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Once again on weak solutions of time inhomogeneous Itô's equations with VMO diffusion and Morrey drift

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Abstract

We prove the existence and weak uniqueness of weak solutions of Itô's stochastic time dependent equations with irregular diffusion and drift terms of Morrey class with mixed norms.

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1 Introduction

This paper is a natural complement of [10] where the drift term was assumed to be the sum of two terms, one of which was bounded in x and L_2 in t, and another one was in a Morrey class for each t with small norm. In this paper we concentrate on the case when the drift is in a Morrey class with respect to (t,x) with mixed norms.

Let $\mathbb{R}^d$ be a $d-$ dimensional Euclidean space of points $x=(x^1,...,x^d)$ with $d\geq 2$. Let $(\Omega,\mathcal{F},P)$ be a complete probability space, carrying a d-dimensional Wiener process w_t. Fix $\delta\in(0,1]$ and denote by $\mathbb{S}_\delta$ the set of $d\times d$ symmetric matrices whose eigenvalues lie in $[\delta,\delta^{-1}]$.

Throughout the article we assume that on $\mathbb{R}^{d+1}=\{(t,x):t\in\mathbb{R},x\in\mathbb{R}^d\}$ we are given Borel $\mathbb{R}^d$-valued function $b=(b^i)$ and $\mathbb{S}_\delta$-valued $\sigma=(\sigma^{ij})$. We are going to investigate the equation

$$x_s=x+\int_0^s\sigma(t+u,x_u)\,dw_u+\int_0^s b(t+u,x_u)\,du. \tag{1.1}$$

We are interested in the so-called weak solutions, that is solutions that are not necessarily $\mathcal{F}^w_s$-measurable, where $\mathcal{F}^w_s$ is the completion of $\sigma(w_u:u\leq s)$. We present sufficient conditions for the equation to have such solutions on appropriate probability spaces and investigate uniqueness of their distributions.

We just reproduced part of the introduction in [10]. The reader interested in learning more about the history of the problem, motivation, and the literature is sent to [1], [3], [12] and to the the introduction in [10].

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Our Morrey type condition on b is stated in terms of mixed norms (different powers of summability with respect t and x). In [3] the Morrey type condition on b is stated in terms of L_p-norms in (t,x) and the weak solvability of (1.1) is proved, if σ is the unit matrix, along with weak uniqueness provided the solutions possess some additional properties. Our result in this respect contains [3] and also allow us to give conditions for unconditional weak uniqueness.

Still our general uniqueness theorem and uniqueness theorems in [12] are conditional. We prove uniqueness only in the class of solutions (which is proved to be nonempty) admitting certain estimates, however, as we said before, there are cases in which we prove unconditional weak uniqueness.

Our σ is not constant or continuous and it is worth saying that restricting the situation to the one when σ and b are independent of time allows one to relax the conditions on b significantly further, see, for instance, [4] and the references therein.

In Remark 3.15 we compare our results with some of those in excellent papers by Röckner and Zhao [12] and [3]. By the way, G. Zhao ([14]) gave an example showing that, if in the definition of $\mathfrak{b}_\rho$ (in Theorem 2.1) we replace r with r^α, $\alpha > 1$, and keep (2.2) (with $k = 0$), weak uniqueness may fail even in the time homogeneous case and unit diffusion. In Example 2.7 we show that in the time inhomogeneous case even the existence may fail.

Here is an example in which we prove existence and (unconditional) weak uniqueness of weak solutions: $|b| = cf$, where the constant $c > 0$ is small enough and

$$f(t,x) = I_{1>t>0,|x|<1}|x|^{-1}\Big(\frac{|x|}{\sqrt{t}}\Big)^{1/(d+1)}, \ \sigma = 2(\delta^{ij}) + I_{x\neq 0}\zeta(x)\sin(\ln|\ln|x||),$$

where ζ is any smooth symmetric $d\times d$-matrix valued function vanishing for $|x| > 1/2$ and satisfying $|\zeta| \le 1$. This example is inadmissible in [12] because b is too singular and σ is not constant but admissible in [3] if σ is constant and yields weak solutions conditionally weakly unique. This example does not fit into the scheme in [10] (barely misses) because of special b.

The paper is organized as follows. In Section 2 we prove the solvability of (1.1) when the drift is the sum of terms with different summability properties. In Section 3 we deal with weak uniqueness and construct the corresponding Markov processes. This time the drift is not split. Section 4 contains a result from [11] used in Section 3.

We conclude the introduction by some notation. We set

$$D_i = \frac{\partial}{\partial x^i}, \quad Du = (D_i u), \quad D_{ij} = D_i D_j, \quad D^2 u = (D_{ij}u), \quad \partial_t = \frac{\partial}{\partial t}.$$

If $\sigma = (\sigma^{i\cdots})$ by $|\sigma|^2$ we mean the sum of squares of all entries.

Introduce

$$B_R(x) = \{y \in \mathbb{R}^d : |x-y| < R\}, \quad B_R = B_R(0),$$
$$C_{\tau,\rho}(t,x) = [t, t+\tau) \times B_\rho(x), \quad C_\rho\ldots = C_{\rho^2,\rho}\ldots, \quad C_\rho = C_\rho(0,0),$$

and let $\mathbb{C}_\rho$ be the collection of $C_\rho(t,x)$.

In the proofs of our results we use various (finite) constants called N which may change from one occurrence to another and depend on the data only in the same way as it is indicated in the statements of the results.

2 Solvability of Itô's equations

Let $d \ge 2$ and let $(\Omega, \mathcal{F}, P)$ be a complete probability space. Let $\mathcal{F}_t, t \ge 0$, be an increasing family of complete σ-fields $\mathcal{F}_t \subset \mathcal{F}$, and let w_t be an $\mathbb{R}^d$-valued Wiener process relative to $\mathcal{F}_t$. Recall that σ is assumed to be $\mathbb{S}_\delta$-valued.

It is well known that, if σ and b are smooth and b is bounded, the solutions of the system

$$x_s = x + \int_0^s \sigma(\mathsf{t}_r, x_r)\,dw_r + \int_0^s b(\mathsf{t}_r, x_r)\,dr, \quad \mathsf{t}_s = t + s \tag{2.1}$$

form a strong Markov process X with trajectories (t_s, x_s).

Define

$$\mathsf{b}_\rho = \sup_{r \le \rho} r^{-1} \sup_{(t,x)\in\mathbb{R}^{d+1}} \sup_{C \in \mathbb{C}_r} E_{t,x} \int_0^{\tau_C} |b(\mathsf{t}_s, x_s)|\,ds,$$

where τ_C is the first exit time of (t_s, x_s) from C.

To continue we need some notation which are somewhat different from what we use in Sections 3 and 4. For $p, q \in [1,\infty)$ and domain $Q \subset \mathbb{R}^{d+1}$ by $L_{p,q}(Q)$ we mean the space of Borel (real-, vector- or matrix-valued) functions on Q with finite norm given by

$$\|f\|^q_{L_{p,q}(Q)} = \|fI_Q\|^q_{L_{p,q}} = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^d} |fI_Q(t,x)|^p\,dx \Big)^{q/p}\,dt$$

if $p \ge q$ and by

$$\|f\|^p_{L_{p,q}(Q)} = \|fI_Q\|^p_{L_{p,q}} = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}} |fI_Q(t,x)|^q\,dt \Big)^{p/q}\,dx$$

if $p \le q$. Set $L_{p,q} = L_{p,q}(\mathbb{R}^{d+1})$. These definitions extend naturally when one or both p, q are infinite. As usual, we write something like $f \in L_{p,q,\mathrm{loc}}$ if $f\zeta \in L_{p,q}$ for any infinitely differentiable ζ with compact support. We write $\|u, v, ...\|_{L_{p,q}}$ to mean the sum of the $L_{p,q}$-norms of what is inside.

By $W^{1,2}_{p,q}(Q)$ we mean the collection of u such that $\partial_t u$, D^2u, Du, $u \in L_{p,q}(Q)$. The norm in $W^{1,2}_p(Q)$ is introduced in an obvious way. We abbreviate $W^{1,2}_{p,q} = W^{1,2}_{p,q}(\mathbb{R}^{d+1})$.

If a Borel $\Gamma \subset \mathbb{R}^{d+1}$, by $|\Gamma|$ we mean its Lebesgue measure and

$$\fint_\Gamma f(t,x)\,dxdt = \frac{1}{|\Gamma|} \int_\Gamma f(t,x)\,dxdt.$$

If $C \in \mathbb{C}_\rho$ we set

$$\# f \|_{L_{p,q}(C)} = \|1\|^{-1}_{L_{p,q}(C)} \|f\|_{L_{p,q}(C)} = N(d)\rho^{-d/p-2/q}\|f\|_{L_{p,q}(C)}.$$

Take the Fabes-Stroock constant $d_0 = d_0(d,\delta) \in (d/2, d)$ introduced in [9] and let us say that (p,q) are *admissible* if

$$p, q \in [1,\infty], \quad \frac{d_0}{p} + \frac{1}{q} \le 1.$$

Also take $m_b = m_b(d,\delta) > 0$ introduced in [9].

Here is a generalization of Corollary 2.14 of [7].

