(\mathbf{0}) = \mathtt{s}\} = \zeta_s$.

5.3. When does macroscopic leaking start?

Write $(w_k)_{k \geq 0}$ for the ARW wired chain on the box $V := [1, L]^d \subset \mathbb{Z}^d$ with initial state $w_0 = 0$ (all sites are empty) and with *uniform driving*: instead of adding particles at a fixed vertex, we add them at a sequence of independent vertices $v_1, v_2, \ldots$ with the uniform distribution on V:

$$w_{k+1} = \mathcal{S}_V(w_k + \delta_{v_{k+1}}), \qquad k \geq 0.$$

When does the wired chain begin to lose a macroscopic number of particles at the boundary? A theorem of Rolla and Tournier partially answers this question. Define

$$\zeta_w := \inf \left\{ t > 0 \, : \, \limsup_L \frac{\mathbb{E}(|w_{tL^d}|)}{L^d} < t \right\}$$

where, as usual, $|w_k|$ denotes the number of particles in w_k.

Theorem 16. *[43, Proposition 3]* $\zeta_w \geq \zeta_c$.

We conjecture $\zeta_w = \zeta_{\mathrm{c}}$, and that the stabilized density has the following simple piecewise linear form.

Conjecture 17 (Hockey stick)**.** *The wired chain on* $[1, L]^d$ *with uniform driving satisfies*

$$\frac{|w_{tL^d}|}{L^d} \to \begin{cases} t, & t \leq \zeta_{\mathrm{c}} \\ \zeta_c, & t \geq \zeta_{\mathrm{c}} \end{cases}$$

in probability as $L \to \infty$*, where* ζ_{c} *is the threshold density of Theorem 8.*

The name for this conjecture comes from the graph of the piecewise linear limit, which has the shape of a hockey stick (Figure 5).

Proposition 18. *If Conjecture 17 holds, then* $\zeta_w = \zeta_{\mathrm{c}}$.

To see this, note that by Theorem 16 it suffices to prove that $\zeta_w \leq \zeta_{\mathrm{c}}$. This can be easily shown arguing by contradiction, we omit the proof.

5.4. Fast mixing

Consider the ARW wired chain in a discrete Euclidean ball $V = \{x \in \mathbb{Z}^d : \sum x_i^2 < L^2\}$ with uniform driving. We highlight the following conjecture from [34] to the effect that this chain mixes immediately after reaching the stationary density ζ_{s}.

Conjecture 19. (Cutoff, [34]) *The ARW wired chain has cutoff in total variation at the time*

$$t_{mix} = \zeta_{\mathrm{s}} \# V.$$

In [34] it is shown that $t_{mix} \leq (1 + o(1))\#V$. The proof uses a coupling between Activated Random Walk and Internal DLA. Bristiel and Salez [7] show that the relaxation time is much smaller: $O(L^{d-1})$ in dimensions $d \neq 2$ and $O(L \log L)$ in dimension 2. They also prove separation cutoff at time $\#V$.

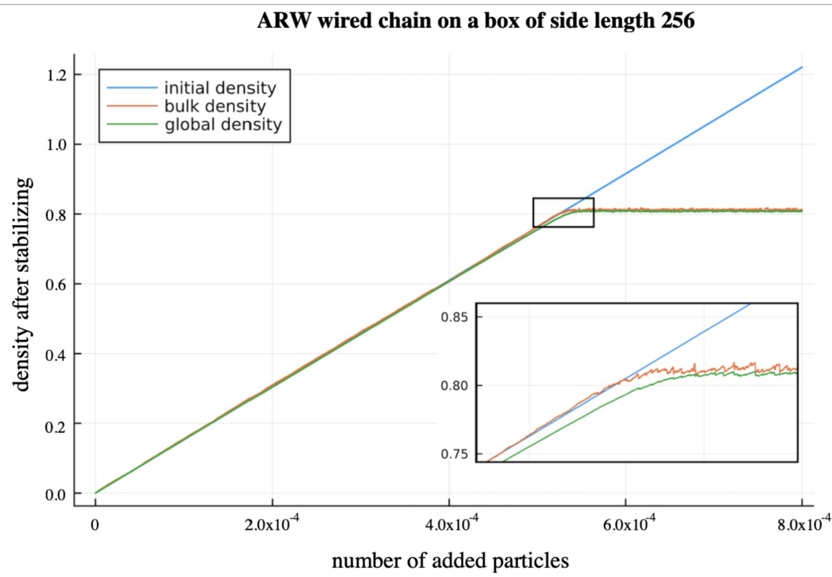


FIG 5. *The hockey stick: As particles are added, the density of the ARW wired chain increases to ζ_c and then flatlines. Here $V \subset \mathbb{Z}^2$ is a box of side length $L = 256$, the sleep rate is $\lambda = 2$, and $\zeta_c \approx 0.813$. "Global density" is the total number of particles divided by L^2. "Bulk density" is the number of particles in a central window of side length $L/2$, divided by $(L/2)^2$.*