Theorem 2.1. *Assume that σ and b are smooth and b is bounded and there is a nonnegative integer k and there are Borel functions $b_i(t,x)$, $0 \le i \le k$, such that $b = \sum_{i=0}^k b_i$, and we are given admissible (p_i, q_i), $i \le k$. Define*

$$\mathfrak{b}_\rho = \sup_{r \le \rho} r \sup_{C \in \mathbb{C}_r} \sum_{i=0}^k \# b_i \|_{L_{p_i,q_i}(C)},$$

introduce $\hat b = \hat b(d,\delta)$ so that $N\hat b = m_b/4$, where $N = N(d,\delta)$ is taken from Theorem 1.1 of [7] and suppose that

$$\mathfrak{b}_{\rho_b} \le \hat b \tag{2.2}$$

holds for some $\rho_b \in (0,\infty)$. Then

$$\mathsf{b}_{\rho_b/2} \le m_b. \tag{2.3}$$

One proves this theorem by repeating the proof of Theorem 1.1 of [7], where the fact that there $k=0$ was not used at all, and also using the argument in Step 1 of the proof of Theorem 1.2 of [7]. After that the same argument as in Corollary 2.14 of [7] yields the result.

Example 2.2. One of situations when $\mathfrak{b}_\rho$ is finite presents when $k=1$, $|b_0(t,x)|\le h_0(x)$, $|b_1(t,x)|\le h_1(t)$ and, say $h_0(x)\le c|x|^{-1}$, where c is sufficiently small, and $h_1\in L_2(\mathbb{R})$. In that case one can take $p_1=d_0, q_1=\infty$, $p_2=\infty, q_2=1$.

Indeed, if $|x_0|\le 2r$, then

$$\fint_{B_r(x_0)}|x|^{-d_0}\,dx\le 2^d\fint_{B_{2r}}|x|^{-d_0}\,dx=N(d)r^{-d_0}.$$

and if $|x_0|\ge 2r$, then $|x|^{-1}\le r^{-1}$ on $B_r(x_0)$ and $\#\|\,|\cdot|^{-1}\|_{L_{d_0}(B_r(x_0))}\le r^{-1}$.

Also

$$\fint_s^{s+r^2}h_1(t)\,dt\le r^{-1}\Big(\int_s^{s+r^2}h_1^2(t)\,dt\Big)^{1/2}$$

and the integral here tends to zero as $r\downarrow 0$ uniformly with respect to s. Therefore, by taking c small enough and taking appropriately small ρ_b we can satisfy (2.2) with any given $\hat b>0$.

In the following theorem we prove the existence of weak solutions of equation (2.1). Somewhat unusual split in its assumption about b is caused by the necessity to use smooth approximations of the b_i's converging to the b_i's in the corresponding norms.

Theorem 2.3. *Suppose that* (2.2) *holds for some* $\rho_b\in(0,\infty)$ *with admissible* p_i,q_i *such that, for each* i*, either (a)* $p_i+q_i<\infty$*, or (b)* $p_i<\infty$*,* $q_i=\infty$ *and* b_i *is independent of* t*. Then*

(i) there is a probability space $(\Omega,\mathcal{F},P)$*, a filtration of* σ*-fields* $\mathcal{F}_s\subset\mathcal{F}$*,* $s\ge 0$*, a process* w_s*,* $s\ge 0$*, which is a* d*-dimensional Wiener process relative to* $\{\mathcal{F}_s\}$*, and an* $\mathcal{F}_s$*-adapted process* x_s *such that (a.s.) for all* $s\ge 0$ *equation* (2.1) *holds with* $(t,x)=(0,0)$*.*

(ii) Furthermore, for any nonnegative Borel g *on* $\mathbb{R}^d$ *and* f *on* $\mathbb{R}^{d+1}$ *and* $T\in(0,\infty)$ *we have*

$$E\int_0^T f(s,x_s)\,ds\le N(d,\delta,T,\rho_b)\|f\|_{L_{p_i,q_i}},\tag{2.4}$$

$$E\int_0^T g(x_s)\,ds\le N(d,\delta,T,\rho_b)\|g\|_{L_{d_0}(\mathbb{R}^d)}.\tag{2.5}$$

Proof. Approximate σ,b by smooth $\sigma^{(\varepsilon)},b^{(\varepsilon)}$, by using mollifying kernel $\varepsilon^{-d-1}\zeta(t/\varepsilon,x/\varepsilon)$, where nonnegative $\zeta\in C_0^\infty(\mathbb{R}^{d+1})$ has unit integral and $\zeta(0)=1$. Then set $b_i^\varepsilon(t,x)=b_i^{(\varepsilon)}(t,x)\zeta(\varepsilon t,\varepsilon x)$ to make the new b_i have compact support. Observe that b_i^ε satisfy (2.2) with the same $\hat b,\rho_b$. Therefore, the corresponding Markov process $(\mathrm{t}_t,x_t^\varepsilon)$ satisfies $\mathsf{b}^\varepsilon_{\rho_b/2}\le m_b$ which makes available all results of [9]. In particular, by Corollary 3.10 of [9] for any $\varepsilon,n>0$ and $r>s\ge 0$

$$E_{0,0}\sup_{u\in[s,r]}|x_u^\varepsilon-x_s^\varepsilon|^n\le N(|r-s|^{n/2}+|r-s|^n),\tag{2.6}$$

where $N=N(n,\rho_b,d,\delta)$. This implies that the $P_{0,0}$-distributions of $x_\cdot^\varepsilon$ are precompact on $C([0,\infty),\mathbb{R}^d)$ and a subsequence as $\varepsilon=\varepsilon_n\downarrow 0$ of them converges to the distribution of a process $x_\cdot^0$ defined on a probability space (the coordinate process on $\Omega=C([0,\infty),\mathbb{R}^d)$ with cylindrical σ-field $\mathcal{F}$ completed with respect to P, that is the limiting distribution of $x_\cdot^\varepsilon$). Furthermore, by Theorem 5.9 (ii) of [9] for any nonnegative Borel g on $\mathbb{R}^d$ and f on

$\mathbb{R}^{d+1}$ and $\varepsilon, T \in (0,\infty)$ we have

$$E_{0,0}\int_0^T f(s,x_s^\varepsilon)\,ds \le N(d,\delta,T,\rho_b)\|f\|_{L_{p_i,q_i}}, \tag{2.7}$$

$$E_{0,0}\int_0^T g(x_s^\varepsilon)\,ds \le N(d,\delta,T,\rho_b)\|g\|_{L_{d_0}(\mathbb{R}^d)}, \tag{2.8}$$

which by continuity is extended to $\varepsilon = 0$ for bounded continuos f and then by the usual measure-theoretic argument for all Borel $f \ge 0$. This proves (ii).

After that arguing as in the proof of Theorem 3.9 of [10] proves assertion (i). Here passing to the limit in the drift term the case (a) we use (2.7) and in the case (b) we use (2.8). The theorem is proved. □

Remark 2.4. Actually as is easy to see, in case (b) the condition that b_i is independent of t can be replaced with the following which is somewhat cumbersome: for any $R \in (0,\infty)$

$$\lim_{\varepsilon\downarrow 0}\int_{B_R}\sup_t |b^{(\varepsilon)} - b|^{d_0}(t,x)\,dx = 0.$$

Remark 2.5. It may look like assertion (i) of Theorem 2.3 is a generalization of Theorem 3.1 (i) of [5] about the solvability of (2.1) with $b \in L_{p,q}$ and $d/p + 1/q \le 1$. However, in the typical case of $k = 0$, along with $b \in L_{p_0,q_0,\text{loc}}$, $d_0/p_0 + 1/q_0 \le 1$, we require (2.2) to hold and, if we ask ourselves what p,q should be in order the inclusion $b \in L_{p,q}$ to imply (2.2), the answer is $d/p + 2/q \le 1$, somewhat disappointing. At the same time in the next example we show that Theorem 3.1 (i) of [5] does not cover all applications of Theorem 2.3.

In assumption (2.2) the size of $\hat{b}$ could not be too large.

Example 2.6. Let

$$b(t,x) = b(x) = -\frac{d}{|x|}\frac{x}{|x|}I_{x\ne 0}, \quad \sigma = \sqrt{2}(\delta^{ij}).$$

Then as is easy to see, for any $p \in (d_0, d)$ and any q the quantity $\rho\, |\!|\!|b\|_{L_{p,q}(C)}$, $\rho > 0, C \in \mathbb{C}_\rho$, is bounded. However, the equation $dx_t = \sigma\,dw_t + b(x_t)\,dt$ with initial condition $x_0 = 0$ does not have any solution.

Indeed, if it does, then by Itô's formula

$$|x_t|^2 = 2d\int_0^t I_{x_s=0}\,ds + 2\sqrt{2}\int_0^t x_t\,dw_t. \tag{2.9}$$

Here the first integral is the time spent at the origin by x_s up to time t. This integral is zero, because by using Itô's formula for $|x_t^1|$, one sees that the local time of x_t^1 at zero exists and is finite, implying that the real time spent at zero is zero.

Then (2.9) says that the local martingale starting at zero which stands on the right is nonnegative. But then it is identically zero, implying the same for x_t. However, $x_t \equiv 0$, obviously, does not satisfy our equation.

At the same time according to Theorem 2.3, the equation $dx_t = \sigma\,dw_t + \varepsilon b(x_t)\,dt$ with initial condition $x_0 = 0$ does have solutions if ε is sufficiently small. Observe that $b \notin L_{p,q,\text{loc}}$ for any $p,q \in (0,\infty)$ satisfying $d/p + 1/q \le 1$, so this example is not covered by Theorem 3.1 (i) of [5].

It turns out that in the definition of $\mathfrak{b}$ one cannot replace r with $r^{1+\alpha}$, no matter how small $\alpha > 0$ is.

Example 2.7. Take numbers α and β satisfying

$$0 < \alpha \le \beta < 1, \quad \alpha + \beta = 1$$

and set

$$b(t,x) = -\frac{1}{t^\alpha |x|^\beta} \frac{x}{|x|} I_{0<|x|\le 1, t\le 1}.$$

Using that $d_0 < d$, it is not hard to find p,q such that $d_0/p + 1/q < 1$ and the quantity $\rho^{1+\alpha} \,\,\text{⫲}b\|_{L_{p,q}(C)}$, $\rho > 0, C \in \mathbb{C}_\rho$, is bounded. However, as we know from [5], the equation $dx_t = dw_t + \varepsilon b(t,x_t)\,dt$ with zero initial condition does not have solutions no matter how small $\varepsilon > 0$ is (actually $\varepsilon = 1$ in [5] but self-similar transformations take care of any $\varepsilon > 0$).

Remark 2.8. If $b \equiv 0$, it turns out that for any admissible (p,q), $R \in (0,\infty)$, $x \in \mathbb{R}^d$ and Borel $f(t,x) \ge 0$

$$E\int_0^\tau f(s,x_s)\,ds \le N(d,\delta)R^2 \,\,\text{⫲}f\|_{L_{p,q}(C_R(0,x))}, \tag{2.10}$$

where τ is the first exit time of (s,x_s) from $C_R(0,x)$.

Indeed, if $R=1$, this follows from (2.4), where we take $T=1$, any appropriate ρ_b and observe that $\tau \le 1$ and we may assume that $f=0$ outside $C_1(0,x)$. The case of general R is treated by parabolic scaling of $\mathbb{R}^{d+1}$.

This simple observation has the following implication in which

$$\mathcal{L}_0 u(t,x) = \partial_t u + (1/2)a^{ij}(t,x)D_{ij}u(t,x), \quad a = \sigma^2.$$

Lemma 2.9. *Let (p,q) be admissible and finite, $x \in \mathbb{R}^d$, $R \in (0,\infty)$, $u \in W^{1,2}_{p,q}(C_R(0,x))$ and $u=0$ on $\partial' C_R(0,x)$ (that is $\big(\partial B_R(x) \times [0,R^2]\big) \cup \big(\bar B_R(x) \times \{R^2\}\big)$). Then*

$$|u(0,0)| \le N(d,\delta)R^2 \,\,\text{⫲}\mathcal{L}_0 u\|_{L_{p,q}(C_R(0,x))}. \tag{2.11}$$

Proof. First note that, since $d_0 > d/2$ we have $d/p + 2/q < 2$ and u is continuous in $\bar C_R(0,x)$ by embedding theorems. Then approximate u in $W^{1,2}_{p,q}$-norm by smooth functions u^n vanishing on $\partial' C_R(0,x)$. By Itô's formula

$$u^n(0,0) = -E\int_0^\tau \mathcal{L}_0 u^n(s,x_s)\,ds.$$

In light of (2.10) estimate (2.11) holds with u^n in place of u. Sending $n \to \infty$ yields (2.11) as is and proves the lemma. □

Here is Itô's formula we have on the basis of Theorem 2.3.

Theorem 2.10. *(i) Suppose that $k=0$ and (2.2) holds for some $\rho_b \in (0,\infty)$ and*

$$p_0, q_0 \in (1,\infty), \quad \frac{1}{2} \le \frac{1}{\beta_0} := \frac{d_0}{p_0} + \frac{1}{q_0} < 1. \tag{2.12}$$

(ii) Let x_t be a solution of equation (2.1) with $(t,x) = (0,0)$ on a probability space such that

(a) for $p = p_0/\beta_0$, $q = q_0/\beta_0$, any $R \in (0,\infty)$ and Borel nonnegative f on $\mathbb{R}^{d+1}$

$$E\int_0^{\tau_R} f(s,x_s)\,ds \le N\|f\|_{L_{p,q}}, \tag{2.13}$$

where N is independent of f and τ_R is the first exit time of (s,x_s) from C_R.

(iii) Let $u \in W^{1,2}_{p,q}(C_R)$ be such that $Du \in L_{r,k}(C_R)$, where $(r,k) = (\beta_0 - 1)^{-1}(p_0,q_0)$. Then, with probability one for all $t \le R^2$,

$$u(t\wedge\tau_R, x_{t\wedge\tau_R}) = u(0) + \int_0^{t\wedge\tau_R} D_i u \sigma^{ik}(s,x_s)\,dw_s^k$$
$$+ \int_0^{t\wedge\tau_R} [\partial_t u(s,x_s) + a^{ij}D_{ij}u(s,x_s) + b^i D_i u(s,x_s)]\,ds$$

and the stochastic integral above is a square-integrable martingale, where τ_R is the first exit time of x_t from B_R.

Proof. The last statement, of course, follows from (2.13) and the fact that $2p \leq r, 2q \leq k$. To prove the rest we approximate u by smooth functions $u^{(\varepsilon)} = \zeta_\varepsilon * u$, where $\zeta_\varepsilon(t,x) = \varepsilon^{-d-2}\zeta(t/\varepsilon^2, x/\varepsilon)$, and $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ has support in $(-1,0)\times B_1$ and unit integral. Since $d/p + 2/q < 2$ ($d < 2d_0$), by embedding theorems $u \in C(\bar{C}_R)$ and, therefore, $u^{(\varepsilon)} \to u$ as $\varepsilon \downarrow 0$ uniformly in any $C_{R'}$ with $R' < R$.

Fix $R' < R$. Then for all sufficiently small $\varepsilon > 0$ by Itô's formula

$$u^{(\varepsilon)}(t \wedge \tau_{R'}, x_{t\wedge\tau_{R'}}) = u^{(\varepsilon)}(0) + \int_0^{t\wedge\tau_{R'}} D_i u^{(\varepsilon)} \sigma^{ik}(s,x_s)\, dw_s^k$$
$$+ \int_0^{t\wedge\tau_{R'}} [\partial_t u^{(\varepsilon)} + a^{ij} D_{ij} u^{(\varepsilon)} + b^i D_i u^{(\varepsilon)}](s,x_s)\, ds. \tag{2.14}$$

We send $\varepsilon \downarrow 0$ and observe that $u^{(\varepsilon)} \to u$ in $W^{1,2}_{p,q}(C_{R'})$ and $Du^{(\varepsilon)} \to Du$ in $L_{r,k}(C_{R'})$. Hence, (2.13) allows us easily to pass to the limit in (2.14), for instance, by using Hölder's inequality we obtain

$$\|gh\|_{L_{p,q}(C_\rho)} \leq \|g\|_{L_{p_0,q_0}(C_\rho)} \|h\|_{L_{r,s}(C_\rho)}, \tag{2.15}$$

implying that

$$E\int_0^{\tau_{R'}} |b|\,|Du - Du^{(\varepsilon)}|(t,x_t)\, dt \leq N \| |b|\,|Du - Du^{(\varepsilon)}| \|_{L_{p,q}(C_{R'})}$$
$$\leq N \|b\|_{L_{p_0,q_0}(C_R)} \|Du - Du^{(\varepsilon)}\|_{L_{r,s}(C_{R'})} \to 0.$$

It follows that (2.14) holds with u in place of $u^{(\varepsilon)}$. After that it only remains to send $R' \uparrow R$ and again use (2.13). The theorem is proved. □

Remark 2.11. The assumption that $Du \in L_{r,s}(C_R)$ looks unrealistic because Sobolev embedding theorems do not provide such high integrability of Du for functions $u \in W^{1,2}_{p,q}(C_R)$. However, if u is in the Morrey class $E^{1,2}_{p,q,\beta}(C_R)$, then $Du \in L_{r,k}(C_R)$ indeed (cf. Remark 4.5).

3 Weak uniqueness and a Markov process

Here we prove a generalization of the Stroock-Varadhan theorem in [13] obtained for σ which is uniformly continuous in x uniformly in t and bounded b. We need an additional assumption on a and can relax conditions imposed on b in Section 2. Since a will have some regularity the range of p_0, q_0 can be substantially extended. Indeed, observe that if $d_0/p_0 + 1/q_0 = 1$, then $1 < d/p_0 + 2/q_0 < 2$ since $d > d_0 > d/2$ (cf. (3.3)).