5.5. Incompressibility

A recurring challenge in proving several of the above conjectures is to show that "dense clumps" are unlikely. We conjecture that clumps denser than the mean in the infinite-volume stationary state $w \sim \pi$ have exponentially small probability. Write $|w|_L := \sum_{x \in [1,L]^d} 1_{\{w(x)=\mathtt{s}\}}$ for the number of particles in the cube $[1, L]^d$.

Conjecture 20. (Incompressibility) *For each $\zeta > \zeta_{\mathrm{s}}$, there is a constant $c = c(\zeta, \lambda) > 0$ such that for $w \sim \pi$*

$$P(|w|_L \geq \zeta L^d) \leq \exp(-cL^d).$$

The ideas introduced in [1, 20] may be useful in proving incompressiblity for ζ sufficiently close to 1.

6. The free Markov chain

Fix a finite connected graph V, an initial configuration $\phi_0 : V \to \{0, \mathtt{s}\}$, and let

$$\phi_{k+1} = \mathcal{S}(\phi_k + \delta_{v_{k+1}})$$

be the configuration of sleeping particles obtained by adding one active particle at a random vertex v_k, and then stabilizing by ARW dynamics in V with sleep rate λ. The vertices $v_1, v_2, \ldots$ are independent with the uniform distribution on V.

Unlike the ARW wired Markov chain in Section 5, particles cannot escape V. So the total number of particles is deterministic: $|\phi_k| = |\phi_0| + k$. As long as this number does not exceed $\#V$, stabilization happens in finite time, but if the number of particles is large then it could take a long time (even exponentially long, [5])! We will define the *threshold time* as the first time k such that ϕ_k takes "too long" to stabilize.

Let $(\phi_k)_{k\geq 0}$ denote the ARW free chain on V initiated from the empty configuration $\phi_0 = 0$. For $k \geq 0$, denote by U_k be the total number of random walk steps needed to stabilize $\phi_k + \delta_{v_{k+1}}$. For any function $f : \mathbb{N} \to \mathbb{R}$, let

$$\tau_f(V) = \inf\{k \geq 0 \,:\, U_k \geq f(\#V)\}.$$

Conjecture 21. (Concentration of the threshold time) *Let $V = \mathbb{Z}_L^d$ be the d-dimensional torus of side-length L. There exists a superlinear*[3] *function $f : \mathbb{N} \to \mathbb{R}$ such that, as $L \uparrow \infty$,*

$$\lim_{L\uparrow\infty} \frac{\tau_f(\mathbb{Z}_L^d)}{L^d} = \zeta_{\mathrm{c}}$$

in probability, where ζ_{c} is the threshold density of Section 4.

A stronger formulation would posit a sharp transition from linear to exponential time:

Question 22. *Is it true that*

$$U_{tL^d} = \begin{cases} O(L^d), & t < \zeta_c \\ \exp(\Omega(L^d)), & t > \zeta_c? \end{cases}$$

7. The wake Markov chain

Fix a finite connected graph V and an initial configuration $\varphi_0 : V \to \{0, \mathtt{s}\}$ with $|\varphi_0| \leq \#V$. Let $\mathcal{W}$ denote the operator that acts on stable particle configurations on V by waking all particles up. The ARW wake Markov chain, supported on stable particle configurations on V, is defined by

$$\varphi_{k+1} = \mathcal{S}(\mathcal{W}(\varphi_k)).$$

So in one time step of the wake chain, we wake all particles up and then stabilize. Note that stabilization is always possible, though it may take a long time, since $|\varphi_k| = |\varphi_0| \leq \#V$ for all $k \geq 0$.

[3]A function $f : \mathbb{N} \to \mathbb{R}$ is said to be superlinear if $f(n)/n \to \infty$ as $n \to \infty$.