An important distinction of the rest of the article from Section 2 is that here (and in Section 4) for $p, q \in [1,\infty)$ and domain $Q \subset \mathbb{R}^{d+1}$ by $L_{p,q}(Q)$ we mean the space of Borel (real-, vector- or matrix-valued) functions on Q with finite norm given in *one of two ways which is fixed throughout the rest of the paper*:

$$\|f\|^q_{L_{p,q}(Q)} = \|fI_Q\|^q_{L_{p,q}} = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^d} |fI_Q(t,x)|^p\, dx \Big)^{q/p} dt \tag{3.1}$$

or

$$\|f\|^p_{L_{p,q}(Q)} = \|fI_Q\|^p_{L_{p,q}} = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}} |fI_Q(t,x)|^q\, dt \Big)^{p/q} dx. \tag{3.2}$$

Naturally, $\,\|\!|\cdot\|_{L_{p,q}(Q)}$ and the spaces $W^{1,2}_{p,q}(Q)$ are now introduced in the same way as in Section 2 but with the new meaning of $L_{p,q}(Q)$.

Fix $p_0, q_0, \beta_0, \beta_0'$ such that

$$\beta_0 \in (1,2), \quad \beta_0' \in (1,\beta_0), \quad p_0, q_0 \in (\beta_0, \infty), \quad \frac{d}{p_0} + \frac{2}{q_0} \geq 1. \tag{3.3}$$

Take an $\alpha \in (0,1)$ and $\theta(d,\delta,p,q,\alpha)$ introduced in Assumption 4.3 and set

$$\tilde{\theta} = \theta(d,\delta,p_0/\beta_0,q_0/\beta_0,\alpha) \wedge \theta(d,\delta,p_0/\beta_0',q_0/\beta_0',\alpha)$$

Also take some $\rho_a, \rho_b \in (0,1]$, take $\check{b}(d,\delta,p,q,\rho_a,\beta_0,\alpha)$ from Theorem 4.4 and set

$$\tilde{b} = \check{b}(d,\delta,p_0/\beta_0,q_0/\beta_0,\rho_a,\beta_0,\alpha) \wedge \check{b}(d,\delta,p_0/\beta_0',q_0/\beta_0',\rho_a,\beta_0',\alpha).$$

Finally, one more restriction on the drift term is related to the following condition:

$$\frac{d}{p_0} + \frac{1}{q_0} \leq 1 \text{ and } \begin{cases} \text{either} & p_0 \geq q_0 \text{ and } L_{p,q} \text{ is defined as in } (3.1), \\ \text{or} & p_0 \leq q_0 \text{ and } L_{p,q} \text{ is defined as in } (3.2). \end{cases} \tag{3.4}$$

Throughout this section we suppose that the following assumption is satisfied unless stated otherwise.

Assumption 3.1. *We have*

$$a^{\sharp}_{x,\rho_a} := \sup_{\substack{\rho \leq \rho_a \\ C \in \mathbb{C}_\rho}} \fint_C |a(t,x) - a_C(t)|\, dxdt \leq \tilde{\theta},$$

where

$$a_C(t) = \fint_C a(t,x)\, dxds \quad (\textit{note} \quad t \quad \textit{and} \quad ds).$$

Introduce

$$\mathfrak{b}_\rho = \sup_{r \leq \rho} r \sup_{C \in \mathbb{C}_r} \Vert b \Vert_{L_{p_0,q_0}(C)}.$$

The parts (a), (b), (c) of the following assumption will be imposed in various combinations.

Assumption 3.2. *(a)* $\mathfrak{b}_{\rho_b} \leq \tilde{b}$*,*

(b) $\mathfrak{b}_{\rho_b} \leq \hat{b} \leq 1$ *and* $N_1 \hat{b} \leq m_b$*, where* N_1 *depending only on* $d,\delta,p_0,q_0,\beta_0,\rho_a,\alpha$ *is taken from* (3.18)*,*

(c) condition (3.4) *is satisfied and* $\mathfrak{b}_{\rho_b} \leq \hat{b} \leq 1$ *and* $N\hat{b} \leq m_b$*, where* $N = N(d,\delta,p_0,q_0)$ *is taken from* (3.14)

The following is very important.

Remark 3.3. Consider equation (2.1) with zero initial data and make the change of variables $x_t = \rho_b y_{\rho_b^{-2} t}$, $B_t = \rho_b w_{\rho_b^{-2} t}$. Then

$$dy_t = \tilde{b}(t,y_t)\, dt + \tilde{\sigma}(t,y_t)\, dB_t, \tag{3.5}$$

where $\tilde{b}(t,x) = \rho_b b(\rho_b^2 t, \rho_b x)$, $\tilde{\sigma}(t,x) = \sigma(\rho_b^2 t, \rho_b x)$, and B_t is a Wiener process.

Taking into account that $\rho_b \leq 1$, it is easy to check that $\tilde{\sigma}$ and $\tilde{b}$ satisfy Assumptions 3.1 and 3.2 with the *same* $\rho_a, \tilde{\theta}, \hat{b}, \tilde{b}$ and 1 in place of ρ_b. At the same time the issues of existence and uniqueness of solutions of (3.5) and (2.1) are equivalent.

This remark shows that without loosing generality in the rest of the article we impose

Assumption 3.4. *We have* $\rho_b = 1$*.*

For $\beta \geq 0$, introduce Morrey's space $E_{p,q,\beta}$ as the set of $g \in L_{p,q,\mathrm{loc}}$ such that

$$\|g\|_{E_{p,q,\beta}} := \sup_{\rho \leq 1, C \in \mathbb{C}_\rho} \rho^\beta \, \|g\|_{L_{p,q}(C)} < \infty. \tag{3.6}$$

Define

$$E^{1,2}_{p,q,\beta} = \{u : u, Du, D^2 u, \partial_t u \in E_{p,q,\beta}\}$$

and provide $E^{1,2}_{p,q,\beta}$ with an obvious norm.

It is important to have in mind that if $\beta < 2$ (our main case) and $u \in E^{1,2}_{p,q,\beta}$, then according to Lemma 2.5 of [11], u is bounded and continuous.

Here is a useful approximation result.

Lemma 3.5 (Lemma 2.3 of [11])**.** *Let $f \in E_{p,q,\beta}$. Define $f^{(\varepsilon)}$ as in the proof of Theorem 2.3. Then for any $C \in \mathbb{C}$ and $\beta' > \beta$*

$$\lim_{\varepsilon \downarrow 0} \|(f - f^{(\varepsilon)}) I_C\|_{E_{p,q,\beta'}} = 0. \tag{3.7}$$

Introduce

$$\mathcal{L}u = \partial_t u + a^{ij} D_{ij} u + b^i D_i u.$$

In the following lemma Assumptions 3.1 and 3.2 are not used.

Lemma 3.6. *Let $x_\cdot$ be a solution of* (2.1) *with* $(t, x) = (0, 0)$*. Set*

$$p = p_0/\beta_0, \quad q = q_0/\beta_0 \tag{3.8}$$

and assume that

(a) for any $R \in (0, \infty)$ and Borel nonnegative f on $\mathbb{R}^{d+1}$

$$E \int_0^{\tau_R} f(s, x_s) \, ds \leq N \|f\|_{E_{p,q,\beta_0}}, \tag{3.9}$$

where N is independent of f and τ_R is the first exit time of (s, x_s) from C_R. Then

(b) for any $R \in (0, \infty)$ and $u \in E^{1,2}_{p,q,\beta_0'}$, with probability one for all $t \geq 0$,

$$u(t \wedge \tau_R, x_{t \wedge \tau_R}) = u(0) + \int_0^{t \wedge \tau_R} D_i u \sigma^{ik}(s, x_s) \, dw^k_s + \int_0^{t \wedge \tau_R} \mathcal{L}u(s, x_s) \, ds \tag{3.10}$$

and the stochastic integral above is a square-integrable martingale.

Proof. By Corollary 5.6 of [8] we have $|Du|^2 \in E_{r/2, s/2, 2(\beta_0' - 1)}$, where $r = p\beta_0'/(\beta_0' - 1)$, $s = q\beta_0'/(\beta_0' - 1)$. Note that

$$2 > \beta_0' > 1, \quad \beta_0'/(\beta_0' - 1) > 2, \quad 2(\beta_0' - 1) < \beta_0, \quad r/2 \geq p, \quad s/2 \geq q.$$

This implies that the last statement of the lemma follows from (3.9).

To prove (3.10), as in the proof of Theorem 2.10 write

$$\begin{aligned} u^{(\varepsilon)}(t \wedge \tau_R, x_{t \wedge \tau_R}) &= u^{(\varepsilon)}(0) + \int_0^{t \wedge \tau_R} D_i u^{(\varepsilon)} \sigma^{ik}(s, x_s) \, dw^k_s \\ &+ \int_0^{t \wedge \tau_R} [\partial_t u^{(\varepsilon)} + a^{ij} D_{ij} u^{(\varepsilon)} + b^i D_i u^{(\varepsilon)}](s, x_s) \, ds. \end{aligned} \tag{3.11}$$

Since $2(\beta_0' - 1) < \beta_0$, by Lemma 3.5 we have $|Du - Du^{(\varepsilon)}|^2 I_{C_R} \to 0$ in E_{p,q,β_0} and the stochastic integral will converge in the mean square sense as $\varepsilon \downarrow 0$ to the one in (3.10) owing to (see (3.9))

$$\begin{aligned} &E \sup_t \Big| \int_0^{t \wedge \tau_R} D_i u^{(\varepsilon)} \sigma^{ik}(s, x_s) \, dw^k_s - \int_0^{t \wedge \tau_R} D_i u \sigma^{ik}(s, x_s) \, dw^k_s \Big|^2 \\ &\leq NE \int_0^{\tau_R} |Du^{(\varepsilon)} - Du|^2(s, x_s) \, ds \leq N \|(Du - Du^{(\varepsilon)})^2 I_{C_R}\|_{E_{p.q.\beta_0}}. \end{aligned}$$

Since u is bounded and continuous (Lemma 2.5 of [11]), we have the convergence of the terms without integrals. Regarding the integrals only the term with b needs to be addressed.

Notice that, thanks to $p = p_0/\beta_0, q = q_0/\beta_0$ and (3.3)

$$\frac{d}{p} + \frac{2}{q} \geq \beta_0 \quad (> 1).$$

Also for any $C \in \mathbb{C}$

$$\text{⫲}b\|_{L_{\beta_0 p, \beta_0 q}(C)} = \text{⫲}b\|_{L_{p_0, q_0}(C)}.$$

This along with (3.9) and Remark 5.8 of [8] imply that

$$\|I_C b|Du - Du^{(\varepsilon)}|\,\|_{E_{p,q,\beta_0}} \leq N\hat{b}\|I_C(u - u^{(\varepsilon)})\|_{E^{1,2}_{p,q,\beta_0}},$$

where N is independent of ε and, owing to the fact that $\beta_0 > \beta_0'$ and Lemma 3.5, the right-hand side tends to zero as $\varepsilon \downarrow 0$. This proves the lemma. ☐

We need the following fact which is a consequence of Theorem 4.4.

Theorem 3.7. *Under Assumptions 3.1 and 3.2 (a) there exists*

$$\lambda_0 = \lambda_0(d, \delta, p_0, q_0, \beta_0, \beta_0', \rho_a, \alpha) > 0$$

such that for $\gamma = \beta_0$ and $\gamma = \beta_0'$, $p = p_0/\gamma, q = q_0/\gamma$, for any $\lambda \geq \lambda_0$, Borel $c(t,x)$ such that $|c| \leq 1$, and $f \in E_{p,q,\gamma}$ there exists a unique solution $u \in E^{1,2}_{p,q,\gamma}$ of

$$\mathcal{L}u - (\lambda + c)u + f = 0. \tag{3.12}$$

Furthermore for any $u \in E^{1,2}_{p,q,\gamma}$ we have

$$\|\lambda u, \sqrt{\lambda} Du, D^2 u, \partial_t u\|_{E_{p,q,\gamma}} \leq N_0 \|\mathcal{L}u - (\lambda + c)u\|_{E_{p,q,\gamma}}, \tag{3.13}$$

where $N_0 = N_0(d, \delta, p_0, q_0, \beta_0, \beta_0', \rho_a, \alpha)$.

Actually, Theorem 4.4 treats only the case of $\gamma = \beta_0$. However, its assumptions are also satisfied if we replace β_0 with β_0'. Then its conclusion holds true with such a replacement as well, but for Theorem 3.7 to hold we need to take the largest of the $\check{\lambda}_0$'s and the N's corresponding to $\gamma = \beta_0$ and $\gamma = \beta_0'$ in Theorem 4.4.

In the following theorem we, in particular, specify the constant N in Assumption 3.2 (c).

Theorem 3.8 (Unconditional and conditional weak uniqueness)**.** *Under Assumptions 3.1 and 3.2 (a)*

(i) If Assumption 3.2 (c) is satisfied, then all solutions of (2.1) *with fixed* (t,x) *(provided they exist) have the same finite-dimensional distributions.*

(ii) Generally, let $y_\cdot$ and $z_\cdot$ be two solutions of (2.1) *with* $(t,x) = (0,0)$ *perhaps on different probability spaces. Assume that for $x_\cdot = y_\cdot$ and $x_\cdot = z_\cdot$ either (a) or (b) of Lemma 3.6 holds.*

Then $x_\cdot$ and $y_\cdot$ have the same finite-dimensional distributions.

Proof. First we prove (ii). Since by Lemma 3.6 (a) implies (b), we only need to show that (b) implies weak uniqueness.

Take bounded Borel c, f on $\mathbb{R}^{d+1}$ such that $0 \leq c \leq 1$. By Theorem 3.7 with $\lambda = \lambda_0$ there is a bounded function u *defined uniquely by* a, b, c, λ, f, such that $u \in E^{1,2}_{p_0/\beta_0', q/\beta_0', \beta_0'} \subset E^{1,2}_{p_0/\beta_0, q/\beta_0, \beta_0'}$ and (3.12) holds. In light of (b) by Itô's formula applied to

$$u(t, x_t) \exp\Big(- \lambda t - \int_0^t c(s, x_s)\, ds \Big)$$

for any finite T we obtain

$$u(0)=Eu(T\wedge\tau_R,x_{T\wedge\tau_R})\exp\Big(-\lambda(T\wedge\tau_R)-\int_0^{T\wedge\tau_R}c(s,x_s)\,ds\Big)$$
$$+E\int_0^{T\wedge\tau_R}f(t,x_t)\exp\Big(-\lambda t-\int_0^t c(s,x_s)\,ds\Big)\,dt,$$

where τ_R is the first exit time of (s,x_s) from C_R. We send here $T,R\to\infty$ taking into account that $\lambda+c\geq\lambda_0>0$, u,f are bounded and $\tau_R\to\infty$. Then we get that

$$E\int_0^\infty f(t,x_t)\exp\Big(-\lambda t-\int_0^t c(s,x_s)\,ds\Big)\,dt$$

is uniquely defined by a,b,c,λ,f (since it equals $u(0)$). For $T>0$ and $f=(\lambda+c)I_{t<T}$ this shows that

$$E\exp\Big(-\lambda T-\int_0^T c(s,x_s)\,ds\Big)$$

is uniquely defined by a,b,c,λ,T. The arbitrariness of c and T certainly proves assertion (ii).

To prove (i), we take any solution of (2.1), say with $(t,x)=(0,0)$. Set

$$\frac{1}{\gamma}:=\frac{d}{p_0}+\frac{1}{q_0}$$

and use Theorem 4.2 of [5] that greatly simplifies in our situation. With $(p_0,q_0)/\gamma$ in place of its (p_0,q_0), in its conditional form this theorem yields that, since for each $t\geq 0$, $\rho\leq 1$ and $C\in\mathbb{C}_\rho$ we have that τ_C, defined as the first exit time of $(t+s,x_{t+s})$, $s\geq 0$, from C, is less than ρ^2 and since

$$\|b\|^{2p_0/(p_0-\gamma d)}_{L_{p_0/\gamma,q_0/\gamma}(C)}\leq\big[N(d)(\hat b/\rho)\rho^{\gamma d/p_0+2\gamma/q_0}\big]^{2p_0/(p_0-\gamma d)}$$
$$=N(d,p_0,q_0)\hat b^{2p_0/(p_0-\gamma d)}\rho^2=N(d,p_0,q_0)\hat b^{2q_0/\gamma}\rho^2,$$

we have with constants N depending only on d,δ,p_0,q_0, that

$$E\Big\{\int_0^{\tau_C}|b(t+s,x_{t+s})|\,ds\mid\mathcal{F}_t\Big\}$$
$$\leq N\big(\rho^2+\|b\|^{2p_0/(p_0-\gamma d)}_{L_{p_0\gamma,q_0/\gamma}(C)}\big)^{\gamma d/(2p_0)}\|b\|_{L_{p_0/\gamma,q_0/\gamma}(C)}$$
$$\leq N\rho^{\gamma d/p_0}(1+\hat b^{2q_0/\gamma})^{\gamma d/(2p_0)}\hat b\rho^{\gamma d/p_0+2\gamma/q_0-1}\leq N(d,\delta,p_0,q_0)\hat b\rho,\tag{3.14}$$

where in the last inequality we used that $\hat b\leq 1$. Hence, the left-hand side is less than $m_b\rho$, provided that $N(d,\delta,p_0,q_0)\hat b\leq m_b$. In that case all results of [9] are available, in particular, estimate (2.13) (implying (3.9) with any β_0) holds for our solution whenever $d_0/p+1/q\leq 1$, that in our case holds with $p=p_0/\beta_0, q=q_0/\beta_0$ if we use $1/\beta_0:=d_0/p_0+1/q_0$ to define β_0, which is in $(1,2)$ owing to $d/p_0+1/q_0\leq 1$, $d/p_0+2/q_0\geq 1$, and $d_0\in(d/2,d)$. This proves the theorem. □

Remark 3.9. It is shown in [2] that assuming $b\in L_{p,q}$ with $d/p_0+1/q_0\leq 1$ alone does not guarantee weak uniqueness even with unit diffusion.

Next, we proceed to proving the existence of weak solutions. Assumptions 3.1, 3.2 (a) and (b), and 3.4 are supposed to hold throughout the rest of this section and we define p,q as in (3.8). We start by drawing consequences from Theorem 3.7.