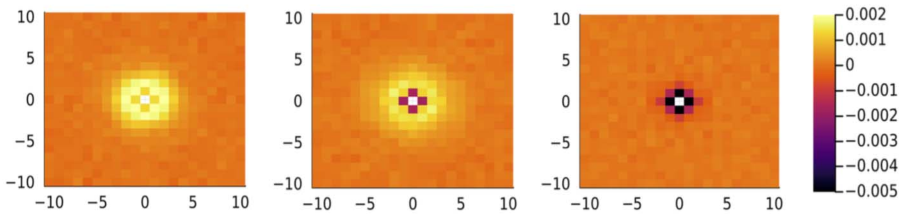


Fig 6. *Site covariance of the ARW Wake Markov Chain on the torus* $\mathbb{Z}_L \times \mathbb{Z}_L$ *with* $L = 2501$*, sleep rate* 1*, and subcritical density* 0.3 *(much less than* $\zeta_c \approx 0.68$*). Left to right: covariances after* $k = 1, 2, 10$ *steps of the wake chain from a uniform random initial condition with* $\lfloor 0.3L^2 \rfloor$ *particles. Each non-central site is shaded according to the covariance between the events that a sleeping particle is located at that site and at the central site. Initial positive correlations with nearby sites become negative after a few time steps.*

7.1. Stationary measure

Take $V = \mathbb{Z}_L^d$ and let $\nu_{L,\zeta}$ denote the stationary measure of the ARW wake chain on V with $\lfloor \zeta L^d \rfloor$ particles. Denote further by $\tilde{\nu}_{L,\zeta}$ the law of the ARW free chain $(\phi_k)_{k\geq 0}$ at time $k = \lfloor \zeta L^d \rfloor$. Note that sleepers configurations drawn according to $\nu_{L,\zeta}$ and $\tilde{\nu}_{L,\zeta}$ have the same density ζ (in fact, the same number of particles). It is natural to conjecture that in the supercritical regime these measures are close.

Conjecture 23. *If* $\zeta > \zeta_c$ *then* $d_{TV}\big(\nu_{L,\zeta}, \tilde{\nu}_{L,\zeta}\big) = o(1)$ *as* $L \to \infty$*, where* d_{TV} *denotes the Total Variation distance.*

This would follow from the following, more general conjecture.

Conjecture 24. (Dense stabilized configurations are hard to distinguish) *Fix the dimension* d *and sleep rate* λ*, and let* ζ_c *be the threshold density of Theorem 8. For each* $\zeta > \zeta_c$ *and* $\epsilon > 0$ *there is an* L_0 *such that for all* $L \geq L_0$ *and any two configurations* $\eta, \tilde{\eta}$ *of active particles on the discrete torus* $\mathbb{Z}_L^d$ *with* $|\eta| = |\tilde{\eta}| \geq \zeta L^d$*, their stabilizations* $\mathcal{S}(\eta), \mathcal{S}(\tilde{\eta})$ *satisfy*

$$d_{\mathrm{TV}}(\mathcal{S}(\eta), \mathcal{S}(\tilde{\eta})) < \epsilon.$$

The underlying mechanism here is that the system takes a long time to stabilize. In particular, it is known that for small enough sleep rate, stabilization takes exponentially many steps in L with high probability: This was proved in dimension 1 by Basu, Ganguly, Hoffman and Richey [5], and recently in all dimensions by Forien and Gaudillière [20, Theorem 3]. Their result plus a coupling argument proves Conjecture 24 for λ small enough: One can couple the trajectories of any two particles in the ARW systems starting from η and $\tilde{\eta}$ so that they will meet prior to stabilization with high probability. Provided all these couplings are successful, the processes stabilize to the same configuration $\mathcal{S}(\eta) = \mathcal{S}(\tilde{\eta})$.

7.2. Mixing time

How long does it take for the ARW wake chain to reach stationarity? We conjecture a transition from slow to instantaneous mixing at the threshold density ζ_c.

Conjecture 25. *Let η be any stable configuration on the d-dimensional torus $\mathbb{Z}_L^d$ on L^d vertices, and denote by $\zeta = |\eta|/L^d$ its particle density. Then the total variation mixing time of the ARW wake chain starting from η is 1 if $\zeta > \zeta_c$, while it is $\Omega(L^2)$ if $\zeta < \zeta_c$, where ζ_c is the threshold density of Theorem 8.*

The first part of this conjecture, fast mixing at high density, would follow directly from Conjecture 24.

8. Hyperuniformity

Experiments suggest that the stationary states for the Activated Random Walk Markov chains introduced in Sections 5, 6, 7 above are *hyperuniform.*

8.1. The wired chain

For a random configuration $\eta \in \{0, \mathtt{s}\}^{\mathbb{Z}^d}$ with $\eta \sim \pi$, write $|\eta|_L$ for the total number of particles in the cube $[1, L]^d$.