Corollary 3.10. *Assume that a, b are smooth and bounded. Take $R \le 1$, smooth f, and let u be the classical solution of*
$$\mathcal{L}u + f = 0 \tag{3.15}$$
in C_R with zero boundary condition on $\partial' C_R$. Then
$$|u| \le N R^{2-\beta_0} \|I_{C_R} f\|_{E_{p,q,\beta_0}}, \tag{3.16}$$
where N depends only on $d, \delta, p_0, q_0, \beta_0, \rho_a, \alpha$.

Indeed, the case $R < 1$ is reduced to $R = 1$ by using parabolic dilations. If $R = 1$, the maximum principle allows us to concentrate on $f \ge 0$ and also shows that $u(t,x)e^{\lambda_0 t}$ is smaller in C_1 than the solution v of
$$\mathcal{L}v - \lambda_0 v + I_{C_1} f e^{\lambda_0 t} = 0$$
in $\mathbb{R}^{d+1}$. Since $\beta_0 < 2$ by embedding theorems we have on C_1
$$u \le v \le N\|v\|_{E^{1,2}_{p,q,\beta_0}} \le N\|I_{C_1} f\|_{E_{p,q,\beta_0}}.$$

Now we can specify the constant N_1 in the definition of $\hat{b}$ in Assumption 3.2 (b). Observe that so far the size of $\hat{b}$ played a role only in assertion (i) of Theorem 3.8.

Corollary 3.11. *Assume that a, b are smooth and bounded and let (t_s, x_s) be the corresponding Markov diffusion process. Then for any $(t,x) \in \mathbb{R}^{d+1}$, $\rho \le 1$, $C \in \mathbb{C}_\rho$, and Borel $f \ge 0$*
$$I(t,x) := E_{t,x} \int_0^{\tau_C} f(t, x_t)\, dt \le N \rho^{2-\beta_0} \|I_C f\|_{E_{p,q,\beta_0}}, \tag{3.17}$$
where τ_C is the first exit time of (t_s, x_s) from C. In particular,
$$E_{t,x} \int_0^{\tau_C} |b(t, x_t)|\, dt \le N_1 \rho \hat{b}, \tag{3.18}$$
and in both estimates N and N_1 depend only on $d, \delta, p_0, q_0, \beta_0, \rho_a, \alpha$.

Indeed, if f is smooth, by Itô's formula, I coincides with the solution of (3.15) in a shifted C_ρ and (3.17) follows from (3.16). For bounded Borel f we use the notation $f^{(\varepsilon)}$ from the proof of Theorem 2.3 and observe that $f^{(\varepsilon)} \to f$ almost everywhere, and the corresponding left-hand sides of (3.17) converge because they are expressed in terms of the Green's function of $\mathcal{L}$. As far as the right-hand sides are concerned, observe that by Minkowski's inequality $\|f^{(\varepsilon)}\|_{E_{p,q,\beta_0}} \le \|f\|_{E_{p,q,\beta_0}}$ and this yields (3.17) with $\|f\|_{E_{p,q,\beta_0}}$ in place of $\|fI_C\|_{E_{p,q,\beta_0}}$. Plugging fI_C in such relation in place of f leads to (3.17) as is. The passage to arbitrary $f \ge 0$ is achieved by taking $f \wedge n$ and letting $n \to \infty$.

To prove (3.18) observe that due to self-similar transformations we may assume that $\rho = 1$, in which case we use (3.17) and the fact that for $r \le 1$ and $C' \in \mathbb{C}_r$
$$r^{\beta_0} \nparallel I_C b\|_{L_{p,q}(C')} \le r^{\beta_0} \nparallel b\|_{L_{p_0,q_0}(C')} \le Nr \nparallel b\|_{L_{p_0,q_0}(C')} \le N\hat{b}.$$

Once N_1 is specified, we have the following.

Corollary 3.12. *Suppose that a, b are smooth and bounded and let (t_s, x_s) be the corresponding Markov diffusion process. Then $\mathrm{b}_1 \le m_b$ and all results from [9] are applicable.*

Corollary 3.13. *Suppose that a, b are smooth and bounded and let (t_s, x_s) be the corresponding Markov diffusion process. Then for any $(t,x) \in \mathbb{R}^{d+1}$, Borel $f \ge 0$, $T \in (0, \infty)$, there exists N depending only on d, δ, p_0, q_0, β_0, ρ_a, α, T, such that*
$$E_{t,x} \int_0^T f(t, x_t)\, dt \le N\|f\|_{E_{p,q,\beta_0}}. \tag{3.19}$$

The proof of this is almost identical to the proof of (3.17) when $\rho = 1$.

Now we abandon the assumption that a and b are smooth and come back to our assumptions (that are supposed to hold throughout the rest of the section) stated before Corollary 3.10. Here is a counterpart of Theorem 2.3.

Theorem 3.14. *(i) There is a probability space* $(\Omega, \mathcal{F}, P)$*, a filtration of* σ*-fields* $\mathcal{F}_s \subset \mathcal{F}$*,* $s \geq 0$*, a process* w_s*,* $s \geq 0$*, which is a* d*-dimensional Wiener process relative to* $\{\mathcal{F}_s\}$*, and an* $\mathcal{F}_s$*-adapted process* x_s *such that (a.s.) for all* $s \geq 0$ *equation* (2.1) *holds with* $(t, x) = (0, 0)$*.*

(ii) Furthermore, for any nonnegative Borel f *on* $\mathbb{R}^{d+1}$ *and* $T \in (0, \infty)$ *we have*

$$E \int_0^T f(s, x_s)\, ds \leq N \|f\|_{E_{p,q,\beta_0}}, \tag{3.20}$$

where N *is the constant from* (3.19)*.*

Proof. As in the proof of Theorem 2.3, approximate σ, b by smooth $\sigma^{(\varepsilon)}, b^{(\varepsilon)}$ and take the corresponding Markov processes $(\mathrm{t}_t, x_t^\varepsilon)$. We noted in Corollary 3.12 that all results of [9] are available for $(\mathrm{t}_t, x_t^\varepsilon)$. In particular, by Corollary 3.10 of [9] for any $\varepsilon, n > 0$ and $r > s \geq 0$ we have (2.6) where $N = N(n, d, \delta)$. This implies that the $P_{0,0}$-distributions of $x_\cdot^\varepsilon$ are precompact on $C([0, \infty), \mathbb{R}^d)$ and a subsequence $\varepsilon = \varepsilon_n \downarrow 0$ of them converges to the distribution of a process $x_\cdot = x_\cdot^0$ defined on a probability space (the coordinate process on $\Omega = C([0, \infty), \mathbb{R}^d)$ with cylindrical σ-field $\mathcal{F}$ completed with respect to P, which is the limiting distribution of $x_\cdot^\varepsilon$). Furthermore, by Theorem 5.1 of [9] for any nonnegative Borel f on $\mathbb{R}^{d+1}$ and $\varepsilon, T \in (0, \infty)$ we have

$$E_{0,0} \int_0^T f(s, x_s^\varepsilon)\, ds \leq N(d, \delta, T) \|f\|_{L_{d+1}(\mathbb{R}^{d+1})}, \tag{3.21}$$

which by continuity is extended to $\varepsilon = 0$ for bounded continuos f and then by the usual measure-theoretic argument for all Borel $f \geq 0$. After that estimate (3.21) also shows that for any bounded Borel f with compact support

$$\lim_{\varepsilon \downarrow 0} E_{0,0} \int_0^T f(s, x_s^\varepsilon)\, ds = E_{0,0} \int_0^T f(s, x_s^0)\, ds. \tag{3.22}$$

Furthermore, one can pass to the limit in (3.20) written for x_s^ε in place of x_s and see that it holds for x_s^0 if f is bounded and continuous. The extension of (3.20) to all Borel nonnegative f is standard and this proves assertion (ii).

Now we prove that assertions (i) holds for $x_\cdot$. Estimate (2.6) implies that for any finite T

$$\lim_{c \to \infty} P(\sup_{s \leq T} |x_s^0| > c) = 0,$$

and estimate (3.20) shows that for any finite c

$$E \int_0^T I_{|x_s^0| \leq c} |b(s, x_s^0)|\, dt < \infty.$$

Hence, with probability one

$$\int_0^T |b(s, x_s^0)|\, dt < \infty.$$

Next, for $0 \leq t_1 \leq ... \leq t_n \leq t \leq s$, bounded continuos $\phi(x(1), ..., x(n))$, and smooth bounded $u(t, x)$ with compact support by Itô's formula we have

$$E_{0,0} \phi(x_{t_1}^\varepsilon, ..., x_{t_n}^\varepsilon) \Big[u(s, x_s^\varepsilon) - u(t, x_t^\varepsilon) - \int_t^s \mathcal{L}^\varepsilon u(r, x_r^\varepsilon)\, dr \Big] = 0,$$

where

$$\mathcal{L}^{\varepsilon} u = \partial_t u + a^{\varepsilon ij} D_{ij} u + b^{\varepsilon i} D_i u, \quad a^{\varepsilon} = (1/2)(\sigma^{(\varepsilon)})^2.$$

Using (3.20), Lemma 3.5, and the fact that u has compact support show that

$$\lim_{\varepsilon_1 \downarrow 0} \lim_{\varepsilon \downarrow 0} E_{0,0} \int_t^s \left| b^{\varepsilon} - b^{(\varepsilon_1)} \right| (r, x_r^{\varepsilon}) |Du(r, x_r^{\varepsilon})| \, dr = 0,$$

$$\lim_{\varepsilon \downarrow 0} E_{0,0} \int_t^s b^{(\varepsilon_1) i}(r, x_r^{\varepsilon}) D_i u(r, x_r^{\varepsilon}) \, dr = E \int_t^s b^{(\varepsilon_1) i}(r, x_r^0) D_i u(r, x_r^0) \, dr,$$

$$\lim_{\varepsilon_1 \downarrow 0} E \int_t^s \left| b - b^{(\varepsilon_1)} \right| (r, x_r^0) |Du(r, x_r^0)| \, dr = 0.$$

After that we easily conclude that

$$E\phi(x_{t_1}^0, ..., x_{t_n}^0) \Big[u(s, x_s^0) - u(t, x_t^0) - \int_t^s \mathcal{L} u(r, x_r^0) \, dr \Big] = 0.$$

It follows that the process

$$u(s, x_s^0) - \int_0^s \mathcal{L} u(r, x_r^0) \, dr$$

is a martingale with respect to the completion of $\sigma\{x_t^0 : t \leq s\}$. Referring to a well-known result from Stochastic Analysis proves assertion (i). The theorem is proved. □

Remark 3.15. In [12] the weak uniqueness is proved in the class of solutions admitting, as they call it, Krylov type estimate when σ is *constant* and we have $p, q \in [1, \infty]$ such that

$$\frac{d}{p} + \frac{2}{q} = 1, \quad \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}^d} |b|^p \, dx \Big)^{q/p} dt \Big)^{1/q} < \infty \tag{3.23}$$

(the Ladyzhenskaya-Prodi-Serrin condition).

Actually, $p = \infty$, $q = 2$ is not allowed in [12], this case fits in [10] where weak existence and conditional weak uniqueness is obtained. In case $p = d, q = \infty$ the comparison of the results in [10] and [12] can be found in [10].

If $p \in [d+1, \infty)$ and we use the norms in (3.1), set $p_0 = p$ and $q_0 := q/2$ ($p \geq q_0$). Then for any $\rho > 0$, and $C \in \mathbb{C}_{\rho}$, by Hölder's inequality we have

$$\|b\|_{L_{p_0,q_0}(C)} \leq \rho^{2/q} \|b\|_{L_{p,q}(C)}, \quad \nparallel b\|_{L_{p_0,q_0}(C)} \leq N(d) \rho^{-1} \|b\|_{L_{p,q}(C)} \tag{3.24}$$

and the last norm tends to zero as $\rho \downarrow 0$. In that case also $d/p_0 + 1/q_0 = 1$. This shows that Assumption 3.2 (c) is satisfied on account of choosing ρ_b small enough.

By Theorem 3.8 (i) we get unconditional weak uniqueness of weak solutions that exist by Theorem 3.14 if, say σ is constant (as in [12] and [3]).

In case $d < p < d+1$ ($q > 2p$), set $p_0 = q_0 = p$. Then by Hölder's inequality we again get the second relation in (3.24), but this time our result is the same (if σ is constant) as in [12] and [3]: weak existence and conditional weak uniqueness.

Interestingly enough, in case $p < d+1$ the estimates (3.24) are still valid and show that Assumption 3.2 (a) is satisfied for an appropriate ρ_b, provided that the inequality in (3.23) is replaced with

$$\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}} |b|^q \, dt \Big)^{p/q} dx < \infty. \tag{3.25}$$

At the same time equations with b satisfying (3.25) are not covered in the literature so far (apart from the author's works) and we present in Remark 3.16 an example related to (3.25).

Remark 3.16. There are examples showing that the assumption of Theorem 3.8 (i) concerning b is satisfied with the norm in $L_{p,q}$ understood as in (3.25) but not as in (3.23) and (3.23) does not hold no matter what p, q are, so that these examples are not covered by the results of [12] or [10]. For instance, take $b(t,x)$ such that $|b| = cf$, where the constant $c > 0$ and

$$f(t,x) = I_{1>t>0,|x|<1}|x|^{-1}\Big(\frac{|x|}{\sqrt{t}}\Big)^{1/(d+1)}.$$

If $p = \infty$, the second condition in (3.23), obviously, is not satisfied. If $p = d, q = \infty$, so that the first condition in (3.23) is satisfied, then

$$\int_{|x|\le 1} f^d(t,x)\,dx = \int_{|x|\le 1/\sqrt{t}} |x|^{-d+d/(d+1)}\,dx \to \infty$$

as $t \downarrow 0$. Thus, the second condition in (3.23) is not satisfied in this case. If $p \in (d,\infty)$ and $t > 0$, $r < 1$, then

$$\int_{|x|\le r} f^p(t,x)\,dx = Nt^{(d-p)/2}\int_0^{r/\sqrt{t}} \rho^{d-1-p+p/(d+1)}\,d\rho =: t^{(d-p)/2} I(r/\sqrt{t}).$$

In order for that integral to converge, we need

$$p < d+1 \quad (q > 2(d+1)), \tag{3.26}$$

and in this case $I(\rho) \sim \rho^{d-p+p/(d+1)}$ as $\rho \to \infty$. Next,

$$\int_0^{r^2}\Big(\int_{|x|\le r} f^p(t,x)\,dx\Big)^{q/p} dt = \int_0^{r^2} t^{(d-p)q/(2p)} I^{q/p}(r/\sqrt{t})\,dt$$
$$= 2r^{2+(d-p)q/p}\int_1^\infty \rho^{-3-(d-p)q/p} I^{q/p}(\rho)\,d\rho.$$

Here the integrand has order of $\rho^{-3+q/(d+1)}$ and $-3+q/(d+1) > -1$ by virtue of (3.26) and (3.23). Therefore, the last integral diverges and condition (3.23) indeed fails to hold.

In contrast to this, although, if norms are understood as in (3.1), the assumption of part (i) of Theorem 3.8 concerning b, obviously, are not satisfied if $d/p_0 + 1/q_0 = 1$ and $p_0 \ge q_0$ (because then $p_0 \ge d+1$), it turns out that they are satisfied with norms from (3.2) for some $q_0 > p_0$ ($d/p_0 + 1/q_0 = 1, q_0 > d+1$) if c is small enough.

Indeed, take $d+1 < q_0 < 2(d+1)$ ($p_0 < d+1$) and note that for $r \le 3$

$$\int_0^{r^2} f^{q_0}(t,x)\,dt = I_{|x|<1}|x|^{-q_0}\int_0^{1\wedge r^2}\Big(\frac{|x|}{\sqrt{t}}\Big)^{q_0/(d+1)} dt$$
$$\le |x|^{2-q_0}\int_0^{r^2/|x|^2} t^{-q_0/(2d+2)}\,dt =: |x|^{2-q_0} J(r^2/|x|^2),$$

where $J(s) \sim s^{1-q_0/(2d+2)}$ as $s \to \infty$. Next,

$$\int_{|x|<r}\Big(\int_0^{r^2} f^{q_0}(t,x)\,dt\Big)^{p_0/q_0} dx$$
$$= N\int_0^r \rho^{d-1+(2-q_0)p_0/q_0} J^{p_0/q_0}(r^2/\rho^2)\,d\rho$$
$$= Nr^{d+(2-q_0)p_0/q_0}\int_0^1 s^{d-1+(2-q_0)p_0/q_0} J^{p_0/q_0}(s^{-2})\,ds.$$

Here in the integral with respect to s

$$d+(2-q_0)p_0/q_0-2\big(1-q_0/(2d+2)\big)p_0/q_0=d-p_0d/(d+1),$$

which is strictly greater than zero and the above integral with respect to ds is finite implying that

$$⫲ f\|_{L_{p_0,q_0}(C_r)}\leq Nr^{-1}. \tag{3.27}$$

If $r\leq 1$, $t\in(-r^2,r^2)$ and $|x|\leq 2r$, then

$$⫲ f\|_{L_{p_0,q_0}(C_r(t,x))}\leq ⫲ f\|_{L_{p_0,q_0}(C_{3r})}\leq Nr^{-1}.$$

In case $r\leq 1$, $t\notin(-r^2,r^2)$ or $|x|>2r$ the left-hand side above is zero. It follows that for small c our b satisfies Assumption 3.2 (a) with $\rho_b=1$. Also $d/p_0+1/q_0=1$. Therefore, due to Theorem 2.3, (2.1) has a solution starting from any point, say if σ is constant, and all solutions starting from the same point have the same finite-dimensional distributions by Theorem 3.8 (i).

On the other hand, the results of [3] still guarantee that there is weak existence and *conditional* weak uniqueness in this example if σ is constant. Recall that our σ is not necessarily constant or even continuous.

By changing the origin we can apply Theorem 3.8 to prove the solvability of (2.1) with any initial data (t,x) and get solutions with the properties as in Theorems 3.14 (ii) weakly unique by Theorem 3.8. For such a solution denote by $P_{t,x}$ the distribution of $(\mathsf{t}_s,x_s), s\geq 0$, ($\mathsf{t}_s=t+s$) on the Borel σ-field $\mathcal{F}$ of $\Omega=C([0,\infty),\mathbb{R}^{d+1})$. For $\omega=(\mathsf{t}_\cdot,x_\cdot)\in\Omega$ set $(\mathsf{t}_s,x_s)(\omega)=(\mathsf{t}_s,x_s)$. Also set $\mathfrak{N}_s=\sigma\{(\mathsf{t}_t,x_t),t\leq s\}$.