Conjecture 26. *Under π, the variance of $|\eta|_L$ is $O(L^\alpha)$ as $L \to \infty$, for some $\alpha < d$.*

For comparison, observe that if the particles are placed in the box $[1, L]^d$ in an i.i.d. fashion, the variance is of order L^d. Thus hyperuniformity implies a kind of rigid repulsion among particles: in order to make the variance of the number of particles in the box $[1, L]^d$ grow sublinearly with its volume L^d, the particle counts $\eta(v)$ for $v \in [1, L]^d$ must have significant negative correlations [22].

Burdzy has proved hyperuniformity in a related particle system called the Meteor Model [8]. The challenge in adapting his proof to ARW lies in adapting the i.i.d. driving of the meteors to the correlated driving that results from active particles waking sleeping particles.

8.2. The free chain

For the free chain the number of particles increases by one at each time step, and we expect hyperuniformity to manifest starting at the threshold density ζ_c. To state a hyperuniformity conjecture for the free chain, we will count particles in a box $B \subset \mathbb{Z}_L^d$. Write $(\phi_k)_{k \geq 0}$ for the ARW free chain on the torus $\mathbb{Z}_L^d$, and write $|\phi_k|_B$ for the total number of particles in B at time k.

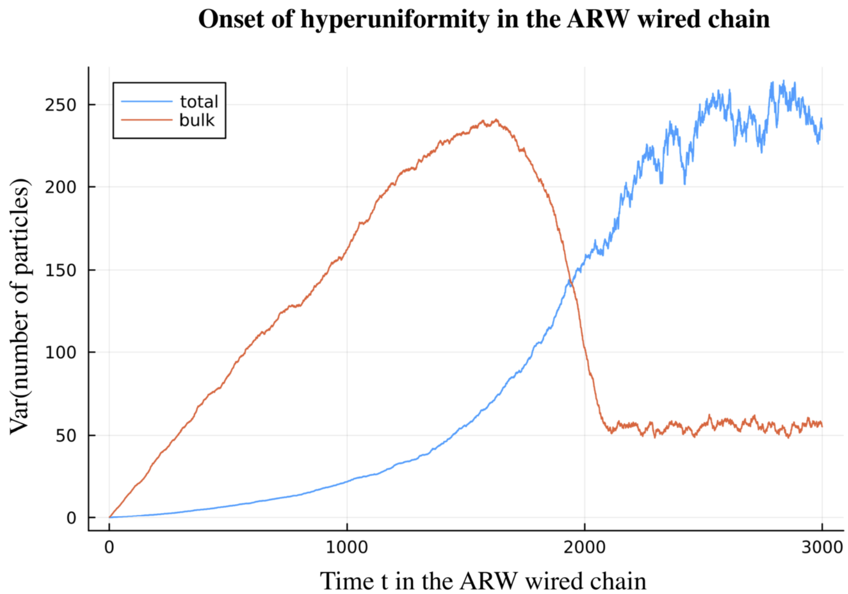


Fig 7. *The onset of hyperuniformity in the ARW wired chain on a square of side length $L = 50$ at sleep rate $\lambda = 2$. The variance $\mathbb{E}|\eta_t|^2 - (\mathbb{E}|\eta_t|)^2$ of the total number of particles in the system increases and then levels off after time $t = \zeta_s L^2$. The variance of the number of particles in the bulk (central square of side length $L/2$) peaks and then drops steeply as t/L^2 approaches the stationary density $\zeta_s \approx 0.81$.*

Conjecture 27. (Onset of hyperuniformity in the free chain) *There exists $\epsilon > 0$ such that for any box $B = [0, \ell_1 - 1] \times \cdots \times [0, \ell_d - 1] \subset \mathbb{Z}_L^d$ we have*

$$\mathrm{Var}(|\phi_{\zeta L^d}|_B) = \begin{cases} \Theta(\ell_1 \dots \ell_d), & \zeta < \zeta_c \\ O((\ell_1 \cdots \ell_d)^{1-\epsilon}), & \zeta \geq \zeta_c. \end{cases}$$

The implied constants depend only on d, λ, ζ.

8.3. The wake chain

Write $\nu_{L,\zeta}$ for the stationary distribution of the ARW wake chain of $\lfloor \zeta L^d \rfloor$ particles on the discrete torus $\mathbb{Z}_L^d$. For a stationary configuration $\eta \sim \nu_{L,\zeta}$, we can once again count the number of particles in a box $B = \prod_{i=1}^d [0, \ell_i - 1]$ and ask whether the variance is linear or sublinear in the size of B.