Theorem 3.17. *The process*

$$X=\{(\mathsf{t}_\cdot,x_\cdot),\infty,\mathfrak{N}_t,P_{t,x}\}\quad(\infty\textit{ is the life time})$$

is strong Markov with strong Feller resolvent for which (2.3) *holds true.*

Proof. Take u from Theorem 3.7 with $\gamma=\beta_0'$, $c=0,\lambda\geq\lambda_0$ and Borel bounded f. By Itô's formula for any (t,x) and $0\leq r\leq s$ we obtain that with $P_{t,x}$-probability one

$$\begin{aligned}u(\mathsf{t}_s,x_s)e^{-\lambda(s\wedge\tau_R)}=u(\mathsf{t}_r,x_r)e^{-\lambda(r\wedge\tau_R)}&+\int_{r\wedge\tau_R}^{s\wedge\tau_R}e^{-\lambda v}\sigma^{ik}D_iu(\mathsf{t}_v,x_v)\,dw_v^k\\&-\int_{r\wedge\tau_R}^{s\wedge\tau_R}e^{-\lambda v}f(\mathsf{t}_v,x_v)\,dv,\end{aligned}\tag{3.28}$$

where τ_R is the first exit time of (t_v,x_v) from C_R

From (3.28) with $r=0$ as in the proof of Theorem 3.8 we obtain

$$E_{t,x}\int_0^\infty e^{-\lambda v}f(\mathsf{t}_v,x_v)\,dv=u(t,x). \tag{3.29}$$

If f is continuous, this implies that the Laplace transform of the continuous in v function $E_{t,x}f(\mathsf{t}_v,x_v)$ is a Borel function of (t,x). Then the function $E_{t,x}f(\mathsf{t}_v,x_v)$ itself is a Borel function of (t,x). Since it is continuous in v, it is Borel with respect to all its arguments. This fact is obtained for bounded continuous f, but by usual measure-theoretic arguments carries it over to all Borel bounded f.

Then take $0\leq r_1\leq...\leq r_m=r$ and continuous f and a bounded Borel function $\zeta\big(x(1),...,x(m)\big)$ on $\mathbb{R}^{md}$ and conclude from (3.28) that

$$\begin{aligned}&E_{t,x}\zeta(x_{r_1},...,x_{r_m})u(\mathsf{t}_r,x_r)e^{-\lambda r}\\&=E_{t,x}\zeta(x_{r_1},...,x_{r_m})\int_r^\infty e^{-\lambda v}f(\mathsf{t}_v,x_v)\,dv.\end{aligned}$$

In light of (3.29) this means that

$$\int_r^\infty E_{t,x}\zeta(x_{r_1},...,x_{r_m})e^{-\lambda v}E_{\mathsf{t}_r,x_r}f(\mathsf{t}_{v-r},x_{v-r})\,dv$$
$$=\int_r^\infty E_{t,x}\zeta(x_{r_1},...,x_{r_m})e^{-\lambda v}f(\mathsf{t}_v,x_v)\,dv.$$

We have the equality of two Laplace's transforms of functions continuous in v. It follows that for $v\geq r$

$$E_{t,x}\zeta(x_{r_1},...,x_{r_m})E_{\mathsf{t}_r,x_r}f(\mathsf{t}_{v-r},x_{v-r})=E_{t,x}\zeta(x_{r_1},...,x_{r_m})f(\mathsf{t}_v,x_v).$$

Again a measure-theoretic argument shows that this equality holds for any Borel bounded f and then the arbitrariness of ζ yields the Markov property of X.

To prove that it is strong Markov it suffices to observe that, owing to (3.29) its resolvent R_λ is strong Feller, that is maps bounded Borel functions into bounded continuous ones.

To deal with (2.3), take, for instance, $(t,x)=(0,0)$ and approximate our (conditionally weakly unique) solution as in the proof of Theorem 2.3 by $x^{\varepsilon}_{\cdot}$. for $R\in(0,\infty),y\in\mathbb{R}^d$, introduce the functional $\gamma_{y,R}(x_\cdot)$ on $C([0,\infty),\mathbb{R}^d)$ as the first exit time of (s,x_s) from $C_R(0,y)$. As is easy to see, $\gamma_{y,R}(x_\cdot)$ is lower semi-continuous. It follows that the same is true for

$$\int_0^{\gamma_{y,R}(x_\cdot)}f(r,x_r)\,dt,$$

as long as a bounded continuous $f(t,x)\geq 0$. It follows that

$$\varliminf_{n\to\infty}E_{0,0}\int_0^{\gamma_{y,R}(x^{\varepsilon_m}_\cdot)}f(r,x^{\varepsilon_m}_r)\,dt\geq E_{0,0}\int_0^{\gamma_{y,R}(x^0_\cdot)}f(r,x^0_r)\,dt. \tag{3.30}$$

In light of (3.20), inequality (3.30) holds for $f=|b|$. If $f=|b|$ and $R\leq\rho_b$, as we have said in the proofs of Theorem 3.8 (i) and as it follows from (3.18), the left-hand side of (3.30) is smaller that m_bR. But then

$$E_{0,0}\int_0^{\tau_R(y)}|b(s,x^0_s)|\,ds\leq m_bR,$$

and this with the possibility to change the origin leads to (2.3). The theorem is proved. □

4 A result from [11]

The content of this section is independent of Sections 2 and 3, however, we borrow some notation from Section 3.

We have p,q,β_0 such that

$$p,q,\beta_0\in(1,\infty),\quad \beta_0\neq 2,\quad \frac{d}{p}+\frac{2}{q}\geq\beta_0. \tag{4.1}$$

Fix some $\rho_a\in(0,\infty)$. Parameters θ and $\check{b}$ below will be specified later.

Assumption 4.1. *We have*

$$a^\sharp_{x,\rho_a}=\sup_{\substack{\rho\leq\rho_a\\ C\in\mathbb{C}_\rho}}\fint_C|a(t,x)-a_C(t)|\,dxdt\leq\theta, \tag{4.2}$$

where

$$a_C(t)=\fint_C a(t,x)\,dxds\quad(\textit{note}\quad t\quad\textit{and}\quad ds).$$

Assumption 4.2. *We have*

$$\mathfrak{b}_1 := \sup_{r\leq 1} r \sup_{C\in\mathbb{C}_r} \text{⫲} b\|_{L_{p\beta_0,q\beta_0}(C)} \leq \check{b}. \tag{4.3}$$

Let us specify θ in (4.2). It is easy to choose $\theta_1(d,\delta,p,q)$ introduced in Lemma 4.5 of [11], so that it is a decreasing function of d, and we suppose it is done. In the following $\alpha \in (0,1)$ is a free parameter.

Assumption 4.3. *For r defined as the least number such that*

$$r \geq (d+2)/\alpha, \quad r \geq p, q$$

and $\Theta(\alpha) = \{(p', q') : p \leq p' \leq r, q \leq q' \leq r\}$ *Assumption 3.1 is satisfied with*

$$\theta = \inf_{\Theta(\alpha)} \theta_1(d+1, \delta, p', q') =: \theta(d, \delta, p, q, \alpha).$$

The fact that this $\theta > 0$ is noted in [11].

Here is the main result of [11] adjusted to our needs. The constant $\nu = \nu(d, \beta_0, p, q)$ below is taken from Remark 2.2 of [11] when $\rho_b = 1$.

Theorem 4.4. *Under the above assumptions there exist*

$$\check{b} = \check{b}(d, \delta, p, q, \rho_a, \beta_0, \alpha) \in (0, 1], \quad \check{\lambda}_0 = \check{\lambda}_0(d, \delta, p, q, \rho_a, \beta_0, \alpha) > 0$$

such that, if (4.3) *holds with this $\check{b}$, then for any $\lambda \geq \check{\lambda}_0$, function $c(t,x)$ such that $|c| \leq 1$ and $f \in E_{p,q,\beta_0}$ there exists a unique $E^{1,2}_{p,q,\beta_0}$-solution u of $\mathcal{L}u - (c+\lambda)u = f$. Furthermore, there exists a constant N depending only on d, δ, p, q, ρ_a, β_0, α, such that*

$$\|\partial_t u, D^2 u, \sqrt{\lambda} Du, \lambda u\|_{E_{p,q,\beta_0}} \leq N \nu^{-1} \|f\|_{E_{p,q,\beta_0}}. \tag{4.4}$$

Remark 4.5. The unique solution u from Theorem 4.4 possesses the following properties

a) obviously, $u \in W^{1,2}_{p,q,\mathrm{loc}}$;

b) by Lemma 2.6 of [11], we have $Du \in L_{r,s,\mathrm{loc}}$, where $(r,s) = (\beta_0 - 1)^{-1}\beta_0(p,q)$;

c) for $\beta_0 < 2$ we have that u is bounded and continuous according to Lemma 2.5 of [11].

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