Question 28. *Is there $\epsilon > 0$ such that the stationary state η of the ARW wake chain satisfies*

$$\mathrm{Var}_{\eta \sim \nu_{L,\zeta}}(|\eta|_B) = \begin{cases} \Theta(\ell_1 \dots \ell_d), & \zeta < \zeta_c \\ O((\ell_1 \cdots \ell_d)^{1-\epsilon}), & \zeta \geq \zeta_c? \end{cases}$$

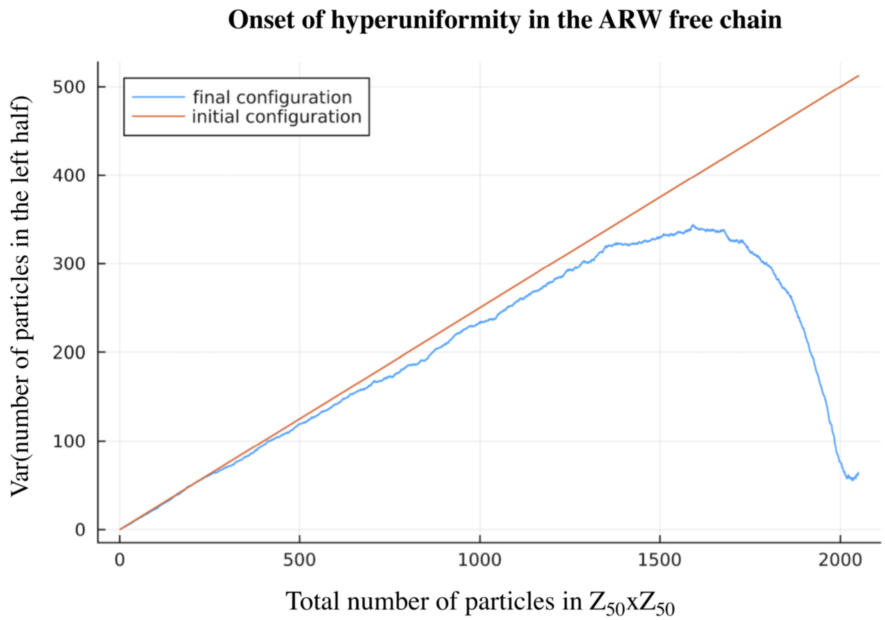


FIG 8. *The onset of hyperuniformity in the ARW free chain on the discrete torus $\mathbb{Z}_L \times \mathbb{Z}_L$ with $L = 50$ and sleep rate $\lambda = 2$. The initial configuration of k active particles at independent uniformly distributed sites, stabilizes to a final configuration of k sleeping particles on the torus. The variance of the number of sleeping particles in the left half $\mathbb{Z}_{L/2} \times \mathbb{Z}_L$ initially increases with k, then peaks around $k = 0.66L^2$, and bottoms out around $\zeta_c L^2$ where $\zeta_c \approx 0.81$.*

Figure 9 tests the case $d = 1$ at density $\zeta = 1/2$ and two different sleep rates: $\zeta_c(0.2) > 1/2$ and $\zeta_c(0.15) \approx 1/2$.

9. Site correlations

To address Question 14 (Is $\pi = \mu = \alpha$?) we performed experiments comparing the site correlations in the ARW free chain, ARW wired chain, and ARW point source aggregate. For the free chain on the torus $\mathbb{Z}_L \times \mathbb{Z}_L$ we are able to average with respect to translation and reflection symmetries of the torus to increase the precision of the numerical estimates. For the wired chain and point source, only reflection symmetries are available, so precision is lower.

9.1. Free site correlations

For small x, y we computed the empirical correlation coefficient

$$\frac{\mathbb{E}(1_f(0,0)1_f(x,y)) - \zeta^2}{\zeta - \zeta^2}$$

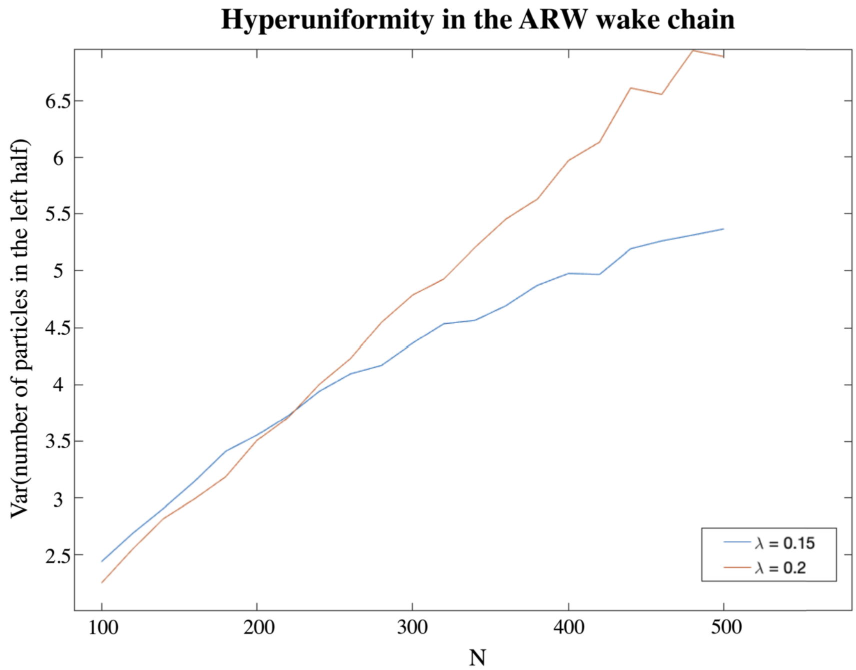


FIG 9. *The ARW wake chain on the cycle $\mathbb{Z}_N$ starting with $N/2$ particles at uniformly random sites and run for $100N$ time steps. The variance of the number of particles on the left half of the cycle grows linearly with N at the subcritical sleep rate $\lambda = 0.2$, but sublinearly with N at the critical sleep rate $\lambda = 0.15$. (Here $\zeta = 1/2$ is less than $\zeta_c(0.2)$ and approximately equal to $\zeta_c(0.15)$.)*

where $1_f(x, y)$ is the indicator of the event that site (x, y) has a sleeping particle in the stabilization of ζL^2 particles started at independent uniform random sites on the torus $\mathbb{Z}_L \times \mathbb{Z}_L$ with side length $L = 63$ at sleep rate $\lambda = 2$ and density $\zeta = 0.81 \approx \zeta_c$. We averaged over 80000 independent samples, and over translation and reflection symmetries of the torus. The results are shown in Table 1.

9.2. Wired site correlations

For small x, y we computed the empirical correlation coefficient

$$\frac{\mathbb{E}(1_w(0,0)1_w(x,y)) - \zeta^2}{\zeta - \zeta^2}$$

where $1_w(x, y)$ is the indicator of the event that site (x, y) has a sleeping particle in the stationary state $\mathcal{S}(1_V)$ of the ARW wired chain on the square $V = [-31, 31]^2 \subset \mathbb{Z}^2$ at sleep rate $\lambda = 2$, and $\zeta = 0.81 \approx \zeta_s$. We averaged over $5 \cdot 10^7$ independent samples, and over the D_8 symmetry of the square lattice. The results are shown in Table 2.

9.3. Point source site correlations

For small x, y we computed the empirical correlation coefficient

$$\frac{\mathbb{E}(1_a(0,0)1_a(x,y)) - \zeta^2}{\zeta - \zeta^2}$$

where $1_a(x, y)$ is the indicator of the event that site (x, y) has a sleeping particle in the stabilization of $n = 3215$ particles started at $(0, 0)$, and $\zeta = 0.81 \approx \zeta_a$. This value of n was chosen to make the total number of particles match the free chain experiment described above. We averaged over 10^5 independent samples, and over the D_8 symmetry of the square lattice. The results are shown in Table 3.

TABLE 1
Short-range correlations for the ARW free Markov chain on the discrete torus $\mathbb{Z}_{63} \times \mathbb{Z}_{63}$.

$y\backslash x$	0	1	2	3	4	5
0	1.0000	−0.0238	−0.0101	−0.0061	−0.0045	−0.0037
1		−0.0139	−0.0082	−0.0056	−0.0044	−0.0037
2			−0.0063	−0.0048	−0.0040	−0.0035
3				−0.0042	−0.0037	−0.0034
4					−0.0034	−0.0032
5						−0.0031

TABLE 2
Short-range correlations for the ARW wired Markov chain on a square of side length 63 in $\mathbb{Z}^2$.

$y\backslash x$	0	1	2	3
0	1.000	−0.021	−0.008	−0.003
1		−0.012	−0.006	−0.003
2			−0.004	−0.002
3				−0.001

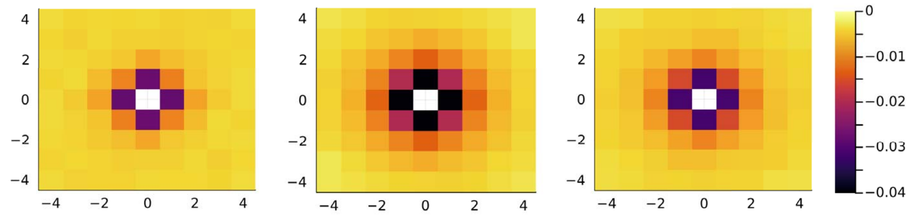
TABLE 3
Short-range correlations for the ARW point source aggregate of 3215 particles in $\mathbb{Z}^2$.

$y\backslash x$	0	1	2
0	1.000	−0.021	−0.007
1		−0.010	−0.006
2			−0.000

Comparing Tables 1 and 2, the spatial decay of correlations is faster in the wired chain than in the free chain. Is this an artifact of the small system size, or are the limiting measures μ and π different? Comparing Tables 2 and 3, the short range correlations are consistent with the measures π and α being equal, but the low precision prevents us from conjecturing this confidently.

9.4. Wake chain site correlations

The ARW wake chain has a family of stationary distributions, one for each density. We wondered whether the site correlations depend on the density. The



$\zeta = 0.2$ $\zeta = 0.6$ $\zeta = 0.7$

FIG 10. *Short-range correlations of the ARW Wake Markov Chain at three different densities $\zeta = 0.2, 0.6, 0.7$. Each non-central site is shaded according to the correlation coefficient between the events that a sleeping particle is located at that site and at the central site.*

answer appears to be yes, as shown in Figure 10. For three different densities $\zeta \in \{0.2, 0.6, 0.7\}$ we computed the empirical correlation coefficient

$$\frac{\mathbb{E}(1_z(0,0)1_z(x,y)) - \zeta^2}{\zeta - \zeta^2}$$

where $1_z(x,y)$ is the indicator of the event that site (x,y) has a sleeping particle in the stationary distribution of the ARW wake chain at density ζ on the torus $\mathbb{Z}_{15} \times \mathbb{Z}_{15}$ with sleep rate 1. After a burn-in period to reach stationarity, we averaged over 100000 subsequent time steps of the ARW wake chain and over translation symmetries of the torus. Short-range correlations are negative at all densities, and nothing notable seems to happen at the threshold density $\zeta_c \approx 0.68$. The strongest nearest-neighbor site correlation occurs at a density somewhat less than ζ_c.

10. Contrasts with the Abelian sandpile model

We close by comparing Activated Random Walk to the Abelian Sandpile, and in particular highlighting which results, among the ones we believe to hold for ARW, are known to fail for the sandpile model.

10.1. Point source

Pegden and Smart [42] proved existence of a limit shape for the point source Abelian Sandpile in $\mathbb{Z}^d$. The scaling limit of the Abelian sandpile on $\mathbb{Z}^2$ obeys a PDE that is not rotationally symmetric [35, 36], so its limit shape from a point source is unlikely to be a Euclidean disk (although it has not been formally proved not to be a disk!). One symptom of the failure of universality in the Abelian Sandpile is the existence of "dense clumps" in the point-source sandpile. These are macroscopic regions whose density is higher than the average density of the whole pile. By contrast, we believe that the point-source ARW stable configuration satisfies a form of incompressibility analogous to that in Conjecture 20, thus making dense clumps unlikely.

10.2. Stationary ergodic

The Abelian Sandpile in $\mathbb{Z}^d$ for $d \geq 2$ has an interval of threshold densities: any density between d and $2d-1$ can be threshold, depending on the law of the initial configuration [18, 41, 16, 19]). By contrast, Activated Random Walk at a given sleep rate has a single threshold density (Theorem 8).

Conjecture 9 fails for the Abelian Sandpile, due to slow mixing: the sandpile stabilization of $\eta_0 + \xi_t$ retains a memory of its initial state η_0 even as $t \uparrow \zeta_c - \zeta_0$ [33]. Terms like "the self-organized critical state" result in lot of confusion in the physics literature on the Abelian Sandpile because there are many such states! One of them, the limit of the uniform recurrent state, is amenable to exact calculations [10, 30, 38, 29], but slow driving from a subcritical state will usually produce a critical state with different properties (e.g. different density).

An anonymous referee raises the interesting question of whether any of these *other* critical states are amenable to exact calculations. The answer is: not that we know of! A natural extension of the uniform recurrent sandpile on a graph G is the distribution that assigns probability proportional to $y^{|r|}$ to each recurrent sandpile r, where $|r|$ is the total number of particles. The resulting partition function is the evaluation $T_G(1, y)$ of the Tutte polynomial of G (as can be seen from the bijection of recurrent sandpiles with spanning trees, taking particle count to external activity [11]). But this evaluation of the Tutte polynomial is known to be $\#P$-hard for all $y \neq 1$ [27], which suggests that the uniform case ($y = 1$) is the only member of this family amenable to exact calculations. There may be exceptions "infinitesimally close" to $y = 1$: For example, the *threshold state* [33] is the limit of slow driving to criticality from initial condition s_0 as $|s_0| \to -\infty$. This limit has a weak form of universality, in that it does not depend on s_0 as long as the total particle count in s_0 tends to $-\infty$. The threshold state can be expressed as the size-biasing of the uniform recurrent state by a statistic called "burst size", which measures how many particles exit the system as a result of the avalanche triggered by adding one particle. In contrast, we do not expect slow driving to criticality from any *fixed* s_0 (including natural-seeming choices like $s_0 \equiv 0$, or $s_0 \sim$ i.i.d. Bernoulli- or Poisson-distributed) to yield a state amenable to exact calculations.

10.3. Wired Markov chain

Fast mixing of ARW stands in contrast to the slow mixing of the Abelian sandpile, where $t_{mix} = \Theta(L^2 \log L)$ for the wired chain on $(\mathbb{Z}/L\mathbb{Z})^2$ [24] and on $[1, L]^2$ [25]. This logarithmic factor is responsible for the discrepancy between the stationary density $\zeta_s = 2.125000$ and the threshold density $\zeta_c = 2.125288$ observed in [15].

To approach Conjectures 11–13 and Question 14 it may be useful to find a combinatorial description of the ARW stationary distribution. An important tool available for the Abelian sandpile, which has no counterpart yet in the ARW setting, is a bijection between recurrent states and spanning trees. This bijection is useful because of the well-developed theory of infinite-volume limits like (5.1) for trees [6]. The bijection from sandpiles to trees plays a starring role in Athreya and Járai's proof that the uniform recurrent sandpile on a finite set $V \subset \mathbb{Z}^d$ has an infinite-volume limit [2], and in Járai and Redig's study of infinite-volume sandpile dynamics [28]. Hutchcroft used the bijection with spanning trees to prove universality results for high-dimensional sandpiles [26].

10.4. Free Markov chain

The simple piecewise linear limit in Conjecture 17 is believed to be false for the Abelian Sandpile due to its slow mixing. There is, however, a weaker conjectured relationship between the sandpile free and wired chains: the threshold time of the free chain coincides with the first time when a macroscopic number of particles exit the wired chain [17].

In spite of all these failures of universality, is there not some sense in which the abelian sandpile is universal? While the uniform spanning tree is universal (for example, paths in the uniform spanning tree of $\mathbb{Z}^2$ scale to SLE(2), and the Peano curve between the tree and its planar dual scales to SLE(8) [32]), the bijections between trees to sandpiles all involve arbitrary choices (e.g. a choice of stabilization procedure [40], or a total ordering of the edges [11]). As a result, natural observables of sandpiles (such as the number of particles, or various measurements of avalanche size) typically do not correspond to natural observables of trees. Nevertheless, there is numerical evidence that avalanche boundaries of two-dimensional sandpiles may scale to SLE(2) [48], and there are intriguing theoretical parallels with logarithmic conformal field theory [47].

Acknowledgments

We thank Ahmed Bou-Rabee, Hannah Cairns, Deepak Dhar, Shirshendu Ganguly, Chris Hoffman, Feng Liang, SS Manna, Pradeep Mohanty, Leonardo Rolla, Vladas Sidoravicius, and Lorenzo Taggi for many inspiring conversations. Thanks to Chris Hoffman for pointing out that Conjecture 11 requires a condition on the boundary of V_n, and that Conjecture 17 requires a condition on the driving. We are also grateful to the anonymous referees for carefully reading our article and for many valuable suggestions which improved the presentation. This project was partly supported by the Funds for joint research Cornell-Sapienza. LL was partly supported by the NSF grant DMS-1105960 and IAS Von-Neumann Fellowship. VS was partly supported by Indam-GNAMPA. We thank Cornell University, Sapienza University, IAS, and ICTS-TIFR for their hospitality.

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